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On the Euler and Lagrange's points of view in rigid body mechanics

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Abstract In this work we take Lagrange's and Euler's important concepts from fluid mechanics and apply them to the movement of a rigid body. By means of two examples, namely motion around a fixed axis and around a fixed point, Lagrangian and Eulerian formulations of the problems are discussed. It is shown that Euler's approach suits better the description of rigid body kinematics, since the linearized equations of motion are simpler than the ones obtained by Lagrange's formulation. This topic is rarely discussed in undergraduate courses on mechanics but it can provide students with a deeper comprehension of the movement of a rigid body and, at the same time, establish a connection with the scope of fluid mechanics.

Keywords Rigid body velocity; Euler's viewpoint; Lagrange's viewpoint

Notation

γ	a generic scalar or vector property of a fluid particle
$P_0(X_0, Y_0, Z_0)$	start position of a point of the rigid body
$P(X, Y, Z)$	end position of a point of the rigid body
$[P]$	a point of the fixed frame
$[p]$	a point of the rigid body
$[R_{\varphi,0,0}]$, $[R_{0,\psi,0}]$, $[R_{0,0,\theta}]$	rotation matrices
$\vec{v}_p$	velocity of the material point p
$\vec{v}_p$	velocity of all rigid body points that pass through P
$[\omega]$	angular velocity matrix
$\vec{\Omega}$	angular velocity vector
s	trajectory arc length
$\mathcal{V}$	control volume

Introduction

Lagrangian and Eulerian viewpoints are concepts commonly covered in fluid mechanics undergraduate courses. Students are taught to describe the physical properties of a flow either by tracking a fluid particle during its motion or by measuring the average properties of a flow of particles that cross the surface of a small control volume.

Given an inertial reference system $OXYZ$, two fixed points $P_0(X_0, Y_0, Z_0)$ and $P(X, Y, Z)$, the scalar or vectorial property γ of the fluid particles is described, according to Lagrangian viewpoint, as a function of the kind

$$\gamma = \gamma(P(P_0, t), t) \tag{1}$$

where $P(P_0, t)$ represents the position of a particle that at instant $t = 0$ was in $P_0(X_0, Y_0, Z_0)$. So, γ is seen, in this case, as a function intrinsically associated with the material point.

In the Eulerian approach, γ is represented by

$$\gamma = \gamma(P, t) \tag{2}$$

and, in this case, it is intrinsically associated with the geometrical fixed position $P(X, Y, Z)$.

The advantages of Euler's viewpoint over Lagrange's in solving a great variety of fluid mechanics problems have been emphasized in the literature. As implicit in equation 2, it is not mandatory that fluid particles keep their material identity during the motion; so, the occurrence of turbulence does not prevent the application of this approach. Moreover, Euler's mathematical model facilitates the linearization of the fluid dynamics equations of motion.

Although comparisons of Euler's and Lagrange's viewpoints are made in fluid mechanics books [1], in the literature concerning rigid body mechanics the subject is seldom approached. Except for one brief attempt [2], the authors ignore those concepts in their explanations of rigid body kinematics. Generally, the equation for the velocity field of a rigid body is presented using algebraic reasoning [3, 4], instead of focusing on the physical phenomenon. However, the lack of information concerning the motion of a moving space (i.e., the body) 'flowing' through a fixed space (i.e., the inertial reference system) hides important physical aspects of the rigid body velocity equation and, as a consequence, prevents students from having a deeper comprehension of its kinematics.

In this paper, we intend to show the application of the above viewpoints in the construction of the rigid body velocity equation, and to demonstrate that the Eulerian approach leads to a natural linearization of the equations of motion; it is thus fitted to a more elementary-level course on rigid body mechanics. In order to do this, we analyze two cases of rigid body motions that are usually discussed in undergraduate courses – otation about a fixed axis and rotation about a fixed point.

Rigid body rotating about a fixed axis

Let OXY be the coordinate system of a plane inertial reference system S, and Oxy be the coordinate system linked to a rigid body E undergoing plane motion around a fixed axis OZ (Fig. 1).

