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Research paper

Asymptotics for positive singular solutions to subcritical sixth order equations ☆

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ABSTRACT

We classify the local asymptotic behavior of positive singular solutions to a class of subcritical sixth order equations on the punctured ball. First, using a version of the integral moving spheres technique, we prove that solutions are asymptotically radially symmetric solutions with respect to the origin. We divide our approach into some cases concerning the growth of nonlinearity. In general, we use an Emden–Fowler change of variables to translate our problem to a cylinder. In the lower critical regime, this is not enough, thus, we need to introduce a new notion of change of variables. The main difficulty is that the cylindrical PDE in this coordinate system is nonautonomous. Nonetheless, we define an associated nonautonomous Pohozaev functional, which can be proved to be asymptotically monotone. In addition, we show *a priori* estimates for these two functionals, from which we extract compactness properties. With these ingredients, we can perform an asymptotic analysis technique to prove our main result.

1. Introduction

We study (classical) positive singular solutions $u \in C^6(B_R^*)$ with $n \geqslant 7$ (which will always be assumed so forth) to the following family of subcritical sixth order PDEs

$$(-\Delta)^3 u = f_p(u) \quad \text{in} \quad B_R^*. \qquad (\mathcal{P}_{6,p,R})$$

Here $B_R^* := B_R \setminus \{0\} \subset \mathbb{R}^n$ is the punctured ball of radius $R > 0$, $(-\Delta)^3 = (-\Delta)\circ(-\Delta)\circ(-\Delta)$ is the tri-Laplacian, and the nonlinearity $f_p \in C^1(B_R)$ is given by

$$f_p(u) := u^p \quad \text{with} \quad p \in (1, 2_\#] \cup (2_\#, 2^\# - 1),$$

where $2_\# := \frac{n}{n-6}$ and $2^\# := \frac{2n}{n-6}$ are, respectively, the lower and upper critical exponents related to the compact Sobolev embedding of $H^3(\mathbb{R}^n)$. We refer to [1,2] for more details on this terminology.

We say that a positive solution $u \in C^6(B_R^*)$ has a removable singularity at the origin if $\lim_{x\to 0} u(x) < +\infty$, that is, it can be continuously extended to the origin; otherwise, we say that it is a non-removable singularity. These are called non-singular and singular solutions, respectively.

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Allowing $R \to +\infty$ in $(\mathcal{P}_{6,p,R})$, we obtain the following blow-up limit PDE

$$(-\Delta)^3 u = f_p(u) \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \qquad (\mathcal{P}_{6,p,\infty})$$

We recall that a non-singular solution $u \in C^6(\mathbb{R}^n)$ to $(\mathcal{P}_{6,p,\infty})$ is said to be stable if

$$\int_{\mathbb{R}^n} |\Delta^3 \varphi|^2 \mathrm{d}x \geqslant p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 \mathrm{d}x \quad \text{for any} \quad \varphi \in H^6(\mathbb{R}^n).$$

Let us mention that S. Luo et al. [3] and A. Harrabi and B. Rahal [4] proved entire stable solutions to $(\mathcal{P}_{6,p,\infty})$ with removable singularities at the origin, which was later generalized by the former authors [5,6] for the case of polyharmonic equations; this result can be stated as

Theorem A. *If $u \in C^6(\mathbb{R}^n)$ is a stable positive non-singular solution to $(\mathcal{P}_{6,p,\infty})$ with $p \in (1, 2^{\#} - 1)$, then $u \equiv 0$.*

It makes sense to divide our analysis into three cases, namely, the Serrin–Lions [1,2] case $p \in (1, 2_{\#})$, Aviles [7] case $p = 2_{\#}$, and the Gidas–Spruck [8] case $p \in (2_{\#}, 2^{\#} - 1)$. Our main result classifies the local behavior of positive solutions to $(\mathcal{P}_{6,p,R})$ in these situations.

Theorem 1. *Let $u \in C^6(B_R^*)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$. Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. Then, it follows*

$$u(x) = (1 + \mathcal{O}(|x|))\overline{u}(|x|) \quad \text{as} \quad x \to 0,$$

where $\overline{u}(r) = ⨍_{\partial B_1} u(r\theta)\mathrm{d}\theta$ is the spherical average of u. Moreover,

(a) *if $p \in (1, 2_{\#})$, then*

$$u(x) \simeq |x|^{6-n} \quad \text{as} \quad x \to 0;$$

(b) *if $p = 2_{\#}$, then*

$$u(x) = (1 + \mathrm{o}(1))\widehat{K}_0(n)^{\frac{n-6}{6}} |x|^{6-n} (\ln |x|)^{\frac{6-n}{6}} \quad \text{as} \quad x \to 0,$$

where

$$\widehat{K}_0(n) = \frac{4}{3}(n-2)(n-4)(n-6)^2; \qquad (1)$$

(c) *if $p \in (2_{\#}, 2^{\#} - 1)$, then*

$$u(x) = (1 + \mathrm{o}(1))K_0(n,p)^{\frac{1}{p-1}} |x|^{-\frac{6}{p-1}} \quad \text{as} \quad x \to 0, \qquad (2)$$

where

$$K_0(n,p) = \gamma_p \left(\gamma_p + 2\right) \left(\gamma_p + 4\right) \left(n - 2 - \gamma_p\right) \left(n - 4 - \gamma_p\right) \left(n - 6 - \gamma_p\right).$$

Remark 2. In a recent paper, it was independently proved by X. Huang, Y. Li, and H. Yang [9] that the super poly-harmonic condition can be removed. We also refer to Q. Ngô and D. Ye [10] for more details on the blow-up limit case $R = +\infty$. We keep this condition in our manuscript because of its natural relations with curvature sign conditions for the upper critical situation $p = 2^{\#} - 1$. They also provided some upper bound estimates that will be important in our methods.

When $p \in (1, 2_{\#})$, we show that solutions behave like the fundamental solution, and so, the origin is a non-removable singularity. When $p \in (2_{\#}, 2^{\#} - 1)$, we prove that solutions to $(\mathcal{P}_{6,p,R})$ behave near the isolated singularity like the homogeneous solutions to the blow-up limit equation $(\mathcal{P}_{6,p,\infty})$, which are classified by Theorem A. In the lower critical case $p = 2_{\#}$, which is the so-called sixth order Serrin exponent [1], we observe that since $K_0(n, 2_{\#}) = 0$, it follows that $(\mathcal{P}_{6,p,\infty})$ does not have non-trivial homogeneous solutions. This explains why this situation has a different blow-up rate near the singularity, which is given by a homogeneous term times a log correction factor.

Now, let us compare our results to the ones in the fourth and second order cases. First, we consider positive solutions $u \in C^4(\mathbb{R}^n \setminus \{0\})$ with $n \geqslant 5$ to the family of fourth order equations

$$(-\Delta)^2 u = f_p(u) \quad \text{in} \quad B_R^*, \qquad (\mathcal{P}_{4,R,p})$$

where $R < +\infty$, $(-\Delta)^2 = (-\Delta)\circ(-\Delta)$ is the bi-Laplacian, and $p \in (1, 2_{**}] \cup (2_{**}, 2^{**} - 1)$, where $2_{**} = \frac{n}{n-4}$ and $2^{**} = \frac{2n}{n-4}$. Notice that $(\mathcal{P}_{4,R,p})$ is subcritical in the sense of the compact Sobolev embedding of $H^2(\mathbb{R}^n)$. On this subject, R. Soranzo [11] for $p \in (1, 2_{**})$, H. Yang [12] and Z. Guo et al. [13] for $p \in (2_{**}, 2^{**} - 1)$, and the first-named author and J. M. do Ó [14] for $p = 2_{**}$ study qualitative properties for positive solutions to $(\mathcal{P}_{4,R,p})$; we have the result below

Theorem B. *Let $u \in C^4(B_R^*)$ be a positive singular solution to $(\mathcal{P}_{4,R,p})$ with $R < +\infty$. Assume that $-\Delta u \geqslant 0$. Then, it follows*

$$u(x) = (1 + \mathcal{O}(|x|))\overline{u}(x) \quad \text{as} \quad x \to 0.$$

Moreover,

(a) *if* $p \in (1, 2_{**})$, *then*

$$u(x) \simeq |x|^{4-n} \quad \text{as} \quad x \to 0;$$

(b) *if* $p = 2_{**}$, *then*

$$u(x) = (1 + \mathrm{o}(1))\widehat{K}_{4,0}(n)^{\frac{n-4}{4}}|x|^{4-n}(\ln|x|)^{\frac{4-n}{4}} \quad \text{as} \quad x \to 0,$$

where

$$\widehat{K}_{4,0}(n) = \frac{(n-2)(n-4)^2}{2};$$

(c) *if* $p \in (2_{**}, 2^{**} - 1)$, *then*

$$u(x) = (1 + \mathrm{o}(1))K_{4,0}(n,p)^{\frac{1}{p-1}}|x|^{-\frac{4}{p-1}} \quad \text{as} \quad x \to 0.$$

where

$$K_{4,0}(n,p) = \frac{4}{p-1}\left(\frac{4}{p-1} + 2\right)\left(n - 2 - \frac{4}{p-1}\right)\left(n - 4 - \frac{4}{p-1}\right).$$

Second, we consider positive singular solutions $u \in C^2(\mathbb{R}^n \setminus \{0\})$ with $n \geqslant 3$ to the second order equation

$$-\Delta u = f_p(u) \quad \text{in} \quad B_R^*, \tag{$\mathcal{P}_{2,R,p}$}$$

where $R < +\infty$, Δ is the Laplacian, and $p \in (1, 2_*] \cup (2_*, 2^* - 1)$, where $2_* = \frac{n}{n-2}$ and $2^* = \frac{2n}{n-2}$ Notice that $(\mathcal{P}_{2,R,p})$ is subcritical in the sense of the compact Sobolev embedding of $H^1(\mathbb{R}^n)$.

It is worth mentioning that all the aforementioned classification results were inspired by the classical theorems of J. Serrin [1] and P.-L. Lions [2], B. Gidas and J. Spruck [8], and P. Aviles [7] on the study of positive singular solutions to the second order semi-linear PDE $(\mathcal{P}_{2,R,p})$

Theorem C. *Let* $u \in C^2(B_R^*)$ *be a positive singular solution to* $(\mathcal{P}_{2,R,p})$ *with* $R < +\infty$. *Then, it follows*

$$u(x) = (1 + \mathcal{O}(|x|))\overline{u}(x) \quad \text{as} \quad x \to 0,$$

Moreover,

(a) *if* $p \in (1, 2_*)$, *then*

$$u(x) \simeq |x|^{2-n} \quad \text{as} \quad x \to 0;$$

(b) *if* $p = 2_*$, *then*

$$u(x) = (1 + \mathrm{o}(1))\widehat{K}_{2,0}(n)^{\frac{n-2}{2}}|x|^{2-n}(\ln|x|)^{\frac{2-n}{2}} \quad \text{as} \quad x \to 0,$$

where

$$\widehat{K}_{2,0}(n) = \frac{(n-2)^2}{2};$$

(c) *if* $p \in (2_*, 2^* - 1)$, *then*

$$u(x) = (1 + \mathrm{o}(1))K_{2,0}(n,p)^{\frac{1}{p-1}}|x|^{-\frac{2}{p-1}} \quad \text{as} \quad x \to 0,$$

where

$$K_{2,0}(n,p) = \frac{2}{p-1}\left(n - 2 - \frac{2}{p-1}\right).$$

This type of asymptotic analysis extends to a rich class of strongly coupled second order systems as studied in [15,16].

The main difference between the asymptotic analysis for the critical and subcritical regimes occurs because of the change in the monotonicity properties of the Pohozaev functional, which classifies the type of stability for singular solutions to $(\mathcal{P}_{6,p,R})$ around a blow-up (shrink-down) limit solution. This method is inspired by Fleming's tangent cone analysis for minimal hypersurfaces [17,18]. In the critical case, it can be shown that the Pohozaev functional becomes constant; that is, blow-up limit solutions are stable. In contrast, in the subcritical case, they are asymptotically stable. This discrepancy is caused by the sign-changing behavior of the coefficients of the tri-Laplacian in Emden–Fowler coordinates (or logarithmic cylindrical coordinates), which are suitable for this problem (see Remark 3.1).

The proof of Theorem 1 is divided into two parts. First, we prove the asymptotic symmetry of singular solutions to $(\mathcal{P}_{6,p,R})$ in the punctured ball. Second, we use some ODE analysis and the monotonicity properties of the Pohozaev functional to study the asymptotic behavior for solutions on the cylinder. The strategy strongly relies on the growth of the subcritical nonlinearity, which, in the lower critical situation $p = 2_\#$, turns out to be far different from the other regime. We should emphasize that the superharmonicity conditions are not necessary in the range $p \in (2_{\#}, 2^{\#}-1)$, which is not the case otherwise. Third, we define two homological-type invariants that satisfy suitable monotonicity properties. This can be used to study the local asymptotic behavior of positive singular solutions near the isolated singularity. We need to subdivide our approach with respect to the growth of the nonlinearity into three cases; namely, Serrin–Lions case, Aviles case, and Gidas–Spruck case.

We remark that in a companion paper, the same authors study the asymptotics for positive singular solutions in the other limit situation $p = 2^{\#}-1$. Indeed, together with [19] and Theorems 1 and A, this would provide a holistic picture of the qualitative behavior of solutions to $(\mathcal{P}_{6,p,R})$ in the broader range $p \in (1, 2^{\#}-1]$.

These results are inspired by the classical literature for semi-linear second order equations due to J. Serrin [1], P.-L. Lions [2], P. Aviles [7], B. Gidas and J. Spruck [8], and L. A. Caffarelli et al. [20] with an improvement given by N. Korevaar et al. [21]. For more results on asymptotic analysis, we refer the interested reader to [22].

We should observe that $(\mathcal{P}_{6,p,R})$, $(\mathcal{P}_{4,R,p})$, and $(\mathcal{P}_{2,R,p})$ are particular cases of a more general class of equations, which we describe as follows. More precisely, for any $N \in \mathbb{N}$ even number, we consider (classical) positive singular solutions $u \in C^N(\mathbb{R}^n \setminus \{0\})$ to the following family of subcritical even order poly-harmonic PDEs

$$(-\Delta)^{N/2} u = f_p(u) \quad \text{in} \quad B_R^*, \tag{$\mathcal{P}_{N,p,R}$}$$

where $0 < R < \infty$, $(-\Delta)^{N/2} = (-\Delta)\circ \ldots \circ(-\Delta)$ is the poly-Laplacian, and the nonlinearity $f_p \in C^1(B_R)$ is given by

$$f_p(u) := |u|^{p-1}u \quad \text{with} \quad p \in \left[\frac{n}{n-N}, \frac{n+N}{n-N}\right) := [2_{N,*}, 2_N^*),$$

where $2_{N,*} := \frac{n}{n-N}$ and $2_N^* := \frac{2n}{n-N}$ are, respectively, the lower and upper critical exponents with respect to the compact Sobolev embedding of $H^{N/2}(\mathbb{R}^n)$.

We observe that most of our arguments can be extended to this higher order setting. In this regard, it is natural to expect a full classification result for the local asymptotic behavior of singular solutions to $(\mathcal{P}_{N,p,R})$ near isolated singularities in the sense of Theorem 1. Namely, there exist three possibilities for this local behavior depending on the growth of the nonlinearity. To summarize this discussion, let us state the following conjecture

Conjecture 3. *Let $N \in \mathbb{N}$ be an even number and $u \in C^N(B_R^*)$ be a positive singular solution to $(\mathcal{P}_{N,p,R})$ with $R < +\infty$ and $p \in (1, 2_N^*-1)$. Assume that $(-\Delta)^j u \geqslant 0$ for any $j \in \{1, \ldots, N/2-1\}$. Then, it follows*

$$u(x) = (1 + \mathcal{O}(|x|))\overline{u}(x) \quad \text{as} \quad x \to 0,$$

Moreover,

(a) *if $p \in (1, 2_N^*)$, then*
$$u(x) \simeq |x|^{N-n} \quad \text{as} \quad x \to 0;$$

(b) *if $p = 2_{N,*}$, then*
$$u(x) = (1+\mathrm{o}(1))\widehat{K}_{N,0}(n)^{\frac{n-N}{N}} |x|^{N-n} (\ln|x|)^{\frac{N-n}{N}} \quad \text{as} \quad x \to 0,$$
where
$$\widehat{K}_{N,0}(n) = \frac{2^{\frac{N}{2}}(\frac{N}{2}-1)!}{N} \prod_{j=0}^{\frac{N}{2}-1} (n-2j)(n-N)^2.$$

(c) *if $p \in (2_{N,*}, 2_N^*-1)$, then*
$$u(x) = (1+\mathrm{o}(1))K_{N,0}(n,p)^{\frac{1}{p-1}} |x|^{-\frac{N}{p-1}} \quad \text{as} \quad x \to 0.$$

Here is our plan for the rest of the paper. In Section 2, we introduce both the autonomous and nonautonomous Emden–Fowler coordinates. In Section 3, we define the associated Pohozaev functionals, and we prove their (asymptotic) monotonicity properties. In Section 4, we prove some *a priori* upper bound estimates. In Section 5, we perform a variant of the integral moving spheres method and prove that solutions are asymptotically radially symmetric. In Section 6, we study the limit values of the Pohozaev functional under blow-up and shrink-down sequences. In Section 7, we use the monotonicity formulas and some asymptotic analysis to prove the classification of the local asymptotic behavior in Theorem 1.

2. Emden–Fowler coordinates

In this section, we define the Emden–Fowler change of variables.

2.1. Autonomous case

We define a classical change of variables, which we have already seen transforms $(\mathcal{P}_{6,p,R})$ into an ODE with constant coefficients.

Definition 2.1. Let us define the sixth order autonomous Emden–Fowler change of variables (or logarithmic cylindrical coordinates) given by

$$v(t,\theta) = r^{\gamma_p} u(r,\sigma), \quad \text{where} \quad t = \ln r, \quad \sigma = \theta = x|x|^{-1}.$$

Let us consider the autonomous Emden–Fowler transformation as follows

$$\mathfrak{F} : C_c^\infty(B_R^*) \to C_c^\infty(\mathcal{C}_T) \quad \text{given by} \quad \mathfrak{F}(u) = e^{\gamma_p t} u(e^t, \theta) := v. \tag{2.1}$$

Using this coordinate system and performing a lengthy computation, we arrive at the following sixth order nonlinear PDE on the cylinder $\mathcal{C}_T := (-\infty, T) \times \mathbb{S}^{n-1}$ with $T = \ln R < +\infty$,

$$-P_{\rm cyl} v = f_p(v) \quad \text{on} \quad \mathcal{C}_T. \tag{$\mathcal{C}_{p,T}$}$$

Here $P_{\rm cyl}$ is the tri-Laplacian written in cylindrical coordinates given by

$$\begin{aligned} P_{\rm cyl} &= \partial_t^{(6)} + K_5(n,p)\partial_t^{(5)} + K_4(n,p)\partial_t^{(4)} + K_3(n,p)\partial_t^{(3)} + K_2(n,p)\partial_t^{(2)} + K_1(n,p)\partial_t + K_0(n,p) \\ &\quad + 2\partial_t^{(4)}\Delta_\theta + J_3(n,p)\partial_t^{(3)}\Delta_\theta + J_2(n,p)\partial_t^{(2)}\Delta_\theta + J_1(n,p)\partial_t\Delta_\theta + J_0(n,p)\Delta_\theta \\ &\quad + 3\partial_t^{(2)}\Delta_\theta^2 + L_1(n,p)\partial_t\Delta_\theta^2 + L_0(n,p)\Delta_\theta^2 + \Delta_\theta^3, \end{aligned} \tag{2.2}$$

where $K_j(n,p), J_j(n,p), L_j(n,p)$ for $j \in \{0,1,2,3,4,5\}$ are constants given by (A.4) and (A.5).

Remark 2.2. The computation to obtain the coefficients in (2.2) and (2.5) is tedious and lengthy. To deal with this, we used Mathematica 13.2. In Appendix, we provide some hints to perform this computation. For more details on this subject for the fourth order setting, we refer the reader to [13,14,23] and the references therein; this is inspired by the pioneering results of [7,20].

2.2. Nonautonomous case

In the lower critical case, because of the vanishing of the coefficient $K_0(n, 2_\#)$, we already that the situation changes dramatically. We define the so-called nonautonomous Emden–Fowler change of variables [7,15,24,25].

