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Eliminating blowing-ups and evanescent waves when using the finite series technique in evaluating beam shape coefficients for some T-matrix approaches, with the example of Gaussian beams

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ABSTRACT

When evaluating beam shape coefficients which encode the description of laser beams, for use in some T-matrix approaches such as generalized Lorenz-Mie theory or Extended Boundary Condition Method for structured beams, by using finite series, blowing-ups are observed. When numerical inaccuracies are ruled out, it has been firmly demonstrated that such blowing-ups correspond to genuine physical phenomena, namely they describe evanescent waves. We propose a method to eliminate these blowing-ups and the corresponding evanescent waves (at least most of them).

1. Introduction

In some T-matrix approaches with arbitrary shaped beam illumination, specifically in Generalized Lorenz-Mie Theory (GLMT), e.g. [1–3], or in Extended Boundary Condition Method (EBCM) [4–6] when applied to structured beams [7,8], illuminating beams may be expressed by an expansion over Vector Spherical Wave Functions (VSWFs) and the expansion coefficients are then expressed in terms of Beam Shape Coefficients (BSCs), e.g. [9–11] and references therein for reviews of these theories and applications.

The BSCs are in general constituted from two sets, and are denoted most usually as $g_{n,TM}^{m}$ and $g_{n,TE}^{m}$ with n ranging from 1 to ∞, m from $(-n)$ to $(+n)$ and in which TM stands for "Transverse Magnetic" while TE stands for "Transverse Electric". There exist several methods to evaluate the BSCs, including (i) the original quadrature method, e.g. [12,13], (ii) the localized approximations which have a long story with a review in [14] to be completed by [15–17], and by a series of paper devoted to the limitations of localized approximations when dealing with beams exhibiting an axicon angle and/or a topological charge, e.g. [18–25], (iii) the use of an Angular Spectrum Decomposition (ASD), e.g. [26] for a review and [27–38], for recent implementations, and (iv) the recently introduced R-quadrature method [38–42].

In the present paper, we are concerned with another technique, named the finite series technique. This technique has been introduced several decades ago [43,44] and has been essentially forgotten for a long time, particularly due to the efficiency of localized approximations in terms of formal flexibility and computing speed. It has recently been set again on the stage due to the above mentioned limitations of localized approximations for beams exhibiting an axicon angle and/or a topological charge, e.g. [45–53].

It has soon been observed that the use of finite series generates blowing-ups in the values of the high-order BSCs, e.g. section 6.7 in [1]. These blowing-ups were attributed to a genuine phenomenon but without any physical explanation of their origins. The renewal of the finite series technique set again the blowing-up issue on the stage. It has then been firmly demonstrated by analytical means that indeed, once numerical inaccuracies are ruled out, blowing-ups are genuine phenomena and, furthermore, it has been demonstrated that they are a signature of the existence of evanescent waves [54,55], and see particularly Eqs. (48)–(49) in [56]. Furthermore, in the case of Gaussian beams characterized by a beam confinement factor s, various criteria to evaluate the critical value of the order of BSCs have been established [54]. These criteria will be discussed as well in the present paper.

Specifically, the present paper presents a method to get rid of these blowing-ups and of the associated evanescent waves (at least most of them). It is interesting to note that the method relies on a mathematical step which, in utmost rigor, is not allowed but which nevertheless provides a correct solution to the issue at stake.

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The paper is organized as follows. Section 2 provides a background and poses a problem to be solved in which BSCs have to be evaluated. Section 3 deals with a first method to the evaluation of these BSCs. Section 4 deals with a second method to the evaluation of the same BSCs. In section 5, we provide a comparison of the results obtained in sections 3 and 4, exhibit the fact that they disagree, one exhibiting the evanescent wave behavior and the other getting rid of them, and explain the reason of this difference of behaviors. Section 6 is a conclusion.

2. Background to a problem to be solved

We recall that Gaussian beams may be described by using a scheme of successive approximations due to Davis [57–59]. In this scheme, using Cartesian coordinates (x, y, z), a x-polarized potential vector is introduced for a beam propagating along the z-direction. The nonzero x -component of the potential vector is expressed in terms of a potential function ψ which is expressed by an infinite series reading as:

$$\psi = \sum_{j=0}^{\infty} s^{2j}\psi_{2j} = \psi_0 + s^2\psi_2 + s^4\psi_4 + \cdots \tag{1}$$

in which s is a small parameter, called the beam confinement factor (or beam shape factor) defining the focusing of the beam, reading as $1/(kw_0)$ in which k is the wavenumber and w_0 the beam waist radius of the beam. Eq. (1) allows one to introduce successive approximations called the first-order, third-order, …approximations. Once an approximation is obtained, the corresponding electric and magnetic fields, **E** and **H** respectively, may be obtained, for instance by using the Lorenz gauge.

