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SELF-SIMILARITY AND LAMPERTI CONVERGENCE FOR FAMILIES OF STOCHASTIC PROCESSES

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Abstract. We define a new type of self-similarity for one-parameter families of stochastic processes, which applies to certain important families of processes that are not self-similar in the conventional sense. This includes Hougaard Lévy processes such as the Poisson processes, Brownian motions with drift and the inverse Gaussian processes, and some new fractional Hougaard motions defined as moving averages of Hougaard Lévy process. Such families have many properties in common with ordinary self-similar processes, including the form of their covariance functions, and the fact that they appear as limits in a Lamperti-type limit theorem for families of stochastic processes.

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1 INTRODUCTION

The purpose of this paper is to extend the notion of self-similarity to one-parameter families of stochastic processes. Recall that a real-valued stochastic process $\{X(t)\colon\ t \geqslant 0\}$ is called *self-similar with Hurst exponent* $H > 0$ if it satisfies the scaling property

$$X(ct) \stackrel{\mathrm{d}}{=} c^H X(t) \quad \text{for } t \geqslant 0, \tag{1.1}$$

for all $c > 0$, where $\stackrel{\mathrm{d}}{=}$ denotes equality of the finite-dimensional distributions (see, e.g., [5, p. 1]). The corresponding process with drift $\mu \in \mathbb{R}$, defined by

$$X(\mu; t) = X(t) + \mu t \quad \text{for } t \geqslant 0, \tag{1.2}$$

does not, however, satisfy (1.1) for $\mu \neq 0$, an example being a Brownian motion with drift. Instead, the family of processes (1.2) satisfies the following scaling property:

$$X\big(\mu c^{H-1}; ct\big) \stackrel{\mathrm{d}}{=} c^H X(\mu; t) \quad \text{for } t \geqslant 0, \tag{1.3}$$

for all $c > 0$, and we shall hence propose (1.3) as a new definition of self-similarity for families of stochastic processes; see Definition 1 below.

We show that there are, in fact, many important families of stochastic processes with finite variance that satisfy (1.3) without having the drift form (1.2) and for values of H that do not necessarily belong to $(0, 1)$. One such example is the class of Hougaard Lévy processes [13], which includes, for example, the family of Poisson processes ($H = 0$), certain gamma compound Poisson processes ($H < 0$), and the family of inverse Gaussian processes ($H = 2$) [19]. A further example is a new class of fractional Hougaard motions defined as moving averages of Hougaard Lévy processes, generalizing fractional Brownian motion. As we shall see, such processes have many properties in common with ordinary self-similar processes, as reflected, for example, in the familiar form of their covariance functions. This represents an important step forward compared with ordinary self-similar processes, where the only processes with finite variance are the fractional Brownian motions with $H \in (0, 1)$.

In Section 2, we present a new notion of self-similarity for families of stochastic processes. We show that their covariance structure is completely determined by the so-called variance function, and we study the special case of power variance functions. In Section 3, we consider self-similarity for natural exponential families of stochastic processes and show the self-similarity of Hougaard Lévy processes. In Section 4, we show the self-similarity of the class of fractional Hougaard motions using their moving average representation. In Section 5, we introduce a Lamperti-type transformation, which transforms a self-similar family into a family of stationary processes. In Section 6, we consider some Lamperti-type limit theorems, where families of self-similar processes appear as limits of suitably scaled families of stochastic processes. Finally, in Section 7, we investigate the case $H = 1$ and its relation with exponential variance functions.

2 SELF-SIMILARITY

2.1 Definition

We consider a family of real-valued stochastic processes $X = \{X(\mu; t)\colon\ \mu \in \overline{\Omega},\ t \geqslant 0\}$, where t denotes time, and the family is indexed by a real parameter $\mu \in \overline{\Omega}$, where $\overline{\Omega}$ is an interval satisfying $\mathbb{R}_+ \subseteq \overline{\Omega}$ and $c\overline{\Omega} \subseteq \overline{\Omega}$ for all $c > 0$. Each process $X(\mu; \cdot) = \{X(\mu; t)\colon\ t \geqslant 0\}$ in the family may live on its own probability space, although, in Section 5, we consider results that require all the member processes to be defined on the same probability space. We allow $\overline{\Omega}$ to contain the values $\pm\infty$ and define $\Omega = \operatorname{int}\overline{\Omega}$, the interior of $\overline{\Omega}$. These conventions are important for the examples that we will discuss. Motivated by the above discussion of processes with drift, we now propose an extended definition of self-similarity for families of stochastic processes.

DEFINITION 1. (i) A family of stochastic processes X is called *self-similar with Hurst exponent* $H \in \mathbb{R}$ and *rate parameter* $\mu \in \overline{\Omega}$ (H-SS) if

$$X\bigl(\mu c^{H-1}; ct\bigr) \overset{\mathrm{d}}{=} c^H X(\mu; t) \quad \text{for } t \geqslant 0, \tag{2.1}$$

for all $c > 0$ and $\mu \in \overline{\Omega}$.

(ii) We say that X has *stationary increments* if each process in the family has stationary increments, that is, for all $s > 0$ and $\mu \in \overline{\Omega}$,

$$X(\mu; s + t) - X(\mu; s) \overset{\mathrm{d}}{=} X(\mu; t) - X(\mu; 0) \quad \text{for } t \geqslant 0. \tag{2.2}$$

The family X is called H-SSSI if it is H-SS and has stationary increments.

When necessary, we refer to definition (2.1) as *general self-similarity*, whereas the conventional definition (1.1) for stochastic processes will be called *strict self-similarity* from now on, a terminology borrowed from stable distributions. The motivation for expanding the use of the term self-similar in this way is that many properties of strictly self-similar processes have immediate analogues for general self-similar families, in particular, for the covariance structure (see Section 2.2).

A family of processes with drift $X = \{X(t) + \mu t\colon\ \mu \in \mathbb{R},\ t \geqslant 0\}$ is, in the general sense, H-SS if and only if the stochastic process $X(\cdot)$ is strictly self-similar with Hurst exponent H. It is clear that general self-similarity constitutes an extension of strict self-similarity, since if X satisfies (2.1), then the distribution of $X(\mu;t) - \mu t$ will generally depend on μ, as illustrated by the examples in Sections 3.2 and 4. Nevertheless, μ plays the role of rate for the process $X(\mu;\cdot)$, in the sense that the dimension of μ is the unit of the process $X(\mu;\cdot)$ per unit of time, thereby providing an extension of the idea of a drift parameter. When applicable, the values $\mu = 0$ and $\mu = \pm\infty$ in (2.1) correspond to the strictly self-similar processes $X(0;\cdot)$ and $X(\pm\infty;\cdot)$, respectively.

An equivalent way of writing definition (2.1) is as follows:

$$c^{-H} X\big(\mu c^{H-1}; ct\big) \overset{\mathrm{d}}{=} X(\mu;t) \quad \text{for } t \geqslant 0, \tag{2.3}$$

for all $c > 0$ and $\mu \in \overline{\Omega}$. Noting that the right-hand side of (2.3) is independent of c, we observe a collapse of the scaled marginal distributions of X onto a single distribution independent of c, much like the notion of universality in turbulence (see [1] and references therein). Taking $c = 1/t$ in (2.3), we obtain the useful relation

$$t^H X\big(\mu t^{1-H}; 1\big) \overset{\mathrm{d}}{\sim} X(\mu;t) \quad \text{for } t > 0, \tag{2.4}$$

for each $\mu \in \overline{\Omega}$, where $\overset{\mathrm{d}}{\sim}$ denotes equality of the marginal distributions for each t. Given $H \in \mathbb{R}$, the left-hand side of (2.3) yields the multiplicative transformation group $\{R_c^H\colon\ c > 0\}$ defined by

$$R_c^H X(\mu;t) = c^{-H} X\big(\mu c^{H-1}; ct\big). \tag{2.5}$$

This is a continuous analogue of the *renormalization group* in the sense of Jona-Lasinio [7] (see also [5, p. 15]) with fixed point given by (2.3), a topic that we shall return to in Section 6.

The general interpretation of definition (2.1) is that a rescaling of $X(\mu;t)$ is equivalent in distribution to a simultaneous rescaling of time t and rate μ, so that nontrivial solutions to (2.1) require that the different processes of the family are suitably linked, for example, by a drift term, as in (1.2), or by means of exponential tilting (see Section 3). In fact, (2.4) suggests that a simple rescaling $t^H X(\mu t^{1-H})$ of a given process $X(\cdot)$ provides a trivial solution to (2.1). Similarly, for $H \neq 1$ and $\mu \neq 0$, we may take $c = |\mu|^{1/(1-H)}$ in (2.3), which yields

$$X(\mu;t) \overset{\mathrm{d}}{=} |\mu|^{H/(H-1)} X\big(\operatorname{sgn}(\mu); |\mu|^{1/(1-H)} t\big) \quad \text{for } t \geqslant 0,$$

where $\operatorname{sgn}(\mu)$ is the sign of μ. Hence, if $X(\cdot)$ is a given stochastic process, then the rescaled family defined by $X(\mu;t) = \mu^{H/(H-1)} X(\mu^{1/(1-H)} t)$ for $\mu, t > 0$ is also a trivial solution to (2.1).

Note that taking $t = 0$ in (2.1) gives $X(\mu c^{H-1}; 0) \overset{\mathrm{d}}{\sim} c^H X(\mu; 0)$, which shows that if $X(\mu;0) \overset{\mathrm{d}}{\sim} 0$ for one value of μ, then the same is the case for all values of μ with the same sign. In most cases, we have $X(\mu;0) \equiv 0$.