Definition 2.3. Let us define the sixth order nonautonomous Emden–Fowler change of variables (or logarithmic cylindrical coordinates) given by

$$w(t,\theta) = r^{6-n}(\ln r)^{\frac{6-n}{6}} u(r,\sigma), \quad \text{where} \quad t = \ln r \quad \text{and} \quad \sigma = \theta = x|x|^{-1}. \tag{2.3}$$

Let us consider the nonautonomous Emden–Fowler transformation as follows

$$\widetilde{\mathfrak{F}} : C_c^\infty(B_R^*) \to C_c^\infty(\mathcal{C}_T) \quad \text{given by} \quad \widetilde{\mathfrak{F}}(u) = e^{(6-n)t} t^{\frac{6-n}{6}} u(e^t, \theta) := w. \tag{2.4}$$

Using this coordinate system and performing a lengthy computation, we arrive at

$$-\widetilde{P}_{\rm cyl} w = t^{-1}|w|^{2_\# - 1} w \quad \text{on} \quad \mathcal{C}_T. \tag{$\widetilde{\mathcal{C}}_T$}$$

Here $\widetilde{P}_{\rm cyl}$ is the tri-Laplacian written in nonautonomous Emden–Fowler coordinates given by

$$\begin{aligned} \widetilde{P}_{\rm cyl} &= \partial_t^{(6)} + \widetilde{K}_5(n,t)\partial_t^{(5)} + \widetilde{K}_4(n,t)\partial_t^{(4)} + \widetilde{K}_3(n,t)\partial_t^{(3)} + \widetilde{K}_2(n,t)\partial_t^{(2)} + \widetilde{K}_1(n,t)\partial_t + \widetilde{K}_0(n,t) \\ &\quad + \widetilde{J}_4(n,t)\partial_t^{(4)}\Delta_\theta + \widetilde{J}_3(n,t)\partial_t^{(3)}\Delta_\theta + \widetilde{J}_2(n,t)\partial_t^{(2)}\Delta_\theta + \widetilde{J}_1(n,t)\partial_t\Delta_\theta + \widetilde{J}_0(n,t)\Delta_\theta \\ &\quad + \widetilde{L}_2(n,t)\partial_t^{(2)}\Delta_\theta^2 + \widetilde{L}_1(n,t)\partial_t\Delta_\theta^2 + \widetilde{L}_0(n,t)\Delta_\theta^2 + \Delta_\theta^3. \end{aligned} \tag{2.5}$$

Here $\widetilde{K}_j(n,t), \widetilde{J}_j(n,t), \widetilde{L}_j(n,t)$ for $j \in \{0,1,2,3,4,5\}$ are functions given by (A.8) and (A.9). Analogously to the standard autonomous case, we also consider a transformation that sends a singular solution to $(\mathcal{P}_{6,p,R})$ with $p = 2_\#$ into solutions to a nonautonomous ODE on the cylinder.

3. Pohozaev functionals

Next, we define two central characters in our studies, the so-called autonomous and nonautonomous Pohozaev functionals. The heuristics are that these Pohozaev functionals classify whether or not the blow-up limit solutions in cylindrical coordinates are (asymptotically) stable equilibrium points of the associated sixth order ODE. More precisely, from the dynamical systems point of view, it works as a Lyapunov functional. Let us remark that to prove the Pohozaev invariant to be well-defined, one needs to use some upper estimates and asymptotic symmetry that will be proved independently in Section 4 (see Lemmas 4.5 and 4.6).

3.1. Autonomous case

Initially, let us define the Pohozaev functional associated to the PDE equation on the cylinder given by $(\mathcal{C}_{p,T})$, which will play a central role in our analysis (for more details on this class of invariants, we refer to [12,23,26,27]).

Remark 3.1. By direct computations, notice that when $p \in (2_\#, 2^\# - 1)$, one has the sign relations

$$K_5(n,p), K_1(n,p), J_3(n,p), L_1(n,p) \geqslant 0, \quad \text{and} \quad K_3(n,p), J_1(n,p) \leqslant 0.$$

In addition, we can explicitly compute these coefficients in the limiting situations $p = 2_\#$ and $p = 2^\# - 1$ (see (A.6) and (A.7)). This change in sign behavior explains why one needs to use distinct methods when the power parameter $p \in (1, +\infty)$ changes.

With respect to Conjecture 3, we speculate that a relationship like this should hold for the coefficients of the poly-Laplacian written in Emden–Fowler coordinates.

We introduce a functional that will be used to classify the local behavior near the isolated singularity. We emphasize that an explicit formula for this functional can be obtained by multiplying $(\mathcal{C}_{p,T})$ by $\partial_t v$ and integrating by parts.

Definition 3.2. For any $v \in C^6(\mathbb{R})$ positive solution to $(\mathcal{C}_{p,T})$ with $p \in (2_\#, 2^\# - 1)$, let us define its *cylindrical Pohozaev functional* by

$$\mathcal{P}_{\rm cyl}(t,v) = \int_{\mathbb{S}_t^{n-1}} \mathcal{H}(t,\theta,v)\mathrm{d}\theta.$$

Here $\mathbb{S}_t^{n-1} = \{t\} \times \mathbb{S}^{n-1}$ is the cylindrical ball with volume element given by $\mathrm{d}\theta = e^{-2t}\mathrm{d}\sigma$, where $\mathrm{d}\sigma_r$ is the volume element of the ball of radius $r > 0$ in $\mathbb{R}^n$. In addition, we set

$$\mathcal{H}(t,\theta,v) := \mathcal{H}_{\rm rad}(t,\theta,v) + \mathcal{H}_{\rm ang}(t,\theta,v),$$

where

$$\begin{aligned}\mathcal{H}_{\rm rad}(t,\theta,v) :=& \left(v^{(5)}v^{(1)} - v^{(4)}v^{(2)} + \frac{1}{2}v^{(3)2}\right) + K_5\left(v^{(4)}v^{(1)} - v^{(3)}v^{(2)}\right) \\ &+ K_4\left(v^{(3)}v^{(1)} - \frac{1}{2}v^{(2)2}\right) + K_3 v^{(2)}v^{(1)} + \frac{K_2}{2}v^{(1)2} + \frac{K_0}{2}v^2 - \frac{|v|^{p+1}}{p+1}\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}_{\rm ang}(t,\theta,v) :=& -J_4\left(\partial_t^{(3)}\nabla_\theta v \partial_t \nabla_\theta v - |\partial_t^{(2)}\nabla_\theta v|^2\right) - \frac{J_2}{2}|\partial_t^{(2)}\nabla_\theta v|^2 - \frac{J_1}{2}|\partial_t^{(2)}\nabla_\theta v|^2 - \frac{J_0}{2}|\nabla_\theta v|^2 \\ &+ \frac{L_2}{2}|\partial_t^{(2)}\Delta_\theta v|^2 + \frac{L_0}{2}|\partial_t^{(2)}\Delta_\theta v|^2 + \frac{1}{2}|\Delta_\theta v|^2.\end{aligned}$$

Remark 3.3. Using the inverse of the Emden–Fowler transformation, one can also construct the *spherical Pohozaev functional* given by $\mathcal{P}_{\rm sph}(r,u) := \left(\mathcal{P}_{\rm cyl}\circ\mathfrak{F}^{-1}\right)(t,v)$. From [22, Proposition A.1], it follows that $\mathcal{P}_{\rm sph}$ also satisfies a monotonicity property.

We deduce a monotonicity formula for the logarithmic cylindrical Pohozaev functional $\mathcal{P}_{\rm cyl}(t,v)$, which will be essential to show that the limit Pohozaev invariant is well-defined as $t \to -\infty$.

Proposition 3.4. *Let $v \in C^6(\mathbb{R})$ be a positive solution to $(\mathcal{C}_{p,T})$ with $p \in (2_\#, 2^\# - 1)$. If $-\infty < t_1 \leqslant t_2 < T$, then $\mathcal{P}_{\rm cyl}(t_2,v) - \mathcal{P}_{\rm cyl}(t_1,v) \leqslant 0$. More precisely, we have the monotonicity formula*

$$\partial_t \mathcal{P}_{\rm cyl}(t,v) = \int_{\mathbb{S}_t^{n-1}} \left(-K_5|\partial_t^{(3)}v|^2 + K_3|\partial_t^{(2)}v|^2 - K_1|\partial_t v|^2 - J_3|\partial_t^{(2)}\nabla_\theta v|^2 - L_1|\partial_t \Delta_\theta v|^2\right)\mathrm{d}\theta < 0. \tag{3.1}$$

In particular, $\mathcal{P}_{\rm cyl}(t,v)$ is nonincreasing, and so $\mathcal{P}_{\rm cyl}(-\infty,v) := \lim_{t\to-\infty}\mathcal{P}_{\rm cyl}(t,v)$ exists.

Proof. Initially, observe that (3.1) follows by a direct computation, which consists of multiplying $(\mathcal{C}_{p,T})$ by $v^{(1)}$, and integrating by parts. Hence, using Remark 3.1, we find $\partial_t \mathcal{P}_{\rm cyl}(t,v) \leqslant 0$, which proves that the cylindrical Pohozaev functional is nonincreasing. Finally, since by Lemma 4.2, the Pohozaev functional is bounded above, the full existence of limit follows.

Remark 3.5. Using the inverse of cylindrical transform, that is, $\mathcal{P}_{\rm sph} = \mathcal{P}_{\rm cyl}\circ\mathfrak{F}^{-1}$, it follows that $\mathcal{P}_{\rm sph}(0,u) = \lim_{r\to0}\mathcal{P}_{\rm sph}(r,u) = \lim_{t\to-\infty}\mathcal{P}_{\rm cyl}(t,v)$. The last equality implies that the Pohozaev invariant is well-defined in the punctured ball when $R \to 0$.

3.2. Nonautonomous case

In the lower critical case $p = 2_\#$, since $K_{0,\#} := K_0(n, 2_\#) = 0$ and $\gamma_{2_\#} = n - 6$, a new cylindrical transformation was defined. Concerning this nonautonomous cylindrical transformation, we compute its associated Pohozaev functional, which in this situation has some time-dependent terms.

Again, let us observe that an explicit formula for this functional can be obtained by multiplying $(\widetilde{\mathcal{C}}_T)$ by $\partial_t w$ and integrating by parts. In this case, a more complicated computation is required due to the appearance of nonautonomous terms.

Definition 3.6. For any $w \in C^6(\mathbb{R})$ be a positive solution to $(\widetilde{C}_T)$, let us define its *cylindrical nonautonomous Pohozaev functional* by

$$\widetilde{\mathcal{P}}_{\mathrm{cyl}}(t,w) = \int_{\mathbb{S}_t^{n-1}} \widetilde{\mathcal{H}}(t,\theta,w)\mathrm{d}\theta.$$

Here

$$\widetilde{\mathcal{H}}(t,\theta,w) := \widetilde{\mathcal{H}}_{\mathrm{rad}}(t,\theta,w) + \widetilde{\mathcal{H}}_{\mathrm{ang}}(t,\theta,w), \tag{3.2}$$

where

$$\begin{aligned}\widetilde{\mathcal{H}}_{\mathrm{rad}}(t,\theta,w) :=\, & t\left(w^{(5)}w^{(1)} - w^{(4)}w^{(2)} + \frac{1}{2}w^{(3)2}\right) + t\widetilde{K}_5\left(w^{(4)}w^{(1)} - w^{(3)}w^{(2)}\right)\\ & + t\widetilde{K}_4\left(w^{(3)}w^{(1)} - \frac{1}{2}w^{(2)2}\right) + t\widetilde{K}_3 w^{(2)}w^{(1)} + \frac{t\widetilde{K}_2}{2}w^{(1)2} + \frac{t\widetilde{K}_0}{2}w^2 - \frac{n-6}{2(n-3)}|w|^{2_\#+1}\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathcal{H}}_{\mathrm{ang}}(t,\theta,w) :=\, & -t\widetilde{J}_4\left(\partial_t^{(3)}\nabla_\theta w\partial_t\nabla_\theta w - |\partial_t^{(2)}\nabla_\theta w|^2\right) - \frac{t\widetilde{J}_2}{2}|\partial_t^{(2)}\nabla_\theta w|^2 - \frac{t\widetilde{J}_1}{2}|\partial_t^{(2)}\nabla_\theta w|^2\\ & - \frac{t\widetilde{J}_0}{2}|\nabla_\theta w|^2 + \frac{t\widetilde{L}_2}{2}|\partial_t^{(2)}\Delta_\theta w|^2 + \frac{t\widetilde{L}_0}{2}|\partial_t^{(2)}\Delta_\theta w|^2 + \frac{t}{2}|\Delta_\theta w|^2.\end{aligned}$$

Remark 3.7. Initially, notice that the nonautonomous Pohozaev invariant is well-defined for any $t \in (-\infty, T)$ since $u \in C^6(\mathbb{R}^n \setminus \{0\})$ is smooth away from the origin. In addition, due to Proposition 5.5, for $R < \infty$, we get that $|w(t,\theta) - \overline{w}(t)| = \mathcal{O}(e^{\beta t})$ as $t \to -\infty$, for some $\beta > 0$, where $\overline{w}(t)$ is the average of $w(t,\theta)$ over $\theta \in \mathbb{S}_t^{n-1}$ with $w = \widetilde{\mathfrak{F}}(u)$ given by (2.4).

In particular, one can find some large $T_0 \gg 1$ and $C > 0$ (independent of t) satisfying

$$\left|\nabla_\theta w(t,\theta)\right| + \left|\Delta_\theta w(t,\theta)\right| + \left|\nabla_\theta\Delta_\theta w(t,\theta)\right| + \left|\Delta_\theta^2 w(t,\theta)\right| \leqslant Ce^{\beta t} \quad \text{in} \quad \mathcal{C}_{T_0}. \tag{3.3}$$

Moreover, from the sharp estimate in Proposition 3.4 and the gradient estimate in Lemma 4.5, it follows

$$\sum_{j=0}^{5}|\partial_t^{(j)} w(t,\theta)| \leqslant C \quad \text{in} \quad \mathcal{C}_{T_0}. \tag{3.4}$$

Before we prove the monotonicity formula, we need to show an auxiliary result. Our strategy follows the same as in [28, Lemmas 4.8 and 4.10]. This argument is inspired in [23, Lemmas 2.1 and 2.2]. As usual, we set $\overline{w}(t) = ⨍_{\mathbb{S}_t^{n-1}}|w(t,\theta)|\mathrm{d}\theta$ denoting the spherical average of w.

From now on, we also denote by $\widehat{K}_j(n) \in \mathbb{R}$ for $j \in \{0,1,2,3,4,5\}$ the dimensional constants given by (A.10). With this notation, we have the auxiliary result below

Lemma 3.8. *Let $w \in C^6(\mathcal{C}_T)$ be a positive solution to $(\widetilde{C}_T)$. Then, one has*

$$\ell^* := \lim_{t\to-\infty} w(t,\theta) = \lim_{t\to-\infty}\overline{w}(t) \in \{0, \widehat{K}_0(n)^{\frac{n-6}{6}}\},$$

and

$$\lim_{t\to-\infty} w^{(j)}(t,\theta) = \lim_{t\to-\infty}\overline{w}^{(j)}(t) = 0 \quad \text{for all} \quad j \in \{1,2,3,4,5\}. \tag{3.5}$$

Proof. Indeed, using Remark 3.7, it is straightforward to check that

$$w(t,\theta) = \overline{w}(t)(1 + \mathcal{O}(\overline{w}(t)e^t)) \quad \text{as} \quad t \to -\infty.$$

Furthermore, the cylindrical transformation of spherical average $\overline{w} = \widetilde{\mathfrak{F}}(\overline{u})$ satisfies,

$$-\widehat{P}_{\mathrm{rad}}\overline{w} = f_\#(\overline{w}) + \mathcal{O}(\overline{w}(t)e^t) \quad \text{as} \quad t \to -\infty, \tag{3.6}$$

where

$$\widehat{P}_{\mathrm{rad}} = \partial_t^{(6)} + \widehat{K}_5(n)\partial_t^{(5)} + \widehat{K}_4(n)\partial_t^{(4)} + \widehat{K}_3(n)\partial_t^{(3)} + \widehat{K}_2(n)\partial_t^{(2)} + \widehat{K}_1(n)\partial_t + \widehat{K}_0(n). \tag{3.7}$$

and

$$f_\#(\overline{w}) = |\overline{w}|^{2_\#-1}\overline{w}$$

We split the rest of the proof into three claims:

Claim 1: Either $\overline{w}(t) = \mathrm{o}(1)$ or $\overline{w}(t) = \widehat{K}_0(n)^{\frac{n-6}{6}} + \mathrm{o}(1)$ as $t \to -\infty$.

Indeed, we define the following Hamiltonian energy associated with (3.6) as follows where

$$\widehat{\mathcal{H}}_{\mathrm{rad}}(t,\overline{w}) := \Bigg[\overline{w}^{(5)}\overline{w}^{(1)} - \overline{w}^{(4)}\overline{w}^{(2)} + \frac{1}{2}w^{(3)^2} + \widehat{K}_5\left(\overline{w}^{(4)}\overline{w}^{(1)} - \overline{w}^{(3)}\overline{w}^{(2)}\right) + \widehat{K}_4\left(\overline{w}^{(3)}\overline{w}^{(1)} - \frac{1}{2}\overline{w}^{(2)^2}\right)$$
$$+\widehat{K}_3\overline{w}^{(2)}\overline{w}^{(1)} + \frac{\widehat{K}_2}{2}\overline{w}^{(1)^2} + \frac{\widehat{K}_0}{2}\overline{w}^2 - \frac{n-6}{2(n-3)}|\overline{w}|^{\frac{2(n-3)}{n-6}}\Bigg](t) + \mathcal{O}(\overline{w}(t)e^t).$$

Now, we observe that the coefficients $\widehat{K}_1(n), \widehat{K}_2(n), \widehat{K}_3(n), \widehat{K}_4(n), \widehat{K}_5(n) \in \mathbb{R}$ are given by (A.10) We also define the associated Pohozaev functional as

$$\widehat{\mathcal{P}}_{\mathrm{rad}}(t,\overline{w}) := \int_{\mathbb{S}^{n-1}_t} \widehat{\mathcal{H}}_{\mathrm{rad}}(t,\overline{w})\mathrm{d}\theta$$

In addition, observe that $\widehat{K}_1(n), \widehat{K}_5(n) \geqslant 0$ and $\widehat{K}_3(n) \leqslant 0$, which by a direct computation shows that

$$\partial_t\widehat{\mathcal{P}}_{\mathrm{rad}}(t,\overline{w}) = \int_{\mathbb{S}^{n-1}_t}\left(-\widehat{K}_5|\overline{w}^{(3)}(t)|^2 + \widehat{K}_3|\overline{w}^{(2)}(t)|^2 - \widehat{K}_1|\overline{w}^{(1)}(t)|^2\right)\mathrm{d}\theta < 0. \tag{3.8}$$

This implies that $\widehat{\mathcal{P}}_{\mathrm{rad}}(t,\overline{w})$ is monotonically nonincreasing on variable t, from which combined with the uniform estimates in Remark 3.7, we conclude that the limit $\lim_{t\to-\infty}\widehat{\mathcal{P}}_{\mathrm{rad}}(t,\overline{w}) < +\infty$ exists.