Hence, consider the motion of the material point $p(x, y)$ of E that departs from the fixed position $P_0(X_0, Y_0)$ of S at instant 0 and arrives at the fixed point $P(X, Y)$ of S at instant t. Coincidence of points p of E and P of S can be achieved if the following relationship between their coordinates, respectively in systems Oxy and OXY, holds:

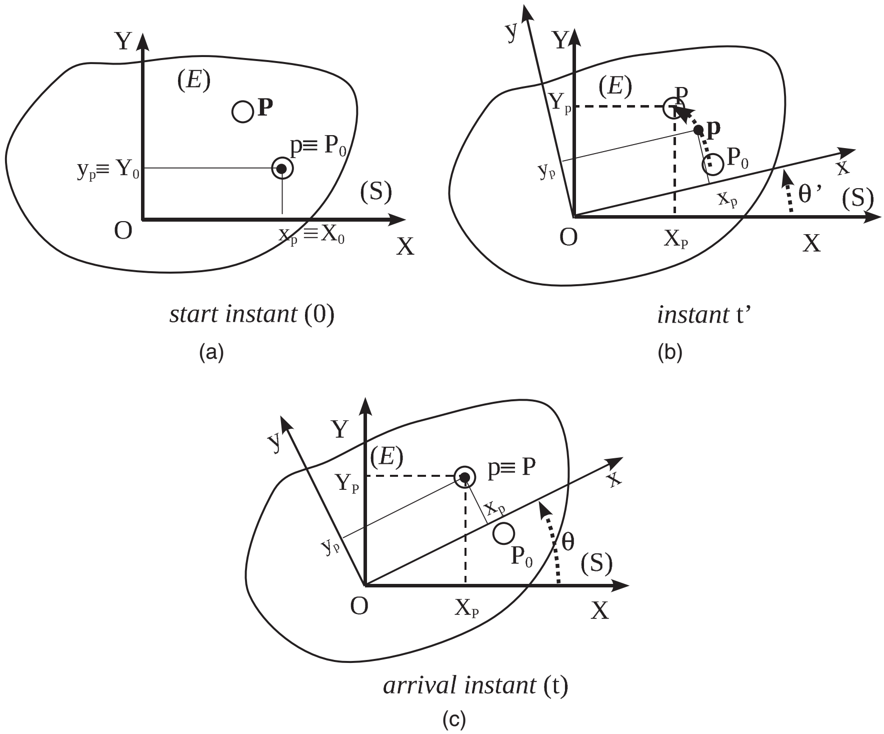

Fig. 1(a,b,c) *Moving (E) and fixed (S) spaces for a body rotating about a fixed axis.*

$$[P]=[R_{0,0,\theta}]\cdot[p] \tag{3}$$

where

$$[R_{0,0,\theta}]=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \tag{4}$$

is the geometrical transformation that moves p to P after time t.

Therefore, adopting Lagrange’s point of view, the velocity of the material point p of E ($\vec{v}_p$) relative to the inertial system S at this instant is given by:

$$\vec{v}_p(t|p\equiv P)=\left[\dot{R}_{0,0,\theta}\right]\cdot[p]+[R_{0,0,\theta}]\cdot[\dot{p}]=\left[\dot{R}_{0,0,\theta}\right]\cdot[p] \tag{5}$$

since $p(x, y)$, as a point fixed to E, does not change its position relative to this reference system during the motion.

Developing equation 5 we finally obtain:

$$\vec{v}_p(t|p\equiv P)=\begin{bmatrix}-\sin\theta\cdot\dot{\theta} & -\cos\theta\cdot\dot{\theta}\\ \cos\theta\cdot\dot{\theta} & -\sin\theta\cdot\dot{\theta}\end{bmatrix}\cdot\begin{bmatrix}x\\ y\end{bmatrix}=\dot{\theta}\cdot\begin{bmatrix}-\sin\theta & -\cos\theta\\ \cos\theta & -\sin\theta\end{bmatrix}\cdot\begin{bmatrix}x\\ y\end{bmatrix} \tag{6}$$

This is the velocity of the material point p of E according to Lagrange’s viewpoint.

It is important to stress that, in the above expression, θ is the necessary *finite* rotation that causes the coincidence of the material point $p(x, y)$ with the fixed point $P(X, Y)$ after time t, considering that its start position coincided with $P_0(X_0, Y_0)$ at departure time, as indicated by equation 1.

Since the length, s, of the trajectory from $P_0(X_0, Y_0)$ to $P(X, Y)$ depends on θ,

$$s = \sqrt{x_p^2 + y_p^2} \cdot \theta \tag{7}$$

the velocity of the material point p becomes a function of the finite angle θ.

On the other hand, in order to describe the velocities of every point p of E that coincides with the fixed point P of S at instant t, it is necessary, first of all, to apply the inverse transformation $[R_{0,0,\theta}]^{-1}$ to P, in such a way that

$$[p] = [R_{0,0,\theta}]^{-1} \cdot [P] \tag{8}$$

represents the position, described in the Oxy system, of any material point p of E coincident with the fixed point P of S at instant t.