The first-order approximation is called the L-approximation (“L” standing for lowest) in which the nonzero field components are E_x, E_z, H_y and H_z given by Eqs. (4.24)–(4.27) in [1]. Neglecting the components E_z and H_z, we obtain a still simpler approximation called the L^- approximation in [1], section 4.1.6. In this approximation which is to be used in the present paper, the electromagnetic fields read as [1]:

$$\mathbf{E}(\mathbf{r}) = \mathbf{e}_x E_0 Q \exp(-Q\rho^2/w_0^2 - ikz) \tag{2}$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{e}_y H_0 Q \exp(-Q\rho^2/w_0^2 - ikz) \tag{3}$$

in which $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors in the x- and y-directions respectively, E_0 and H_0 are field strengths, $\rho^2 = x^2 + y^2$, and Q, which is defined in a way slightly differently from in [1], reads as:

$$Q = \frac{1}{1 - 2iz/(kw_0^2)} \tag{4}$$

The radial components of the electric and magnetic fields may be obtained in spherical coordinates (r, θ, φ) from Eqs. (3.10) and (3.19) in [1]. Together with Eqs. (2) and (3), we then obtain:

$$\begin{aligned} E_r(\mathbf{r}) &= E_0 Q \exp(-Qr^2\sin^2\theta/w_0^2 - ikr\cos\theta)\sin\theta\cos\varphi \\ &= \frac{E_0}{r}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} c_n^{pw} n(n+1) g_{n,TM}^m j_n(kr) P_n^{|m|}(\cos\theta)\exp(im\varphi) \end{aligned} \tag{5}$$

$$\begin{aligned} H_r(\mathbf{r}) &= H_0 Q \exp(-Qr^2\sin^2\theta/w_0^2 - ikr\cos\theta)\sin\theta\sin\varphi \\ &= \frac{H_0}{r}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n} c_n^{pw} n(n+1) g_{n,TE}^m j_n(kr) P_n^{|m|}(\cos\theta)\exp(im\varphi) \end{aligned} \tag{6}$$

in which $j_n(kr)$ are spherical Bessel functions of the first kind, $P_n^{|m|}(\cos\theta)$ are associated Legendre functions defined according to Hobson’s notation [60], and c_n^{pw}, with “pw” standing for “plane wave”, are plane wave coefficients given by Eq. (3.3) in [1], reading as:

$$c_n^{pw} = \frac{1}{k}(-i)^{n+1}\frac{2n+1}{n(n+1)} \tag{7}$$

Introducing $R = kr$ and rearranging, Eqs. (5)–(6) may be rewritten as:

$$\begin{aligned} &RQ\exp(-Qs^2R^2\sin^2\theta - iR\cos\theta)\sin\theta\cos\varphi \\ &= \sum_{n=1}^{\infty}\sum_{m=-n}^{+n}(-i)^{n+1}(2n+1)g_{n,TM}^m j_n(R)P_n^{|m|}(\cos\theta)\exp(im\varphi) \end{aligned} \tag{8}$$

$$\begin{aligned} &RQ\exp(-Qs^2R^2\sin^2\theta - iR\cos\theta)\sin\theta\sin\varphi \\ &= \sum_{n=1}^{\infty}\sum_{m=-n}^{+n}(-i)^{n+1}(2n+1)g_{n,TE}^m j_n(R)P_n^{|m|}(\cos\theta)\exp(im\varphi) \end{aligned} \tag{9}$$

We then multiply both sides by $\exp(-im'\varphi)$ and carry out the integration over the interval $[0, 2\pi]$, using classical expressions reading as:

$$\int_0^{2\pi} e^{i(m-m')\varphi}d\varphi = 2\pi\delta_{mm'} \tag{10}$$

$$\int_0^{2\pi}\sin\varphi e^{i(m-m')\varphi}d\varphi = i\pi(\delta_{m,m'+1} - \delta_{m,m'-1}) \tag{11}$$

$$\int_0^{2\pi}\cos\varphi e^{i(m-m')\varphi}d\varphi = \pi(\delta_{m,m'+1} + \delta_{m,m'-1}) \tag{12}$$

to obtain:

$$\begin{aligned} &RQ\exp(-Qs^2R^2\sin^2\theta - iR\cos\theta)\sin\theta(\delta_{m,1} + \delta_{m,-1}) \\ &= 2\sum_{n=1}^{\infty}(-i)^{n+1}(2n+1)g_{n,TM}^m j_n(R)P_n^{|m|}(\cos\theta) \end{aligned} \tag{13}$$

$$\begin{aligned} &-iRQ\exp(-Qs^2R^2\sin^2\theta - iR\cos\theta)\sin\theta(\delta_{m,1} - \delta_{m,-1}) \\ &= 2\sum_{n=1}^{\infty}(-i)^{n+1}(2n+1)g_{n,TE}^m j_n(R)P_n^{|m|}(\cos\theta) \end{aligned} \tag{14}$$