Let $u \in C^6(\mathbb{R}^n \setminus \{0\})$ be a nonnegative solution to $(\mathcal{P}_{6,p,\infty})$ with $p = 2_\#$. such that $u = \widetilde{\mathfrak{F}}^{-1}(w)$, and define

$$\widehat{\mathcal{P}}_{\mathrm{rad}}(r,u) := \int_{\mathbb{S}^{n-1}_t} \widehat{\mathcal{H}}_{\mathrm{rad}}(t,\overline{w})\mathrm{d}\theta$$

Notice that we have

$$\widehat{\mathcal{P}}_{\mathrm{rad}}(0,u) := \lim_{r\to 0^+}\widehat{\mathcal{P}}_{\mathrm{rad}}(r,u) = \lim_{t\to-\infty}\widehat{\mathcal{P}}_{\mathrm{rad}}(t,\overline{w}).$$

For any $\lambda > 0$ and $\tau = \ln\lambda$, define the rescaled solution

$$u_\lambda(x) := (\ln\lambda)^{\frac{n-6}{6}}\lambda^{n-6}u(\lambda x)$$

or using nonautonomous Emden–Fowler change of variables

$$\overline{w}_\tau(t) := \tau^{\frac{n-6}{6}}e^{(n-6)\tau}\overline{w}(t+\tau).$$

It is not hard to verify that $u_\lambda \in C^6(B^*_{1/\lambda})$ is also a nonnegative solution to $(\mathcal{P}_{6,p,R})$ with $R = \lambda^{-1}$ and $p = 2_\#$. Moreover, we have

$$\widehat{\mathcal{P}}_{\mathrm{rad}}(r,u_\lambda) = \widehat{\mathcal{P}}_{\mathrm{rad}}(t,\overline{w}_\tau) = \widehat{\mathcal{P}}_{\mathrm{rad}}(t+\tau,\overline{w}) = \widehat{\mathcal{P}}_{\mathrm{rad}}(\lambda r,u),$$

from which we derive the following scaling invariance

$$\widehat{\mathcal{P}}_{\mathrm{rad}}(r,u_\lambda) = \widehat{\mathcal{P}}_{\mathrm{rad}}(\lambda r,u_\lambda).$$

Now, let us compute the possible asymptotic values of $\widehat{\mathcal{P}}_{\mathrm{rad}}(0,u)$. Using Lemma 4.6, it follows that the family $\{u_\lambda\}_{\lambda>0} \subset C^{6,\alpha}(B_{1/2\lambda})$ is uniformly bounded in $C^{6,\alpha}(K)$ on every compact set $K \subset B_{1/2\lambda}$ and for some $\alpha \in (0,1)$. Hence, one can find a nonnegative function $u_0 \in C^6(\mathbb{R}^n \setminus \{0\})$ solving $(\mathcal{P}_{6,p,\infty})$ with $p = 2_\#$ and subfamily, still denoted the same, satisfying

$$\lim_{\lambda\to 0}\|u_\lambda - u_0\|_{C^6_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})} = 0.$$

Moreover, by [9, Theorem 1.2], we know that $u_0 \in C^6(\mathbb{R}^n\setminus\{0\})$ satisfies $-\Delta u_0 \geqslant 0$ and $\Delta^2 u_0 \geqslant 0$ in $\mathbb{R}^n\setminus\{0\}$, which, by the maximum principle, yields that either $u_0 \equiv 0$ or $u_0 > 0$ in $\mathbb{R}^n\setminus\{0\}$. Therefore, it is not hard to check that this blow-up limit $u_0 \in C^6(\mathbb{R}^n \setminus \{0\})$ is radially symmetric with respect to the origin. Moreover, by the scaling invariance of the Pohozaev functional, we find

$$\widehat{\mathcal{P}}_{\mathrm{rad}}(r,u_0) = \lim_{\lambda\to 0}\widehat{\mathcal{P}}_{\mathrm{rad}}(r,u_\lambda) = \lim_{\lambda\to 0}\widehat{\mathcal{P}}_{\mathrm{rad}}(\lambda r,u) = \widehat{\mathcal{P}}_{\mathrm{rad}}(0,u) \quad \text{for any} \quad r > 0. \tag{3.9}$$

Finally, we set $w_0 = \widetilde{\mathfrak{F}}(u_0)$ and notice that it solves

$$-\widehat{P}_{\mathrm{rad}}\overline{w}_0 = f_\#(\overline{w}_0) \quad \text{in} \quad \mathbb{R}.$$

In addition, from (3.9) it follows that $\widehat{\mathcal{P}}_{\mathrm{rad}}(t,\overline{w}_0) = \widehat{\mathcal{P}}_{\mathrm{rad}}(r,u_0)$ is a constant, which combined with (3.8) implies

$$\partial_t\widehat{\mathcal{P}}_{\mathrm{rad}}(t,\overline{w}_0) = \int_{\mathbb{S}^{n-1}_t}\left(-\widehat{K}_5|\overline{w}_0^{(3)}(t)|^2 + \widehat{K}_3|\overline{w}_0^{(2)}(t)|^2 - \widehat{K}_1|\overline{w}_0^{(1)}(t)|^2\right)\mathrm{d}\theta \equiv 0.$$

From this, we conclude that either $\overline{w}_0 \equiv 0$ or $\overline{w}_0 \equiv \widehat{K}_0(n)^{\frac{n-6}{6}}$, which together with (3.8) yields

$$\widehat{\mathcal{P}}_{\mathrm{rad}}(0,u) \in \left\{0, \left(\frac{1}{2} - \frac{n-6}{2(n-3)}\right)\widehat{K}_0(n)^{\frac{n-3}{3}}|\mathbb{S}^{n-1}|\right\}.$$

At last, we have two cases to analyze:

Case 1: If $\widehat{\mathcal{P}}_{\mathrm{rad}}(0,u) = 0$, then $u_0 \equiv 0$.

Since the blow-up limit $u_0 \in C^6(\mathbb{R}^n \setminus \{0\})$ is unique, we conclude

$$\lim_{\lambda\to 0}\|u_\lambda\|_{C^6_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})} = 0.$$

Therefore, we easily get

$$\lim_{|x|\to 0}(\ln|x|)^{\frac{n-6}{6}}|x|^{n-6}u(x)=0,$$

which finished the first case.

Case 2: If $\widehat{\mathcal{P}}_{\rm rad}(0,u)=\frac{3|\mathbb{S}^{n-1}|}{2(n-3)}\widehat{K}_0(n)^{\frac{n-3}{3}}$, then $u_0(x)\equiv \widehat{K}_0(n)^{\frac{n-6}{6}}(\ln|x|)^{\frac{6-n}{6}}|x|^{6-n}$.

In this case, we also have uniqueness of $u_0\in C^6(\mathbb{R}^n\setminus\{0\})$, so we find

$$\lim_{\lambda\to 0}\|u_\lambda-\widehat{K}_0(n)^{\frac{n-6}{6}}(\ln|x|)^{\frac{6-n}{6}}|x|^{6-n}\|_{C^6_{\rm loc}(\mathbb{R}^n\setminus\{0\})}=0.$$

In particular, we have

$$\|u_\lambda-\widehat{K}_0(n)^{\frac{n-6}{6}}\|_{C^6(\mathbb{S}^{n-1})}\quad\text{as}\quad\lambda\to 0,$$

from which we quickly get

$$\lim_{|x|\to 0}(\ln|x|)^{\frac{n-6}{6}}|x|^{n-6}u(x)=\widehat{K}_0(n)^{\frac{n-6}{6}},$$

and so the second case is proved

The proof of the first claim is concluded.

Claim 2:$\overline{w}^{(j)}(t)=o(1)$ as $t\to-\infty$ for all $j\geqslant 1$.

Indeed, let us prove the following claim that $\lim_{t\to-\infty}\overline{w}^{(4)}(t)=0$ and $\lim_{t\to-\infty}\overline{w}^{(2)}(t)=0$.

Let us define the function $Y_1(\overline{w})=\overline{w}^{(5)}+\widehat{K}_5\overline{w}^{(4)}+\widehat{K}_4\overline{w}^{(3)}+\widehat{K}_3\overline{w}^{(2)}+\widehat{K}_2\overline{w}^{(1)}$, which satisfies $Y_1^{(2)}(\overline{w})=\overline{w}^{(1)}g_\#(\overline{w})$, where $g_\#(\overline{w})=f_\#(\overline{w})-\widehat{K}_0\overline{w}$. Now, we have to consider two cases:

Case 1: $g_\#^{(1)}(\ell_0)\neq 0$

In this case, $g_\#^{(1)}(\overline{w}(t))$ has a sign for $-t\gg 1$ large enough, that is, either $g_\#^{(1)}(\overline{w}(t))\geqslant 0$ or $g_\#^{(1)}(\overline{w}(t))\leqslant 0$ for large $-t\gg 1$. Also, since $\overline{w}^{(1)}\geqslant 0$, it follows $\lim_{t\to-\infty}\operatorname{sign}(Y_1^{(2)}(\overline{w}(t)))\neq 0$, which implies that $Y_1^{(2)}(\overline{w}(t))$ has a sign for $-t\gg 1$ large enough; thus by a comparison principle $Y_1(\overline{w}(t))$ also does. Furthermore, because $Y_1(\overline{w}(t))$ has a sign for $-t\gg 1$ large enough, we find the following limit exists:

$$\lim_{t\to-\infty}\left(\overline{w}^{(4)}(t)+\widehat{K}_5\overline{w}^{(3)}(t)+\widehat{K}_4\overline{w}^{(2)}(t)+\widehat{K}_3\overline{w}^{(1)}(t)+\widehat{K}_2\overline{w}(t)\right):=\ell_1.$$

Consequently, we get

$$\lim_{t\to-\infty}\left(\overline{w}^{(4)}(t)+\widehat{K}_5\overline{w}^{(3)}(t)+\widehat{K}_4\overline{w}^{(2)}(t)+\widehat{K}_3\overline{w}^{(1)}(t)\right):=\ell_1-K_2\ell_0. \tag{3.10}$$

In addition, it is not hard to check that there exists $\mu_1,\mu_2>0$ such that the following decomposition holds

$$\left(\overline{w}^{(4)}(t)+\widehat{K}_5\overline{w}^{(3)}(t)+\widehat{K}_4\overline{w}^{(2)}(t)+\widehat{K}_3\overline{w}^{(1)}(t)\right)=(L_{\mu_1}\circ L_{\mu_2})\overline{w}(t),$$

where

$$L_{\mu_1}=-\partial_t^{(2)}+\mu_1\quad\text{and}\quad L_{\mu_2}=-\partial_t^{(2)}+\mu_2.$$

Now defining

$$Y_2(\overline{w}):=L_{\mu_2}(\overline{w}),$$

we get $\lim_{t\to-\infty}L_{\mu_1}(Y_2(\overline{w}))\neq 0$, that is, $Y_2(\overline{w}(t))$ has a sign for $-t\gg 1$ large enough. Hence, by a comparison principle $Y_2(\overline{w}(t))$ also does. Moreover, since $Y_2(\overline{w}(t))$ has a sign for $-t\gg 1$ large enough, we get that the following limit exists

$$\lim_{t\to-\infty}Y_2(\overline{w}(t)):=\ell_2.$$

Therefore, we conclude

$$\lim_{t\to-\infty}\overline{w}^{(2)}(t)=\ell_2+\mu_2\ell_0,$$

which by the boundedness of $\overline{w}$, proves that $\overline{w}^{(2)}(t)\to 0$ as $t\to-\infty$. Finally, going back to (3.10), we find $\overline{w}^{(4)}(t)\to 0$ as $t\to-\infty$.

Case 2: $g_\#^{(1)}(\ell_0)=0$

In this case, we can define

$$\widetilde{Y}_1(\overline{w})=\overline{w}^{(5)}+\widehat{K}_5\overline{w}^{(4)}+\widehat{K}_4\overline{w}^{(3)}+\widehat{K}_3\overline{w}^{(2)}+\frac{\widehat{K}_2}{2}\overline{w}^{(1)}$$

and proceed as before to conclude the proof.

At last, using that $\overline{w}^{(2)}(t)\to 0$ and $\overline{w}^{(4)}(t)\to 0$ as $t\to-\infty$, one can prove that $\overline{w}^{(1)}(t)\to 0$, $\overline{w}^{(3)}(t)\to 0$, and $\overline{w}^{(5)}(t)\to 0$ as $t\to-\infty$. Since, we can write (3.6) as

$$\overline{w}^{(6)}=-K_5\overline{w}^{(5)}-K_4\overline{w}^{(4)}-K_3\overline{w}^{(3)}-K_2\overline{w}^{(2)}-K_1\overline{w}^{(1)}-g_\#(\overline{w})+\mathcal{O}(\overline{w}(t)e^t)\quad\text{in}\quad\mathbb{R},$$

it holds $\lim_{t\to-\infty}\overline{w}^{(6)}(t):=\ell_3=g_\#(\ell_0)$. Therefore, $\overline{w}$ is bounded, we find $\ell_3=0$, and thus $\ell_0\in\{0,\widehat{K}_0(n)^{\frac{n-6}{6}}\}$. By this discussion, it is easy now to see that (3.5) holds; this concludes the proof of this case.

At last, Claims 1 and 2 combined gives the proof of the lemma.

To prove the full existence of $\widetilde{\mathcal{P}}_{\rm cyl}(-\infty, w)$, we shall verify the estimates in the next lemma, which is a sixth order version of [7, Lemma 3.2]. Due to the appearance of higher order derivative terms, we give different proof to the ones in the lemma quoted above. Namely, our proof is based on Lemma 3.8 combined with a simple L'Hôpital rule. Initially, by differentiating $\widetilde{\mathcal{H}}(t,\theta,w)$ with respect to t, integrating by parts over $\mathbb{S}^{n-1}_t$ (using differentiation under the integral sign one can even omit the dependence on t), and using $(\widetilde{\mathcal{C}}_T)$, we find

$$\partial_t \widetilde{\mathcal{P}}_{\rm cyl}(t,w) = \Xi_{\rm rad}(t,w) + \Xi_{\rm ang}(t,w).$$

Here, by direct differentiating (3.2) with respect to t, one has

$$\Xi_{\rm rad}(t,w) := \int_{\mathbb{S}^{n-1}} \partial_t \widetilde{\mathcal{H}}_{\rm rad}(t,\theta,w)\mathrm{d}\theta \quad \text{and} \quad \Xi_{\rm ang}(t,w) := -\int_{\mathbb{S}^{n-1}} \partial_t \widetilde{\mathcal{H}}_{\rm ang}(t,\theta,w)\mathrm{d}\theta. \tag{3.11}$$

Although we do not use the angular part of the Pohozaev functional we introduced above, we need to study its radial part. We observe that it is only necessary to compute the asymptotic behavior of this functional for $t \to -\infty$. This is inspired by the computations in [7, Section 3] and [14, Section 4].

Lemma 3.9. *Let $w \in C^6(\mathbb{R})$ be a positive solution to $(\widetilde{\mathcal{C}}_T)$. Then, one has*

$$\begin{aligned}\partial_t \widetilde{\mathcal{H}}_{\rm rad}(t,\theta,\overline{w}) = {} & \overline{w}^{(5)}\overline{w}^{(1)} - \overline{w}^{(4)}\overline{w}^{(2)} + \mathfrak{p}_5 \overline{w}^{(4)}\overline{w}^{(1)} - \mathfrak{p}_5 \overline{w}^{(3)}\overline{w}^{(2)} + \mathfrak{p}_2 \overline{w}^{(3)} w^{(1)} + \mathfrak{p}_4 \overline{w}^{(2)}\overline{w}^{(1)} \\ & + \mathfrak{p}_3 (\overline{w}^{(3)})^2 - \mathfrak{p}_2 (\overline{w}^{(2)})^2 + \mathfrak{p}_1 (\overline{w}^{(1)})^2 + \mathfrak{p}_0 \overline{w}^2 + \mathcal{O}(\overline{w}(t)e^t) \quad \text{as} \quad t \to -\infty,\end{aligned}$$

where the real-valued functions $\mathfrak{p}_j(n,\cdot) : (-\infty, \ln R) \to \mathbb{R}$ for $j \in \{0,1,2,3,4,5\}$ satisfy

$$\mathfrak{p}_0(n,t) := \frac{1}{36} n(n-6)(3n^3 - 48n^2 + 228n - 320)t^{-2} + \mathcal{O}(t^{-4}), \tag{3.12}$$

$$\mathfrak{p}_1(n,t) := 8(n-2)(n-4)(n-6)t + \frac{(n-6)^2 n(n^2 - 24n + 68)}{12t} + \mathcal{O}(t^{-2}), \tag{3.13}$$

$$\mathfrak{p}_2(n,t) := -\frac{1}{2}(3n^2 - 42n + 124) + \mathcal{O}(t^{-2}), \tag{3.14}$$

$$\mathfrak{p}_3(n,t) := \frac{1}{2}(6n - 35) + \mathcal{O}(t^{-1}), \tag{3.15}$$

$$\mathfrak{p}_4(n,t) := -\frac{1}{2}(n^3 - 30n^2 + 212n - 408) + \mathcal{O}(t^{-2}), \tag{3.16}$$

$$\mathfrak{p}_5(n,t) := -\frac{3}{2}(n-6). \tag{3.17}$$

as $t \to -\infty$.

Proof. On one hand, a direct computation shows

$$\begin{aligned}\partial_t \widetilde{\mathcal{H}}_{\rm rad}(t,\theta,\overline{w}) = {} & \overline{w}^{(5)}\overline{w}^{(1)} - \overline{w}^{(4)}\overline{w}^{(2)} + \frac{1}{2}(\widetilde{K}_5 + t\widetilde{K}_5^{(1)})(\overline{w}^{(4)}\overline{w}^{(1)} - \overline{w}^{(3)}\overline{w}^{(2)}) \\ & + \frac{1}{2}(\widetilde{K}_4 + t\widetilde{K}_4^{(1)})(\overline{w}^{(3)}\overline{w}^{(1)} - \overline{w}^{(2)^2}) + \frac{1}{2}(\widetilde{K}_3 + t\widetilde{K}_3^{(1)})\overline{w}^{(2)}\overline{w}^{(1)} \\ & + \frac{1}{2}(1 - 2t\widetilde{K}_5)(\overline{w}^{(3)})^2 + \frac{1}{2}(\widetilde{K}_2 + t\widetilde{K}_2^{(1)})(\overline{w}^{(1)})^2 + \frac{1}{2}(\widetilde{K}_0 + t\widetilde{K}_0^{(1)})\overline{w}^2 \\ & + \left(t\overline{w}^{(6)} + t\widetilde{K}_5\overline{w}^{(5)} + t\widetilde{K}_4\overline{w}^{(4)} + t\widetilde{K}_3\overline{w}^{(3)} + t\widetilde{K}_2\overline{w}^{(2)} + t\widetilde{K}_0\overline{w} - f_\#(\overline{w})\right)\overline{w}^{(1)}.\end{aligned}$$

On the other hand, since $w \in C^6(\mathbb{R})$ is a positive solution to $(\widetilde{\mathcal{C}}_T)$, we get

$$t\overline{w}^{(6)} + t\widetilde{K}_5\overline{w}^{(5)} + t\widetilde{K}_4\overline{w}^{(4)} + t\widetilde{K}_3\overline{w}^{(3)} + t\widetilde{K}_2\overline{w}^{(2)} + t\widetilde{K}_0\overline{w} - f_\#(\overline{w}) = -t\widetilde{K}_1\overline{w}^{(1)} + \mathcal{O}(\overline{w}(t)e^t) \quad \text{as} \quad t \to -\infty.$$

From this, we conclude that the coefficient functions $\mathfrak{p}_j(n,\cdot) : (-\infty, \ln R) \to \mathbb{R}$ for $j \in \{0,1,2,3,4,5\}$ are given in terms of nonautonomous coefficients and their derivatives. More precisely, we find

$$\mathfrak{p}_5(n,t) := \frac{1}{2}[\widetilde{K}_5(n,t) + t\widetilde{K}_5^{(1)}(n,t)],$$

$$\mathfrak{p}_4(n,t) := \frac{1}{2}[\widetilde{K}_3(n,t) + t\widetilde{K}_3^{(1)}(n,t)],$$

$$\mathfrak{p}_3(n,t) := \frac{1}{2}[1 - 2t\widetilde{K}_5(n,t)],$$

$$\mathfrak{p}_2(n,t) := \frac{1}{2}[\widetilde{K}_4(n,t) + t\widetilde{K}_4^{(1)}(n,t)],$$

$$\mathfrak{p}_1(n,t) := \frac{1}{2}[\widetilde{K}_2(n,t) + t\widetilde{K}_2^{(1)}(n,t) - 2t\widetilde{K}_1(n,t)],$$

$$\mathfrak{p}_0(n,t) := \frac{1}{2}[\widetilde{K}_0(n,t) + t\widetilde{K}_0^{(1)}(n,t)].$$

A direct computation using the formulas above finishes the proof.

Now we use the asymptotic estimate proved in Lemma 3.8 to prove the result below

Lemma 3.10. *Let $w \in C^6(\mathbb{R})$ be a positive solution to $(\widetilde{C}_T)$. Then,* $\lim_{t\to-\infty} \Xi_{\rm rad}(t,w) = 0$.

Proof. Notice that

$$\Xi_{\rm rad}(t,w) =: I_{10} + I_9 + I_8 + I_7 + I_6 + I_5 + I_4 + I_3 + I_2 + I_1 + I_0 + \mathcal{O}(\bar{w}(t)e^t).$$

Next, we estimate each term of the last identity separately by steps.

Step 1: $I_0 := \lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_0(n,t)w(t,\theta)^2 \mathrm{d}\theta = 0$.

Using (3.12), we find that $\lim_{t\to-\infty} \mathfrak{p}_0(n,t) = 0$, which by means of (3.4) yields

$$\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_0(n,t)w(t,\theta)^2 \mathrm{d}\theta = \left(\lim_{t\to-\infty} \mathfrak{p}_0(n,t)\right)\left(\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} w(t,\theta)^2 \mathrm{d}\theta\right) = 0.$$

This gives us the desired conclusion.

Step 2: $I_1 := \lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_1(n,t)w^{(1)}(t,\theta)^2 \mathrm{d}\theta = 0$.