As $[R_{0,0,\theta}]$ is an orthogonal matrix, the following relationship holds:

$$[R_{0,0,\theta}]^{-1} = [R_{0,0,\theta}]^T \tag{9}$$

Consequently, the velocity of all material points of E that pass through P, henceforth designated as $\vec{v}_p$, is obtained by substituting equations 8 and 9 into equation 5, to give:

$$\vec{v}_P = \left[\dot{P}\right] = \left[\dot{R}_{0,0,\theta}\right] \cdot [p] = \left[\dot{R}_{0,0,\theta}\right] \cdot [R_{0,0,\theta}]^T \cdot [P] \tag{10}$$

Developing the above expression, we finally obtain:

$$\vec{v}_P = \dot{\theta} \cdot \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} = \dot{\theta} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} \tag{11}$$

Noticing that $\dot{\theta}$ corresponds to the angular velocity ω_z of E relative to the inertial reference system S and that the product of the skew matrix

$$[\omega] = \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \tag{12}$$

and the position vector $\vec{P} = (P - O)$ can be written as a cross-vector product,

$$\begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} = \vec{\omega} \wedge (P - O) \tag{13}$$

we finally obtain

$$\vec{v}_P = \vec{\omega} \wedge (P - O) \tag{14}$$

This, according to Euler's point of view, represents the velocity of points of E that pass through point P fixed at the inertial reference system S at instant t.

By comparing equations 6 and 14, one can see that the Eulerian approach naturally gives rise to a linear representation of the velocity of a point of a rigid body, whereas linearization of Lagrange's formulation (equation 6) requires the assumption that arc length s between points $P_0(X_0, Y_0)$ and $P(X, Y)$ is sufficiently small to make $\sin\theta \approx 0$ and $\cos\theta \approx 1$, so that equations 6 and 14 are alike.

Rigid body rotating about a fixed point

Let now $OXYZ$ be the coordinate system of an inertial reference system S and $Oxyz$ the coordinate system linked to a rigid body E undergoing rotation about a fixed point O (Fig. 2).

To describe the position of a point, $p(x, y, z)$, of E that coincided with fixed point $P(X_0, Y_0, Z_0)$ at start time, and at time t occupies the position of fixed point $P(X, Y, Z)$ of S, we apply the following geometrical transformations:

$$[P] = [R]\cdot[p] = [R_{0,0,\theta}]\cdot[R_{0,\psi,0}]\cdot[R_{\varphi,0,0}]\cdot[p] \tag{15}$$

where $[R_{\varphi,0,0}]$, $[R_{0,\psi,0}]$, $[R_{0,0,\theta}]$ represent, respectively, the rotation matrices about axes X, y_1 and z_3 (see Fig. 2), φ, ψ and θ are the respective rotation angles and R is the resultant rotation matrix.

These rotation matrices are given by:

$$R_{\varphi,0,0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi & c\varphi \end{bmatrix}, R_{0,\psi,0} = \begin{bmatrix} c\psi & 0 & s\psi \\ 0 & 1 & 0 \\ -s\psi & 0 & c\psi \end{bmatrix}, R_{0,0,\theta} = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16a–c}$$

where c and s stand respectively for $cos(.)$ and $sin(.)$.

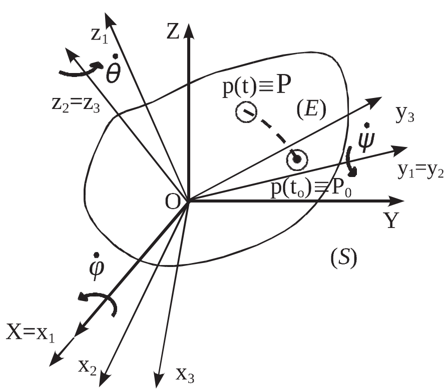

Fig. 2 *Moving (E) and fixed (S) spaces for a body rotating about a fixed point.*

Consequently, the resultant rotation matrix is:

$$R=\begin{bmatrix}1 & 0 & 0\\ 0 & c\varphi & -s\varphi\\ 0 & s\varphi & c\varphi\end{bmatrix}\cdot\begin{bmatrix}c\psi & 0 & s\psi\\ 0 & 1 & 0\\ -s\psi & 0 & c\psi\end{bmatrix}\cdot\begin{bmatrix}c\theta & -s\theta & 0\\ s\theta & c\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}c\psi\cdot c\theta & -c\psi\cdot s\theta & s\psi\\ s\varphi\cdot s\psi\cdot c\theta+c\varphi\cdot s\theta & -s\varphi\cdot s\psi\cdot s\theta+c\varphi\cdot c\theta & -s\varphi\cdot c\psi\\ -c\varphi.\cdot s\psi\cdot c\theta+s\varphi\cdot s\theta & c\varphi\cdot .s\psi\cdot s\theta+s\varphi\cdot c\theta & c\varphi\cdot c\psi\end{bmatrix} \quad (17)$$