2.2 Covariance structure

Let X be an H-SSSI family of stochastic processes. If $X(\mu;1)$ has finite second moments for all $\mu \in \Omega$ (second-moment assumption), then (2.4) implies that $X(\mu;t)$ has finite second moments for all $t > 0$ and $\mu \in \Omega$. We shall now explore the covariance structure for such families. Most of the results in the following require $H \neq 1$, and the case $H = 1$ will be considered separately in Section 7. The next result concerns the structure of the first moment.

Proposition 1. *Let X be an H-SSSI family of stochastic processes with $H \neq 1$, and assume that $X(\mu;t)$ has finite expectation for all $\mu \in \Omega$ and $t \geqslant 0$. If the function $t \mapsto \mathbb{E}[X(\mu;t)]$ is continuous on $[0,\infty)$ for each*

$\mu \in \Omega$, then there exist functions a and b such that

$$\mathbb{E}\big[X(\mu;t)\big] = a\big(\operatorname{sgn}(\mu)\big)|\mu|^{H/(H-1)} + b\big(\operatorname{sgn}(\mu)\big)|\mu|t \quad \textit{for } t \geqslant 0 \textit{ and } \mu \in \Omega. \tag{2.6}$$

If $\Omega = \mathbb{R}$ and $0 \leqslant H < 1$, then the continuity of the function $\mu \mapsto \mathbb{E}[X(\mu;t)]$ implies $a \equiv 0$.

Proof. For given $\mu \in \Omega$, the assumption of stationary increments (2.2) implies that, for $s, t \geqslant 0$,

$$\mathbb{E}\big[X(\mu;s+t)\big] + \mathbb{E}\big[X(\mu;0)\big] = \mathbb{E}\big[X(\mu;s)\big] + \mathbb{E}\big[X(\mu;t)\big].$$

Using standard results for the Cauchy functional equation (see [4, p. 4]), we see that the continuity of $\mathbb{E}[X(\mu;\cdot)]$ implies that

$$\mathbb{E}\big[X(\mu;t)\big] = a(\mu) + b(\mu)t \quad \text{for } t \geqslant 0,$$

for suitable functions a and b defined on Ω. In view of (2.1), we obtain, for all $c, t > 0$ and $\mu \in \Omega$,

$$a\big(\mu c^{H-1}\big) + b\big(\mu c^{H-1}\big)ct = c^H a(\mu) + c^H b(\mu)t,$$

which implies that

$$a\big(\mu c^{H-1}\big) = c^H a(\mu) \quad \text{and} \quad b\big(\mu c^{H-1}\big) = c^{H-1} b(\mu).$$

By inserting $\mu = \pm 1$ and $x = \pm c^{H-1}$ into these equations, we obtain $a(x) = |x|^{H/(H-1)} a(\operatorname{sgn}(x))$ and $b(x) = |x| b(\operatorname{sgn}(x))$, in agreement with (2.6) for $\mu \neq 0$.

If $\Omega = \mathbb{R}$, so that $0 \in \Omega$, then we have already seen in Section 2.1 that the process $X(0;\cdot)$ is strictly self-similar, in which case it is well known that $X(0;0) \equiv 0$. This is in agreement with the limit of (2.6) as $\mu \to 0$, provided that either $a \equiv 0$ or $|\mu|^{H/(H-1)} \to 0$ as $\mu \to 0$, the latter being the case outside the interval $0 \leqslant H \leqslant 1$. Hence, in the case $0 \leqslant H < 1$, the continuity of the function $\mu \mapsto \mathbb{E}[X(\mu;t)]$ implies $a \equiv 0$. This completes the proof. □

The constant term $a(\operatorname{sgn}(\mu))|\mu|^{H/(H-1)}$ may be removed from the processes by subtraction. Thus, if the family X satisfies the conditions of Proposition 1, we define the corresponding *centered family* X_0 by

$$X_0(\mu;t) = X(\mu;t) - a\big(\operatorname{sgn}(\mu)\big)|\mu|^{H/(H-1)}. \tag{2.7}$$

It is easy to check that the family X_0 is again H-SSSI, now with $a(\operatorname{sgn}(\mu)) \equiv 0$. In many cases, the mean function (2.6) for the centered family X_0 takes the following simple form:

$$\mathbb{E}\big[X_0(\mu;t)\big] = \mu t \quad \text{for } t \geqslant 0 \text{ and } \mu \in \Omega, \tag{2.8}$$

corresponding to $b(\mu) = \mu$. The latter form may be assumed without loss of generality (i.e., up to a rescaling of μ) in the important case $\Omega = \mathbb{R}_+$, whereas for $\Omega = \mathbb{R}$, this requires $b(1) = b(-1)$. From now on, we assume that the family X is centered and satisfies (2.8), unless otherwise stated.

We shall now derive the covariance structure of the family X under the second-moment assumption, which is done in much the same way as for strictly self-similar processes. Let us introduce the *variance function* $V : \Omega \to \mathbb{R}_+$ defined by

$$V(\mu) = \operatorname{Var}\big[X(\mu;1)\big] \quad \text{for } \mu \in \Omega. \tag{2.9}$$

Consider a given value of $\mu \in \Omega$. The marginal variance of the process may be obtained by calculating the variance on both sides of (2.4), giving

$$\operatorname{Var}\big[X(\mu;t)\big] = t^{2H} V\big(\mu t^{1-H}\big) = V_H(\mu;t), \tag{2.10}$$

say, for all $t > 0$. As is well known [17], the whole covariance structure of X may now be expressed in terms of the variance (2.10). Thus, for $s, t > 0$, we obtain

$$\operatorname{Var}\big[X(\mu;t) - X(\mu;s)\big] = \operatorname{Var}\big[X(\mu;s)\big] + \operatorname{Var}\big[X(\mu;t)\big] - 2\operatorname{Cov}\big[X(\mu;s), X(\mu;t)\big]. \tag{2.11}$$

From the stationarity of the increments together with (2.10), Eq. (2.11) gives the following covariance function for the process $X(\mu;\cdot)$ for $s, t > 0$:

$$\operatorname{Cov}\big[X(\mu;s), X(\mu;t)\big] = \frac{1}{2}\big[V_H(\mu;s) + V_H(\mu;t) - V_H\big(\mu;|t-s|\big)\big]. \tag{2.12}$$

2.3 Power variance functions

Let us consider the important case where a centered H-SSSI family of stochastic processes X has power variance function $V(\mu) = \sigma^2\mu^p$ for some $p \in \mathbb{R}$ and $\sigma^2 > 0$, where p is called the *power parameter*. Here we take the interior of the domain for μ to be $\Omega_0 = \mathbb{R}$ for $p = 0$ and $\Omega_p = \mathbb{R}_+$ for $p \neq 0$. In this case, the variance (2.10) becomes

$$\operatorname{Var}\big[X(\mu;t)\big] = \sigma^2\mu^p t^{2H+p(1-H)} = \sigma^2\mu^p t^{2-D} \quad \text{for } t > 0,$$

say, where D is the *fractal dimension* defined by

$$D = (H-1)(p-2), \tag{2.13}$$

with domain $D \in [0,2]$ (see Lemma 1 below). The parameter D is useful in the following discussion of the covariance structure of the processes.

The covariance function (2.12) now takes the form

$$\operatorname{Cov}\big[X(\mu;s), X(\mu;t)\big] = \sigma^2\mu^p R_D(s,t) \quad \text{for } s, t > 0, \tag{2.14}$$

where

$$R_D(s,t) = \frac{1}{2}\big[s^{2-D} + t^{2-D} - |t-s|^{2-D}\big] \quad \text{for } s, t > 0.$$

Compared with (2.12), the covariance function (2.14) is now a product of the variance function $\sigma^2\mu^p$ and a function depending on s and t only. Let us also consider the correlation between two nonoverlapping increments, which may be expressed in terms of $r = \sqrt{s/t}$:

$$\operatorname{Corr}\big[X(\mu;s), X(\mu;s+t) - X(\mu;s)\big] = \frac{1}{2}\big[\big(r^{-1} + r\big)^{2-D} - r^{2-D} - r^{-2+D}\big] \quad \text{for } s, t > 0. \tag{2.15}$$

The next two basic lemmas appear to be new (but not surprising), except that the "if" part of Lemma 1 is well known; see the references given in the proof.