It holds $\lim_{t\to-\infty} \mathfrak{p}_1(n,t) = -\infty$. However, by (3.13), one has $\mathfrak{p}_1(n,t) = \widetilde{\mathfrak{p}}_1(n,t) + \widehat{\mathfrak{p}}_1(n,t)$, where

$$\widetilde{\mathfrak{p}}_1(n,t) = \frac{(n-6)^2 n(n^2-24n+68)}{12} t^{-1} + \mathcal{O}(t^{-2}) \quad \text{and} \quad \widehat{\mathfrak{p}}_1(n,t) = 8(n-2)(n-4)(n-6)t.$$

With this notation, we set

$$\begin{aligned}\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_1(n,t)w^{(1)}(t,\theta)^2 \ \mathrm{d}\theta &= \lim_{t\to-\infty} \left[\widetilde{\mathfrak{p}}_1(n,t) \int_{\mathbb{S}^{n-1}} w^{(1)}(t,\theta)^2 \ \mathrm{d}\theta\right] + \lim_{t\to-\infty} \left[\widehat{\mathfrak{p}}_1(n,t) \int_{\mathbb{S}^{n-1}} w^{(1)}(t,\theta)^2 \ \mathrm{d}\theta\right] \\ &:= \widetilde{I}_1 + \widehat{I}_1.\end{aligned}$$

Since $\lim_{t\to\infty} \tilde{\mathfrak{p}}_1(n,t) = 0$, we see that $\tilde{I}_1 = 0$. Moreover, to estimate $\widehat{I}_1$, we use the L'Hôpital rule as follows

$$\begin{aligned}\widehat{I}_1 &= \lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \widehat{\mathfrak{p}}_1(n,t)w^{(1)}(t,\theta)^2 \ \mathrm{d}\theta \\ &= \lim_{t\to-\infty} \frac{8(n-2)(n-4)(n-6)}{\partial_t\left[\left(\int_{\mathbb{S}^{n-1}} w^{(1)}(t,\theta)^2 \mathrm{d}\theta\right)^{-1}\right]} \\ &= -4(n-2)(n-4)(n-6) \lim_{t\to-\infty} \frac{\left(\int_{\mathbb{S}^{n-1}} w^{(1)}(t,\theta)^2 \mathrm{d}\theta\right)^2}{\int_{\mathbb{S}^{n-1}} w^{(1)}(t,\theta)w^{(2)}(t,\theta)\mathrm{d}\theta} \\ &\leqslant \lim_{t\to-\infty} \frac{w^{(1)}(t,\theta)^2}{w^{(2)}(t,\theta)} \\ &= \lim_{t\to-\infty} \frac{6\overline{w}^{(3)}(t)^2 + 8\overline{w}^{(2)}(t)\overline{w}^{(4)}(t) + 2\overline{w}^{(1)}(t)\overline{w}^{(5)}(t)}{\overline{w}^{(6)}(t)} \\ &= \lim_{t\to-\infty} \frac{6\overline{w}^{(3)}(t)^2 + 8\overline{w}^{(2)}(t)\overline{w}^{(4)}(t) + 2\overline{w}^{(1)}(t)\overline{w}^{(5)}(t)}{-\widehat{K}_5(n)\overline{w}^{(5)}(t) - \widehat{K}_4(n)\overline{w}^{(4)}(t) - \widehat{K}_3(n)\overline{w}^{(3)}(t) - \widehat{K}_2(n)\overline{w}^{(2)}(t) - \widehat{K}_1(n)\overline{w}^{(1)}(t) - \widehat{K}_0(n) - f_\#(\overline{w}(t))} \\ &= 0\end{aligned}$$

where we used (3.6), (3.7), and (3.5) in the last three steps. Since $\widehat{I}_1 \geqslant 0$, the proof is concluded.

Step 3: $I_2 := \lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_2(n,t)w^{(2)}(t,\theta)^2 \mathrm{d}\theta = 0$.

As before, by (3.14), it holds that $\lim_{t\to-\infty} \mathfrak{p}_2(n,t) < \infty$ and using

$$\lim_{t\to\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_2(n,t)w^{(2)}(t,\theta)^2 \mathrm{d}\theta = \left(\lim_{t\to-\infty} \mathfrak{p}_2(n,t)\right)\left(\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} w^{(2)}(t,\theta)^2 \mathrm{d}\theta\right) = 0,$$

the proof of this step follows promptly.

Step 4: $I_3 := \lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_3(n,t)w^{(2)}(t,\theta)w^{(1)}(t,\theta) \mathrm{d}\theta = 0$.

In fact, using (3.15), one has that $\lim_{t\to-\infty} \mathfrak{p}_3(n,t) < +\infty$. Thus,

$$\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_3(n,t)w^{(2)}(t,\theta)w^{(1)}(t,\theta) \mathrm{d}\theta = \left(\lim_{t\to-\infty} \mathfrak{p}_3(n,t)\right)\left(\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} w^{(2)}(t,\theta)w^{(1)}(t,\theta) \mathrm{d}\theta\right) = 0,$$

where we used Lemma 3.8. This concludes the argument.

Step 5: $I_4 := -\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_4(n,t)w^{(2)}(t,\theta)^2 \mathrm{d}\theta = 0$.

Indeed, by (3.16), we obtain that $\lim_{t\to-\infty} \mathfrak{p}_4(n,t) < +\infty$. Thus,

$$-\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_4(n,t)w^{(2)}(t,\theta)^2 \mathrm{d}\theta = -\left(\lim_{t\to-\infty} \mathfrak{p}_4(n,t)\right)\left(\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} w^{(2)}(t,\theta)^2 \mathrm{d}\theta\right) = 0,$$

where we used Lemma 3.8. This concludes the argument.

Step 6: $I_5 := \lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_4(n,t)w^{(3)}(t,\theta)w^{(1)}(t,\theta) \mathrm{d}\theta = 0$.

Using the same argument as before, we find

$$\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_4(n,t)w^{(3)}(t,\theta)w^{(1)}(t,\theta) \mathrm{d}\theta = \left(\lim_{t\to-\infty} \mathfrak{p}_4(n,t)\right)\left(\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} w^{(3)}(t,\theta)w^{(1)}(t,\theta) \mathrm{d}\theta\right) = 0,$$

which finishes this step.

Step 7: $I_6 := -\lim_{t\to-\infty} \int_{\mathbb{S}^{n-1}} \mathfrak{p}_5(n,t)w^{(3)}(t,\theta)w^{(2)}(t,\theta) \mathrm{d}\theta = 0$.

Indeed, by (3.17), we obtain that $\lim_{t\to-\infty}\mathfrak{p}_4(n,t) < +\infty$. Thus,

$$-\lim_{t\to-\infty}\int_{\mathbb{S}^{n-1}}\mathfrak{p}_5(n,t)w^{(3)}(t,\theta)w^{(2)}(t,\theta)\mathrm{d}\theta = -\left(\lim_{t\to-\infty}\mathfrak{p}_5(n,t)\right)\left(\lim_{t\to-\infty}\int_{\mathbb{S}^{n-1}}w^{(3)}(t,\theta)w^{(2)}(t,\theta)\mathrm{d}\theta\right) = 0,$$

where we used Lemma 3.8. This concludes the argument.

Step 8: $I_7 := \lim_{t\to-\infty}\int_{\mathbb{S}^{n-1}}\mathfrak{p}_5(n,t)w^{(4)}(t,\theta)w^{(1)}(t,\theta)\mathrm{d}\theta = 0$.

Using the same argument as before, we find

$$\lim_{t\to-\infty}\int_{\mathbb{S}^{n-1}}\mathfrak{p}_5(n,t)w^{(4)}(t,\theta)w^{(1)}(t,\theta)\mathrm{d}\theta = \left(\lim_{t\to-\infty}\mathfrak{p}_5(n,t)\right)\left(\lim_{t\to-\infty}\int_{\mathbb{S}^{n-1}}w^{(4)}(t,\theta)w^{(1)}(t,\theta)\mathrm{d}\theta\right) = 0,$$

which finishes this step.

Step 9: $I_8 := \lim_{t\to-\infty}\int_{\mathbb{S}^{n-1}}w^{(3)}(t,\theta)^2\mathrm{d}\theta = 0$.

It follows directly by Lemma 3.8.

Step 10: $I_9 := \lim_{t\to-\infty}\int_{\mathbb{S}^{n-1}}w^{(4)}(t,\theta)w^{(2)}(t,\theta)\mathrm{d}\theta = 0$.

It follows directly by Lemma 3.8.

Step 11: $I_{10} := \lim_{t\to-\infty}\int_{\mathbb{S}^{n-1}}w^{(5)}(t,\theta),w^{(1)}(t,\theta)\mathrm{d}\theta = 0$.

It follows directly by Lemma 3.8.

Finally, putting together all the steps the proof is concluded.

The next proposition is the monotonicity of this new Pohozaev functional.

Proposition 3.11. *Let $w \in C^6(\mathbb{R})$ be a positive solution to $(\widetilde{C}_T)$. Then, one has*

(i) *if $n = 7, 8$, then there exists $T_* \gg 1$ such that $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(t,w)$ is non-decreasing for $t > T_*$.*
(ii) *if $n \geqslant 9$, then there exists $T^* \gg 1$ such that $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(t,w)$ is nonincreasing for $t > T^*$.*
Moreover, $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(-\infty,w) := \lim_{t\to-\infty}\widetilde{\mathcal{P}}_{\mathrm{cyl}}(t,w)$ exists.

Proof. Initially, using Remark 3.7 and Lemma 3.8 one can find $-t_0 \gg 1$ such that $\mathrm{sign}(\partial_t\widetilde{\mathcal{H}}_{\mathrm{rad}}(t,w)) = \mathrm{sign}(\mathfrak{p}_0(n,t)|w|^2)$ for $|t| > |t_0|$. Let us denote by $\mathfrak{p}_0(n,t) = \hat{\mathfrak{p}}_0(n)t^{-2}$, where $\hat{\mathfrak{p}}_0(n) \in \mathbb{Z}[n]$ as $\hat{\mathfrak{p}}_0(n) = 3n^3 - 48n^2 + 228n - 320$.

On one hand, by (3.12) since $\hat{\mathfrak{p}}_0(n) < 0$ for $n \geqslant 9$, we directly verify that there exists $-t_1 \gg 1$ sufficiently large such that $\mathfrak{p}_0(n,t) < 0$ for $t| > |t_1|$, which, by taking $T^* := \max\{t_1,t_0\}$, implies that $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(t,w)$ is nonincreasing for $|t| > T^*$. On the other hand, since $\hat{\mathfrak{p}}_0(n) > 0$ for $n = 7, 8$, there exists $-t_1 \gg 1$ sufficiently large such that $\mathfrak{p}_0(n,t) > 0$ for $|t| > t_2$; thus, setting $T_* := \max\{t_0,t_2\}$, we get that $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(t,w)$ is nondecreasing for $|t| > T_*$. Since $\lim_{t\to-\infty}\Xi_{\mathrm{ang}}(t,w) = 0$ exists and $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(t,w)$ is uniformly bounded as $t \to -\infty$, we deduce that $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(-\infty,w)$ exists. The proof is concluded.

4. A priori upper bounds

This subsection is devoted to providing a priori upper bounds for positive singular solutions to $(\mathcal{P}_{6,p,R})$ with $p \in (1, 2^\# - 1)$. Our strategy relies on the classification of the limit solutions to $(\mathcal{P}_{6,p,R})$ combined with a blow-up argument. When $p \in (2_\#, 2^\# - 1)$, most of our arguments in this section are similar in spirit to the ones in [9], so we omit them here. Nevertheless, for $p = 2_\#$, our proof is completely new and based on [11]. Notice that by scaling invariance, we may assume $R = 1$ without loss of generality. Also, all constants in this section depend only on n and p.

Lemma 4.1. *Let $u \in C^6(B_1) \cap C(\bar{B}_1)$ be a positive non-singular solution to $(\mathcal{P}_{6,p,R})$ with $p \in (1, 2^\# - 1)$ and $R = 1$. Then, there exists $C > 0$ (depending only on n and p) such that*

$$u(x) \leqslant C(1-|x|)^{-\gamma_p} \quad \text{in} \quad B_1. \tag{4.1}$$

Proof. See [9, Lemma 3.1].

Using the last lemma, we can prove the auxiliary result below

Lemma 4.2. *Let $u \in C^6(B_1^*)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p \in (1, 2^\# - 1)$ and $R = 1$. Then, there exists $C > 0$ (depending only on n and p) such that*

$$|u(x)| \leqslant C|x|^{-\gamma_p} \quad \text{in} \quad B_{1/2}^*.$$

Proof. Fixing $x_0 \in B_{1/2}^*$, let us define $r = \frac{1}{2}|x_0|$. Since $\bar{B}_r(x_0) \subset B_1^*$, the rescaled function given by

$$\widetilde{u}_r(x) = r^{\gamma_p}u(rx + x_0) \quad \text{in} \quad \bar{B}_1.$$

is well-defined. Furthermore, since u is a positive singular solution to $(\mathcal{P}_{6,p,R})$, we obtain that $\widetilde{u}_r$ is a positive non-singular solution to

$$(-\Delta)^3\widetilde{u}_r = f_p(\widetilde{u}_r) \quad \text{in} \quad B_1,$$

which is continuous up to the boundary. Therefore, we can apply Lemma 4.1 to $\widetilde{u}_r$, which, by taking $x = 0$ in (4.1), provides $|\widetilde{u}_r(0)| \leqslant C$. Hence, by rewriting in terms of u, we get $|u(x_0)| \leqslant Cr^{-\gamma_p}$. At last, since $x_0 \in B^*_{1/2}$ is arbitrary and $r = \frac{1}{2}|x_0|$, the proof is finished.

As a consequence, we have the following Harnack-type estimate

Corollary 4.3. *Let $u \in C^6(B^*_1)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p \in (1, 2^\# - 1)$ and $R = 1$. Then, there exists $C > 0$ (depending only on n and p) such that*

$$\sum_{j=0}^{5} |x|^{\gamma_p+j}|D^{(j)}u(x)| \leqslant C \quad \text{in} \quad B^*_{1/2}.$$

Proof. See [9, Theorem 1.1].

The last lemma is a Harnack-type inequality

Lemma 4.4. *Let $u \in C^6(B^*_1)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p \in (1, 2^\# - 1)$ and $R = 1$. Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. Then, there exist $C > 0$ and $r \in (0, 1/16)$ (depending only on n and p) such that*

$$\sup_{B_{r/2}\setminus \bar{B}_{3r/2}} u \leqslant C \inf_{B_{r/2}\setminus \bar{B}_{3r/2}} u \quad \text{in} \quad B^*_{1/16}.$$

Proof. See [9, Proposition 4.2].

We also establish some auxiliary results about upper bounds near the isolated singularity.

Lemma 4.5. *Let $u \in C^6(B^*_1)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p \in (2_\#, 2^\# - 1)$ and $R = 1$, and $v = \mathfrak{F}^{-1}(u)$ be its autonomous Emden–Fowler transformation given by (2.1). Then, there exists $C > 0$ (depending only on n and p) such that*

$$|v| + |v^{(1)}| + |v^{(2)}| + |v^{(3)}| + |v^{(4)}| + |v^{(5)}| + |\nabla_\theta v| + |\Delta_\theta v| + |\nabla_\theta \Delta_\theta v| + |\Delta^2_\theta v| \leqslant C \quad \text{in} \quad \mathcal{C}_{\ln 2}.$$

Proof. First, by Lemma 4.2, we know that $v \in C^6(\mathcal{C}_T)$ is uniformly bounded. Moreover, using Corollary 4.3, one can find $C > 0$ such that

$$|v^{(1)}| + |\nabla_\theta v| \leqslant C \sum_{j=0}^{1} |x|^{\gamma_p}|D^{(j)}u(x)| \leqslant C,$$

$$|v^{(2)}| + |\Delta_\theta v| \leqslant C \sum_{j=0}^{2} |x|^{\gamma_p+j}|D^{(j)}u(x)| \leqslant C,$$

$$|v^{(3)}| + |\nabla_\theta \Delta_\theta v| \leqslant C \sum_{j=0}^{4} |x|^{\gamma_p+j}|D^{(j)}u(x)| \leqslant C,$$

$$|v^{(4)}| + |\Delta^2_\theta v| \leqslant C \sum_{j=0}^{4} |x|^{\gamma_p+j}|D^{(j)}u(x)| \leqslant C,$$

$$|v^{(5)}| \leqslant C \sum_{j=0}^{5} |x|^{\gamma_p+j}|D^{(j)}u(x)| \leqslant C,$$

for $0 < |x| < 1/2$, which by a direct rescaling argument proves the lemma.

When $p = 2_\#$, we have the asymptotic upper bound below. Our approach here follows the same lines of [11, Theorem 5]. Nonetheless, we write a short proof for the convenience of the reader.

Lemma 4.6. *Let $u \in C^6(B^*_R)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p = 2_\#$. Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. Then, there exist $C_0(n) > 0$ and $0 < r_0 < R$ such that*

$$|\overline{u}(x)| \leqslant C_0(n)|x|^{6-n}(\ln|x|)^{\frac{6-n}{6}} \quad \text{for} \quad 0 < |x| < r_0,$$

where $\overline{u}(r) = \fint_{\partial B_R} |u(r\theta)|\mathrm{d}\theta$ denotes the spherical average of u.

Proof. Note that for each $0 < r < R$ with $r = |x|$, we get that the spherical average $\overline{u} \in C^6(\mathbb{R})$ satisfies the following nonautonomous ODE

$$r^{-6}\partial_r^{(6)} + M_5(n,r)\partial_r^{(5)} + M_4(n,r)\partial_r^{(4)} + M_3(n,r)\partial_r^{(3)} + M_2(n,r)\partial_r^{(2)} + M_1(n,r)\partial_r - f_\#(u) \geqslant 0, \tag{4.2}$$

where the coefficients $M_j(n,\cdot):(0,R)\to\mathbb{R}$ are given by (A.2) and (A.3). Next, setting

$$\psi_0=\overline{u},\quad \psi_1=-\Delta\overline{u},\quad \text{and}\quad \psi_2=\Delta^2\overline{u},$$

we can reformulate $(\mathcal{P}_{6,p,R})$ as the following system

$$\begin{cases}-\left(r^{n-1}\psi_0^{(1)}(r)\right)^{(1)}=r^{n-1}\psi_0(r)\\ -\left(r^{n-1}\psi_1^{(1)}(r)\right)^{(1)}=r^{n-1}\psi_1(r) \qquad \text{for } r\in(0,R)\\ -\left(r^{n-1}\psi_2^{(1)}(r)\right)^{(1)}=r^{n-1}f_\#(\overline{u}(r)).\end{cases} \tag{4.3}$$

In what follows, the proof will be divided into some steps

Step 1: Either $u\in C(B_1)\cap H^3(B_1)$ is a continuous weak solution to $(\mathcal{P}_{6,p,R})$ with $p=2_\#$, or $\lim_{r\to0}\overline{u}(r)=\lim_{r\to0}-\Delta\overline{u}(r)\lim_{r\to0}\Delta^2\overline{u}(r)=+\infty$. In particular,

$$\overline{u}^{(1)}(r)\leqslant0,\quad (-\Delta\overline{u})^{(1)}(r)\leqslant0,\quad \text{and}\quad (\Delta^2\overline{u})^{(1)}(r)\leqslant0\quad \text{for}\quad r\in(0,R).$$

Suppose that $\overline{u}\in C^1(B_1^*)$ does not have a removable singularity at the origin. Then $\overline{u}$ must be unbounded in B_1, and, in this case, also a distribution solution to $(\mathcal{P}_{6,p,R})$ in the entire B_1. Since $f_p(\overline{u})\in L^{n/(6-\delta)}(B_1)$ for some $\delta\in(0,6)$, by bootstrap argument, u may be extended to a continuous weak solution to $(\mathcal{P}_{6,p,R})$, which is a contradiction.

First, we claim that $\overline{u}(r)\to+\infty$ as $r\to0$. In fact, suppose by contradiction that $\liminf_{r\to0}\overline{u}(r)<+\infty$. Then there exist two real sequences $\{m_k\}_{k\in\mathbb{N}},\{m_k\}_{k\in\mathbb{N}}$ such that $M_k>m_k\to0$ as $k\to\infty$ and u assumes local maxima at M_k, local minima at m_k for all $k\in\mathbb{N}$. Moreover $\overline{u}(M_k)\to+\infty$ as $k\to\infty$ while $\{\overline{u}(m_k)\}_{k\in\mathbb{N}}\subset\mathbb{R}$ remains bounded. Since $-\Delta u\geqslant0$, by the maximum principle, we get

$$\overline{u}(M_k)\geqslant\min\{\overline{u}(m_k),\overline{u}(m_{k+1})\}\quad \text{for all}\quad k\in\mathbb{N},$$

which contradicts the boundedness of $\{\overline{u}(m_k)\}_{k\in\mathbb{N}}\subset\mathbb{R}$. Using the same argument, it is also clear that $-\Delta\overline{u}(r)\to+\infty$ and $\Delta^2\overline{u}(r)\to+\infty$ as $r\to0$.

By integrating (4.3) on (ρ,r), where $0<\rho\ll R$ small enough, we find

$$r^{n-1}\psi_j^{(1)}(r)\leqslant\rho^{n-1}\psi_j^{(1)}(\rho)<0\quad \text{for}\quad j\in\{0,1,2\}.$$

This completes the proof of Step 1.

To continue with the proof, we need to introduce some auxiliary functions. Namely, we define the functions $\Psi_0,\Psi_1,\Psi_2:(0,R)\to\mathbb{R}$ given by

$$\Psi_0(r)=r\psi_0(r),\quad \Psi_1(r)=r\psi_1(r)+(n-2)\psi_1^{(1)},\quad \Psi_2(r)=r\psi_2+(n-2)\psi_2^{(1)},$$

which satisfy

$$\begin{cases}-\Delta\Psi_0=\Psi_1\\ -r^{-1}\Psi_1^{(1)}=\Psi_2\\ -r^{-1}\Psi_2^{(1)}=(-\Delta)^3\Psi_0.\end{cases} \tag{4.4}$$

Step 2. There exists $0<r_0\ll1$ and $0<\rho<r_0$, it follows $\lim_{r\to0}r^{n-2(2-j)}\psi_j(r)=0$ and $\Psi_j(r)\geqslant0$ in $r\in(0,\rho)$ for $j\in\{0,1,2\}$.