Adopting Lagrange's point of view, the velocity of the material point *p*, when it coincides with *P*, is given by:

$$\vec{v}_p(t|p\equiv P)=\left[\dot{R}\right]\cdot[p]+[R]\cdot[\dot{p}]=\left[\dot{R}\right]\cdot[p] \quad (18)$$

Differentiating expression 17 with respect to time and in relation to the inertial reference frame *S*, we obtain:

$$\dot{R}=\left[\begin{matrix}-\dot{\psi}\cdot s\psi\cdot c\theta-\dot{\theta}\cdot c\psi\cdot s\theta\\ \dot{\varphi}\cdot c\varphi\cdot s\psi\cdot c\theta+\dot{\psi}\cdot s\varphi\cdot c\psi\cdot c\theta-\dot{\theta}\cdot s\varphi\cdot c\psi\cdot c\theta-\dot{\varphi}\cdot s\varphi\cdot s\theta+\dot{\theta}\cdot c\varphi\cdot c\theta\\ \dot{\varphi}\cdot s\varphi\cdot s\psi\cdot c\theta-\dot{\psi}\cdot c\varphi\cdot c\psi\cdot c\theta+\dot{\theta}\cdot c\varphi\cdot s\psi\cdot s\theta+\dot{\varphi}\cdot c\varphi\cdot s\theta+\dot{\theta}\cdot s\varphi\cdot c\theta\end{matrix}\right.$$
$$\begin{matrix}\dot{\psi}\cdot s\psi\cdot s\theta-\dot{\theta}\cdot c\psi\cdot c\theta\\ -\dot{\varphi}\cdot c\varphi\cdot s\psi\cdot s\theta-\dot{\psi}\cdot s\varphi\cdot c\psi\cdot s\theta-\dot{\theta}\cdot s\varphi\cdot s\psi\cdot c\theta-\dot{\varphi}\cdot s\varphi\cdot c\theta-\dot{\theta}\cdot c\varphi\cdot s\theta\\ -\dot{\varphi}\cdot s\varphi\cdot s\psi\cdot s\theta+\dot{\psi}\cdot c\varphi\cdot c\psi\cdot s\theta+\dot{\theta}\cdot c\varphi\cdot s\psi\cdot c\theta+\dot{\varphi}\cdot c\varphi\cdot c\theta-\dot{\theta}\cdot s\varphi\cdot s\theta\end{matrix} \quad (19)$$
$$\left.\begin{matrix}\dot{\psi}\cdot c\psi\\ -\dot{\varphi}\cdot c\varphi\cdot c\psi+\dot{\psi}\cdot s\varphi\cdot s\psi\\ -\dot{\varphi}\cdot s\varphi\cdot c\psi-\dot{\psi}\cdot c\varphi\cdot s\psi\end{matrix}\right]$$

Substituting expression 19 into 17 we finally obtain the velocity of the material point *p* of *E*. That is, according to the Lagrange's point of view:

$$\vec{v}_p=\left[\begin{matrix}-\dot{\psi}\cdot s\psi\cdot c\theta-\dot{\theta}\cdot c\psi\cdot s\theta\\ \dot{\varphi}\cdot c\varphi\cdot s\psi\cdot c\theta+\dot{\psi}\cdot s\varphi\cdot c\psi\cdot c\theta-\dot{\theta}\cdot s\varphi\cdot c\psi\cdot c\theta-\dot{\varphi}\cdot s\varphi\cdot s\theta+\dot{\theta}\cdot c\varphi\cdot c\theta\\ \dot{\varphi}\cdot s\varphi\cdot s\psi\cdot c\theta-\dot{\psi}\cdot c\varphi\cdot c\psi\cdot c\theta+\dot{\theta}\cdot c\varphi\cdot s\psi\cdot s\theta+\dot{\varphi}\cdot c\varphi\cdot s\theta+\dot{\theta}\cdot s\varphi\cdot c\theta\end{matrix}\right.$$
$$\begin{matrix}\dot{\psi}\cdot s\psi\cdot s\theta-\dot{\theta}\cdot c\psi\cdot c\theta\\ -\dot{\varphi}\cdot c\varphi\cdot s\psi\cdot s\theta-\dot{\psi}\cdot s\varphi\cdot c\psi\cdot s\theta-\dot{\theta}\cdot s\varphi\cdot s\psi\cdot c\theta-\dot{\varphi}\cdot s\varphi\cdot c\theta-\dot{\theta}\cdot c\varphi\cdot s\theta\\ -\dot{\varphi}\cdot s\varphi\cdot s\psi\cdot s\theta+\dot{\psi}\cdot c\varphi\cdot c\psi\cdot s\theta+\dot{\theta}\cdot c\varphi\cdot s\psi\cdot c\theta+\dot{\varphi}\cdot c\varphi\cdot c\theta-\dot{\theta}\cdot s\varphi\cdot s\theta\end{matrix} \quad (20)$$
$$\left.\begin{matrix}\dot{\psi}\cdot c\psi\\ -\dot{\varphi}\cdot c\varphi\cdot c\psi+\dot{\psi}\cdot s\varphi\cdot s\psi\\ -\dot{\varphi}\cdot s\varphi\cdot c\psi-\dot{\psi}\cdot c\varphi\cdot s\psi\end{matrix}\right]\cdot\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