These Eqs. (13)–(14) imply that the non-zero values for the BSCs are for $m = \pm 1$, and that they satisfy a relation allowing one to define uni-index BSCs g_n reading as:

$$\frac{g_n}{2} = g_{n,TM}^1 = g_{n,TM}^{-1} = ig_{n,TE}^1 = -ig_{n,TE}^{-1} \tag{15}$$

This relation is typical of on-axis Gaussian beams, e.g. Eq. (6.3) in [1], more generally of on-axis axisymmetric beams of the first kind. Such beams are defined by the fact that the z-component S_z of the time-averaged Poynting vector does not depend on the azimuthal coordinate φ, in a coordinate system in which the axis z is the symmetry axis of the beam. Such beams may be viewed as z-axisymmetric, e.g. Eq. (1) in [61], and Eqs. (6) and (7) in [62]. It has recently been demonstrated that such beams are actually circularly symmetric, i.e. all components of the Poynting vector does not depend on φ [63].

As a result, Eqs. (13)–(14) may be expressed using a single equation reading as:

$$RQ\exp(-Qs^2R^2\sin^2\theta - iR\cos\theta)\sin\theta = \sum_{n=1}^{\infty}(-i)^{n+1}(2n+1)g_n j_n(R)P_n^1(\cos\theta) \tag{16}$$

In the next step, which is typical of the finite series approach, we specify a xy-plane location, i.e. $\theta = \pi/2$, leading to:

$$R\exp(-s^2R^2) = \sum_{n=1}^{\infty}(-i)^{n+1}(2n+1)g_n j_n(R)P_n^1(0) \tag{17}$$

When $\theta = \pi/2$, $P_n^m(\cos\theta)$ is nonzero only for $(n-m)$ even, which again is a typical result of the finite series approach in which the case $(n-m)$ odd requires another treatment (see Appendix A). For $(n-m)$ even, we then set $n = 2p+1$, leading to:

$$x\exp(-s^2x^2) = \sum_{p=0}^{\infty} a_p j_{2p+1}(x) \tag{18}$$

in which we conveniently change R (related to the physical problem in hand) to x (to insist on the mathematical problem in hand), and to:

$$a_p = (-1)^{p+1}(4p+3)g_{2p+1}P^1_{2p+1}(0) \tag{19}$$

We then use [60]:

$$P^m_n(0) = (-1)^{(n+m)/2}\frac{(n+m-1)!!}{2^{(n-m)/2}(\frac{n-m}{2})!},\ (n-m)\ \text{even} \tag{20}$$

and a classical expression for $n!! = 1.3.5...n$ reading as:

$$\Gamma(\frac{1}{2}+n) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \tag{21}$$

to obtain:

$$P^1_{2p+1}(0) = (-1)^{p+1}\frac{2\Gamma(p+3/2)}{\sqrt{\pi}p!} \tag{22}$$

so that Eq. (19) becomes:

$$a_p = (4p+3)\frac{2\Gamma(p+3/2)}{\sqrt{\pi}p!}g_{2p+1} \tag{23}$$

The problem is then to evaluate the expansion coefficients a_p of Eq. (18) from which we can obtain the uni-index BSCs g_n, n odd, using Eq. (23).

3. First method

We provide two approaches to illustrate the first method used to evaluate the expansion coefficients.

3.1. First approach

Let us introduce the function $h(x)$ reading as:

$$h(x) = x^{3/2}\exp(-s^2x^2) \tag{24}$$

and introduce its Maclaurin expansion reading as:

$$x^{3/2}\exp(-s^2x^2) = \sum_{q=0}^{\infty}\frac{(-s^2)^q}{q!}x^{2q+3/2} \tag{25}$$

We next use a Bessel expansion reading as [64]:

$$x^q = \frac{2^{q+1}}{\sqrt{\pi}}\sum_{p=0}^{\infty}\frac{(q+2p+1/2)\Gamma(q+p+1/2)}{p!}j_{q+2p}(x) \tag{26}$$

leading to:

$$(x/2)^\mu = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(\mu+2k+1/2)\Gamma(\mu+k+1/2)}{k!}j_{\mu+2k}(x) \tag{27}$$

We may afterward use:

$$j_n(x) = \sqrt{\frac{\pi}{2x}}J_{n+1/2}(x) \tag{28}$$

in which $J_{n+1/2}(.)$ denotes Bessel functions [65], allowing one to rewrite Eq. (27) as:

$$(x/2)^\mu = \sum_{k=0}^{\infty}\frac{(\mu+2k)\Gamma(\mu+k)}{k!}J_{\mu+2k}(x) \tag{29}$$