Lemma 1. *The function R_D is nonnegative definite if and only if $0 \leqslant D \leqslant 2$.*

Proof. Let us first take $p = 0$ in (2.13), corresponding to $D = 2(1-H)$, in which case, we observe that

$$R_{2(1-H)}(s,t) = \frac{1}{2}\big[s^{2H} + t^{2H} - |t-s|^{2H}\big]$$

has the same functional form as the covariance function of a standard fractional Brownian motion, which is known to be nonnegative definite for $0 \leqslant H \leqslant 1$ [5, 17]. It follows that R_D is nonnegative definite for any $D \in [0,2]$. Let us now show the necessity of this condition. Since the correlation (2.15) must be less than or equal to 1, we obtain for $r = 1$ that $(2^{2-D} - 1 - 1)/2 \leqslant 1$, which is equivalent to $D \geqslant 0$. Now consider the case $D > 2$, where the function on the right-hand side of (2.15) behaves asymptotically as $-r^{-2+D}/2$ as $r \to \infty$. For large r, this is in contradiction with (2.15) being a correlation, so that we must have $D \leqslant 2$. □

Lemma 2. *Let X be an H-SSSI family with power variance function. Then*:

1. *The correlation* (2.15) *is positive for* $0 \leqslant D < 1$ *and negative for* $1 < D \leqslant 2$.
2. *The process* $X(\mu; \cdot)$ *has uncorrelated increments if and only if* $D = 1$, *where* (2.14) *becomes*

$$\operatorname{Cov}\bigl[X(\mu; s), X(\mu; t)\bigr] = \frac{1}{2}\sigma^2\mu^p \min\{s, t\}.$$

Proof. The correlation (2.15) is positive if and only if

$$\bigl(r^{-1} + r\bigr)^{2-D} > r^{2-D} + r^{-2+D}.$$

In terms of the strictly convex function $f(x) = \log(\mathrm{e}^x + \mathrm{e}^{-x})$, this is equivalent to the inequality

$$(2 - D)f(x) > f\bigl(x(2 - D)\bigr). \tag{2.16}$$

Using the convexity of f along with the fact that $f(x)$ behaves asymptotically as $|x|$ for $|x|$ large, we find that inequality (2.16) is satisfied for $0 \leqslant D < 1$, whereas the opposite inequality holds for $1 < D \leqslant 2$, proving item 1. The correlation (2.15) is zero if $D = 1$, which by item 1 is also necessary, proving item 2. □

It is convenient at this point to introduce the parameter $\alpha \in [-\infty, \infty)$ defined by the following one-to-one transformation of $p \in (-\infty, \infty]$:

$$\alpha = \alpha(p) = 1 + (1 - p)^{-1}, \tag{2.17}$$

with the conventions that $\alpha(1) = -\infty$ and $\alpha(\infty) = 1$ (see also [9, Chap. 4]), where α is useful in connection with Tweedie distributions. In the following, we shall often use α instead of p whenever convenient. One reason for our interest in the parameter α comes from the connection with α-stable distributions in the case $\alpha \in (0, 2]$; see Section 3.2, where we discuss a class of self-similar families of Lévy processes. By item 2 of Lemma 2, we find that the correlation between two nonoverlapping increments (2.15) is zero if and only if $H = 1/\alpha$, with the convention that $\alpha = -\infty$ corresponds to $H = 0$.

We may now express the domain for the Hurst exponent H, for given α, as an interval with endpoints 1 and $(2 - \alpha)/\alpha$, which follows from the domain $0 \leqslant D \leqslant 2$ via (2.13) and (2.17). The domain for H is summarized in Table 1, along with the sign of the correlation (2.15) (item 1 of Lemma 2). We note, in particular, that negative values of H in the interval $((2 - \alpha)/\alpha, 0)$ are allowed when $\alpha < 0$. By comparison, in the case of strict self-similarity, the Hurst parameter is restricted to the domain $H > 0$ and to the smaller domain $0 < H \leqslant 1$ under second-moment assumptions (see [17]).

Table 1. The sign of the correlation (2.15) as a function of α and H

α	Positive correlation	Negative correlation
$\alpha \notin [0, 1]$	$1/\alpha < H \leqslant 1$	$(2 - \alpha)/\alpha \leqslant H < 1/\alpha$
$\alpha \in (0, 1)$	$1 \leqslant H < 1/\alpha$	$1/\alpha < H \leqslant (2 - \alpha)/\alpha$

The importance of power variance functions will become clear in Sections 3.2 and 4, where we give examples of non-Gaussian processes with independent increments and with correlated increments, respectively, giving rise to H-SSSI families corresponding to many of the possible combinations of H and p discussed above. We note in this connection that if $D = 0$, then either $H = 1$ (cf. Section 7), or $p = 2$ ($\alpha = 0$), in which case the covariance function (2.14) takes the form

$$\mathrm{Cov}\big[X(\mu; s), X(\mu; t)\big] = \sigma^2\mu^2 st \quad \text{for } s, t > 0. \tag{2.18}$$

We return briefly to this case at the end of Section 3.2.

3 NATURAL EXPONENTIAL FAMILIES AND HOUGAARD LÉVY PROCESSES

3.1 Natural exponential families of Lévy processes

We now consider the case of a natural exponential family of Lévy processes. In this subsection, we let $X = \{X_t\colon t \geqslant 0\}$ denote a given Lévy process defined on a probability space $(\mathcal{X}, \mathcal{F}, \mathbb{P})$, satisfying $X_0 = 0$. Let κ denote the cumulant generating function of X_1,

$$\kappa(\theta) = \log \mathbb{E}\big[\mathrm{e}^{\theta X_1}\big],$$

with effective domain $\Theta = \{\theta \in \mathbb{R}\colon \kappa(\theta) < \infty\}$, an interval containing 0. We assume that Θ is nondegenerate, i.e., $\Theta \neq \{0\}$. Following [11, p. 8], for each $t \geqslant 0$, we define the class of probability measures $\{\mathbb{P}_\theta^t\colon \theta \in \Theta\}$ on the space $(\mathcal{X}, \mathcal{F}_t)$ by

$$\frac{\mathrm{d}\mathbb{P}_\theta^t}{\mathrm{d}\mathbb{P}^t} = \exp\big[\theta X_t - t\kappa(\theta)\big], \tag{3.1}$$

where $\{\mathcal{F}_t\colon t \geqslant 0\}$ is a suitably defined right-continuous filtration, and $\mathbb{P}^t$ denotes the marginal distribution of X_t. Küchler and Sørensen [11, p. 8] show that for each $\theta \in \Theta$, this defines a unique probability measure $\mathbb{P}_\theta$ on $(\mathcal{X}, \mathcal{F})$ such that $\mathbb{P}_\theta^t$ is the restriction of $\mathbb{P}_\theta$ to $\mathcal{F}_t$. The family $\{\mathbb{P}_\theta\colon \theta \in \Theta\}$ is called the *natural exponential family generated by* X. In fact, for each $\theta \in \Theta$, X is a Lévy process on the probability space $(\mathcal{X}, \mathcal{F}, \mathbb{P}_\theta)$. We also say that $\mathbb{P}_\theta$ is obtained from $\mathbb{P}$ by *exponential tilting*.

We note that for each $t \geqslant 0$, $\{\mathbb{P}_\theta^t\colon \theta \in \Theta\}$ is a natural exponential family of probability measures on $\mathbb{R}$ with cumulant generating functions $s \mapsto t[\kappa(\theta + s) - \kappa(\theta)]$. The cumulants of X_t are hence obtained by differentiating $t\kappa(\theta)$. In particular, the mean is

$$\mathbb{E}_\theta[X_t] = t\dot{\kappa}(\theta) \quad \text{for } \theta \in \Theta,$$

where from now on dots denote derivatives. Here we let $\dot{\kappa}(\theta)$ be defined by continuity at the boundary of Θ, if necessary, such that the domain for the rate parameter $\mu = \dot{\kappa}(\theta)$ is $\overline{\Omega} = \dot{\kappa}(\Theta)$. Since $\dot{\kappa}$ is a strictly increasing function, we may parameterize the family $\{\mathbb{P}_\theta\colon \theta \in \Theta\}$ by the rate parameter μ instead of θ. This provides a connection with the original notation introduced in Section 2. We hence let $X(\mu; \cdot)$ denote the version of the process X corresponding to the probability measure $\mathbb{P}_{\dot{\kappa}^{-1}(\mu)}$.

The variance of X_t is

$$\mathrm{Var}_\theta[X_t] = t\ddot{\kappa}(\theta),$$

and the family hence has variance function V defined by

$$V(\mu) = \mathrm{Var}\big[X(\mu; 1)\big] = \ddot{\kappa} \circ \dot{\kappa}^{-1}(\mu) \quad \text{for } \mu \in \Omega.$$

The corresponding covariance function is

$$\mathrm{Cov}\big[X(\mu; t), X(\mu; t + s)\big] = \mathrm{Var}\big[X(\mu; t)\big] = tV(\mu) \quad \text{for } s, t > 0,\ \mu \in \Omega. \tag{3.2}$$

It is well known from the theory of natural exponential families (see [9, Chap. 2]) that the natural exponential family $\{\mathbb{P}_\theta^t\colon \theta \in \Theta\}$ is characterized among all natural exponential families by its variance function, i.e., the function expressing the variance $tV(\mu)$ as a function of the mean $m = \mu t$, in other words, the function $m \mapsto tV(m/t)$. It follows that the variance function V, together with its domain Ω, characterizes the natural exponential family of Lévy processes $\{\mathbb{P}_\theta\colon \theta \in \Theta\}$. We now use this tool to characterize the so-called Hougaard natural exponential families of Lévy processes.

3.2 Self-similarity of Hougaard Lévy processes

Returning to the previous notation, we let $X = \{X(\mu; t)\colon \mu \in \overline{\Omega},\ t \geqslant 0\}$ denote the natural exponential family of Lévy processes considered in Section 3.

Let us denote the marginal distribution of $t^{-1}X(\mu;t)$ by the symbol $\mathrm{ED}(\mu, t)$, where $\mathrm{ED}(\mu, t)$ is an *exponential dispersion model* in the sense of Jørgensen [9, Chap. 3]. This distribution has mean μ and variance $t^{-1}V(\mu)$ for $t > 0$ and $\mu \in \Omega$. The above characterization of the family X in terms of the variance function V may now be rephrased by saying that the variance function V with domain Ω characterizes the exponential dispersion model $\mathrm{ED}(\mu, t)$ among all exponential dispersion models (see [9, Chap. 3]).