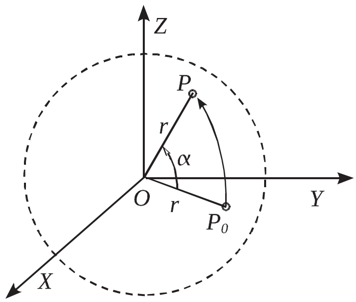

Fig. 3 *Circular trajectory between points P_0 and P.*

We must stress that, in equation 20, φ, ψ and θ are the finite rotations strictly necessary to move material point $p(x, y, z)$ from $P_0(X_0, Y_0, Z_0)$ at instant 0 to position $P(X, Y, Z)$ at instant t. As shown in Fig. 3, the trajectory arc length s, between points $P_0(X_0, Y_0, Z_0)$ and $P(X, Y, Z)$, is given by:

$$s = r \cdot \alpha = |(P-O)| \cdot 2 \cdot \sin^{-1}\left(\frac{1}{2} \cdot \frac{|(P-P_O)|}{|(P-O)|}\right) \tag{21}$$

Therefore, according to equation 15, s depends on the finite angles φ, ψ, θ, and, consequently, the velocity $\vec{v}_p$ of the material point p becomes a nonlinear function of those angles.

Then, following the same rationale followed in the previous section for rotation about an axis, we determine the velocities of all material points of E that pass through P using the equation below:

$$\vec{v}_P = \left[\dot{P}\right] = \left[\dot{R}\right] \cdot [p] = \left[\dot{R}\right] \cdot [R]^T \cdot [P] \tag{22}$$

Adopting the notation

$$[\omega] = \left[\dot{R}\right] \cdot [R]^T \tag{23}$$

and calculating every term of $[\omega]$ by applying expression 19, we obtain:

$$\begin{aligned}\omega_{11} &= \left(-\dot{\psi} \cdot s\psi \cdot c\theta - \dot{\theta} \cdot c\psi \cdot s\theta\right) \cdot c\psi \cdot c\theta + \left(\dot{\psi} \cdot s\psi \cdot s\theta - \dot{\theta} \cdot c\psi \cdot c\theta\right) \cdot (-\dot{\psi} \cdot c\psi \cdot s\theta) \\ &\quad + \dot{\psi} \cdot c\psi \cdot s\psi = -\dot{\psi} \cdot s\psi \cdot c\psi + \dot{\psi} \cdot c\psi \cdot s\psi = 0\end{aligned} \tag{24}$$

$$\begin{aligned}\omega_{12} &= \left(-\dot{\psi} \cdot s\psi \cdot c\theta - \dot{\theta} \cdot c\psi \cdot s\theta\right) \cdot (s\varphi \cdot s\psi \cdot c\theta + c\varphi \cdot s\theta) + \left(\dot{\psi} \cdot s\psi \cdot s\theta - \dot{\theta} \cdot c\psi \cdot c\theta\right) \cdot \\ &\quad (-s\varphi \cdot s\psi \cdot s\theta + c\varphi \cdot c\theta) + \dot{\psi} \cdot c\psi \cdot (-s\varphi \cdot c\psi) = -\dot{\psi} \cdot s\varphi - \dot{\theta} \cdot c\varphi \cdot c\psi\end{aligned} \tag{25}$$