Since $u\in L^1_{\mathrm{loc}}(B_R)$ is a distribution solution to $(\mathcal{P}_{6,p,R})$ with $p=2_\#$, it follows that $\liminf_{r\to0}r^{n-2(2-j)}\psi_j(r)=0$ for $j\in\{0,1,2\}$.

We start with the case $j=2$. Indeed, let us denote by $\psi_0^*(r^{-1})=r^{n-6}\psi_0(r)$ the sixth order Kelvin transform $\psi_0=u$ with respect to ∂B_1. Hence, by [11, Lemma 3], we find that $-\Delta^3\psi_0^*\geqslant0$ in $\mathcal{I}^*(B_R):=\mathbb{R}^n\setminus B_1^*$. Then, by Step 1, ψ_0^* is monotone near $+\infty$, which implies that $r^{n-6}\psi_0(r)$ is monotone near the origin. Hence $\lim_{r\to0}r^{n-6}\psi_0(r)>0$ exists and is positive for small $0<r\ll1$.

To verify the case $j=1$, let us denote by $\psi_1^*(r^{-1})(r^{-1})=r^{n-4}\overline{u}(r)$ the fourth order Kelvin transform of $\psi_1=\Delta\overline{u}$ with respect to ∂B_1. Hence, by [11, Lemma 3], we find that $\Delta^2\psi_1^*\geqslant0$ in $\mathcal{I}^*(B_R)$. Then, from Step 1, we conclude that ψ_1^* is monotone near $+\infty$, which implies that $r^{n-4}\psi_1(r)$ is monotone near the origin. Hence $\lim_{r\to0}r^{n-4}\psi_1(r)>0$ exists and is positive for small $0<r\ll1$.

Finally, for the case $j=0$, let us denote by $\psi_2^*(r^{-1})=r^{n-2}\overline{u}(r)$ the second order Kelvin transform of $\psi_2=\Delta^2u$ with respect to ∂B_1. Hence, by [11, Lemma 3], we find that $-\Delta\psi_2^*\geqslant0$ in $\mathcal{I}^*(B_R)$. Then, from Step 1, we conclude that ψ_2^* is monotone near $+\infty$, which implies that $r^{n-2}\psi_2(r)$ is monotone near the origin. Hence $\lim_{r\to0}r^{n-2}\psi_2(r)>0$ exists and is positive for small $0<r\ll1$.

The proof of Step 2 is concluded.

Step 3. $\lim_{r\to0}r^{n-2}\Psi_0(r)=0$ and $\Psi_0^{(1)}(r)+(n-2)\Psi_0(r)\geqslant0$.

Finally, from the case $j=2$, it holds that $0\leqslant-r\overline{u}^{(1)}(r)\leqslant(n-6)\overline{u}(r)$ in $(0,\rho)$. Thus, multiplying the last inequality by r^{n-2} and letting $r\to0$, we get that $\lim_{r\to0}r^{n-1}\overline{u}^{(1)}(r)=0$, and so

$$\lim_{r\to0}r^{n-2}\Psi_0(r)=\lim_{r\to0}\left(r^{n-1}\overline{u}^{(1)}(r)+(n-6)r^{n-2}\overline{u}(r)\right)=0.$$

Next, for $0<r_0\ll1$ small given by Step 2, it follows from (4.4) that $-\Delta\Psi_0\geqslant0$ and $\Psi_0\geqslant0$ in $B^*_{r_0}$. Therefore, by considering $\Psi_0^*(r^{-1})=r^{n-2}\Psi_0(r)$, we obtain $-\Delta\Psi_0^*\geqslant0$ and $\Psi_0^*\geqslant0$ in $\mathcal{I}^*(B_R)$. In addition, $\Psi_0^*(r^{-1})\to0$. Consequently, a direct application of the maximum principle, yields $\Psi_0^*(r^{-1})\leqslant0$ for all $r^{-1}>r_0^{-1}$, which in turn proves completes the proof of Step 3.

Step 4. $\Psi_0^{(1)}(r)\leqslant0$ for $0<r\ll1$, then $\Psi_0(r)\to+\infty$ as $r\to0$.

We observe that from Step 3 it follows that $-(r^{n-1}\Psi_0^{(1)})^{(1)}\geqslant0$ for $r\in(0,\rho_1)$, which by integrating on (ε,r), with $r\in(0,\rho_1)$, yields

$$-r^{n-1}\Psi_0^{(1)}(r)+\varepsilon^{n-1}\Psi_0^{(1)}(\varepsilon)\geqslant0.$$

Hence, by letting $\varepsilon \to 0$ in the last inequality we find that $\Psi_0^{(1)}(r) \leqslant 0$ for $r \in (0, \rho_1)$. At last, suppose by contradiction that there exists $C > 0$ such that $\Psi_0(r) \leqslant C$ for all $r > 0$, which imply

$$\Psi_0(r) = r^{7-n}(r^{n-6}\overline{u}(r))^{(1)} \leqslant C.$$

On the other hand, integrating the last inequality on $(0, r)$, we would obtain $\overline{u}(r) \leqslant C(n-6)^{-1}$, which is a contradiction of the origin is a non-removable singularity and proves Step 4.

Step 5. $|\overline{u}(x)| \leqslant C_0(n)|x|^{6-n}(\ln|x|)^{\frac{6-n}{6}}$ for $0 < |x| < r_0$.

Let us set

$$\varphi_0(r) = r^{n-6}\overline{u}(r), \quad \varphi_1(r) = r^{n-4}\varphi_0(r), \quad \text{and} \quad \varphi_2(r) = r^{n-2}\varphi_1(r).$$

Whence, using Step 1, we get that $6\varphi_0^{(1)}(r) \geqslant 0$, $\varphi_1^{(1)}(r)$ and $\varphi_2^{(2)}(r) \geqslant 0$ in $(0, r_0)$. Now, since $\Psi_1(r) = r^{4-n}\varphi_1^{(1)}(r)$, one has from Step 2 the following holds

$$-\left(r\Psi_0^{(1)}(r) + (n-2)\Psi_0(r)\right)^{(1)} = r^{4-n}\varphi_1^{(1)}(r),$$

which, by integrating on (r, r_0), yields

$$-\left(r\Psi_0^{(1)}(r_0) + (n-2)\Psi_0(r_0)\right) + r\Psi_0^{(1)}(r) + (n-2)\Psi_0(r) \geqslant \varphi_1(r)\left(r^{4-n} - r_0^{4-n}\right).$$

Now, since $\varphi_0(r) \to 0$ as $r \to 0$, one can find $r_1 \in (0, r_0)$ such that

$$\left(r\Psi_0^{(1)}(r_0) + (n-2)\Psi_0(r_0)\right) - \varphi_1(r)r_0^{4-n} \geqslant 0, \quad \text{in} \quad (0, r_1),$$

which, since from Step 5 implies $\Psi_0^{(1)}(r) \leqslant 0$ for r small, one can find $C > 0$ such that

$$\Psi_0(r) \geqslant Cr^{4-n}\varphi_0(r) \quad \text{in} \quad (0, r_1),$$

and so $\varphi_0^{(1)}(r) \geqslant Cr^{-1}\varphi_1(r)$. Hence, since $\psi_1(r) \geqslant Cf_\#(u(r))$, we get $\varphi_1(r) \geqslant Cf_\#(\varphi_0(r))$, which yields

$$\varphi_0^{(1)}(r) \geqslant Cr^{-1}f_\#(\varphi_0(r)) \quad \text{in} \quad r \in (0, r_1).$$

Up to a rescaling, we may assume that the last inequality holds for $r \in (0, 1)$. Thus, by a direct integration, we obtain $\varphi_0(r)^{-\frac{6}{n-6}} \geqslant C \ln r$ as $r \to 0$, which in turn leads to

$$\overline{u}(r)r^{n-6}(\ln r)^{\frac{n-6}{6}} \leqslant C \quad \text{as} \quad r \to 0.$$

This completes the proof of the lemma.

5. Asymptotic radial symmetry

We prove the first part of Theorem 1 asserting that solutions to $(\mathcal{P}_{6,p,R})$ are radially symmetric about the origin. This symmetry will later be used to convert the singular PDE into a non-singular ODE on the cylinder.

Before that, we need to establish some preliminaries.

5.1. Kelvin transform

Later we will employ the moving spheres technique, which is based on the sixth order Kelvin transform of a real valued function. To define the Kelvin transform, we need to establish the concept of inversion about a sphere $\partial B_\mu(x_0)$, which is a map $\mathcal{I}_{x_0,\mu} : \mathbb{R}^n \to \mathbb{R}^n \setminus \{x_0\}$ given by $\mathcal{I}_{x_0,\mu}(x) = x_0 + K_{x_0,\mu}(x)^2(x - x_0)$, where $K_{x_0,\mu}(x) = \mu/|x - x_0|$.

Definition 5.1. For any $u \in C^6(B_R^*)$, let us consider the sixth order Kelvin transform about the sphere with center at $x_0 \in \mathbb{R}^n$ and radius $\mu > 0$ defined by

$$u_{x_0,\mu}(x) = K_{x_0,\mu}(x)^{n-6}u\left(\mathcal{I}_{x_0,\mu}(x)\right).$$

Lemma 5.2. *If $u \in C^6(B_R^*)$ is a solution to $(\mathcal{P}_{6,p,R})$, then $u_{x_0,\mu} \in C^6(B_R^* \setminus \{x_0\})$ is a solution to*

$$(-\Delta)^3 u_{x_0,\mu} = K_{x_0,\mu}(x)^{(n-6)p-(n+6)}f_p(u_{x_0,\mu}) \quad \text{in} \quad B_R^* \setminus \{x_0\}.$$

Proof. It directly follows from the facts that the tri-Laplacian is conformally covariant and the Kelvin transform is a conformal diffeomorphism.

5.2. Integral representation

Now we use a Green identity to transform the sixth order differential system ($\mathcal{P}_{6,p,R}$) into an integral system. In this way, we can avoid using the classical form of the maximum principle, and a sliding method is available [29,30], which will be used to classify solutions. In this setting is also possible to prove regularity through a barrier construction.

We start with the following result

Lemma 5.3. *Let $u \in C^6(B_R^*)$ be a positive solution to ($\mathcal{P}_{6,p,R}$) with $p \in (1,+\infty)$. Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. Then, $u \in L^p(B_1)$. In particular, if $p \in (2_{\#},+\infty)$, then $u \in L^1(B_1)$ is a distribution solution to ($\mathcal{P}_{6,p,R}$), that is, for all positive $\phi \in C_c^\infty(B_1)$, one has*

$$\int_{B_1} u(-\Delta)^3\phi \mathrm{d}x = \int_{B_1} f_p(u)\phi \mathrm{d}x.$$

Proof. See the proof of [12, Lemma 3.1] (see also [31, Theorem 3.7]).

The next result uses the Green identity to convert ($\mathcal{P}_{6,p,R}$) into an integral system. We divide this results in two cases, namely $R < +\infty$ and $R = +\infty$.

Lemma 5.4. *Let $u \in C^6(B_R^*)$ be a positive solution to ($\mathcal{P}_{6,p,R}$) with $p \in (1,+\infty)$. Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$.*

(i) *If $R < +\infty$, then (up to constant) there exists $r_0 > 0$ such that*

$$u(x) = \int_{B_{r_0}} |x-y|^{6-n} f_p(u)\mathrm{d}y + \psi(x), \tag{5.1}$$

where $\psi > 0$ satisfies $(-\Delta)^3\psi = 0$ in B_{r_0}. Moreover, one can find a constant $C(\widetilde{r}) > 0$ such that $\|\nabla \ln \psi\|_{C^0(B_{\widetilde{r}})} \leqslant C(\widetilde{r})$ for all $0 < \widetilde{r} < r_0$.

(ii) *If $R = +\infty$, then (up to constant), it follows*

$$u(x) = \int_{\mathbb{R}^n} |x-y|^{6-n} f_p(u)\mathrm{d}y.$$

Proof. For (i), the proof is a simple adaptation of [22, Lemma 2.3]. For (ii) see [32, Theorem 4.3] (see also [33]).

5.3. Sliding technique

Now we use the preliminary results to run an integral moving spheres technique. We use an asymptotic moving spheres technique in the same spirit of [15]. Although our proofs are almost the same we include it here for the sake of completeness.

Proposition 5.5. *Let $u \in C^6(B_R^*)$ be a positive singular solution to ($\mathcal{P}_{6,p,R}$) with $p \in (1, 2^\# - 1)$. Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. Then,*

$$u(x) = (1 + \mathcal{O}(|x|))\overline{u}(x) \quad \text{as} \quad x \to 0.$$

Proof. Initially, if the origin is a removable singularity, then the conclusion is clear. Hence, we suppose that the origin is a non-removable singularity.

We divide the proof into some claims.

Claim 1: There exists small $0 < \varepsilon \ll 1$ such that for any $z \in B^*_{\varepsilon/2}$, it holds

$$u_{z,r} \leqslant u \quad \text{in} \quad B_1 \setminus \big(B_r(z) \cup \{0\}\big) \quad \text{for} \quad 0 < r \leqslant |z|. \tag{5.2}$$

Indeed, the proof follows almost the same lines as the one in [22, Lemma 3.2], so we omit it.

In the next claim, we provide some estimates to be used later in the proof.

Claim 2: There exists $z \in B^*_{\varepsilon/2}$, $0 < r < |z|$ and $\mu_* \gg 1$ large such that

$$\frac{y_\mu}{|y_\mu|^2} - z = \left(\frac{r}{\left|y/|y|^2 - z\right|}\right)^2 \left(\frac{y}{|y|^2} - z\right) \quad \text{and} \quad \frac{|y_\mu|}{|y|} \leqslant \frac{1}{r}\left|\frac{y}{|y|^2} - z\right| \quad \text{for any} \quad \mu > \mu_*. \tag{5.3}$$

Here $y_\mu = y + 2(\mu - y\cdot\mathbf{e})\mathbf{e}$ is the reflection of y about the hyperplane $\partial H_\mu(\mathbf{e})$, where $H_\mu(\mathbf{e}) = \{x : \langle x,\mathbf{e}\rangle > \mu\}$ and $\mathbf{e} \in \mathbb{S}^{n-1}$. In other words, $y_\mu|y_\mu|^{-2}$ is the reflection point of $y|y|^{-2}$ about $\partial B_r(z)$.

As matter of fact, choosing $r = |z|$, it involves an elementary computation, as follows

$$z = \frac{y}{|y|^2} + \frac{\left|y_\mu\right|^2}{|y|^2 - \left|y_\mu\right|^2}\left(\frac{y}{|y|^2} - \frac{y_\mu}{\left|y_\mu\right|^2}\right) = \frac{(y - y_\mu)}{|y|^2 - \left|y_\mu\right|^2}.$$

Next, we establish a comparison involving the Kelvin transform of a component solution with itself.

Claim 3: For any $\mu > \frac{1}{\varepsilon}$ and $\mathbf{e} \in \partial B_1$, if $\langle x,\mathbf{e}\rangle > \mu$ and $|y_\mu| > 1$, it holds $u_{0,1}(y) \leqslant u_{0,1}\left(y_\mu\right)$.

In fact, to prove the last inequality, let us note first that $y \in B_{1/\varepsilon}$, if and only if, $y|y|^{-2} \in B_\varepsilon$. Now given $y \in \mathbb{R}^n$ such that $\langle y,\mathbf{e}\rangle > \mu$, $|y_\mu| > 1$ and $0 < r < |z| < \varepsilon/2$ satisfying (5.3). Let us define $x = y|y|^{-2}$ and $x_{z,r} = y_\mu|y_\mu|^{-2}$. Then, since $\langle y,\mathbf{e}\rangle > \mu > \varepsilon^{-1}$

and $\| y_\mu| > 1$, we have $x \in B_r(z)$ and $x_{z,r} \in B_1 \setminus B_r(z)$. Hence, using (5.2) and (5.3), we find

$$u_{0,1}(y) \leqslant u_{0,1}(y_\mu),$$

which proves the claim.

Ultimately, using Claim 3, we invoke [20, Theorem 6.1 and Corollary 6.2] to find $C > 0$, independent of $\varepsilon > 0$, such that if $|y| \geqslant |x| + C\varepsilon^{-1}$, it follows $u_{0,1}(y) \leqslant u_{0,1}(x)$. Therefore, since $u_{0,1}$ is positive and satisfies $-\Delta u_{0,1} \geqslant 0$ and $\Delta^2 u_{0,1} \geqslant 0$, the last inequality implies

$$u_{0,1}(|x|) = \left(1 + \mathcal{O}\left(\frac{1}{R}\right)\right)\left(\inf_{\partial B_R} u_{0,1}\right) \quad \text{as} \quad R \to +\infty,$$

uniformly on ∂B_R, which in terms of u implies the desired asymptotic radial symmetry with respect to the origin, and the proof is concluded.

6. The limiting Pohozaev levels

After proving the radial symmetry of singular solutions to $(\mathcal{P}_{6,p,R})$, we shall classify them in the blow-up and shrink-down limit. The idea is to use a blow-up/shrink-down analysis, which comes from tangent cone techniques from minimal hypersurface theory, and will be described in the sequel.

For any $u \in C^6(B_R^*)$ solution to $(\mathcal{P}_{6,p,R})$ and $\lambda > 0$, let us define the following λ-rescaling solution given by

$$\widehat{u}_\lambda(x) := \lambda^{\gamma_p} u(\lambda x), \tag{6.1}$$

where we recall that $\gamma_p = 6(p-1)^{-1}$. Notice that the λ-rescaled solution is still a positive solution to $(\mathcal{P}_{6,p,R})$ with $R = \lambda^{-1}$. Moreover, we get the following scaling invariance

$$\mathcal{P}_{\rm sph}(r, \widehat{u}_\lambda) = \mathcal{P}_{\rm sph}(\lambda r, u). \tag{6.2}$$

This follows by directly using the inverse cylindrical transform as in Remark 3.5. Besides, by a blow-up (resp. shrink-down) solution u_0 (resp. u_∞) to $(\mathcal{P}_{6,p,R})$, we mean the limit $u_0 := \lim_{\lambda\to 0} \widehat{u}_\lambda$ (resp. $u_\infty := \lim_{\lambda\to+\infty} \widehat{u}_\lambda$). In fact, utilizing some a priori estimates and the compactness of the family $\{\widehat{u}_\lambda\}_{\lambda>0} \subset C^{6,\alpha}_{\rm loc}(\mathbb{R}^n)$ for some $\alpha \in (0,1)$, these limits will be proven to exist. Next, we study the limit Pohozaev functional both as $r \to 0$ (blow-up) and $r \to +\infty$ (shrink-down), this will give the desired information about the asymptotic behavior for solutions to $(\mathcal{P}_{6,p,R})$.

Here is our main result of this section:

Proposition 6.1. *Let $u \in C^6(\mathbb{R}^n \setminus \{0\})$ be a positive solution to $(\mathcal{P}_{6,p,\infty})$ with $p \in (1, 2^\# - 1)$. Assume that u is homogeneous of degree $-\gamma_p$.*

(a) *If $p \in (1, 2_\#]$, then $u \equiv 0$;*
(b) *If $p \in (2_\#, 2^\# - 1)$, then $\mathcal{P}_{\rm cyl}(r,u)$ converges both as $r \to 0$ and $r \to +\infty$, namely*

$$\{\mathcal{P}_{\rm sph}(0,u), \mathcal{P}_{\rm sph}(+\infty,u)\} = \{-\omega_{n-1}\ell_p^*, 0\}, \quad \text{where} \quad \ell_p^* = \frac{p-1}{2(p+1)} K_0(n,p)^{\frac{p+1}{p-1}}. \tag{6.3}$$

First, we prove that the invariance of the Pohozaev invariant is equivalent to the homogeneity of the blow-up limit solutions to $(\mathcal{P}_{6,p,\infty})$. A similar result can also be found in [18], where a different type of functional is considered.

Lemma 6.2. *Let $u \in C^6(\mathbb{R}^n \setminus \{0\})$ be a positive solution to $(\mathcal{P}_{6,p,\infty})$ with $p \in (2_\#, 2^\# - 1)$. Then, $\mathcal{P}_{\rm sph}(r, \mathcal{U})$ is constant, if and only if, for any $r \in (r_1, r_2)$ with $0 < r_1 \leqslant r_2 < +\infty$, u is homogeneous of degree $-\gamma_p > 0$ in $B_{r_2} \setminus \bar{B}_{r_1}$, that is, it holds*

$$u(x) = |x|^{-\gamma_p} u\left(\frac{x}{|x|}\right) \quad \text{in} \quad B_{r_2} \setminus \bar{B}_{r_1}.$$

Proof. Notice that if $p \neq 2^\# - 1$, then $K_0(n,p) \neq 0$. Thus, supposing that $\mathcal{P}_{\rm sph}(r,u)$ is constant for $r_1 < r < r_2$, together with Remark 3.3 yields that $\partial_\nu u = \gamma_p r^{-1} u$ on ∂B_r for any $r_1 < r < r_2$, where ν is the unit normal pointing towards the origin. Therefore, u is homogeneous of degree $-\gamma_p$ in $B_{r_2} \setminus \bar{B}_{r_1}$, which concludes the proof.