Following [8] and [13], we define a *Hougaard Lévy process* $S_p(\mu; \cdot)$ to be a natural exponential family of Lévy process with power variance function $V(\mu) = \sigma^2\mu^p$ (see also [6], [9, Chap. 4], and [18]). We define the *Tweedie distribution* $\mathrm{Tw}_p(\mu, t)$ to be the corresponding exponential dispersion model such that

$$t^{-1}S_p(\mu; t) \overset{\mathrm{d}}{\sim} \mathrm{Tw}_p(\mu, t)$$

with mean μ and variance $t^{-1}\sigma^2\mu^p$ (note that we suppress the parameter σ^2 in the notation). The parameter domains are $p \in \Delta = \mathbb{R} \setminus (0, 1)$, $\sigma^2 > 0$, and $\mu \in \overline{\Omega}_p$, where $\overline{\Omega}_p$ is defined by

$$\overline{\Omega}_p = \begin{cases} [0, \infty) & \text{for } p < 0, \\ \mathbb{R} & \text{for } p = 0, \\ \mathbb{R}_+ & \text{for } 1 \leqslant p \leqslant 2, \\ (0, \infty] & \text{for } p > 2. \end{cases}$$

The form of the Lévy measure of the Tweedie distribution was derived by Jørgensen and Martínez [10] (see also [11, p. 12]).

The Tweedie distribution satisfies a scaling property for all $p \in \Delta$, $\mu \in \Omega_p$, and $t > 0$, namely,

$$c\,\mathrm{Tw}_p(\mu, t) = \mathrm{Tw}_p\bigl(c\mu, c^{p-2}t\bigr) \quad \text{for } c > 0, \tag{3.3}$$

which characterizes the Tweedie model among all exponential dispersion models (see [9, Chap. 4]). The Hougaard Lévy process $S_p(\mu; t)$ is hence characterized by the following distribution of the increments:

$$S_p(\mu; t) \overset{\mathrm{d}}{\sim} t\,\mathrm{Tw}_p(\mu, t) = \mathrm{Tw}_p\bigl(\mu t, t^{p-1}\bigr) \tag{3.4}$$

with variance $t\sigma^2\mu^p$. We shall now characterize the Hougaard Lévy processes in terms of self-similarity. Recall that the parameter α is defined as a function of p by (2.17) and note that $p \in \Delta$ corresponds to the domain $\alpha \in [-\infty, 1) \cup (1, 2]$.

Theorem 1. *The family of Hougaard Lévy processes $S_p(\mu; \cdot)$ may be characterized as follows. Let X be a natural exponential family of Lévy processes. Then X has power variance function with power $p \in \Delta \setminus 2$ if and only if X is self-similar with Hurst exponent*

$$H = \frac{1}{\alpha}, \tag{3.5}$$

with the convention that $\alpha = -\infty$ $(p = 1)$ corresponds to $H = 0$.

Note that the condition $H = 1/\alpha$ is in agreement with the condition for uncorrelated increments of Lemma 2, item 2. There are three main cases of Hougaard Lévy processes [9, Chap. 4]:

1. Processes generated by extreme α-stable processes: $p \leqslant 0$ $(1 < \alpha \leqslant 2) \Leftrightarrow 1/2 \leqslant H < 1$.
2. Compound Poisson processes (Poisson process for $p = 1$): $1 \leqslant p < 2$ $(-\infty \leqslant \alpha < 0) \Leftrightarrow H \leqslant 0$.
3. Processes generated by positive α-stable processes: $p > 2$ $(0 < \alpha < 1) \Leftrightarrow H > 1$.

Proof. Let us first show that the Hougaard Lévy process $S_p(\mu; t)$ is $1/\alpha$-SS. Since a Hougaard Lévy process has stationary and independent increments, it suffices to consider the marginal distribution of $S_p(\mu; t)$, so that we must show

$$S_p(\mu c^{H-1}; ct) \stackrel{\mathrm{d}}{\sim} c^H S_p(\mu; t) \quad \text{for } t > 0$$

with $H = 1/\alpha$. Using (3.4), we see that the left-hand side has marginal distribution

$$S_p(\mu c^{H-1}; ct) \stackrel{\mathrm{d}}{\sim} \mathrm{Tw}_p(\mu t c^H, (ct)^{p-1}).$$

The scaling property (3.3) yields the following marginal distribution for the right-hand side:

$$c^H S_p(\mu; t) \stackrel{\mathrm{d}}{\sim} c^H \mathrm{Tw}_p(\mu t, t^{p-1}) = \mathrm{Tw}_p(\mu t c^H, c^{H(p-2)} t^{p-1}) = \mathrm{Tw}_p(\mu t c^H, (ct)^{p-1}),$$

where we have used the fact that $H(p-2) = p-1$, which follows from (2.17) and (3.5). This implies that $S_p(\mu; t)$ is H-SSSI with H given by (3.5).

To show the reverse implication, let us assume that the exponential family of Lévy processes X is self-similar with Hurst exponent $H = 1/\alpha$. By (3.2) the variance of the process is $\mathrm{Var}[X(\mu; t)] = tV(\mu)$, and comparing with (2.10), we obtain the equation

$$tV(\mu) = t^{2H} V(\mu t^{1-H}). \tag{3.6}$$

By taking $\mu = 1$ in (3.6) and redefining μ to be $\mu = t^{1-H}$, we obtain the following solution:

$$V(\mu) = V(1)\mu^{(1-2H)/(1-H)},$$

and using (3.5) and (2.17), we find the power to be $(1-2H)/(1-H) = p$. Hence, V is a power variance function with power parameter p and $\sigma^2 = V(1)$. □

The Hougaard Lévy process corresponding to $p = 2$ (excluded from Theorem 1) is the gamma Lévy process $S_2(\mu; t)$ corresponding to $\alpha = 0$. The gamma process is not H-SS for any finite value of H but satisfies the following scaling property for each $a > 0$:

$$S_2(a\mu; t) \stackrel{\mathrm{d}}{=} aS_2(\mu; t) \quad \text{for } t > 0. \tag{3.7}$$

This corresponds to a limiting case of (2.1) obtained by fixing $a = c^H$ and letting H tend to infinity. We may hence interpret (3.7) as representing a kind of self-similarity with infinite Hurst exponent.

4 FRACTIONAL HOUGAARD MOTION

We shall now consider stochastic integration with respect to a Lévy process, which, in turn, will be used to construct a class of fractional Hougaard motions as moving averages of Hougaard Lévy processes. Some of the results in the following are parallel to results of Marquardt [15], but unlike Marquardt, who considers Lévy processes with zero mean, finite variance, and no Brownian component, we shall consider stochastic integration with respect to an arbitrary Lévy process.

4.1 Stochastic integration with respect to a Lévy process

We now consider stochastic integration with respect to a Lévy process X. We follow the approach of Barndorff-Nielsen and Thorbjørnsen [2], which has the advantage that we work directly with the cumulant transform (log characteristic function) and the cumulants, although there are more general approaches to stochastic integration with respect to a Lévy process available (see, e.g., [16]).

We shall define the stochastic integral

$$\int_a^b f(u)\,\mathrm{d}X(u), \tag{4.1}$$

for a nonrandom real function f. Initially, we consider the case where $0 \leqslant a < b < \infty$, after which we extend the integration interval to the whole real line. We denote the cumulant transform of an infinitely divisible random variable Y by

$$C(z;Y) = \log \mathbb{E}\big[\mathrm{e}^{\mathrm{i}zY}\big].$$

Due to the infinite divisibility, the characteristic function $\mathbb{E}[\mathrm{e}^{\mathrm{i}zY}]$ has no zeroes, and hence C is well defined by means of the principal branch of the complex logarithm. Recall that

$$\mathbb{E}[Y] = -\mathrm{i}\dot{C}(0;Y) \quad \text{and} \quad \mathrm{Var}[Y] = -\ddot{C}(0;Y),$$

provided that C is twice differentiable at zero, where $\dot{C}(z;Y)$ and $\ddot{C}(z;Y)$ denote the first and second derivatives of C, respectively, with respect to z.