$$\begin{aligned}\omega_{13} &= \left(-\dot{\psi} \cdot s\psi \cdot c\theta - \dot{\theta} \cdot c\psi \cdot s\theta\right) \cdot (-c\varphi \cdot s\psi \cdot c\theta + s\varphi \cdot s\theta) + \left(\dot{\psi} \cdot s\psi \cdot s\theta - \dot{\theta} \cdot c\psi \cdot c\theta\right) \cdot \\ &\quad (c\varphi \cdot s\psi \cdot s\theta + s\varphi \cdot c\theta) + \dot{\psi} \cdot c\psi \cdot c\varphi \cdot c\psi = \left(-\dot{\psi} \cdot c\varphi - \dot{\theta} \cdot s\varphi \cdot c\psi\right)\end{aligned} \tag{26}$$

$$\omega_{21}=\left(\dot{\varphi}\cdot c\varphi\cdot s\psi\cdot c\theta+\dot{\psi}\cdot s\varphi\cdot c\psi\cdot c\theta-\dot{\theta}\cdot s\varphi\cdot s\psi\cdot s\theta-\dot{\varphi}\cdot s\varphi\cdot s\theta+\dot{\theta}\cdot c\varphi\cdot c\theta\right)\cdot c\psi\cdot c\theta$$
$$+\left(-\dot{\varphi}\cdot c\varphi\cdot s\psi\cdot s\theta-\dot{\psi}\cdot s\varphi\cdot c\psi\cdot s\theta-\dot{\theta}\cdot s\varphi\cdot s\psi\cdot c\theta-\dot{\varphi}\cdot s\varphi\cdot c\theta-\dot{\theta}\cdot c\varphi\cdot s\theta\right)\cdot$$
$$(-c\psi\cdot s\theta)+(-\dot{\varphi}\cdot c\varphi\cdot c\psi+\dot{\psi}\cdot s\varphi\cdot s\psi)\cdot s\psi=\dot{\psi}\cdot s\varphi+\dot{\theta}\cdot c\varphi\cdot c\psi \tag{27}$$

$$\omega_{22}=\left(\dot{\varphi}\cdot c\varphi\cdot s\psi\cdot c\theta+\dot{\psi}\cdot s\varphi\cdot c\psi\cdot c\theta-\dot{\theta}\cdot s\varphi\cdot s\psi\cdot s\theta-\dot{\varphi}\cdot s\varphi\cdot s\theta+\dot{\theta}\cdot c\varphi\cdot c\theta\right)\cdot$$
$$(s\varphi\cdot s\psi\cdot c\theta+c\varphi\cdot s\theta)+\left(-\dot{\varphi}\cdot c\varphi\cdot s\psi\cdot s\theta-\dot{\psi}\cdot s\varphi\cdot c\psi\cdot c\theta-\dot{\theta}\cdot s\varphi\cdot s\psi\cdot c\theta\right.$$
$$\left.-\dot{\varphi}\cdot s\varphi\cdot c\theta-\dot{\theta}\cdot c\varphi\cdot s\theta\right)\cdot(-s\varphi\cdot s\psi\cdot s\theta+c\varphi\cdot c\theta)=0 \tag{28}$$

$$\omega_{23}=\left(\dot{\varphi}\cdot c\varphi\cdot s\psi\cdot c\theta+\dot{\psi}\cdot s\varphi\cdot c\psi\cdot c\theta-\dot{\theta}\cdot s\varphi\cdot s\psi\cdot s\theta-\dot{\varphi}\cdot s\varphi\cdot s\theta+\dot{\theta}\cdot c\varphi\cdot c\theta\right)\cdot$$
$$(-c\varphi\cdot s\psi\cdot c\theta+s\varphi\cdot s\theta)+\left(-\dot{\varphi}\cdot c\varphi\cdot s\psi\cdot s\theta-\dot{\psi}\cdot s\varphi\cdot c\psi\cdot s\theta-\dot{\theta}\cdot s\varphi\cdot s\psi\cdot c\theta\right.$$
$$\left.-\dot{\varphi}\cdot s\varphi\cdot c\theta-\dot{\theta}\cdot c\varphi\cdot s\theta\right)\cdot(c\varphi\cdot s\psi\cdot s\theta+s\varphi\cdot c\theta)$$
$$+(-\dot{\varphi}\cdot c\varphi\cdot c\psi+\dot{\psi}\cdot s\varphi\cdot s\psi)\cdot c\varphi\cdot c\psi=-\dot{\varphi}-\dot{\theta}\cdot s\psi \tag{29}$$