The following lemma provides an upper bound estimate for singular solutions to $(\mathcal{P}_{6,p,R})$.

Lemma 6.3. *Let $u \in C^6(\mathbb{R}^n \setminus \{0\})$ be a positive singular solution to $(\mathcal{P}_{6,p,\infty})$ with $p \in (1, 2^\# - 1)$. Then, it follows*

$$u(x) \leqslant \left(\frac{p-1}{2n}\right)^{-\frac{1}{p-1}} |x|^{-\gamma_p} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.$$

Proof. By [10, Theorem 6.1], we know that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$ in $\mathbb{R}^n \setminus \{0\}$, which, by using the extended maximum principle [34, Theorem 1], gives us

$$\liminf_{x\to 0} u(x) > 0.$$

Considering $\varphi = u^{1-p}$, a direct computation, provides

$$\Delta\varphi \geqslant \frac{p}{p-1}\frac{|\nabla\varphi|^2}{\varphi} + p - 1 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.$$

Thus, for any $r > 0$, let us consider the auxiliary function $\widetilde{\varphi}(x) = \varphi(x) - \frac{p-1}{2n}|x|^2$, which satisfies $-\Delta\widetilde{\varphi} \leqslant 0$ in B_r^*. Furthermore, 6.3 implies that $\widetilde{\varphi}$ is bounded close to the origin, and thus, again, by the extended maximum principle, we find

$$0 \leqslant \limsup_{x\to 0} \widetilde{\varphi}(x) \leqslant \sup_{\partial B_r} \widetilde{\varphi} = \sup_{\partial B_r} \varphi - \frac{p-1}{2n} r^2,$$

which yields

$$\inf_{\partial B_r} u \leqslant \left(\frac{p-1}{2n}\right)^{-\frac{1}{p-1}} r^{-\gamma_p}.$$

Finally, a direct application of Proposition 5.5 finishes this proof.

As a consequence of this uniform upper bound, we prove the compactness of the family $\{\widehat{u}_\lambda\}_{\lambda>0} \subset C^{6,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, for some $\alpha \in (0,1)$, which provides the existence of both blow-up and shrink-down limits for the scaling family defined by (6.1).

Lemma 6.4. *Let $u \in C^6(\mathbb{R}^n \setminus \{0\})$ be a positive singular solution to $(\mathcal{P}_{6,p,\infty})$ with $p \in (1, 2^\# - 1)$. Then, $\{\widehat{u}_\lambda\}_{\lambda>0} \subset C^{6,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ is locally uniformly bounded for some $\alpha \in (0,1)$.*

Proof. If the origin is a removable singularity of $\widehat{u}_\lambda$ for all $\lambda > 0$, according to Theorem A, u is trivial and the conclusion follows.

On the other hand, assuming that the origin is a removable singularity, Lemma 6.3 provides that $\{\widehat{u}_\lambda\}_{\lambda>0}$ is globally bounded in $\mathbb{R}^n \setminus \{0\}$. Thus, we know that $\{\widehat{u}_\lambda\}_{\lambda>0}$ is uniformly bounded in each compact subset of $K \subset \mathbb{R}^n \setminus \{0\}$. Moreover, since for each $\lambda > 0$, the scaling $\widehat{u}_\lambda$ also satisfies $(\mathcal{P}_{6,p,\infty})$, it follows from standard elliptic estimates that $\{\widehat{u}_\lambda\}_{\lambda>0}$ is uniformly bounded in $C^{6,\alpha}(K)$, for some $\alpha \in (0,1)$, which concludes the proof.

Recall that $\mathcal{P}_{\mathrm{sph}}(r,u)$ is the Pohozaev functional introduced in Remark 3.3, which by Proposition 3.4 is monotonically nonincreasing in $r > 0$ when $p \in (1, 2^\# - 1)$.

Lemma 6.5. *Let $u_0 \in C^6(\mathbb{R}^n \setminus \{0\})$ (or $u_\infty \in C^6(\mathbb{R}^n \setminus \{0\})$) be a positive singular blow-up (or shrink-down) solution to $(\mathcal{P}_{6,p,R})$ under the family of scalings $\{\widehat{u}_\lambda\}_{\lambda>0} \subset C^6(\mathbb{R}^n \setminus \{0\})$. Then, $\mathcal{P}_{\mathrm{sph}}(r,u_0) \equiv \mathcal{P}_{\mathrm{sph}}(0,u)$ (or $\mathcal{P}_{\mathrm{sph}}(r,u_\infty) \equiv \mathcal{P}_{\mathrm{sph}}(\infty,u)$) is constant for all $r > 0$. In particular, both u_0 and u_∞ are homogeneous of degree $-\gamma_p$.*

Proof. Let $\{\lambda_k\}_{k\in\mathbb{N}} \subset (0,+\infty)$ be a blow-up sequence such that $\lambda_k \to 0$, and $u_0 \in C^{6,\alpha}(\mathbb{R}^n \setminus \{0\})$ be its blow-up limit, that is, $\widehat{u}_{\lambda_k} \to u_0$ in $C^{6,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ as $k \to +\infty$, for some $\alpha \in (0,1)$. Now, using Proposition 3.4 and Lemma 6.4, one concludes that there exists the limiting level $\mathcal{P}_{\mathrm{sph}}(0,u) := \lim_{r\to 0} \mathcal{P}_{\mathrm{sph}}(r,u)$. Moreover, due to the scaling invariance of the Pohozaev functional in (6.2), for any $r > 0$, it follows

$$\mathcal{P}_{\mathrm{sph}}(r,u_0) = \lim_{k\to+\infty} \mathcal{P}_{\mathrm{sph}}(r,\widehat{u}_{\lambda_k}) = \lim_{k\to+\infty} \mathcal{P}_{\mathrm{sph}}(r\lambda_k,u) = \mathcal{P}_{\mathrm{sph}}(0,u),$$

which finishes the proof of the first assertion. Now, we can check that the homogeneity follows from Lemma 6.2. Finally, notice that the same argument can readily be employed, replacing the blow-up limit by the shrink-down limit, so we omit it here.

Lemma 6.6. *Let $u \in C^6(\mathbb{R}^n \setminus \{0\})$ be a positive solution to $(\mathcal{P}_{6,p,\infty})$ with $p \in (1, 2^\# - 1)$. Assume that u is homogeneous of degree $-\gamma_p$.*

(a) *If $p \in (1, 2_\#]$, then $u \equiv 0$.*
(b) *If $p \in (2_\#, 2^\# - 1)$, then either $u \equiv 0$, or $u \equiv K_0(n,p)^{\frac{1}{p-1}}|x|^{-\gamma_p}$.*

Proof. Since u is homogeneous of degree $-\gamma_p$, the Emden–Fowler transformation $v = \mathfrak{F}(u)$ given by (2.1) satisfies

$$-\mathscr{L}_\theta v + f_p(v) = 0 \quad \text{on} \quad \mathbb{S}^{n-1}_t, \tag{6.4}$$

where $\mathscr{L}_\theta := \Delta_\theta^3 + L_0(n,p)\Delta_\theta^2 + J_0(n,p)\Delta_\theta v + K_0(n,p)$. Now we divide the proof into two cases:

Case 1: $p \in (1, 2_\#]$.

Initially, one can verify that $\mathscr{L}_\theta(v) \leqslant 0$ on $\mathbb{S}^{n-1}_t$. Next, observe that $\mathscr{L}_\theta$ is the composition of three elliptic operators. This, together with [9, Theorem 1.2], implies that $v \in C^6(C_T)$ does not attain any strict local minimum on $\mathbb{S}^{n-1}_t$. Therefore, since $\mathbb{S}^{n-1}_t$ is a compact manifold, it follows that v is constant, which yields $v \equiv v_0$. Nevertheless, using that $K_0(n,p) \leqslant 0$, any positive constant solution to (6.4) is trivial. By using the inverse of the Emden–Fowler transformation, it holds that u is trivial on ∂B_1, which by the superharmonicity property, implies that u is trivial in the whole domain. This conclusion finishes the proof of the first case, and so part (a) of the lemma follows.

Case 2: $p \in (2_\#, 2^\# - 1)$.

Assume that u is a nontrivial limit solution in the punctured space. Hence, since each component of u is positive and satisfies $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$, it quickly follows that $u > 0$ in $\mathbb{R}^n \setminus \{0\}$. By homogeneity, the origin is a non-removable singularity of $u \in C^6(\mathbb{R}^n \setminus \{0\})$. Hence, by Proposition 5.5, u is radially symmetric; thus, $u \equiv u_0$ is a positive constant. Moreover, by (6.4), it holds $u_0 = K_0(n,p)^{\frac{1}{p-1}}$, which, by using the homogeneity of u and Lemma 6.2, finishes the proof of the second case, and so (b) holds.

At last, we can prove the main result of this part.

Proof of Proposition 6.1. Let $u_0 \in C^6(\mathbb{R}^n \setminus \{0\})$ and $u_\infty \in C^6(\mathbb{R}^n \setminus \{0\})$ be, respectively, a blow-up and a shrink-down limit of u. According to Lemma 7.5 both u_0 and u_∞ are homogeneous of degree γ_p. In what follows, we divide the rest of the proof into two cases:

Case 1: $p \in (1, 2_\#]$.

Here, it follows from Lemma 6.6(a) that both u_0 and u_∞ are trivial, which, by Lemma 6.5, provides $\mathcal{P}_{\rm sph}(0,u) = \mathcal{P}_{\rm sph}(+\infty, u) = 0$. In addition, using the monotonicity property of the Pohozaev functional, we find $\mathcal{P}_{\rm sph}(r,u) = 0$ for all $r > 0$. Hence, by Lemma 6.2, u is homogeneous of degree $-\gamma_p$. Therefore, the proof of (a) of Proposition 6.1 is now an immediate consequence of Lemma 6.6(a).

Case 2: $p \in (2_\#, 2^\# - 1)$.

Initially, by Lemmas 6.5 and 6.6(b), any blow-up u_0 is either trivial or has the form (2). If u_0 is trivial, then clearly $\mathcal{P}_{\rm sph}(r,u_0) = 0$ for all $r > 0$, which combined with Lemma 6.5 implies that $\mathcal{P}_{\rm sph}(0,u) = 0$. Otherwise, a simple computation shows $\mathcal{P}_{\rm sph}(r,u_0) = -\omega_n \ell_p^*$ for all $r > 0$. Therefore, using again Lemma 6.5, we have $\mathcal{P}_{\rm sph}(0,u) = -\ell_p^*$. Since the converse trivially follows, we obtain that $\mathcal{P}_{\rm sph}(0,u) \in \{-\omega_n \ell_p^*, 0\}$. Moreover, $\mathcal{P}_{\rm sph}(0,u) = 0$, if and only if, all the blow-ups are trivial, whereas $\mathcal{P}_{\rm sph}(0,u) = -\omega_n \ell_p^*$, if and only if, all the blow-ups are of the form (2). In the case of shrink-down u_∞ solution, the strategy is similar, so we omit it. These conclusions finish the proof of Case 2, and therefore the proposition holds.

7. Local asymptotic behavior

In this section, we present the proof of Theorem 1. First, the asymptotic symmetry result from Proposition 5.5 allows us to migrate to an ODE setup. Second, we prove some universal upper bound estimates, not depending on the superharmonic assumption. However, we should emphasize that in the rest of the argument, there is a significant change of behavior of radial solutions $(\mathcal{C}_{p,T})$ for distinct values of the power $p \in (1, 2^\# - 1]$. This difference occurs due to the change of sign of the coefficients in the tri-Laplacian written in cylindrical coordinates. These signs control the Lyapunov stability of the solutions to the linearized operator around a limit blow-up solution, and so the asymptotic behavior of the local solutions near the isolated singularity.

We divide our argument into three subsections, where we prove, respectively, the local behavior near the isolated singularity for the situations: $p \in (1, 2^\# - 1)$ in Section 7.1, $p = 2_\#$ in Section 7.3, and $p \in (2_\#, 2^\# - 1)$ in Section 7.2.

7.1. Serrin–Lions case

We prove Theorem 1(a). The asymptotic analysis for this case is straightforward. We are based on the approach given by [15]. In the sequel, we aim to prove the following proposition:

Proposition 7.1. *Let $u \in C^6(B_R^*)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p \in (1, 2_\#)$. Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. Then, there exist $C_1, C_2 > 0$ (depending on u) such that $C_1|x|^{6-n} \leqslant u(x) \leqslant C_2|x|^{6-n}$ for $0 < |x| \ll 1$, or equivalently, $u(x) \simeq |x|^{6-n}$ as $x \to 0$.*

First, we prove an upper bound estimate based on a Green identity from Lemma 5.4(i).

Lemma 7.2. *Let $u \in C^6(B_R^*)$ be a positive solution to $(\mathcal{P}_{6,p,R})$ with $p \in (1, 2_\#)$. Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. Then, there exists $C_2 > 0$, depending only on u, such that $u(x) \leqslant C_2|x|^{4-n}$ as $x \to 0$.*

Proof. Initially, by Lemma 5.3, we have that $u \in L^p(B_1)$. Moreover, since $p \in (1, 2_\#)$, and $u \in C^6(B_R^*)$ satisfies the Harnack inequality in Lemma 4.4, it follows

$$u(x) = \mathrm{o}\left(|x|^{-\gamma_p}\right) \quad \text{as} \quad x \to 0,$$

which by $n - 6 < \gamma_p$, implies that for any $n - 6 < q < \gamma_p$, there exists $0 < r_q < 1$ depending only on n p, and q such that

$$u(x) < |x|^{-q} \quad \text{in} \quad B_{r_q}^*,$$

where in the last claim we have used a blow-up argument. Now taking $r_q > 0$ as before, and using Lemma 5.3 again, we get that $(-\Delta)^3 u = f_p(u) \in L^1(B_1)$. Thus, using (5.1), we decompose

$$u(x) = |x|^{6-n} - \int_{B_{r_q}} |x-y|^{6-n}(-\Delta)^3 u(y)\mathrm{d}y + \psi(x) \quad \text{in} \quad B_{r_q}^*, \tag{7.1}$$

where $\psi \in C^\infty(B_1)$ is such that $(-\Delta)^3\psi = 0$ in B_{r_q}. Nevertheless, using (5.1) that there exists $C_q > 0$, depending only on n, p, and q such that

$$\left|\int_{B_{r_q}} |x-y|^{6-n}(-\Delta)^3 u(y)\mathrm{d}y\right| \leqslant \int_{B_{r_q}} |x-y|^{6-n}|y|^{-pq}\mathrm{d}y \leqslant C_q|x|^{6-n}.$$

Hence, fixing $n - 6 < q < \gamma_p$ and choosing suitable $r_q > 0$ and $C_q > 0$ on the last inequality, the proof follows directly from (7.1).

Second, we give a sufficient condition to classify whether the origin is a removable singularity or non-removable singularity.

Lemma 7.3. *Let $u \in C^6(B_R^*)$ be a positive solution to $(\mathcal{P}_{6,p,R})$ with $p \in (1, 2_\#)$. Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. If*

$$u(x) = \mathrm{o}\left(|x|^{6-n}\right) \quad \text{as} \quad x \to 0, \tag{7.2}$$

then, the origin is a removable singularity.

Proof. By (7.2), we get $u \in L^q(B_1)$ for any $q \in [1, 2_\#)$. Moreover, since $p \in (1, 2_\#)$ and $|(-\Delta)^3 u| \leqslant |u|^p$, it follows $(-\Delta)^3 u \in L^{q/p}(B_1)$ for any $q \in [1, 2_\#)$. Whence, we can use standard elliptic theory combined with a bootstrap argument to find $u \in W^{5,Q}(B_1)$ for any $Q \in (1, +\infty)$. In particular, it holds from Morrey's embedding that $u \in C^{4,\alpha}(B_1)$ for any $\alpha \in (0,1)$. Therefore, $u \in C^6(B_R)$, that is, it must have a removable singularity at the origin.

Now we are in a position to prove our main result of this part.

Proof of Proposition 7.1. Suppose by contradiction that $u \in C^6(B_R^*)$ has a non-removable singularity at the origin, that is, $u \in C^6(B_R)$. Then, using Lemma 7.3, we get that u does not satisfy (7.2), that is, there exists $\rho > 0$ and $\{r_k\}_{k \in \mathbb{N}}$ such that $r_k \to 0$ as $k \to +\infty$ satisfying

$$\sup_{\partial B_{r_k}} u \geqslant \rho r_k^{6-n}.$$

On the other hand, by the Harnack inequality in Lemma 4.4, there exists $c_1 > 0$ satisfying $\inf_{\partial B_{r_k}} u \geqslant c_1 \rho r_k^{6-n}$, where $c_1 > 0$ depends only on n, and p. Taking $0 < \rho \ll 1$ smaller to ensure that there exists $c_2 \rho \leqslant \inf_{\partial B_{1/2}} u$, it follows from the maximum principle that

$$u(x) \geqslant c_2 \rho |x|^{6-n} \quad \text{in} \quad B_{1/2}^*,$$

which proves the asymptotic lower bound estimate in this case, and together with Lemma 7.2, the proof of the proposition is concluded.

7.2. Gidas–Spruck case

The objective of this subsection is to prove Theorem 1(c). Our strategy is based on the monotonicity formula for the Pohozaev functional in cylindrical coordinates (see Proposition 3.4), which relies on the strategy given in [6,9,12]. More precisely, we show that the local models near the origin are the limit blow-up solutions, whose limits are provided by its image under the action of the spherical Pohozaev functional. Finally, to prove the removability of the singularity theorem, we use a technique relying on the regularity lifting method from [35].

We will prove the result below

Proposition 7.4. *Let $u \in C^6(B_R^*)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p \in (2_\#, 2^\# - 1)$. Then,*

$$u(x) = (1 + \mathrm{o}(1)) K_0(n,p)^{\frac{1}{p-1}} |x|^{-\gamma_p}.$$

Now we use this rescaled family $\{\widehat{u}_\lambda\}_{\lambda > 0} \subset C^{6,\alpha}(B_1^*)$, for some $\alpha \in (0,1)$, to obtain the blow-up limit for $(\mathcal{P}_{6,p,R})$. This allows us to study the limiting values for the Pohozaev functional, by using the classification results from Theorem A.

Lemma 7.5. *Let $u \in C^6(B_R^*)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p \in (2_\#, 2^\# - 1)$ and $v = \mathfrak{F}(u)$ be its autonomous Emden–Fowler transformation given by (2.3). Then, $\mathcal{P}_{\mathrm{cyl}}(-\infty, v) \in \{-\ell_p^*, 0\}$, where ℓ_p^* is given by (6.3). Moreover, it follows*

(i) $\mathcal{P}_{\mathrm{cyl}}(-\infty, v) = 0$, *if and only if,*

$$u(x) = \mathrm{o}\left(|x|^{-\gamma_p}\right) \quad \text{as} \quad x \to 0. \tag{7.3}$$

(ii) $\mathcal{P}_{\mathrm{cyl}}(-\infty, v) = -\ell_p^*$, *if and only if,*

$$u(x) = (1 + \mathrm{o}(1)) K_0(n,p)^{\frac{1}{p-1}} |x|^{-\gamma_p} \quad \text{as} \quad x \to 0.$$

Proof. Initially, by Lemma 4.2, for any $K \subset B_{1/2\lambda}$ compact subset, the family $\{\widehat{u}_\lambda\}_{\lambda > 0} \subset C^{6,\alpha}(B_1^*)$ is uniformly bounded, for some $\alpha \in (0,1)$. Then, by standard elliptic theory, there exists a positive function $u_0 \in C^{6,\alpha}(\mathbb{R}^n \setminus \{0\})$, such that, up to a subsequence, we have that $\|\widehat{u} - u_0\|_{C^{6,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})}$ as $\lambda \to 0$, where u_0 satisfies the blow-up limit system $(\mathcal{P}_{6,p,R})$. Moreover, by [9, Theorem 1.2], we know that $u_0 \in C^6(\mathbb{R}^n \setminus \{0\})$ satisfies $-\Delta u_0 \geqslant 0$ and $\Delta^2 u_0 \geqslant 0$ in $\mathbb{R}^n \setminus \{0\}$, which, by the maximum principle, yields that either $u_0 \equiv 0$ or $u_0 > 0$ in $\mathbb{R}^n \setminus \{0\}$. Therefore, by Theorem A, the blow-up limit u_0 is radially symmetric with respect to the origin. Furthermore, by the scaling invariance of the Pohozaev functional, we get

$$\mathcal{P}_{\mathrm{sph}}(r, u_0) = \lim_{\lambda \to 0} \mathcal{P}_{\mathrm{sph}}(r, \widehat{u}_\lambda) = \lim_{\lambda \to 0} \mathcal{P}_{\mathrm{sph}}(\lambda r, u_0) = \mathcal{P}_{\mathrm{sph}}(0, u_0). \tag{7.4}$$

In addition, since $v_0 = \mathfrak{F}(u_0)$ satisfies $(\mathcal{C}_{p,T})$, by (7.4), we get that $\mathcal{P}_{\mathrm{cyl}}(t, v_0) = \mathcal{P}_{\mathrm{sph}}(r, u_0)$ is a constant. Consequently, by the monotonicity formula in Proposition 3.4, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{P}_{\mathrm{cyl}}(t, v_0) = \left[-K_5(n,p) {v_0^{(3)}}^2 + K_3(n,p) {v_0^{(2)}}^2 - K_1(n,p) {v_0^{(1)}}^2 \right] \equiv 0.$$

Moreover, since $K_5(n,p), K_1(n,p) > 0$ and $K_3(n,p) < 0$, we find that $v_0^{(1)} \equiv 0$ in $\mathbb{R}$, and so v_0 is constant, which can be directly computed, namely either $v_0 = 0$ or $v_0 = K_0(n,p)^{\frac{1}{p-1}}$. Moreover, by (7.4), it follows $\mathcal{P}_{\rm cyl}(0,v_0) \in \{-\ell_p^*, 0\}$ and $\mathcal{P}_{\rm sph}(0,u_0) \in \{-\omega_{n-1}\ell_p^*, 0\}$.