Lemma 3. *Assume that $0 \leqslant a < b < \infty$. Let X be a Lévy process, and let the function $f : [a,b] \to \mathbb{R}$ be continuous. Then the stochastic integral* (4.1) *exists as the limit, in probability, of approximating Riemann sums. The distribution of the random variable* (4.1) *is infinitely divisible and has the cumulant transform given by*

$$C\Bigg(z; \int_a^b f(u)\,\mathrm{d}X(u)\Bigg) = \int_a^b C\big(zf(u); X(1)\big)\,\mathrm{d}u \quad \textit{for } z \in \mathbb{R}. \tag{4.2}$$

If $X(t)$ has finite second moments, the mean and variance of the random variable (4.1) *are*

$$\mathbb{E}\Bigg[\int_a^b f(u)\,\mathrm{d}X(u)\Bigg] = \mathbb{E}\big[X(1)\big] \int_a^b f(u)\,\mathrm{d}u, \tag{4.3}$$

$$\mathrm{Var}\Bigg[\int_a^b f(u)\,\mathrm{d}X(u)\Bigg] = \mathrm{Var}\big[X(1)\big] \int_a^b f^2(u)\,\mathrm{d}u. \tag{4.4}$$

Proof. The stochastic integral (4.1) and the result (4.2) follow from Lemma 2.5 of Barndorff-Nielsen and Thorbjørnsen [2]. By differentiating (4.2) and putting $z = 0$, we obtain

$$\mathbb{E}\Bigg[\int_a^b f(u)\,\mathrm{d}X(u)\Bigg] = -\mathrm{i}\int_a^b f(u)\dot{C}\big(0; X(1)\big)\,\mathrm{d}u = \mathbb{E}\big[X(1)\big]\int_a^b f(u)\,\mathrm{d}u,$$

where we have used the fact that $-\mathrm{i}\dot{C}(0; X(1)) = \mathbb{E}[X(1)]$. Differentiating (4.2) twice and putting $z = 0$, we obtain

$$\operatorname{Var}\left[\int_a^b f(u)\,\mathrm{d}X(u)\right] = -\int_a^b f^2(u)\ddot{C}\big(0; X(1)\big)\,\mathrm{d}u = \operatorname{Var}\big[X(1)\big]\int_a^b f^2(u)\,\mathrm{d}u,$$

where we have used the fact that $-\ddot{C}(0; X(1)) = \operatorname{Var}[X(1)]$. This shows the last two results. In particular, the two integrals on the right-hand sides of (4.3) and (4.4) are finite, each being the integral of a continuous function on a compact interval. □

We shall now extend the integration interval to the whole real line. To this end, we first need to extend the time domain of the process X to $\mathbb{R}$. Since we are dealing with a Lévy process, we have that $X(0) \equiv 0$. We then extend the process to the negative half-axis by assuming that

$$-X(-t) = X(0) - X(-t) \stackrel{\mathrm{d}}{=} X(t) \quad \text{for } t \geqslant 0, \tag{4.5}$$

where the processes $\{X(t)\colon\ t > 0\}$ and $\{X(t)\colon\ t < 0\}$ are assumed independent (see [17]).

Proposition 2. *Assume that $-\infty \leqslant a < b \leqslant \infty$. Let $X(t)$ be a Lévy process on $\mathbb{R}$, and let $f : (a, b) \to \mathbb{R}$ be continuous on $(a, b) \setminus \{0\}$. Assume that*

$$\forall z \in \mathbb{R}\colon \quad \int_a^b \big|C\big(zf(u); X(1)\big)\big|\,\mathrm{d}u < \infty. \tag{4.6}$$

Then the stochastic integral (4.1) *exists as the limit, in probability, of the sequence*

$$\left\{\int_{a_n}^{b_n} f(u)\,\mathrm{d}X(u)\right\}_{n\in\mathbb{N}},$$

where a_n and b_n are arbitrary sequences in (a, b) such that $a_n \leqslant b_n$ for all n and $a_n \downarrow a$ and $b_n \uparrow b$ as $n \to \infty$. The distribution of the random variable (4.1) *is infinitely divisible and has the cumulant transform given by* (4.2)*. If $X(t)$ has finite second moments, then the mean and variance of the random variable* (4.1) *are given by* (4.3) *and* (4.4)*, respectively.*

Proof. First assume that $0 \leqslant a < b \leqslant \infty$. In this case, the stochastic integral and the result (4.2) follow from Proposition 2.6 of Barndorff-Nielsen and Thorbjørnsen [2]. In the case $-\infty \leqslant a < b \leqslant 0$, we make the substitution $v = -u$ and rewrite the integral as follows:

$$\int_a^b f(u)\,\mathrm{d}X(u) = \int_{-a}^{-b} f(-v)\,\mathrm{d}X(-v) = \int_{-b}^{-a} f(-v)\,\mathrm{d}\big[-X(-v)\big]. \tag{4.7}$$

In view of (4.5), the process $\{-X(-t)\colon\ t \geqslant 0\}$ is a Lévy process on the positive half-line. Since $0 \leqslant -b < -a \leqslant \infty$, the right-most integral of (4.7) is defined according to the first case just considered, thereby providing the required definition of the left-most integral of (4.7). In the case where a is negative and b is positive, we split the integral in two parts,

$$\int_a^b f(u)\,\mathrm{d}X(u) = \int_a^0 f(u)\,\mathrm{d}X(u) + \int_0^b f(u)\,\mathrm{d}X(u),$$

and use the fact that both integrals on the right-hand side of the equation are now properly defined. The remainder of the results follows from Lemma 3 by elementary arguments. □

4.2 Fractional Hougaard motion

We shall now define a class of fractional Hougaard motions by means of stochastic integration with respect to a Hougaard Lévy process. This approach is similar to the moving average representation of a fractional Brownian motion, where the integration is with respect to an ordinary Brownian motion [14]. Fractional Hougaard motions provide our main examples of non-Lévy self-similar families of stochastic processes.

Following [3, pp. 56–59] and [17], we define the weight function $w_h(t,u)$ for $t \geqslant 0$ and $u \in \mathbb{R}$ by

$$w_h(t,u) = \begin{cases} (t-u)^h - (-u)^h & \text{for } u < 0, \\ (t-u)^h & \text{for } 0 \leqslant u < t, \\ 0 & \text{for } t \leqslant u, \end{cases}$$

where h is a real constant. Using the notation $u_+ = \max\{0,u\}$, we may also express the weight function in the more compact form

$$w_h(t,u) = (t-u)_+^h - (-u)_+^h. \tag{4.8}$$

For a given $p \in \Delta$, we let $\{S_p(\mu;t)\colon\ t \in \mathbb{R}\}$ denote the Hougaard Lévy process defined in Section 3.2, extended to the whole real line by means of (4.5). For simplicity, we take $\sigma^2 = 1$, and we recall from (2.17) that $\alpha = 1 + (1-p)^{-1}$.

Proposition 3. *Assume that $p \in \Delta \setminus \{1,2\}$ and $H < 1/\alpha$ and that $H > 0$ for $\alpha > 0$. Define the* fractional Hougaard motion *$S_{p,H}(\mu;t)$ by the stochastic integral*

$$S_{p,H}(\mu;t) = \int_{-\infty}^{\infty} w_h(t,u)\,\mathrm{d}S_p(\mu;u) \quad \text{for } t \geqslant 0, \tag{4.9}$$

where $h = H - 1/\alpha < 0$ and $\mu \in \overline{\Omega}_p$. Then the family $S_{p,H}(\mu;t)$ is H-SSSI with cumulant transform

$$C\big(z; S_{p,H}(\mu;t)\big) = \frac{\alpha-1}{\alpha}\mu^{\alpha/(\alpha-1)} \int_{-\infty}^{\infty} \left\{ \left[1 + \frac{\mathrm{i}z w_h(t,u)}{(\alpha-1)\mu^{1/(\alpha-1)}} \right]^{\alpha} - 1 \right\} \mathrm{d}u \quad \text{for } z \in \mathbb{R}. \tag{4.10}$$

For $1/\alpha - 1 < H < 1/\alpha$, the process $S_{p,H}(\mu;t)$ has mean zero, and for $1/\alpha - 1/2 < H < 1/\alpha$, the process has the variance given by

$$\mathrm{Var}\big[S_{p,H}(\mu;t)\big] = \mu^p t^{1+2(H-1/\alpha)} \int_{-\infty}^{\infty} w_h^2(1,v)\,\mathrm{d}v.$$

Proof. Our method of proof is inspired by Beran [3, pp. 56–59], and we shall hence show self-similarity by a direct argument, rather than via the covariance function, say, as is usually done for a fractional Brownian motion. We first note that the assumption $p \in \Delta \setminus \{1,2\}$ implies that α is finite and $\alpha \neq 0, 1$. We need to check that the integral (4.6) is finite for the function $f(u) = w_h(t,u)$. Let us define the complex analytic function κ_α by

$$\kappa_\alpha(x) = \frac{\alpha-1}{\alpha}\left(\frac{x}{\alpha-1}\right)^{\alpha} \quad \text{for } \operatorname{Re}\frac{x}{\alpha-1} > 0$$

[9, p. 131], where the domain may be extended to the imaginary axis when $\alpha > 0$ and to the whole complex plane when $\alpha = 2$. Then $S_p(\mu; 1)$ has the cumulant transform

$$C\big(z; S_p(\mu; 1)\big) = \kappa_\alpha(\theta + \mathrm{i}z) - \kappa_\alpha(\theta) = \kappa_\alpha(\theta)\left[\left(1 + \frac{\mathrm{i}z}{\theta}\right)^\alpha - 1\right],$$

where $\theta = (\alpha - 1)\mu^{1/(\alpha-1)}$. The integral (4.6) hence takes the form

$$\int_{-\infty}^{\infty} \big|C\big(zw_h(t,u); S_p(\mu; 1)\big)\big| \,\mathrm{d}u = \int_{-\infty}^{\infty} \big|\kappa_\alpha\big(\theta + \mathrm{i}zw_h(t,u)\big) - \kappa_\alpha(\theta)\big| \,\mathrm{d}u.$$

In the limit $u \to -\infty$, a Taylor expansion of κ_α yields

$$\big|\kappa_\alpha\big(\theta + \mathrm{i}zw_h(t,u)\big) - \kappa_\alpha(\theta)\big| = O\big(\big|w_h(t,u)\big|\big),$$

which implies integrability in this limit, due the assumption that $h = H - 1/\alpha < 0$. For u near zero or t, the integrability of $|\kappa_\alpha(\theta + \mathrm{i}zw_h(t,u)) - \kappa_\alpha(\theta)| = O(|w_h(t,u)|^\alpha)$ requires $\alpha h = \alpha H - 1 > -1$ or, equivalently, $H\alpha > 0$, which is satisfied by assumption for $\alpha > 0$ and is trivially satisfied for $\alpha < 0$ due to the condition $H < 1/\alpha$. By (4.2) we find that the process $S_{p,H}(t)$ has the cumulant transform

$$\begin{aligned} C\big(z; S_{p,H}(\mu; t)\big) &= \int_{-\infty}^{\infty} C\big(zw_h(t,u); S_p(\mu; 1)\big) \,\mathrm{d}u = \int_{-\infty}^{\infty} \big[\kappa_\alpha\big(\theta + \mathrm{i}zw_h(t,u)\big) - \kappa_\alpha(\theta)\big] \,\mathrm{d}u \\ &= \kappa_\alpha(\theta) \int_{-\infty}^{\infty} \left\{\left[1 + \frac{\mathrm{i}z}{\theta} w_h(t,u)\right]^\alpha - 1\right\} \mathrm{d}u, \end{aligned}$$

which implies (4.10) in view of the above definition of θ.