We may then return to Eq. (25) and express $x^{2q+3/2}$ using the Bessel series of Eq. (29) to obtain:

$$x^{3/2}\exp(-s^2x^2) = 2^{3/2}\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2q+2k+3/2)\Gamma(2q+k+3/2)}{k!}\frac{(-4s^2)^q}{q!} \times J_{2q+2k+3/2}(x) \tag{30}$$

which, with $p = q+k$, may be rewritten as:

$$x^{3/2}\exp(-s^2x^2) = 2^{3/2}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\frac{(2p+3/2)\Gamma(q+p+3/2)}{(p-q)!}\frac{(-4s^2)^q}{q!}J_{2p+3/2}(x) = 2^{3/2}\sum_{p=0}^{\infty}(2p+3/2)\sum_{q=0}^{p}\frac{\Gamma(p+q+3/2)}{q!(p-q)!}(-4s^2)^qJ_{2p+3/2}(x) \tag{31}$$

Using Eq. (28), Eq. (31) may be given the form of Eq. (18) with:

$$a_p = \frac{4}{\sqrt{\pi}}(2p+3/2)\sum_{q=0}^{p}\frac{\Gamma(p+q+3/2)}{q!(p-q)!}(-4s^2)^q \tag{32}$$

which is the first solution to the evaluation of the expansion coefficients a_p. The correctness of this result has been numerically checked with high precision calculations as follows. Relying on Eq. (18), and defining:

$$f(x) = \left|x\exp(-s^2x^2) - \sum_{p=0}^{p_{\max}}a_pj_{2p+1}(x)\right| \tag{33}$$

it is indeed found that $f(x) \to 0$ when $p_{\max} \to \infty$ for various values of s and x.

3.2. Second approach

Let us again consider Eq. (18) and implement Eq. (28) to obtain:

$$x^{3/2}\exp(-s^2x^2) = \sqrt{\frac{\pi}{2}}\sum_{p=0}^{\infty}a_pJ_{2p+3/2}(x) \tag{34}$$

We are now going to apply the Neumann Expansion Theorem (NET) which is the basic theorem used in the finite series technique to the evaluation of BSCs, see Chapter 5 in [1] and [64]. For this, Eq. (34) must be given the form of Eq. (5.2) in [1] reading as:

$$x^{1/2}g(x) = \sum_{n=0}^{\infty}c_nJ_{n+1/2}(x) \tag{35}$$

This is done by introducing:

$$g(x) = x\exp(-s^2x^2) \tag{36}$$

so that Eq. (34) becomes:

$$x^{1/2}g(x) = \sqrt{\frac{\pi}{2}}\sum_{p=0}^{\infty}a_pJ_{2p+3/2}(x) \tag{37}$$

which is to be compared with Eq. (35). Setting $n = 2p+1$ in the r.h.s. of Eq. (37) and equating with the r.h.s. of Eq. (35), we obtain:

$$\sqrt{\frac{\pi}{2}}\sum_{n=1,\ \text{odd}}^{\infty}a_{(n-1)/2}J_{n+1/2}(x) = \sum_{n=0}^{\infty}c_nJ_{n+1/2}(x) \tag{38}$$

so that:

$$\begin{cases} c_n = 0,\ n\ \text{even} \\ c_n = \sqrt{\frac{\pi}{2}}a_{(n-1)/2},\ n\ \text{odd} \end{cases} \tag{39}$$

that is to say:

$$\begin{cases} c_{2p} = 0 \\ c_{2p+1} = \sqrt{\frac{\pi}{2}}a_p \end{cases} \tag{40}$$

The next step requires us to obtain the expansion coefficients b_n defined by the expansion, see Eq. (5.3) in [1]:

$$g(x) = \sum_{n=0}^{\infty}b_nx^n \tag{41}$$

From Eqs. (36) and (41), we then have:

$$g(x) = \sum_{n=0}^{\infty}\frac{(-s^2)^nx^{2n+1}}{n!} = \sum_{n=0}^{\infty}b_nx^n \tag{42}$$

leading to:

$$\begin{cases} b_n = 0,\ n\ \text{even} \\ b_n = (-s^2)^{(n-1)/2}/[(n-1)/2]!,\ n\ \text{odd} \end{cases} \tag{43}$$

The relationship between the coefficients c_n of Eqs. (39)–(40) and the coefficients b_n of Eq. (43) are given by the NET, see Eq. (5.4) of [1]:

$$c_n = (n + \frac{1}{2}) \sum_{m=0}^{\leq n/2} 2^{n-2m+1/2} \frac{\Gamma(n-m+1/2)}{m!} b_{n-2m} \tag{44}$$

Since we only deal with non-zero values of c_n, i.e. those for n odd as shown in Eq. (40), we have:

$$c_{2p+1} = \sqrt{\frac{\pi}{2}} a_p = (2p + \frac{3}{2}) \sum_{m=0}^{p} 2^{2p-2m+3/2} \frac{\Gamma(2p-m+3/2)}{m!} b_{2p-2m+1} \tag{45}$$

which, setting $q = (p - m)$ becomes:

$$c_{2p+1} = \sqrt{\frac{\pi}{2}} a_p = (2p + \frac{3}{2}) \sum_{q=0}^{p} 2^{2q+3/2} \frac{\Gamma(p+q+3/2)}{(p-q)!} b_{2q+1} \tag{46}$$

in which we only need the coefficients b_n with n odd, rewritten from Eq. (43) with $n = 2q + 1$, reading as:

$$b_{2q+1} = (-s^2)^q / q! \tag{47}$$

Inserting Eq. (47) into Eq. (46), we obtain:

$$\sqrt{\frac{\pi}{2}} a_p = 2^{3/2} (2p + \frac{3}{2}) \sum_{q=0}^{p} \frac{\Gamma(p+q+3/2)(-4s^2)^q}{q!(p-q)!} \tag{48}$$

which identifies with Eq. (32) as it should.