Finally, if $\mathcal{P}_{\rm sph}(0,u_0) = 0$, then, by uniqueness of the limit $u_0 \equiv 0$. Whence, we conclude that $\|\hat{u}_\lambda\|_{C^{6,\alpha}(K)} \to 0$ for any sequence of $\lambda \to 0$, for some $\alpha \in (0,1)$, which straightforwardly provides (7.3). Otherwise, we have

$$u_0 \equiv K_0(n,p)^{\frac{1}{p-1}} |x|^{-\gamma_p},$$

which proves (ii) of this lemma and finishes the proof.

Next, we use the last lemma to prove the removable singularity theorem. Our proof is based on regularity lifting methods combined with the De Giorgi–Nash–Moser iteration technique.

Lemma 7.6. *Let $u \in C^6(B_R^*)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p \in (2_\#, 2^\# - 1)$. If*

$$u(x) = \mathrm{o}\left(|x|^{-\gamma_p}\right) \quad \text{as} \quad x \to 0,$$

then the origin is a removable singularity.

Proof. Without loss of generality, let us consider $R = 1$. The proof will be divided into some claims.

Claim 1: If $u(x) = \mathrm{o}(|x|^{-\gamma_p})$ as $x \to 0$, then

$$\int_{B_{1/2}} u^{n\gamma_p^{-1}} \mathrm{d}x < +\infty. \tag{7.5}$$

In fact, let us consider $\phi(|x|) = |x|^{\zeta_p}$, where $\zeta_p = -2\gamma_p(2^\# - 1)(n - 2_\#)(p-1)^{-1}$. Then, a direct computation, provides

$$\Delta^3\phi(x) = \zeta_p(\zeta_p - 2)(\zeta_p - 3)(\zeta_p + n - 2)(\zeta_p + n - 4)(\zeta_p + n - 6)|x|^{\zeta_p - 6} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\},$$

which, since $\zeta_p + n - 6 = \gamma_p > 0$, it follows that

$$A_p := \zeta_p(\zeta_p - 2)(\zeta_p - 3)(\zeta_p + n - 2)(\zeta_p + n - 4)(\zeta_p + n - 6) > 0.$$

Thus, we can write

$$\frac{(-\Delta)^3\phi}{\phi} = \frac{A_p}{|x|^6} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \tag{7.6}$$

For any $0 < \varepsilon \ll 1$, let us consider $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$ with $0 \leqslant \eta_\varepsilon \leqslant 1$ a cut-off function satisfying

$$\eta_\varepsilon(x) = \begin{cases} 0, & \text{for } \varepsilon \leqslant |x| \leqslant 1/2 \\ 1, & \text{for } |x| \leqslant \varepsilon/2 \text{ or } |x| \geqslant 3/4, \end{cases} \tag{7.7}$$

and $|D^{(j)}\eta_\varepsilon(x)| \leqslant C\varepsilon^{-j}$ for $j \in \{0,1,2,3,4,5,6\}$. Defining $\xi_\varepsilon = \eta_\varepsilon\phi$, multiplying $(\mathcal{P}_{6,p,R})$ by ξ_ε and integrating by parts in B_1, we obtain

$$\int_{B_1} \eta_\varepsilon u\phi\left(\frac{(-\Delta)^3\phi}{\phi} - f_p(u)\right)\mathrm{d}x = -\int_{B_1} u\mathfrak{T}\left(\eta_\varepsilon, \phi\right)\mathrm{d}x, \tag{7.8}$$

where $\mathfrak{T}_\varepsilon : C_c^\infty(B_1) \to C_c^\infty(B_1)$ is defined by

$$\mathfrak{T}_\varepsilon(\phi) = \mathfrak{T}\left(\eta_\varepsilon, \phi\right) = 6\nabla\eta_\varepsilon\nabla\Delta^2\phi - 15\Delta\eta_\varepsilon\Delta^2\phi + 20\nabla\Delta\eta_\varepsilon\nabla\Delta\phi - 15\Delta^2\eta_\varepsilon\Delta\phi + 6\nabla\Delta^2\eta_\varepsilon\nabla\phi - \phi\Delta^3\eta_\varepsilon.$$

Using Lemma 4.2 combined with the estimates on the cut-off function (7.7) and its derivatives, there exist $c_1, c_2 > 0$, independent of ε, satisfying the following estimates,

$$\left|\int_{B_1} u\mathfrak{T}_\varepsilon(\phi)\mathrm{d}x\right| \leqslant c_1 + c_2\varepsilon^n\varepsilon^{\zeta_p - 6}\varepsilon^{-\gamma_p} < +\infty,$$

which implies that the right-hand side of (7.8) is uniformly bounded. In addition, assumption (7.5) yields that $u^{p-1}(x) = \mathrm{o}(1)|x|^{-6}$ as $x \to 0$, which together with (7.6) and (7.9) provides that there exists $C > 0$ satisfying

$$\int_{B_1} \eta_\varepsilon u(x)|x|^{\zeta_p - 6}\mathrm{d}x \leqslant C. \tag{7.9}$$

Therefore, by Lemma 4.2, it holds

$$\int_{\{\varepsilon \leqslant |x| \leqslant 1/2\}} u(x)^{n\gamma_p^{-1}}\mathrm{d}x \leqslant C\int_{\{\varepsilon \leqslant |x| \leqslant 1/2\}} u(x)|x|^{\zeta_p - 6}\mathrm{d}x \leqslant C\int_{B_1} \eta_\varepsilon u(x)|x|^{\zeta_p - 6}\mathrm{d}x < +\infty, \tag{7.10}$$

where the last inequality comes from (7.9). Finally, passing to the limit as $\varepsilon \to 0$ in (7.10), the proof of Claim 1 follows by applying the dominated convergence theorem.

Claim 2: If (7.5) holds, then $u \in L^q(B_1)$ for all $q > 2^\#$.

Indeed, by Lemma 5.4(i), there exist a Green function with homogeneous Dirichlet boundary conditions $G_3(x,y)$ and $\psi \in L^\infty_{\mathrm{loc}}(B_{1/2})$ with $\Delta^2\psi = 0$ and $\Delta\psi = 0$ such that

$$u(x) = \int_{B_1} G_2(x,y)(-\Delta)^3 u \mathrm{d}x + \psi(x) \quad \text{in} \quad B_{1/2}.$$

More precisely, $G_3(x,y)$ is a distributional solution to the Dirichlet problem

$$\begin{cases} \Delta^3 G_2(x,y) = \delta_x(y) & \text{in} \quad B_{1/2} \\ G_3(x,y) = \partial_\nu G_3(x,y) = \Delta G_3(x,y) = 0 & \text{on} \quad \partial B_{1/2}, \end{cases}$$

and there exists positive constant $C_n > 0$ such that

$$0 < G_3(x,y) \leqslant \widetilde{G}_3(|x-y|) := C_n|x-y|^{6-n} \quad \text{for} \quad x, y \in B_{1/2} \quad \text{and} \quad |x-y| > 0,$$

where $\widetilde{G}_3(x,y) = C_n|x-y|^{6-n}$ is the fundamental solution to Δ^3 in $\mathbb{R}^n$. Recall that $u \in C^6(B_1^*)$ satisfies

$$(-\Delta)^3 u = V(x)u \quad \text{in} \quad B_1^*, \tag{7.11}$$

where $V(x) = u(x)^{p-1}$. Moreover, using (7.5), we find that $V \in L^{n/6}(B_{1/2})$.

Let us consider $Z = C_c^\infty(B_{1/4})$, $X = L^{2^{**}}(B_{1/4})$ and $Y = L^p(B_{1/4})$ for $q \in (2^\#, +\infty)$ as in [35, Theorem 3.3.1]. Hence, it is well-defined the following inverse operator

$$(Tu)(x) = \int_{B_{1/4}} \widetilde{G}_3(x,y)u(y)\mathrm{d}y.$$

We also consider the operator $T_M := \widetilde{G}_3 * V_M$, which applied in both sides of (7.11), provides $u = T_M u + \widetilde{T}_M u$, where

$$(T_M u)(x) = \int_{B_{1/4}} \widetilde{G}_3(x,y)V_M(y)u(y)\mathrm{d}y \quad \text{and} \quad (\widetilde{T}_M u)(x) = \int_{B_{1/4}} \widetilde{G}_3(x,y)\widetilde{V}_M(y)u(y)\mathrm{d}y.$$

Here, for $M > 0$, we define $\widetilde{V}_M(x) = V(x) - V_M(x)$, where

$$V_M(x) = \begin{cases} V(x), \text{ if } |V(x)| \geqslant M, \\ 0, \text{ otherwise.} \end{cases}$$

Now we can run the regularity lifting method, which is divided into two steps.

Step 1: For $q \in (2_\#, +\infty)$, there exists $M \gg 1$ large such that $T_M : L^q(B_{1/4}) \to L^q(B_{1/4})$ is a contraction.

In fact, for any $q \in (2_\#, +\infty)$, there exists $m \in (1, n/6)$ such that $q = nm/(n-6\,m)$. Then, by the Hardy–Littlewood–Sobolev and Hölder inequalities [36], for any $u \in L^q(\mathbb{R}^n)$, we get

$$\|T_M u\|_{L^q(B_{1/4})} \leqslant \|\widetilde{G}_3 * V_M u\|_{L^q(B_{1/4})} \leqslant C\|V_M\|_{L^{n/6}(B_{1/4})}\|u\|_{L^q(B_{1/4})}.$$

Since $V_M \in L^{n/6}(B_{1/4})$ it is possible to choose a large $M \gg 1$ satisfying $\|V_M\|_{L^{n/6}(B_{1/4})} < 1/2C$. Therefore, we arrive at

$$\|T_M u\|_{L^q(B_{1/4})} \leqslant 1/2\|u\|_{L^q(B_{1/4})},$$

which implies that T_M is a contraction.

Step 2: For any $q \in (2_\#, +\infty)$, it follows that $\widetilde{T}_M u \in L^q(B_{1/4})$.

Indeed, for any $q \in (2_\#, +\infty)$, we pick $1 < m < n/6$ satisfying $q = nm/(n-6\,m)$. Since $\widetilde{V}_M$ is bounded, we get

$$\|\widetilde{T}_M\|_{L^q(B_{1/4})} = \|\widetilde{G}_3 * \widetilde{V}_M u\|_{L^q(B_{1/4})} \leqslant C\|\widetilde{V}_M u\|_{L^m(B_{1/4})} \leqslant C\|u\|_{L^m(B_{1/4})}.$$

However, using (7.5), we have that $u \in L^q(B_{1/4})$ for $q \in (1, n\gamma_p^{-1})$. Besides, $q = (p-2)n\gamma_p^{-1}$ when $m = n\gamma_p^{-1}$. Thus, we obtain that $u \in L^q(B_{1/4})$ for

$$\begin{cases} 1 < q < \infty, & \text{if } p \geqslant 2, \\ 1 < q \leqslant (2-p)^{-1}n\gamma_p^{-1}, & \text{if } 1 < p < 2. \end{cases}$$

Now we can repeat the argument for $m = (p-2)n\gamma_p^{-1}$ to get that $u \in L^q(B_{1/4})$ for

$$\begin{cases} 1 < q < \infty, & \text{if } p \geqslant 2, \\ 1 < q \leqslant (2-p)^{-1}n\gamma_p^{-1}, & \text{if } 1 < p < 2. \end{cases}$$

Therefore, by proceeding inductively as in [12, Lemma 3.8], the proof of the claim follows. Ultimately, combining Steps 1 and 2, we can apply [35, Theorem 3.3.1] to show that $u \in L^p(B_{1/4})$ for all $p > 2^\#$. In particular, the proof of the claim is finished.

Now, by the Morrey's embedding theorem, it follows that $u \in C^{0,\alpha}(B_{1/4})$, for some $\alpha \in (0,1)$. Finally using Schauder estimates, one gets that $u \in C^{6,\alpha}(B_{1/4})$. In particular, the singularity at the origin is removable, which concludes the proof of the lemma.

Proof of Proposition 7.4. Suppose by contradiction that $u \in C^6(\mathbb{R}^n \setminus \{0\})$ has a non-removable singularity at the origin, then by Lemma 7.6, u does not satisfy (7.3). Therefore, the proof follows as a consequence of Lemma 7.5.

7.3. Aviles case

Finally, we prove Theorem 1(b). The asymptotic analysis for the lower critical exponent, $p = 2_\#$ exhibits its subtlety. First, since $\gamma(2_\#) = n-6$, one would expect the singular solutions to $(\mathcal{P}_{6,p,R})$ to have the same behavior as the fundamental solution to the tri-Laplacian near the origin; thus, the isolated singularity would be removable.

Our objective is to prove the proposition below

Proposition 7.7. *Let $u \in C^6(B_R^*)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p = 2_\#$. Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. Then,*

$$u(x) = (1+\mathrm{o}(1))\widehat{K}_0(n)^{\frac{n-6}{6}}|x|^{6-n}(\ln|x|)^{\frac{6-n}{6}},$$

where $\widehat{K}_0(n) := -\lim_{t\to-\infty} t\widetilde{K}_0(n,t)$ is given by (1).

As in the autonomous case, we use the limiting energy levels $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(-\infty, w)$ to classify the local behavior near the isolated singularity.

Lemma 7.8. *Let $u \in C^6(B_R^*)$ be a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p = 2_\#$ and $w = \widetilde{\mathfrak{F}}(v)$ be its nonautonomous Emden–Fowler transformation given by (2.3). Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. Then, $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(-\infty, w) \in \{-\ell_\#^*, 0\}$, where $-\ell_\#^* = \frac{3}{2(n-3)}\widehat{K}_0(n)^{\frac{n-3}{3}}$. Moreover, it follows*

(i) $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(-\infty, w) = 0$ *if and only if,*

$$u(x) = \mathrm{o}\left(|x|^{6-n}(\ln|x|)^{\frac{6-n}{6}}\right) \quad \text{as} \quad x \to 0. \tag{7.12}$$

(ii) $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(-\infty, w) = \ell_\#^*$ *if and only if,*

$$u(x) = (1+\mathrm{o}(1))\widehat{K}_0(n)^{\frac{n-6}{6}}|x|^{6-n}(\ln|x|)^{\frac{6-n}{6}} \quad \text{as} \quad x \to 0.$$

Proof. First, combining (3.3) with Proposition 3.11 and Lemma 3.10, we find

$$\widetilde{\mathcal{P}}_{\mathrm{cyl}}(-\infty, w) = \lim_{t\to-\infty}\int_{\mathbb{S}_t^{n-1}}\left(\frac{n-6}{2(n-3)}|w(t,\theta)|^{2_\#+1} + \widehat{K}_0(n)|w(t,\theta)|^2\right)\mathrm{d}\theta.$$

Furthermore, by (3.3), we see that for any $\{t_k\}_{k\in\mathbb{N}}$ such that $t_k \to +\infty$ as $k \to -\infty$, it follows that $\{w(t_k,\theta)\}_{k\in\mathbb{N}}$ converges to a limit, which is independent of $\theta \in \mathbb{S}_t^{n-1}$. Hence, up to subsequence, there exists $w_0 \in \mathbb{R}$ such that $w(t_k,\theta) \to w_0$ uniformly on $\theta \in \mathbb{S}_t^{n-1}$, which gives us

$$\widetilde{\mathcal{P}}_{\mathrm{cyl}}(-\infty, w_0) = \frac{n-6}{2(n-3)}|w_0|^{2_\#+1} + \widehat{K}_0(n)|w_0|^2. \tag{7.13}$$

Thus, since the right-hand side of the last equation has at most three nonnegative roots, the limit $w_0 \in \mathbb{R}$, under the uniform convergence of $w(t,\theta)$ on $\mathbb{S}_t^{n-1}$ as $t \to -\infty$, is unique. Finally, multiplying $(\widetilde{\mathcal{C}}_T)$ by w, integrating both sides over $(-\infty, t_0,)\times\mathbb{S}^{n-1}$, and using (3.3),(3.4) and Lemma 3.10, it follows

$$\left|\int_{-\infty}^{t_0}\frac{1}{t}\int_{\mathbb{S}_t^{n-1}}\left(\widehat{K}_0(n) - |w(t,\theta)|^{2_\#-1}\right)|w(t,\theta)|^2\mathrm{d}\theta\mathrm{d}t\right| < +\infty.$$

Now since $\lim_{t\to-\infty} w(t,\theta) = w_0$ uniformly on $\mathbb{S}_t^{n-1}$, we get either $w_0 = 0$ or $w_0 = \widehat{K}_0(n)^{\frac{n-6}{6}}$, which by substituting into (7.13), implies that either $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(-\infty, w) = 0$ if and only if, $w_0 = 0$, or $\widetilde{\mathcal{P}}_{\mathrm{cyl}}(-\infty, w) = \ell_\#^*$, otherwise. The proof trivially follows by applying the inverse $\widetilde{\mathfrak{F}}^{-1}$ of the nonautonomous cylindrical transform.

Now we are left to show that, if Lemma 7.8(i) holds, then the singularity at the origin is removable. Here, we are based on the barriers construction in [30] (see also [22]), which is available due to the integral representation (5.1).

Lemma 7.9. *Let $u \in C^6(B_R^*)$ be a positive solution to $(\mathcal{P}_{6,p,R})$ with $p = 2_\#$. Assume that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. If*

$$u(x) = \mathrm{o}\left(|x|^{6-n}(\ln|x|)^{\frac{6-n}{6}}\right) \quad \text{as} \quad x \to 0,$$

then, the origin is a removable singularity.