Now let us calculate the mean and variance of $S_{p,H}(\mu; t)$ using (4.3) and (4.4). Using well-known results about the integral of w_h (e.g., [3, p. 59, formula (2.22)]), we obtain that the mean is finite for $1/\alpha - 1 < H < 1/\alpha$ and

$$\mathbb{E}\big[S_{p,H}(\mu; t)\big] = \mathbb{E}\big[S_p(\mu; 1)\big] \int_{-\infty}^{\infty} w_h(t,u) \,\mathrm{d}u = 0.$$

To calculate the variance, we use the following scaling relation for the weight function w_h:

$$w_h(t,u) = t^h w_h\big(1, ut^{-1}\big). \tag{4.11}$$

Using the substitution $v = ut^{-1}$ and with $h = H - 1/\alpha$, we obtain for $1/\alpha - 1/2 < H < 1/\alpha$,

$$\mathrm{Var}\big[S_{p,H}(\mu; t)\big] = \mu^p \int_{-\infty}^{\infty} w_h^2(t,u) \,\mathrm{d}u = \mu^p t^{2h} \int_{-\infty}^{\infty} w_h^2\big(1, ut^{-1}\big) \,\mathrm{d}u = \mu^p t^{1+2(H-1/\alpha)} \int_{-\infty}^{\infty} w_h^2(1,v) \,\mathrm{d}v.$$

In order to show that the family of fractional Hougaard motions $S_{p,H}(\mu; t)$ is H-SS, we shall use the fact that the Hougaard Lévy process $S_p(\mu; t)$ is $1/\alpha$-SS with $\alpha = 1 + (1-p)^{-1}$. Consider (4.9) with the arguments

μc^{H-1} and ct. We note that the function w_h satisfies the following scaling relation similar to (4.11):

$$w_h(ct, u) = c^h w_h\big(t, uc^{-1}\big) \quad \text{for } c > 0. \tag{4.12}$$

In view of (4.12), we obtain

$$S_{p,H}\big(\mu c^{H-1}; ct\big) = \int_{-\infty}^{\infty} w_h(ct, u)\, \mathrm{d}S_p\big(\mu c^{H-1}; u\big) = c^h \int_{-\infty}^{\infty} w_h\big(t, uc^{-1}\big)\, \mathrm{d}S_p\big(\mu c^{H-1}; u\big).$$

Making the substitution $v = uc^{-1}$, we have

$$S_{p,H}\big(\mu c^{H-1}; ct\big) = c^h \int_{-\infty}^{\infty} w_h(t, v)\, \mathrm{d}S_p\big(\mu c^{H-1}; cv\big).$$

Recalling that $h = H - 1/\alpha$ and using the $1/\alpha$-SS property of $S_p(\mu; t)$, we have that this implies

$$S_{p,H}\big(\mu c^{H-1}; ct\big) \stackrel{\mathrm{d}}{=} c^{H-1/\alpha} \int_{-\infty}^{\infty} w_h(t, v) c^{1/\alpha}\, \mathrm{d}S_p(\mu; v) = c^H S_{p,H}(\mu; t).$$

Hence, $S_{p,H}(\mu; t)$ is self-similar with Hurst exponent H.

To show that $S_{p,H}(\mu; t)$ has stationary increments, we observe, using the expression (4.8) for w_h, that for any $s, t > 0$,

$$\begin{aligned}
&S_{p,H}(\mu; t+s) - S_{p,H}(\mu; t) \\
&\quad = \int_{-\infty}^{\infty} \big[w_h(t+s, u) - w_h(t, u)\big]\, \mathrm{d}S_p(\mu; u) = \int_{-\infty}^{\infty} \big[(t+s-u)_+^h - (t-u)_+^h\big]\, \mathrm{d}S_p(\mu; u) \\
&\quad \stackrel{\mathrm{d}}{=} \int_{-\infty}^{\infty} \big[(s-v)_+^h - (-v)_+^h\big]\, \mathrm{d}S_p(\mu; v) = S_{p,H}(\mu; s),
\end{aligned}$$

where, in the second-last equality, we have used the substitution $v = u - t$. Since $S_{p,H}(\mu; 0) \equiv 0$, this is the desired result, completing the proof. □

As already mentioned, fractional Hougaard motion is our main example of general self-similarity outside of the Lévy process case. However, the processes $S_{p,H}(\mu; s)$ do not have the same marginal distribution as the Hougaard Lévy processes of Section 3.2. In particular, for $p > 1$, $S_{p,H}(\mu; s)$ has support on the whole real line rather than on $[0, \infty)$ or $(0, \infty)$. This follows from the fact that the parameter h is restricted to negative values, which implies that $w_h(t, u)$ is negative for $u < 0$ and positive for $0 < u < t$, so that $S_{p,H}(\mu; s)$ has support on the whole real line.

5 LAMPERTI TRANSFORMATIONS

We shall now introduce an extension of the *Lamperti transformation* to the case of general self-similarity. Consider a strictly H-SS stochastic process $X(t)$ and recall that the Lamperti transformation yields a new strictly stationary stochastic process $\{Y(t)\colon\ t \in \mathbb{R}\}$ by means of the following exponential scaling transformation:

$$Y(t) = \mathrm{e}^{-tH} X\big(\mathrm{e}^t\big) \quad \text{for } t \in \mathbb{R}; \tag{5.1}$$

see [12] and [5, p. 11]. Now, let X be a general H-SS family, and define the family of processes $Y(\mu;t)$ for $\mu \in \overline{\Omega}$ by

$$Y(\mu;t) = \mathrm{e}^{-tH} X\big(\mu \mathrm{e}^{t(H-1)}; \mathrm{e}^t\big) \quad \text{for } t \in \mathbb{R}. \tag{5.2}$$

Equation (5.2) may be obtained from (2.3) by taking $t = 1$ and then substituting $c = \mathrm{e}^t$. The fact that the right-hand side of (2.3) does not depend on c suggests that $Y(\mu;t)$ might be stationary. By the self-similarity of X we obtain, for each $\mu \in \overline{\Omega}$,

$$Y(\mu;t) = \mathrm{e}^{-tH} X\big(\mu \mathrm{e}^{t(H-1)}; \mathrm{e}^t\big) \overset{\mathrm{d}}{\sim} X\big(\mathrm{e}^{-t(H-1)} \mu \mathrm{e}^{t(H-1)}; \mathrm{e}^{-t}\mathrm{e}^t\big) = X(\mu;1) \quad \text{for } t \in \mathbb{R}.$$

This result shows that the marginals of $Y(\mu;t)$ are stationary as functions of t for each μ.

Note that in the special case of a process with drift, $X(\mu;t) = X(t) + \mu t$, say, the process (5.2) takes the form

$$Y(\mu;t) = \mathrm{e}^{-tH} X\big(\mathrm{e}^t\big) + \mu = Y(t) + \mu \quad \text{for } t \in \mathbb{R}, \tag{5.3}$$

which is simply the process (5.1) with a shift $\mu \in \mathbb{R}$. If we assume that the drift term μt is added pathwise to $X(t)$, then the shift μ is similarly added pathwise to $Y(t)$. In this case, (5.2) is equivalent to the conventional Lamperti transformation. In the special case where X is a Brownian motion with drift, the corresponding family Y consists of shifted Ornstein–Uhlenbeck processes, and (5.3) hence represents a kind of generalized Ornstein–Uhlenbeck processes.

We now assume that all the processes $X(\mu;\cdot)$ are defined on the same probability space, making $\{X(\mu;t)\colon t \geqslant 0,\ \mu \in \overline{\Omega}\}$ a random field. Let us now assume that (2.3) holds in the sense of equality of the finite-dimensional distributions of the two random fields. In this case, the stationarity of the process $Y(\mu;\cdot)$ may be shown as follows:

$$\begin{aligned} Y(\mu;s+t) &= \mathrm{e}^{-(s+t)H} X\big(\mu \mathrm{e}^{(s+t)(H-1)}; \mathrm{e}^{s+t}\big) = \mathrm{e}^{-tH}\mathrm{e}^{-sH} X\big(\mu \mathrm{e}^{s(H-1)}\mathrm{e}^{t(H-1)}; \mathrm{e}^s\mathrm{e}^t\big) \\ &\overset{\mathrm{d}}{=} \mathrm{e}^{-tH} X\big(\mu \mathrm{e}^{t(H-1)}; \mathrm{e}^t\big) = Y(\mu;t) \quad \text{for } t \in \mathbb{R}, \end{aligned}$$

for all $\mu \in \overline{\Omega}$.