4. Second method

We multiply Eq. (18) by $j_{2n+1}(x)$ and integrate from 0 to ∞, leading to:

$$\int_0^\infty x \exp(-s^2 x^2) j_{2n+1}(x) dx = \int_0^\infty \sum_{p=0}^{\infty} a_p j_{2p+1}(x) j_{2n+1}(x) dx \tag{49}$$

Interchanging the integral and the summation, we obtain:

$$\int_0^\infty x \exp(-s^2 x^2) j_{2n+1}(x) dx = \sum_{p=0}^{\infty} a_p \int_0^\infty j_{2p+1}(x) j_{2n+1}(x) dx \tag{50}$$

so that, from Eqs. (49) and (50), we obtain:

$$\int_0^\infty \sum_{p=0}^{\infty} a_p j_{2p+1}(x) j_{2n+1}(x) dx = \sum_{p=0}^{\infty} a_p \int_0^\infty j_{2p+1}(x) j_{2n+1}(x) dx \tag{51}$$

But we have [65]:

$$\int_0^\infty j_n(x) j_m(x) dx = \frac{\sin[(n-m)(\pi/2)]}{n(n+1) - m(m+1)} \text{ for } n \neq m \tag{52}$$

$$\int_0^\infty j_n(x) j_m(x) dx = \frac{\pi}{2(2n+1)} \text{ for } n = m \tag{53}$$

In Eq. (51), we have to deal with $n = 2p + 1$ and $m = 2n + 1$, so that Eqs. (52) and (53) reduce to:

$$\int_0^\infty j_{2p+1}(x) j_{2n+1}(x) dx = \frac{\pi}{2(4n+3)} \delta_{np} \tag{54}$$

Inserting Eq. (54) into Eq. (50), we readily obtain:

$$a_p = \frac{2(4p+3)}{\pi} \int_0^\infty x \exp(-s^2 x^2) j_{2p+1}(x) dx \tag{55}$$

which is the second solution to the evaluation of the expansion coefficients a_p.

5. Discussion

The two solution of Eqs. (32) and (55) are compared in the figure, using logarithmic scales. Furthermore, absolute values are used because the series may become alternating, e.g. the Table in Appendix C. The results for four values of the beam confinement factor s are displayed, namely $1/10$, $1/13$, $1/17$ and $1/20$. We observe that the first solution of Eq. (32) displayed with a blue line and blue symbols blows up in contrast with the second solution of Eq. (55). These behaviors may be checked analytically, see Appendix B.

The ranges chosen for p depend on s and have been chosen to visually exhibit the blowing-ups, from 0 to 40 when $s = 1/10$, up to from 0 to more than 80 for $s = 1/20$. From [54], we extract two successful criteria to the evaluation of the critical values $n_c = 2p_c + 1$ for the onset of the blowing-ups (see Fig. 1).

One criterion is theoretical and reads as $p_c = int[e/(16s^2)]$ leading to $p_c = 17, 29, 49$ and 68 for $s = 1/10, 1/13, 1/17$ and $1/20$ respectively. The second criterion is empirical and can read $p_c = int[3/(16s^2) - 1/2]$ leading to $p_c = 18, 31, 54$ and 74 respectively. Both criteria agree reasonably well with the critical values available from the figure. It should be noted that, for $p < p_c$, the two solutions seem to provide the same results. This is actually not true. The results are different although the differences are so small that they cannot be seen on the figure, see however the Table in Appendix C. These differences are 0 only for $s = 0$.

From a physical point of view the blowing-ups correspond to the presence of evanescent waves, see [54,55] , [56], while the other results correspond to the elimination of the evanescent waves (at least most of them).

The reason for the disagreement between the two solutions is the fact that the interchange between the integral and the summation carried out in the step from Eq. (49) is not allowed (see Appendix C). By nevertheless carrying out the interchange, we therefore force the mathematics to produce a solution in which the interchange is allowed, and "indirectly" eliminate the evanescent waves (or at least most of them). We may demonstrate that this last interchange is indeed allowed using the Dominated Convergence Theorem [66], but such a demonstration may be omitted since it is a direct consequence of the procedure used to obtain the second solution of Eq. (55), in which Eq. (51) is indeed used to force it.