$$\omega_{31}=\left(\dot{\varphi}\cdot s\varphi\cdot s\psi\cdot c\theta-\dot{\psi}\cdot c\varphi\cdot c\psi\cdot c\theta+\dot{\theta}\cdot c\varphi\cdot s\psi\cdot s\theta+\dot{\varphi}\cdot c\varphi\cdot s\theta+\dot{\theta}\cdot s\varphi\cdot c\theta\right)\cdot$$
$$c\psi\cdot c\theta+\left(-\dot{\varphi}\cdot s\varphi\cdot s\psi\cdot s\theta+\dot{\psi}\cdot c\varphi\cdot c\psi\cdot s\theta+\dot{\theta}\cdot c\varphi\cdot s\psi\cdot c\theta+\dot{\varphi}\cdot c\varphi\cdot c\theta\right.$$
$$\left.-\dot{\theta}\cdot s\varphi\cdot s\theta\right)\cdot(-c\psi\cdot s\theta)+(-\dot{\varphi}\cdot s\varphi\cdot c\psi-\dot{\psi}\cdot c\varphi\cdot s\psi)\cdot s\psi=-\dot{\psi}\cdot c\varphi+\dot{\theta}\cdot s\varphi\cdot c\psi \tag{30}$$

$$\omega_{32}=\left(\dot{\varphi}\cdot s\varphi\cdot s\psi\cdot c\theta-\dot{\psi}\cdot c\varphi\cdot c\psi\cdot c\theta+\dot{\theta}\cdot c\varphi\cdot s\psi\cdot s\theta+\dot{\varphi}\cdot c\varphi\cdot s\theta+\dot{\theta}\cdot s\varphi\cdot c\theta\right)\cdot$$
$$(s\varphi\cdot s\psi\cdot c\theta+c\varphi\cdot s\theta)+\left(-\dot{\varphi}\cdot s\varphi\cdot s\psi\cdot s\theta+\dot{\psi}\cdot c\varphi\cdot c\psi\cdot s\theta+\dot{\theta}\cdot c\varphi\cdot s\psi\cdot c\theta\right.$$
$$\left.+\dot{\varphi}\cdot c\varphi\cdot c\theta-\dot{\theta}\cdot s\varphi\cdot s\theta\right)\cdot(-s\varphi\cdot s\psi\cdot s\theta+c\varphi\cdot c\theta)+$$
$$(-\dot{\varphi}\cdot s\varphi\cdot c\psi-\dot{\psi}\cdot c\varphi\cdot s\psi)\cdot(-s\varphi\cdot c\psi)=\dot{\varphi}+\dot{\theta}\cdot s\psi \tag{31}$$

$$\omega_{33}=\left(\dot{\varphi}\cdot s\varphi\cdot s\psi\cdot c\theta-\dot{\psi}\cdot c\varphi\cdot c\psi\cdot c\theta+\dot{\theta}\cdot c\varphi\cdot s\psi\cdot s\theta+\dot{\varphi}\cdot c\varphi\cdot s\theta+\dot{\theta}\cdot s\varphi\cdot c\theta\right)\cdot$$
$$(-c\varphi\cdot s\psi\cdot c\theta+s\varphi\cdot s\theta)+\left(-\dot{\varphi}\cdot s\varphi\cdot s\psi\cdot s\theta+\dot{\psi}\cdot c\varphi\cdot c\psi\cdot s\theta+\dot{\theta}\cdot c\varphi\cdot s\psi\cdot c\theta\right.$$
$$\left.+\dot{\varphi}\cdot c\varphi\cdot c\theta-\dot{\theta}\cdot s\varphi\cdot s\theta\right)\cdot(c\varphi\cdot s\psi\cdot s\theta+s\varphi\cdot c\theta)+$$
$$(-\dot{\varphi}\cdot s\varphi\cdot c\psi-\dot{\psi}\cdot c\varphi\cdot s\psi)\cdot c\varphi\cdot c\psi=0 \tag{32}$$

Therefore, $[\omega]$ is given by:

$$[\omega]=\begin{bmatrix} 0 & -\dot{\theta}\cdot c\varphi\cdot c\psi-\dot{\psi}\cdot s\varphi & \dot{\psi}\cdot c\varphi-\dot{\theta}\cdot s\varphi\cdot c\psi \\ \dot{\theta}\cdot c\varphi\cdot c\psi+\dot{\psi}\cdot s\varphi & 0 & -\dot{\varphi}-\dot{\theta}\cdot s\psi \\ -\dot{\psi}\cdot c\varphi+\dot{\theta}\cdot s\varphi\cdot c\psi & \dot{\varphi}+\dot{\theta}\cdot s\psi & 0 \end{bmatrix} \tag{33}$$

Equation 33, after substituted into equation 22, gives:

$$\vec{v}_P = [\omega]\cdot[P] = \begin{bmatrix} 0 & -\dot{\theta}\cdot c\varphi\cdot c\psi - \dot{\psi}\cdot s\varphi & \dot{\psi}\cdot c\varphi - \dot{\theta}\cdot s\varphi\cdot c\psi \\ \dot{\theta}\cdot c\varphi\cdot c\psi + \dot{\psi}\cdot s\varphi & 0 & -\dot{\varphi} - \dot{\theta}\cdot s\psi \\ -\dot{\psi}\cdot c\varphi + \dot{\theta}\cdot s\varphi\cdot c\psi & \dot{\varphi} + \dot{\theta}\cdot s\psi & 0 \end{bmatrix}\cdot\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{34}$$

Comparison of equations 20 and 34 exposes, again, the simplicity of rigid body velocity point formulation according to Euler's point of view.