Let us now turn to the inverse Lamperti transformation. Suppose that $\{Y(\mu;t)\colon \mu \in \overline{\Omega},\ t \in \mathbb{R}\}$ is a family of strictly stationary processes. For a given $H \in \mathbb{R}$, we define the family X by

$$X(\mu;t) = t^H Y\big(\mu t^{1-H}; \log t\big) \quad \text{for } t > 0, \tag{5.4}$$

for all $\mu \in \overline{\Omega}$. In the special case where $Y(\mu;t) = Y(t) + \mu$, namely a stationary process $Y(t)$ plus a shift $\mu \in \mathbb{R}$, X has the form $X(\mu;t) = t^H Y(\log t) + \mu t$, which is a strictly H-SS process plus a drift term.

The discussion of the self-similarity of X is similar to be above discussion for the ordinary Lamperti transformation (5.2). Let us assume that $\{Y(\mu;t)\colon \mu \in \overline{\Omega},\ t \in \mathbb{R}\}$ is a t-stationary random field, in the sense that

$$Y(\mu;t+s) \overset{\mathrm{d}}{=} Y(\mu;t) \quad \text{for } \mu \in \overline{\Omega} \text{ and } t \in \mathbb{R},$$

for all $s \in \mathbb{R}$, where $\overset{\mathrm{d}}{=}$ here means equality of the finite-dimensional distributions of the two random fields. Under this assumption, we obtain, for X defined by (5.4),

$$\begin{aligned} c^H X(\mu;t) &= (ct)^H Y\big(\mu t^{1-H}; \log t\big) = (ct)^H Y\big(\mu c^{H-1}(ct)^{1-H}; \log t\big) \\ &\overset{\mathrm{d}}{=} (ct)^H Y\big(\mu c^{H-1}(ct)^{1-H}; \log c + \log t\big) = X\big(\mu c^{H-1}; ct\big), \end{aligned}$$

which shows that the family X is H-SS.

6 LAMPERTI-TYPE LIMIT THEOREMS

We shall now consider some generalizations of Lamperti's limit theorem [12], according to which all self-similar stochastic processes appear as large-sample limits of suitably scaled stochastic processes.

Lamperti's fundamental limit theorem says that if we are given a stochastic process $Y(t)$ and a positive measurable function $A(c)$ satisfying

$$A(c) \to \infty \quad \text{as } c \to \infty \tag{6.1}$$

and, with $\xrightarrow{\mathrm{d}}$ denoting convergence of the finite-dimensional distributions,

$$A(c)^{-1} Y(ct) \xrightarrow{\mathrm{d}} X(t) \quad \text{for } t \geqslant 0 \tag{6.2}$$

as $c \to \infty$, where the stochastic process $X(t)$ has nondegenerate marginal distributions for all $t > 0$, then $X(t)$ is H-SS for some $H > 0$, and all H-SS processes appear as limits of this form. Moreover, the function A is regularly varying with index H, that is,

$$A(c) = c^H L(c), \tag{6.3}$$

where L is a slowly varying function. See, e.g., [5, p. 14] for a proof.

Let us first discuss an infinitely divisible asymptotic version of Lamperti's limit theorem. Hence, let us replace condition (6.1) by

$$A(c) \to \infty \quad \text{as } c \downarrow 0,$$

and let us assume that (6.2) now holds for $c \downarrow 0$. Then a straightforward modification of Lamperti's proof shows that $X(t)$ is again H-SS for some $H > 0$, and (6.3) holds with L slowly varying at zero.

Let us consider the processes with drift corresponding to $Y(t)$ and $X(t)$, namely, $Y(\mu; t) = Y(t) + \mu t$ and $X(\mu; t) = X(t) + \mu t$, say. Then (6.2) implies that

$$A(c)^{-1} Y\bigl(\mu A(c) c^{-1}; ct\bigr) = A(c)^{-1} Y(ct) + \mu t \xrightarrow{\mathrm{d}} X(t) + \mu t = X(\mu; t) \quad \text{for } t \geqslant 0$$

as $c \to \infty$ or $c \downarrow 0$, where $X(\mu; t)$ is a general self-similar family of stochastic processes.

These results motivate the following question. Suppose we are given families of stochastic processes $Y(\mu; t)$ and $X(\mu; t)$, and a function $A(c)$ such that

$$A(c)^{-1} Y\bigl(\mu A(c) c^{-1}; ct\bigr) \xrightarrow{\mathrm{d}} X(\mu; t) \quad \text{for } t \geqslant 0 \text{ as } c \to \infty \text{ or } c \downarrow 0. \tag{6.4}$$

Under which conditions does this imply that the family X is self-similar? We shall not answer this question in full generality here, but clearly (6.4) includes as special cases the different cases of convergence discussed so far. The following partial result shows a case where (6.4) applies to non-Lévy processes.

Let us replace $A(c)$ by c^H in (6.4), which yields the condition

$$c^{-H} Y\bigl(\mu c^{H-1}; ct\bigr) \xrightarrow{\mathrm{d}} X(\mu; t) \quad \text{for } t \geqslant 0 \text{ as } c \to \infty \text{ or } c \downarrow 0, \tag{6.5}$$

which corresponds to the convergence of the action of the renormalization group (2.5) to the fixed point (2.3). We then have the following Lamperti-type limit theorem, based on an adaptation of the original proof, as given by Embrechts and Maejima [5, p. 14].

Theorem 2. *Consider a family of stochastic processes* $\{X(\mu; t)\colon\ t \geqslant 0\}$ *indexed by* $\mu \in \overline{\Omega}$.

1. *If there exists a family of stochastic processes* $Y(\mu; t)$ *such that* (6.5) *holds, then the process* $X(\mu; t)$ *is* H*-SS for each* $\mu \in \overline{\Omega}$.
2. *If* $X(\mu; t)$ *is* H*-SS, then there exists a family* $Y(\mu; t)$ *satisfying* (6.5).

Proof. We first show item 1 in the case $c \to \infty$. The proof in the case $c \downarrow 0$ is similar. By (6.5), for any $t_1, \ldots, t_k > 0$ and for continuity points $y_1, \ldots, y_k$ of $\{X(\mu; t_j),\ 1 \leqslant j \leqslant k\}$ and $\{X(\mu b^{H-1}; bt_j),\ 1 \leqslant j \leqslant k\}$, where $b > 0$, we obtain

$$\lim_{c\to\infty} \mathbb{P}\big[c^{-H} Y\big(\mu c^{H-1}; ct_j\big) \leqslant y_j,\ 1 \leqslant j \leqslant k\big] = \mathbb{P}\big[X(\mu; t_j) \leqslant y_j,\ 1 \leqslant j \leqslant k\big]. \tag{6.6}$$

Replacing t_j by bt_j, μ by μb^{H-1}, and y_j by $y_j b^H$, we obtain

$$\lim_{c\to\infty} \mathbb{P}\big[c^{-H} Y\big(\mu (cb)^{H-1}; (cb)t_j\big) \leqslant y_j b^H,\ 1 \leqslant j \leqslant k\big] = \mathbb{P}\big[X\big(\mu b^{H-1}; bt_j\big) \leqslant y_j b^H,\ 1 \leqslant j \leqslant k\big]. \tag{6.7}$$

Since the limits on the left-hand sides of (6.6) and (6.7) have the same value, due to (6.5), it follows by comparing the two right-hand sides that $X(\mu; t) \overset{\mathrm{d}}{=} b^{-H} X(\mu b^{H-1}; bt)$ for $t \geqslant 0$, which is the self-similarity of X.

Item 2 is trivially obtained by taking $Y(\mu; t) \equiv X(\mu; t)$. This completes the proof. □

7 EXPONENTIAL VARIANCE FUNCTIONS AND $H = 1$

Much as in the case of strict self-similarity, the value $H = 1$ for the Hurst exponent corresponds to a degenerate family of processes, at least when second moments are finite. In fact, if the family X is SSSI in the sense of Definition 1, the scaling (2.1) with $H = 1$ becomes

$$X(\mu; ct) \overset{\mathrm{d}}{=} cX(\mu; t) \quad \text{for } t \geqslant 0. \tag{7.1}$$

Under the second-moment assumptions, this implies that $\mathrm{Cov}[X(\mu; s), X(\mu; t)] = V(\mu)st$ (compare with (2.18)). For $t = 1$, it follows, in turn, that

$$\begin{aligned}\mathrm{Var}\big[X(\mu; s) - sX(\mu; 1)\big] &= \mathrm{Var}\big[X(\mu; s)\big] + s^2\, \mathrm{Var}\big[X(\mu; 1)\big] - 2s\, \mathrm{Cov}\big[X(\mu; s), X(\mu; 1)\big] \\ &= V(\mu)s^2 + V(\mu)s^2 - 2sV(\mu)s = 0.\end{aligned}$$

This implies that $X(\mu; \cdot)$ is a degenerate, straight-line processes $X(\mu; s) \equiv sX(\mu; 1)$ a.s. for each $\mu \in \Omega$.

Let us instead propose an extended definition of self-similarity for the case $H = 1$. We say that a family of stochastic processes X is *self-similar with Hurst exponent* $H = 1$ (1-SSSI) if there exists a function $f : \mathbb{R}_+ \to \overline{\Omega}$ such that, for all $c > 0$ and $\mu \in \overline{\Omega}$,

$$X\big(\mu + f(c); ct\big) \overset{\mathrm{d}}{=} c\big[X(\mu; t) + tf(c)\big] \quad \text{for } t > 0. \tag{7.2}$$

The interpretation of this definition is that a location change for the rate and a simultaneous rescaling of time is equivalent to a location and scale change for the process. In the special case $f(c) \equiv 0$, we obtain the degenerate case (7.1) discussed above.