As a last word, let us mention that, although the obtained second solution is physically of interest, we have to insist on the fact that it is mathematically incorrect. Indeed, we may numerically check that Eq. (18) is not satisfied when we use Eq. (55) to specify the values of a_p 's. This is maybe the most important message of the present paper, namely that a mathematically incorrect procedure may be used to produce a physically correct solution of interest. Actually, any set of BSCs generates a genuine electromagnetic field, even if these coefficients were tossed, and therefore generates a physically correct solution. But, in the present case, the solution obtained is not only physically correct but it is also *of interest* since it eliminates the evanescent waves (or at least most of them).

6. Conclusion

Usually, physicists do not care too much with interchanges between integrals and summations. This is usually justified by the fact that equations, in physics, most often do not exhibit "pathological" features which would prevent such interchanges to be carried out. The present work provides a counter-example.

But there is an interesting consequence to be noted. Namely, by using an "incorrect" mathematical step, we force the mathematics to provide a solution which is physically a "correct" solution, by eliminating evanescent waves (or at least most of them).

It must be furthermore noted that the blowing-ups observed with the correct solution do not imply a blowing-up of the associated physical quantities, in particular do not imply a blowing-up of the associated fields. This is because, actually, when dealing with physical quantities, BSCs which blow-up are multiplied by functions decreasing to zero fast enough to compensate for the blowing-ups. We then have obtained two solutions which, both of them, are physically correct. Comparisons between physical quantities, using either of the two solutions would then allow one to characterize the influence of the evanescent waves [67].

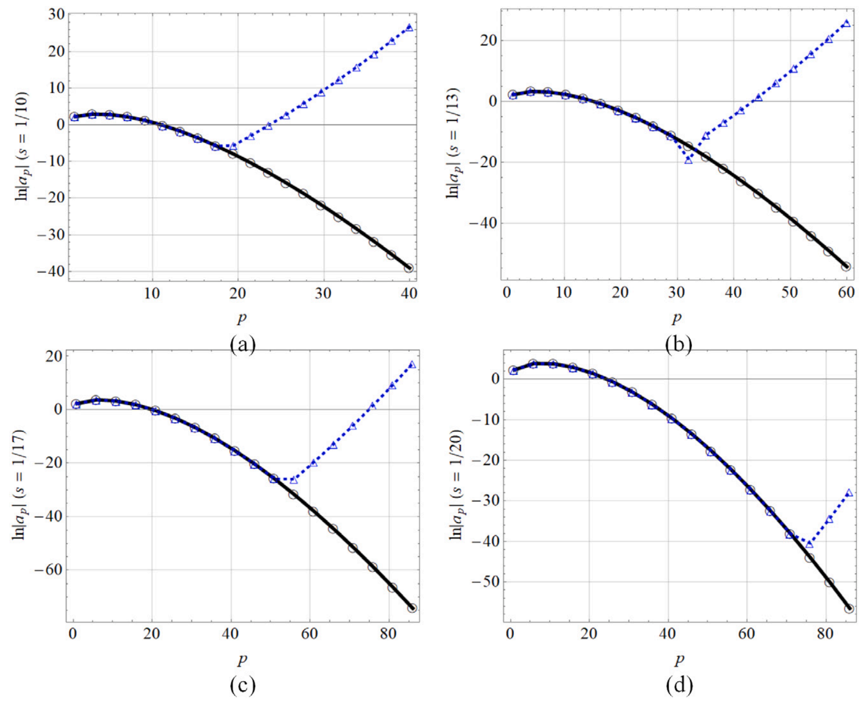

Fig. 1. Expansion coefficients a_p for various values of the beam confinement parameter. Blue: blowing-up solution. Black: blowing-up eliminated.

CRediT authorship contribution statement

Gérard Gouesbet: Writing – original draft, Formal analysis, Data curation, Conceptualization. **Jianqi Shen:** Writing – review & editing, Formal analysis, Data curation, Conceptualization. **Leonardo André Ambrosio:** Writing – review & editing, Funding acquisition, Formal analysis, Data curation, Conceptualization.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

The paper deals with the case $(n-m)$ even. The case $(n-m)$ odd is not considered in this paper because it would behave in a completely similar way, and then would unnecessarily increase the size of the paper. The similarity between the two cases is obvious from two equations below, providing the expressions of BSCs using finite series for on-axis Gaussian beams at an order of approximation called L^-, see [1], page 167:

$$g_{2p+1} = \sum_{j=0}^{p} \frac{p!}{j!(p-j)!} \frac{\Gamma(p+j+3/2)}{\Gamma(p+3/2)} (-4s^2)^j \tag{56}$$

$$g_{2p+2} = \sum_{j=0}^{p} \frac{p!}{j!(p-j)!} \frac{\Gamma(p+j+5/2)}{\Gamma(p+5/2)} (-4s^2)^j \tag{57}$$

Appendix B

In a first step, we analytically demonstrate that the second solution of Eq. (55) does not blow up. Using Eq. (28), Eq. (55) may be rewritten as:

$$a_p = (4p+3)\frac{\sqrt{2}}{\pi} \int_0^\infty \exp(-s^2x^2) J_{p+3/2}(x) x^{1/2} dx \tag{58}$$