If, by hypothesis, φ, ψ and θ are small rotations, i.e., $\varphi \approx 0 \quad \psi \approx 0 \quad \theta \approx 0$, $[\omega]$ assumes its well known linear form:

$$[\omega] = \begin{bmatrix} 0 & -\dot{\theta} & \dot{\psi} \\ \dot{\theta} & 0 & -\dot{\varphi} \\ -\dot{\psi} & \dot{\varphi} & 0 \end{bmatrix} \tag{35}$$

Consequently, equation 22 becomes:

$$\vec{v}_P = [\omega]\cdot[P] = \begin{bmatrix} 0 & -\dot{\theta} & \dot{\psi} \\ \dot{\theta} & 0 & -\dot{\varphi} \\ -\dot{\psi} & \dot{\varphi} & 0 \end{bmatrix}\cdot\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -\dot{\theta}\cdot Y + \dot{\psi}\cdot Z \\ \dot{\theta}\cdot X - \dot{\varphi}\cdot Z \\ -\dot{\psi}\cdot X + \dot{\varphi}\cdot Y \end{bmatrix} \tag{36}$$

Since the product of the skewed matrix $[\omega]$ by the vector $[P]$ can be represented by the cross-vector product of $\vec{\Omega}$ and $[P]$, where

$$\vec{\Omega} = \dot{\varphi}\vec{i} + \dot{\psi}\vec{j} + \dot{\theta}\vec{k} \tag{37}$$

according to Euler's viewpoint, the velocity of points of E that pass through point P, fixed relative to the inertial reference system S, is given by:

$$\vec{v}_P = [\omega]\cdot[P] = \vec{\Omega} \wedge (P - O). \tag{38}$$

It is important to emphasize that linearized equation 38, the well known velocity formula according to Eulerian approach, implicitly assumes the existence of an infinitesimal control volume, $\mathcal{V}$, around the fixed point, P (see Fig. 4), where the

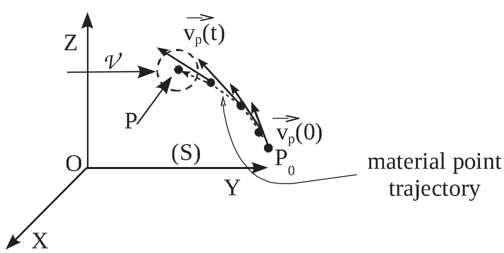

Fig. 4 *Eulerian control volume and Lagrangian trajectory.*

velocity, $\vec{v}_p$, of all material points p of E converge to $\vec{v}_p$ as soon as the volume of $\mathcal{V}$ tends to zero. That is:

$$\vec{v}_P = \lim_{\mathrm{vol}(V)\to 0} \vec{v}_p \tag{39}$$

On the other hand, equation 39 means, according to the Lagrangian viewpoint, that the trajectory of the material point, p, becomes small enough to allow linearization of equation 20 in the same way that was done for equation 34; as a consequence, both viewpoints converge to the same result:

$$\vec{v}_p = \vec{v}_P \tag{40}$$

as one should expect.

Conclusions

The comparison of Lagrangian and Eulerian viewpoints is fundamental in fluid mechanics. Nevertheless, even in fluid mechanics undergraduate literature, this important subject is frequently disregarded. As a consequence, students, in general, cannot properly interpret the meaning of the velocity field of particles in a moving fluid that occupy a control volume at a given instant, a fact that is implicit in the Eulerian representation of flow.

The basic rigid body kinematics usually taught in undergraduate courses makes natural use of the Eulerian concepts, although little effort is employed in making it explicit to the students. The presentation of both the Lagrangian and Eulerian viewpoints in courses covering either rigid or deformable mechanics would certainly increase student understanding of motion in general. In the present paper, the formulations of Euler and Lagrange have been presented, and it has been shown that they lead to the same results once proper hypotheses are assumed – infinitesimal control volume and infinitesimal trajectory length. This can provide a student with insight into both points of view of rigid body kinematics.

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