Proposition 4. *Let X be a 1-SSSI family of stochastic processes satisfying* (7.2) *with a nonconstant f. Assume that $X(\mu; t)$ has finite expectation for all $\mu \in \Omega$ and $t \geqslant 0$. If the function $t \mapsto \mathbb{E}[X(\mu; t)]$ is continuous on $[0, \infty)$, then up to a translation of μ there exist constants a and $b \neq 0$ such that*

$$\mathbb{E}\big[X(\mu; t)\big] = a\mathrm{e}^{b\mu} + \mu t \quad \textit{for } t \geqslant 0 \textit{ and } \mu \in \Omega. \tag{7.3}$$

When $a \neq 0$, this implies that

$$f(c) = b^{-1} \log c \quad \textit{for } c > 0. \tag{7.4}$$

Proof. By the proof of Proposition 1, the stationarity of the increments for X implies that

$$\mathbb{E}\big[X(\mu;t)\big] = a(\mu) + b(\mu)t$$

for suitable functions a and b. In view of (7.2), this implies that the functions a, b, and f satisfy

$$a\big(\mu + f(c)\big) + b\big(\mu + f(c)\big)ct = ca(\mu) + \big[b(\mu) + f(c)\big]ct$$

for $c, t > 0$. By comparing the left- and right-hand linear functions of t, we obtain the equations

$$a\big(\mu + f(c)\big) = ca(\mu), \tag{7.5}$$
$$b\big(\mu + f(c)\big) = b(\mu) + f(c). \tag{7.6}$$

Since f is not constant, the solution to (7.5) is $a(\mu) = a\mathrm{e}^{b\mu}$, say, where $a = a(0)$ and $b \neq 0$ are constants. When $a \neq 0$, this implies $f(c) = b^{-1}\log c$. From (7.6) we obtain $b(\mu) = b(0) + \mu$, where we may take $b(0) = 0$ up to a translation of μ. $\square$

When the family X satisfies the conditions of Proposition 4, we define the centered family X_0 by analogy with (2.7), that is,

$$X_0(\mu;t) = X(\mu;t) - a\mathrm{e}^{b\mu}.$$

It is easy to check that the family X_0 is again self-similar, now with $a = 0$. The mean function (7.3) for X_0 then takes the form

$$\mathbb{E}\big[X_0(\mu;t)\big] = \mu t \quad \text{for } t \geqslant 0 \text{ and } \mu \in \Omega. \tag{7.7}$$

Unless otherwise stated, we assume from now on that the 1-SSSI family X is centered.

Let us now derive the covariance function of an 1-SSSI family X, assuming that the rate μ and variance function V satisfy (7.7) and (2.9), respectively. A simple rearrangement of (7.2) with $c = t^{-1}$ yields

$$X(\mu;t) \stackrel{\mathrm{d}}{\sim} tX\big(\mu + f\big(t^{-1}\big);1\big) - tf\big(t^{-1}\big) \quad \text{for } t > 0,$$

which gives the following expression for the variance of the process:

$$\operatorname{Var}\big[X(\mu;t)\big] = t^2 V\big(\mu + f\big(t^{-1}\big)\big) = V_1(\mu;t), \tag{7.8}$$

say. The covariance function now looks similar to the general case, namely,

$$\operatorname{Cov}\big[X(\mu;s), X(\mu;t)\big] = \frac{1}{2}\big[V_1(\mu;s) + V_1(\mu;t) - V_1\big(\mu;|t-s|\big)\big]$$

for $s, t > 0$.

Following [9, Chap. 4], we consider the family of infinitely divisible exponential dispersion models $\mathrm{Tw}_\infty(\mu,b,t)$, indexed by the parameter $b \in \mathbb{R}$ and with domain $\Omega = \mathbb{R}$ for the rate parameter μ. For given b, the model $\mathrm{Tw}_\infty(\mu,b,t)$ is defined by the variance function $V(\mu) = \sigma^2\mathrm{e}^{b\mu}$ for $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. The case $b = 0$ corresponds to the normal distribution with constant variance function, but for $b \neq 0$, we may consider $\mathrm{Tw}_\infty(\mu,b,t)$ to be a Tweedie model with power parameter $p = \infty$. For $b > 0$ ($b < 0$), the model $\mathrm{Tw}_\infty(\infty,b,t)$ ($\mathrm{Tw}_\infty(-\infty,b,t)$) is an extreme 1-stable distribution, and we hence obtain the extended domain $\overline{\Omega}_b = (-\infty,\infty]$ for $b > 0$ and $\overline{\Omega}_b = [-\infty,\infty)$ for $b < 0$.

The model $\mathrm{Tw}_\infty(\mu,b,t)$ satisfies two separate transformation properties [9, Chap. 4], where the first is a translation property,

$$c + \mathrm{Tw}_\infty(\mu,b,t) = \mathrm{Tw}_\infty\big(c+\mu,b,t\mathrm{e}^{bc}\big) \quad \text{for } c \in \mathbb{R}, \tag{7.9}$$

and the second is a scaling property,

$$c\,\mathrm{Tw}_\infty(\mu,b,t) = \mathrm{Tw}_\infty\big(c\mu, bc^{-1}, tc^{-2}\big) \quad \text{for } c>0. \tag{7.10}$$

The latter leads us to define $S_\infty(\mu,b;t)$ as the Lévy process with marginal distributions

$$S_\infty(\mu,b;t) \overset{\mathrm{d}}{\sim} t\,\mathrm{Tw}_\infty(\mu,b,t) = \mathrm{Tw}_\infty\big(\mu t, bt^{-1}, t^{-1}\big) \quad \text{for } t>0. \tag{7.11}$$

Depending on the sign of b, we find that $S_\infty(\infty,b;t)$ ($b>0$), respectively $S_\infty(-\infty,b;t)$ ($b<0$), are extreme 1-stable Lévy processes. The exponential family of stochastic processes $S_\infty(\mu,b;t)$ for $\mu \in \overline{\Omega}_b$ may hence be generated from the respective processes $S_\infty(\pm\infty,b;t)$ by exponential tilting. As is the case for ordinary Hougaard Lévy processes, the families $S_\infty(\mu,b;t)$ may be characterized by self-similarity, as we shall now see.

Theorem 3. *Let X be a nondegenerate exponential family of Lévy processes. Then X is self-similar with Hurst exponent $H=1$ in the sense* (7.2) *if and only if X is a Hougaard family of Lévy processes $S_\infty(\mu,b;t)$ with $b\neq 0$.*

Proof. Let us first show that the family of process $S_\infty(\mu,b;t)$ for $\mu\in\overline{\Omega}_b$ is 1-SSSI and satisfies the relation

$$S_\infty\big(\mu+b^{-1}\log c, b; ct\big) \overset{\mathrm{d}}{=} c\big[S_\infty(\mu,b;t)+tb^{-1}\log c\big] \quad \text{for } t>0, \tag{7.12}$$

corresponding to (7.2) and (7.4) (with $f(c)=b^{-1}\log c$). Since we have a Lévy process, it is enough to consider the marginal distribution of $S_\infty(\mu,b;t)$. Using (7.11) and (7.9), we find that the left-hand side of (7.12) has the marginal distribution

$$S_\infty\big(\mu+b^{-1}\log c, b; ct\big) \overset{\mathrm{d}}{\sim} \mathrm{Tw}_\infty\big(ct\big(\mu+b^{-1}\log c\big), b(ct)^{-1}, (ct)^{-1}\big).$$

The right-hand side of (7.12) follows from (7.11), (7.9), and (7.10):

$$\begin{aligned} c\big[S_\infty(\mu,b;t)+tb^{-1}\log c\big] &\overset{\mathrm{d}}{\sim} c\,\mathrm{Tw}_\infty\big(\mu t+tb^{-1}\log c, bt^{-1}, t^{-1}c\big) \\ &= \mathrm{Tw}_\infty\big(ct\big(\mu+b^{-1}\log c\big), b(ct)^{-1}, (ct)^{-1}\big). \end{aligned}$$

This implies the 1-SS property (7.12), as desired.

To show that self-similarity with $H=1$ characterizes the Hougaard Lévy process, let us consider a natural exponential family of Lévy processes X with variance function V such that (7.2) is satisfied. We know that V is positive and analytic on Ω, due to the exponential family assumption, so f is not identically zero. Since we are dealing with a Lévy process, the variance of the process is $\mathrm{Var}\,X(\mu;t)=tV(\mu)$. Comparing this with (7.8), we obtain the equation $tV(\mu)=t^2V(\mu+f(t^{-1}))$ or, equivalently, with $s=t^{-1}$,

$$V\big(\mu+f(s)\big)=sV(\mu). \tag{7.13}$$

If $f(c)\equiv 0$, we obtain the degenerate process (7.1), which is ruled out by assumption. Hence, by the same type of arguments as used in connection with (7.5), we write the solution to (7.13) as

$$V(\mu)=V(0)\mathrm{e}^{b\mu} \quad \text{for } \mu\in\mathbb{R}, \tag{7.14}$$

for some $b\neq 0$, and, consequently, $f(s)=b^{-1}\log s$. Taking $\sigma^2=V(0)$, we see that the variance function (7.14) characterizes the family $\mathrm{Tw}_\infty(\mu,b,t)$, and, hence, the family of Hougaard Lévy processes $S_\infty(\mu,b;t)$, completing the proof. □

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