From Eq. (6.631.1) of [68], with $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(n+t) > -1$, we have:

$$\begin{aligned} &\int_0^\infty \exp(-ax^2) J_n(bx) x^t dx \\ &= \frac{b^n \Gamma(\frac{n+t+1}{2})}{2^{n+1} a^{(n+t+1)/2} \Gamma(n+1)} [{}_1F_1(\frac{n+t+1}{2}; n+1; \frac{-b^2}{4a})] \\ &= \frac{b^n \Gamma(\frac{n+t+1}{2})}{2^{n+1} a^{(n+t+1)/2} \Gamma(n+1)} \exp(\frac{-b^2}{4a}) {}_1F_1(\frac{n-t+1}{2}; n+1; \frac{b^2}{4a}) \end{aligned} \tag{59}$$

in which ${}_1F_1$ is a hypergeometric function, also called the Kummer's function of the first kind, and in which we have used the Kummer's transformation [69]:

$${}_1F_1(a; b; x) = \exp(x) {}_1F_1(b-a; b; -x) \tag{60}$$

Letting $a = s^2$, $b = 1$, $n = 2p + 3/2$ and $t = 1/2$, we then obtain:

$$a_p = \frac{2^{-2p-1} s^{-2p-3}}{\sqrt{\pi}} \frac{\Gamma(p+3/2)}{\Gamma(2p+3/2)} \exp(\frac{-1}{4s^2}) {}_1F_1(p+1; 2p+5/2; \frac{1}{4s^2}) \tag{61}$$

We then use a classical expression reading as:

$$ {}_1F_1(a;b;z) = \sum_{q=0}^{\infty} \frac{(a)_q z^q}{(b)_q q!} \tag{62} $$

in which $(c)_q$ is the Pochhammer symbol defined as:

$$ (c)_0 = 1 \tag{63} $$

$$ (c)_q = c(c+1)(c+2)...(c+q-1),\ q \neq 0 \tag{64} $$

so that the hypergeometric function of Eq. (61) may be rewritten as:

$$ {}_1F_1(p+1;2p+5/2;\frac{1}{4s^2}) = \sum_{q=0}^{\infty} \frac{(p+1)_q}{(2p+5/2)_q}(\frac{1}{4s^2})^q \frac{1}{q!} \tag{65} $$

Similarly:

$$ {}_1F_1(p+1;2p+2;\frac{1}{4s^2}) = \sum_{q=0}^{\infty} \frac{(p+1)_q}{(2p+2)_q}(\frac{1}{4s^2})^q \frac{1}{q!} \tag{66} $$

from which we readily deduce:

$$ {}_1F_1(p+1;2p+5/2;\frac{1}{4s^2}) <_1 F_1(p+1;2p+2;\frac{1}{4s^2}) \tag{67} $$

But, from [69], we have:

$$ {}_1F_1(\nu,2\nu,z) = \Gamma(\nu+1/2)\exp(z/2)(z/4)^{-\nu+1/2} I_{\nu-1/2}(z/2) \tag{68} $$

in which $I_\alpha(x)$ is the modified Bessel function satisfying the relation:

$$ I_\alpha(x) = \sum_{s=0}^{\infty} \frac{1}{s!(s+\alpha)!}(\frac{x}{\alpha})^{2s+\alpha} \tag{69} $$

.

Using Eqs. (61) and (68), and the inequality of (67), we may then establish another inequality reading as:

$$ a_p < b_p = \frac{2^{2p+1}s^{-2}}{\sqrt{\pi}} \frac{[\Gamma(p+3/2)]^2}{\Gamma(2p+3/2)} \exp(\frac{-1}{8s^2}) I_{p+1/2}(\frac{1}{8s^2}) \tag{70} $$

Using $\Gamma(x+1) = x\Gamma(x)$ and Eq. (69), we finally find that, for p sufficiently large:

$$ \frac{b_{b+1}}{b_p} \approx \frac{I_{p+3/2}(\frac{1}{8s^2})}{I_{p+1/2}(\frac{1}{8s^2})} < 1 \tag{71} $$

meaning that the b_ps, and therefore the a_ps according to Eq. (70), do not blow up.

In a second step, we analytically demonstrate that the first solution of Eq. (32) indeed blows up. To begin with, this equation is rewritten as:

$$ \begin{aligned} a_p &= \frac{2(4p+3)}{\sqrt{\pi}} \frac{\Gamma(p+3/2)}{p!} \sum_{q=0}^{p} \binom{p}{q} \frac{\Gamma(p+q+3/2)}{\Gamma(p+3/2)}(-4s^2)^q \\ &= \frac{2(4p+3)}{\sqrt{\pi}} \frac{\Gamma(p+3/2)}{p!} \sum_{q=0}^{p} \binom{p}{q} (p+3/2)_q (-4s^2)^q \\ &= \frac{4}{p!\sqrt{\pi}} \frac{4p+3}{2p+3} \Gamma(p+5/2) \sum_{q=0}^{p} \binom{p}{q} (p+3/2)_q (-4s^2)^q \end{aligned} \tag{72} $$

in which $\binom{p}{q}$ is the binomial coefficient and $(a)_q$ is again the Pochhammer coefficient.