Proof. For any $\delta > 0$, we choose $0 < \rho \ll 1$ such that $u(x) \leqslant \delta|x|^{-\gamma_p}$ in B_ρ^*. Fixing $\varepsilon > 0$, $\kappa \in (0, \gamma_p)$ and $M \gg 1$ to be chosen later, we define

$$\varsigma(x) = \begin{cases} M|x|^{-\kappa} + \varepsilon|x|^{6-n-\kappa}, & \text{if } 0 < |x| < \rho, \\ u(x), & \text{if } \rho < |x| < 2. \end{cases}$$

Notice that for every $0 < \kappa < n-6$ and $0 < |x| < 2$, there exists $C > 0$ such that

$$\int_{\mathbb{R}^n}|x-y|^{6-n}|y|^{-6-\kappa}\mathrm{d}y = |x|^{6-n}\int_{\mathbb{R}^n}\left||x|^{-1}x - |x|^{-1}y\right|^{6-n}|y|^{\kappa-6}\mathrm{d}y \leqslant C\left(\frac{1}{n-6-\kappa} + \frac{1}{\kappa} + 1\right)|x|^{-\kappa},$$

which, for $0 < |x| < 2$ and $0 < \delta \ll 1$, yields
$$\int_{B_\rho} u^{2_\#-1}(y)\varsigma(y)|x-y|^{6-n}\mathrm{d}y \leqslant \delta^{2_\#-1}\int_{\mathbb{R}^n}\varsigma(y)|x-y|^{n-6}|y|^{-6}\mathrm{d}y \leqslant C\delta^{2_\#-1}\varsigma(x) < \frac{1}{2}\varsigma(x).$$
Moreover, for $0 < |x| < \rho$ and $\bar{x} = \rho x|x|^{-1}$, we get
$$\int_{B_2\setminus B_\rho} u^{2_\#-1}(y)\varsigma(y)|x-y|^{6-n}\mathrm{d}y = \int_{B_2\setminus B_\rho}\frac{|\bar{x}-y|^{n-6}}{|x-y|^{n-6}}\frac{u^{2_\#}(y)}{|\bar{x}-y|^{n-6}}\mathrm{d}y \leqslant 2^{n-6}\max_{\partial B_\rho} u.$$
The last inequality implies that for $0 < |x| < \tau$ and $M \geqslant \max_{\partial B_\rho} u$,
$$\psi(x) + \int_{B_2}\frac{u^{2_\#-1}(y)\varsigma(y)}{|x-y|^{6-n}}\mathrm{d}y \leqslant \psi(x) + 2^{n-6}\max_{\partial B_\rho} u + \frac{1}{2}\varsigma(x) < \varsigma(x).$$

We show that ς can be taken as a barrier for any u. Namely, we claim that $u(x) \leqslant \varsigma(x)$ in B_ρ^*. In fact, suppose by contradiction that the conclusion is not true. Then, since $u(x) \leqslant \delta|x|^{-\gamma_p}$ in B_ρ^*, by the definition of ς, there exists $\widetilde{\tau} \in (0,\rho)$, depending on ε, such that $\varsigma \geqslant u$ in B_ρ^* and $\varsigma > u$ close to the boundary ∂B_ρ. Let us consider $\bar{\tau} := \inf\{\tau > 1 : \tau\psi > u \text{ in } B_\rho^*\}$. Then, we have that $\bar{\tau} \in (1,+\infty)$ and there exists $\bar{x} \in B_\rho \setminus \bar{B}_{\widetilde{\tau}}$ such that $\bar{\tau}\varsigma(\bar{x}) = u(\bar{x})$ and, for $0 < |x| < \tau$, it follows
$$\bar{\tau}\varsigma(x) \geqslant \int_{B_2} u^{2_\#-1}(y)\bar{\tau}\varsigma(y)|x-y|^{6-n}\mathrm{d}y + \bar{\tau}\psi(x) \geqslant \int_{B_2} u^{2_\#-1}(y)\bar{\tau}\varsigma(y)|x-y|^{6-n}\mathrm{d}y + \psi(x),$$
which gives us
$$\bar{\tau}\varsigma(x) - u(x) \geqslant \int_{B_2} u^{2_\#-1}(y)(\bar{\tau}\varsigma(y) - u(y))|x-y|^{6-n}\mathrm{d}y.$$
Finally, by evaluating the last inequality at $\bar{x} \in B_\rho \setminus \bar{B}_{\widetilde{\tau}}$, we get a contradiction.

At last, we find $u(x) \leqslant \varsigma(x) \leqslant M|x|^{-\kappa} + \varepsilon|x|^{6-n-\kappa}$ in B_ρ^*, which yields that $u^{2_\#-1} \in L^p(B_\rho^*)$ for some $p > n/6$. Hence, standard elliptic regularity concludes the proof of the lemma.

Ultimately, the proof of the main result in this section is merely a consequence of the last results.

Proof of Proposition 7.7. Suppose that $u \in C^6(\mathbb{R}^n \setminus \{0\})$ is a positive singular solution to $(\mathcal{P}_{6,p,R})$ with $p = 2_\#$, then by Lemma 7.9, u does not satisfy (7.12). Therefore, the proof follows as a consequence of Lemma 7.8.

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Appendix. The general Emden–Fowler change of coordinates

In this appendix, using the software Mathematica 13.2, we compute the coefficients of the bi-Laplacian written in cylindrical coordinates (see also [12,23]). More generally, let us consider the following change of coordinates
$$u(r) = \rho(r)v(t) \quad \text{with} \quad t = \psi(r), \tag{A.1}$$
where $\rho, \psi : \mathbb{R} \to \mathbb{R}$ are smooth functions, and ψ is a smooth diffeomorphism. Here we adopt the notations $\psi_r = \mathrm{d}\psi/\mathrm{d}r$, $\rho_r = \mathrm{d}\rho/\mathrm{d}r$, $\psi_r^{(j)} = \mathrm{d}^j\psi/\mathrm{d}r^j$, and $\rho_r^{(j)} = \mathrm{d}^j\rho/\mathrm{d}r^j$ $\partial_t^{(j)} = \partial^j/\partial t^j$ (resp. $\partial_r^{(j)} = \partial^j/\partial r^j$) with the convention $\partial_t^{(0)}$ equals the identity operator on $C^\infty(\mathbb{R})$, and we omit u, v when it is convenient. Now the idea is to express the operator $\partial_r^{(j)}$ for $j \in \mathbb{N}$ in terms of $\partial_t^{(\ell)}$ for $\ell \in \{1,\dots,j\}$, that is,
$$\partial_r^{(j)} = \sum_{\ell=0}^{j} c_{j\ell}(\rho,\psi)\partial_t^{(\ell)},$$
where $c_{j\ell} : \mathbb{R}^{2(\ell+1)+1} \to \mathbb{R}$ are the coefficient functions, depending on ρ, ψ and all theirs derivative until ℓ-th order. Notice that $C = (c_{j\ell})_{j\ell}$ is a lower triangular matrix, *i.e.*, $c_{j\ell} \equiv 0$ when $j < \ell$. For this manuscript, we always choose $\psi(r) = -\ln r$. The choice of the other function depends on the growth of the nonlinearity, namely we either choose
$$\rho(r) = r^{6/(p-1)} \quad \text{when} \quad p \in (2_\#, 2^\# - 1)$$
or
$$\rho(r) = (\ln r)^{6-n} r^{\frac{6-n}{6}} \quad \text{when} \quad p = 2_\#$$
for the autonomous and nonautonomous cases, respectively. This turns (A.1) into the classical logarithm cylindrical change of coordinates.

Now let us derive the explicit formula for the coefficients of the tri-Laplacian in autonomous and nonautonomous Emden–Fowler coordinates, respectively. We consider the cylinder $\mathcal{C}_R := (0,R) \times \mathbb{S}^{n-1}$ and $(-\Delta)^3_{\mathrm{sph}}$ the tri-Laplacian written in spherical (polar) coordinates given by

$$\begin{aligned}\Delta^3_{\rm sph} &= r^{-6}\partial_r^{(6)} + M_5(n,r)\partial_r^{(5)} + M_4(n,r)\partial_r^{(4)} + M_3(n,r)\partial_r^{(3)} + M_2(n,r)\partial_r^{(2)} + M_1(n,r)\partial_r \\ &\quad + 2r^{-2}\partial_r^{(4)}\Delta_\sigma + N_3(n,r)\partial_r^{(3)}\Delta_\sigma + N_2(n,r)\partial_r^{(2)}\Delta_\sigma + N_1(n,r)\partial_r\Delta_\sigma + N_0(n,r)\Delta_\sigma \\ &\quad + 3r^{-4}\partial_r^{(2)}\Delta_\sigma^2 + O_1(n,r)\partial_r\Delta_\sigma^2 + O_0(n,r)\Delta_\sigma^2 + r^{-6}\Delta_\sigma^3,\end{aligned}$$

where Δ_σ denotes the Laplace–Beltrami operator in $\mathbb{S}^{n-1}$ and

$$\begin{aligned}M_5(n,r) &= 3(n-1)r^{-1}\\ M_4(n,r) &= 3(n-1)(n-3)r^{-2}\\ M_3(n,r) &= (n-1)(n-3)(n-8)r^{-3}\\ M_2(n,r) &= -3(n-1)(n-3)(n-5)r^{-4}\\ M_1(n,r) &= 3(n-1)(n-3)(n-5)r^{-5}\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}N_3(n,r) &= 2(n-7)r^{-3}\\ N_2(n,r) &= 2(n^2-n-3)r^{-4}\\ N_1(n,r) &= 6(7n-23)r^{-5}\\ N_0(n,r) &= 8(n-1)(n-5)r^{-6}\\ O_1(n,r) &= 3(n-5)r^{-5}\\ O_0(n,r) &= -2(3n-16)r^{-6}\end{aligned} \tag{A.3}$$

are its coefficients in this coordinate system. Notice that they can be computed recursively in terms of the derivatives of the coefficients of bi-Laplacian written in polar coordinates.

A.1. Autonomous case

Let us recall the sixth order autonomous Emden–Fowler change of variables (or logarithmic cylindrical coordinates) given by

$$v(t,\theta) = r^{\gamma_p}u(r,\sigma), \quad \text{where} \quad t = \ln r \quad \text{and} \quad \sigma = \theta = x|x|^{-1}.$$

Using this change of variables and performing a lengthy computation, we arrive at the following sixth order operator on the cylinder,

$$\begin{aligned}P^6_{\rm cyl} &= \partial_t^{(6)} + K_5(n,p)\partial_t^{(5)} + K_4(n,p)\partial_t^{(4)} + K_3(n,p)\partial_t^{(3)} + K_2(n,p)\partial_t^{(2)} + K_1(n,p)\partial_t + K_0(n,p)\\ &\quad + 2\partial_t^{(4)}\Delta_\theta + J_3(n,p)\partial_t^{(3)}\Delta_\theta + J_2(n,p)\partial_t^{(2)}\Delta_\theta + J_1(n,p)\partial_t\Delta_\theta + J_0(n,p)\Delta_\theta\\ &\quad + 3\partial_t^{(2)}\Delta_\theta^2 + L_1(n,p)\partial_t\Delta_\theta^2 + L_0(n,p)\Delta_\theta^2 + \Delta_\theta^3,\end{aligned}$$

where

$$\begin{aligned}K_\ell(n,p) &= \sum_{j=0}^{6} N_\ell(n)c_{j\ell}(n,p) \quad \text{for} \quad \ell \in \{0,1,2,3,4,5\},\\ J_\ell(n,p) &= \sum_{j=0}^{4} M_\ell(n)c_{j\ell}(n,p) \quad \text{for} \quad \ell \in \{0,1,2,3,4\},\end{aligned}$$

and

$$L_\ell(n,p) = \sum_{j=0}^{2} O_\ell(n)c_{j\ell}(n,p) \quad \text{for} \quad \ell \in \{0,1\}.$$

More explicitly, we can use Mathematica 13 to obtain

$$\begin{aligned}K_0(n,p) &= -24(p-1)^{-6}(p+2)(2p+1)(np-6p-n)(np-4p-2-n)(np-2p-n-4)\\ K_1(n,p) &= 4(p-1)^{-5}(np-6p-n-6)(2n^2p^4-12np^4+16p^4+10n^2p^3-108np^3+224p^3\\ &\quad -15n^2p^2-36np^2+492p^2-8n^2p+120np+224p+11n^2+36n+16)\\ K_2(n,p) &= -2(p-1)^{-4}(3n^3p^4-48n^2p^4+228np^4-320p^4-3n^3p^3-78n^2p^3+996np^3-2392p^3-9n^3p^2\\ &\quad +198n^2p^2+180np^2-4296p^2+15n^3p+30n^2p-1020np-2392p-6n^3-102n^2-384n-320)\\ K_3(n,p) &= -(p-1)^{-3}(np-6p-6-n)(n^2p^2-24np^2+68p^2-2n^2p-12np+224p+n^2+36n+68)\\ K_4(n,p) &= (p-1)^{-2}(3n^2p^2-42np^2+124p^2-6n^2p-6np+292p+3n^2+48n+124)\\ K_5(n,p) &= 3(p-1)^{-1}(np-6p-6-n)\end{aligned} \tag{A.4}$$

and

$$
\begin{aligned}
J_0(n,p) &= 4(p-1)^{-4}(2n^2p^4 - 12np^4 + 10p^4 - 5n^2p^3 - 24np^3 + 218p^3 + 21n^2p^2 + 72np^2 - 192p^2 \\
&\quad - 35n^2p - 132np + 1094p + 17n^2 + 96n - 482) \\
J_1(n,p) &= -2(p-1)^{-3}(n^2p^3 - 24np^3 + 86p^3 + 9n^2p^2 + 24np^2 + 90p^2 - 21n^2p - 84np + 966p \\
&\quad + 11n^2 + 84n - 278) \\
J_2(n,p) &= 2(p-1)^{-2}(n^2p^2 - 4np^2 + 29p^2 - 2n^2p - 10np + 176p + n^2 + 14n + 11) \\
J_3(n,p) &= 2(p-1)^{-1}(np - 13p - n - 11) \\
L_0(n,p) &= -(p-1)^{-2}(3np^2 16p^2 - 3np + 22p + 6n + 16) \\
L_1(n,p) &= 3(p-1)^{-1}(np - 6p - 6 - n).
\end{aligned} \tag{A.5}
$$

A.2. Upper critical case

When $p = 2^\# - 1$, it follows

$$
\begin{aligned}
K_0^\#(n) &= -2^{-8}(n-6)^2(n-2)^2(n+2)^2 \\
K_1^\#(n) &= 0 \\
K_2^\#(n) &= 2^{-4}(3n^4 - 24n^3 + 72n^2 - 96n + 304) \\
K_3^\#(n) &= 0 \\
K_4^\#(n) &= -2^{-2}(3n^2 - 12n + 44) \\
K_5^\#(n) &= 0 \\
J_0^\#(n) &= 2^{-3}(3n^4 - 18n^3 - 192n^2 + 1864n - 3952) \\
J_1^\#(n) &= -2^{-1}(3n^3 + 3n^2 - 244n + 620) \\
J_2^\#(n) &= 2n^2 + 13n - 68 \\
J_3^\#(n) &= -2(n+1) \\
L_0^\#(n) &= -2^{-2}(3n^2 - 12n - 20) \\
L_1^\#(n) &= 0.
\end{aligned} \tag{A.6}
$$

A.3. Lower critical case

When $p = 2_\#$, it follows

$$
\begin{aligned}
K_{0,\#}(n) &= 0 \\
K_{1,\#}(n) &= -8(n-6)(n-4)(n-2) \\
K_{2,\#}(n) &= -2(3n^3 - 48n^2 + 228n - 320) \\
K_{3,\#}(n) &= -(n-6)(n^2 - 24n + 68) \\
K_{4,\#}(n) &= 3n^2 - 42n + 124 \\
K_{5,\#}(n) &= -3(n-6) \\
J_{0,\#}(n) &= 2(n^4 - 8n^3 - 39n^2 + 470n - 964) \\
J_{1,\#}(n) &= -2(3n^3 - 16n^2 - 62n + 278) \\
J_{2,\#}(n) &= 2(4n^2 - 19n + 11) \\
J_{3,\#}(n) &= -2(3n - 11) \\
L_{0,\#}(n) &= -2(3n - 16) \\
L_{1,\#}(n) &= -3(n-6).
\end{aligned} \tag{A.7}
$$

A.4. Nonautonomous case

Let us recall the sixth order nonautonomous Emden–Fowler change of variables (or logarithmic cylindrical coordinates) given by

$$
w(t,\theta) = r^{6-n}(\ln r)^{\frac{6-n}{6}} u(r,\sigma), \quad \text{where} \quad t = \ln r \quad \text{and} \quad \sigma = \theta = x|x|^{-1}.
$$

Using this coordinate system and performing a lengthy computation, we arrive at the following sixth order nonautonomous operator PDE on the cylinder

$$
\begin{aligned}
\widetilde{P}_{\rm cyl} &= \partial_t^{(6)} + \widetilde{K}_5(n,t)\partial_t^{(5)} + \widetilde{K}_4(n,t)\partial_t^{(4)} + \widetilde{K}_3(n,t)\partial_t^{(3)} + \widetilde{K}_2(n,t)\partial_t^{(2)} + \widetilde{K}_1(n,p)\partial_t + \widetilde{K}_0(n,t) \\
&\quad + 2\partial_t^{(4)}\Delta_\theta + \widetilde{J}_3(n,t)\partial_t^{(3)}\Delta_\theta + \widetilde{J}_2(n,t)\partial_t^{(2)}\Delta_\theta + \widetilde{J}_1(n,t)\partial_t\Delta_\theta + \widetilde{J}_0(n,t)\Delta_\theta
\end{aligned}
$$

$$+\ 3\partial_t^{(2)}\Delta_\theta^2 + \widetilde{L}_1(n,t)\partial_t\Delta_\theta^2 + \widetilde{L}_0(n,t)\Delta_\theta^2 + \Delta_\theta^3,$$

where

$$\widetilde{K}_\ell(n,t) = \sum_{j=0}^{6} N_\ell(n)\tilde{c}_{j\ell}(n,t) \quad \text{for} \quad \ell \in \{0,\dots,5\}$$

$$\widetilde{J}_\ell(n,t) = \sum_{j=0}^{4} M_\ell(n)\tilde{c}_{j\ell}(n,t) \quad \text{for} \quad \ell \in \{0,\dots,4\}$$

and

$$\widetilde{L}_\ell(n,t) = \sum_{j=0}^{2} O_\ell(n)\tilde{c}_{j\ell}(n,t) \quad \text{for} \quad \ell \in \{0,1\}.$$

More explicitly, we have

$$\begin{aligned}
\widetilde{K}_0(n,t) &= \frac{4(n-6)(n^3-12n^2+44n-48)}{3t} - \frac{(n-6)n(3n^3-48n^2+228n-320)}{18t^2}\\
&+ \frac{(n-6)n(n^4-24n^3+32n^2+864n-2448)}{216t^3} + \frac{(n-6)n(3n^4+12n^3-416n^2-792n+8928)}{1296t^4}\\
&+ \frac{(n-6)n(n^4+30n^3+180n^2-1080n-7776)}{2592t^5} + \frac{(n-6)n(n^4+60n^3+1260n^2+10800n+31104)}{46656t^6}\\
\widetilde{K}_1(n,t) &= -8(n^3-12n^2+44n-48) + \frac{2(n^4-66n^3+516n^2-1688n^3+1920)}{3t}\\
&- \frac{(n-6)^2n(n^2-24n+68)}{12t^2} - \frac{n(3n^4-42n^3+16n^2+1512n-4464)}{54t^3}\\
&- \frac{5(n-6)^2n(n^2+18n+72)}{432t^4} - \frac{n(n^4+30n^3+180n^2-1080n-7776)}{1296t^5}\\
\widetilde{K}_2(n,t) &= -(6n^3-96n^2+456n-640) + \frac{(n-6)^2(n^2-24n+68)}{2t}\\
&+ \frac{n(3n^3-60n^2+376n-744)}{6t^2} + \frac{5(n-6)^2n(n+6)}{36t^3} + \frac{5n(n^3+12n^2-36n-432)}{432t^4}\\
\widetilde{K}_3(n,t) &= -(n^3-30n^2+212n-408) - \frac{(6n^3-120n^2+752n-3496)}{3t} - \frac{5(n-6)^2n}{6t^2} - \frac{5n(n^2-36)}{54t^3}\\
\widetilde{K}_4(n,t) &= (3n^2-42n+124) + \frac{5(n-6)^2}{2t} + \frac{5n(n-6)}{12t^2}\\
\widetilde{K}_5(n,t) &= -3(n-6) + \frac{n-6}{t}
\end{aligned} \tag{A.8}$$

and

$$\begin{aligned}
\widetilde{J}_0(n,t) &= 2(n^4-8n^3-39n^2+470n-964) + \frac{(3n^4-34n^3+34n^2+650n-1668)}{3t}\\
&+ \frac{n(4n^3-43n^2+125n-66)}{18t^2} + \frac{n(n^3-11n^2-108n+396)}{108t^3}\\
&+ \frac{n(n^3+12n^2-36n-432)}{648t^4}\\
\widetilde{J}_1(n,t) &= -(6n^3-32n^2-124n+556) + \frac{2(4n^3-43n^2+125n-66)}{3t}\\
&- \frac{n(3n^2-29n+66)}{6t^2} - \frac{n(n^2-36)}{27t^3}\\
\widetilde{J}_2(n,t) &= (8n^2-38n+22) + \frac{(3n^2-29n+66)}{t} + \frac{n(n-6)}{3t^2}\\
\widetilde{J}_3(n,t) &= -\frac{4(n-6)}{3} - (6n-22)t\\
\widetilde{L}_0(n,t) &= -\frac{(72n-384)}{12} + \frac{(n-6)^2}{2t} + \frac{n(n-6)}{12t^2}\\
\widetilde{L}_1(n,t) &= -3(n-6) - \frac{n-6}{t}.
\end{aligned} \tag{A.9}$$

Finally, we set

$$\begin{aligned}
\widehat{K}_0(n) &= \frac{4(n-6)(n^3-12n^2+44n-48)}{3}\\
\widetilde{K}_1(n) &= \frac{2(n^4-66n^3+516n^2-1688n^3+1920)}{3}
\end{aligned}$$

$$\widehat{K}_2(n) = \frac{(n-6)^2(n^2-24n+68)}{2}$$
$$\widehat{K}_3(n) = -\frac{(6n^3-120n^2+752n-3496)}{3}$$
$$\widehat{K}_4(n) = \frac{5(n-6)^2}{2}$$
$$\widehat{K}_5(n) = n-6 \tag{A.10}$$

to be the coefficients of the so-called asymptotic nonautonomous cylindrical Paneitz operator given by (3.7).

Data availability

No data was used for the research described in the article.

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