We then introduce the generalized Bessel polynomial $y_p(x;a)$ which, according to Grosswald [70], see as well Eq. (29) in [55], reads as:

$$ y_p(x;a) = \sum_{q=0}^{p} \binom{p}{q} (p+a-1)_q (\frac{x}{2})^q \tag{73} $$

allowing one to deal again with Eq. (72) to rewrite it as:

$$ \begin{aligned} a_p &= \frac{2(4p+3)}{\sqrt{\pi}} \frac{\Gamma(p+3/2)}{p!} \sum_{q=0}^{p} \binom{p}{q} (p+3/2)_q (-4s^2)^q \\ &= \frac{2(4p+3)}{\sqrt{\pi}} \frac{\Gamma(p+3/2)}{p!} y_p(-8s^2;5/2) \end{aligned} \tag{74} $$

But, when p is large, $y_p(x;a)$ tends to the expression of Eq. (75) given below, see again Grosswald [70] and as well Eq. (31) in [55]:

$$ y_p(x;a) = (\frac{2x}{e}p)^p 2^{a-3/2} e^{1/x}[1+O(1/p)] \tag{75} $$

so that, for p large enough, we have:

$$ a_p \approx \frac{4(4p+3)}{\sqrt{\pi}} \frac{\Gamma(p+3/2)}{p!} (\frac{-16s^2}{e}p)^p \exp(\frac{-1}{8s^2}) \tag{76} $$

leading to:

$$ \frac{a_{p+1}}{a_p} \approx \frac{-16s^2}{e} p \tag{77} $$

implying the blowing-up of the coefficients a_p.

Appendix C

Let IS be:

$$ IS = \int_0^\infty \sum_{p=0}^{\infty} a_p j_{2p+1}(R) j_{2n+1}(R) dR \tag{78} $$

and SI be:

$$ SI = \sum_{p=0}^{\infty} a_p \int_0^\infty j_{2p+1}(R) j_{2n+1}(R) dR \tag{79} $$

both quantities being taken from Eq. (51). In IS, the summation S is carried out first and the integral I afterward. In SI, the integral is carried out first and the summation afterward.

We also define the difference:

$$ D = IS - SI \tag{80} $$

All these quantities are displayed in the Table below, using for a_p the first solution of Eq. (32), setting the beam confinement parameter s to 1/10, and running the computations with 3000 digits.

p	IS	SI	D
1	2.120575	2.120575	1.38E−10
2	2.194795	2.194795	−1.65E−10
3	1.953446	1.953446	2.11E−10
4	1.547777	1.547777	−2.93E−10
5	1.109633	1.109633	4.36E−10
6	0.72692	0.72692	−6.97E−10
7	0.438198	0.438198	1.20E−09
8	0.244416	0.244416	−2.19E−09
9	0.126731	0.126731	4.29E−09
10	0.061333	0.061333	−8.94E−09
11	0.027806	0.027806	1.98E−08
12	0.011848	0.011848	−4.65E−08
13	0.004759	0.004759	1.15E−07
14	0.001807	0.001807	−3.02E−07
15	0.00065	0.000649	−8.34E−07
16	0.000222	0.000225	−2.42E−06
17	7.23E−05	6.49E−05	7.37E−06
18	2.24E−05	4.59E−05	−2.349E−05
19	6.66E−06	−7.2E−05	7.8332E−05
20	1.89E−06	0.000275	−0.0002727
21	5.15E−07	−0.00099	0.00098976
22	1.35E−07	0.00374	−0.0037395
23	3.39E−08	−0.01469	0.01468854
24	8.20E−09	0.059904	−0.0599044
25	1.29E−09	−0.25336	0.25335919
26	4.32E−10	1.109987	−1.1099872
27	9.41E−11	−5.03193	5.031922733
28	1.98E−11	23.5799	−23.579905
29	4.05E−12	−114.109	114.109102

We observe that, for $p > 1$, IS decreases steadily when p increases while SI decreases steadily up to $p = 18$. From this value $p = 18$, SI

becomes alternating with absolute values increasing steadily, exhibiting a blowing-up. For small values of p up to 14, IS and SI perfectly agree up to 6 digits and afterward disagree more and more. We recall that we found that the critical value p_c was found to be 17 using the theoretical criterion and 18 using the empirical criterion. These values are well confirmed by the table. Apart of these remarks, the most important message is that interchanging the integral and the summation as in Eq. (51) is indeed forbidden. Doing this, we then generate another solution, which eliminates the evanescent waves, and for which the interchange is allowed since it is forced to obtain the second solution, which does not blow-up.

Data availability

No data was used for the research described in the article.

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