

Tight-minimal dichotomies in Banach spaces

by

ALEJANDRA C. CÁCERES-RIGO and VALENTIN FERENCZI

Abstract. We extend the methods used by V. Ferenczi and Ch. Rosendal to obtain the “third dichotomy” in the program of classification of Banach spaces up to subspaces, in order to prove that a Banach space E with an admissible system of blocks $(\mathcal{D}_E, \mathcal{A}_E)$ contains an infinite-dimensional subspace with a basis which is either \mathcal{A}_E -tight or \mathcal{A}_E -minimal. In this setting we obtain, in particular, dichotomies regarding subsequences of a basis, and as a corollary, we show that every normalized basic sequence $(e_n)_n$ has a subsequence which satisfies a tightness property or is spreading. Other dichotomies between notions of minimality and tightness are demonstrated, and the Ferenczi–Godefroy interpretation of tightness in terms of Baire category is extended to this new context.

Contents

1. Introduction	2
1.1. A sketch of the proof	5
2. Preliminaries	5
2.1. A law and Ramsey-like theorems	7
3. Admissible sets and families	7
3.1. Definitions and notations	7
3.2. Properties of admissible sets	10
3.3. Admissible families	12
4. Embeddings and minimality	16
5. Interpretations for the set of blocks	17
5.1. Blocks as nonzero F-linear combinations	17
5.2. Blocks as vectors of the basis	19
5.3. Blocks as signed elements of the basis	20
6. Summary of types of minimality	20
7. Results on \mathcal{A} -tightness	24
7.1. Notions of tightness	24
8. Games for tightness	28
9. Games for minimality	37
9.1. An auxiliary minimal game	39

2020 *Mathematics Subject Classification*: Primary 46B20; Secondary 46B03, 03E15.

Key words and phrases: tight bases, minimal spaces, spreading bases, dichotomies on Banach spaces.

Received 19 December 2023; revised 20 December 2024.

Published online 6 April 2025.

10. Tight-minimal dichotomies	40
10.1. Corollaries from the \mathcal{A} -tight-minimal dichotomy	44
10.2. Corollaries from the \mathcal{A} -tight-minimal dichotomy: subsequences	45
References	46

1. Introduction. In this paper, when we refer to a Banach space, we mean a separable infinite-dimensional Banach space. Subspaces of Banach spaces are assumed to be infinite-dimensional and closed, unless stated otherwise. In [12] W. T. Gowers began the *Classification Program of Banach spaces up to subspaces*. The program aims to classify Banach spaces into “inevitable” classes, using dichotomies between two opposite inevitable classes of Banach spaces. Conditions for a class to be considered of interest for the program were given by Gowers: the classes must be *inevitable*, that is, every Banach space must belong to a class. A class must be *hereditary* for closed subspaces or, if the property that determines the class is defined for basic sequences, then the class must be hereditary for block subspaces. Two different classes must be *disjoint*. The property that determines the class *must give additional information* about the space of operators defined over the space or over its subspaces.

A Banach space X is decomposable if it can be written as the direct sum of two closed infinite-dimensional subspaces, otherwise X is said indecomposable. A Banach is said hereditarily indecomposable (or HI) if all its infinite-dimensional subspaces are indecomposable. Gowers showed a *first dichotomy* (see [11]) giving the first two examples of inevitable classes: every Banach space has a separable subspace that is either hereditarily indecomposable, or has an unconditional basis. In [12] a *second dichotomy* was proved: Any Banach space contains a subspace with a basis such that no pair of disjointly supported block subspaces are isomorphic, or any two block subspaces have isomorphic subspaces. In résumé, in [12], four inevitable classes were presented.

Later V. Ferenczi and Ch. Rosendal [10] proved three new dichotomies. They refined the list of inevitable classes into six main classes and 19 secondary classes. The main result in [10] is a *third dichotomy*, which contrasts the dual notions of minimality and tightness and is central for the present work:

THEOREM 1.1 (Third dichotomy, [10]). *If E is a Banach space, then E contains a subspace with a basis which is either tight or minimal.*

It is well known that a Banach space is minimal if it can be isomorphically embedded in any of its subspaces. Suppose that E is a Banach space with a Schauder basis $(e_n)_n$. A Banach space Y is tight in E (see [10] for the definition and an extensive study of this notion) if there is a sequence $(I_n)_n$

of successive finite subsets of \mathbb{N} such that for every infinite subset A of \mathbb{N} , Y cannot be isomorphically embedded in $[e_n : n \notin \bigcup_{i \in A} I_i]$. A basis $(e_n)_n$ is tight for E if any Banach space Y is tight in E , and E is tight if it has a tight basis.

A useful characterization of tightness was given in [7] using Baire category: Y is tight in $E = [e_n]_n$ if and only if the set of indices $A \subseteq \mathbb{N}$ for which Y can be embedded in $[e_n : n \in A]$ is meager in $\mathbb{P}(\mathbb{N})$ (after the natural identification of $\mathbb{P}(\mathbb{N})$ with the Cantor space 2^ω , via characteristic functions).

Tightness is an opposite notion to minimality: it is clear that a tight space cannot be minimal, nor can a minimal space have a tight subspace. In both definitions, of tight and minimal spaces, the underlying embedding is an isomorphic embedding. We say that $Y = [y_n]_n$ isomorphically embeds in $E = [e_n]_n$ if $(y_n)_n$ is equivalent to a (basic) sequence $(x_n)_n$ in E . After a standard perturbation argument, one can ask that such a basic sequence is a sequence of finitely supported vectors of E . One can consider different forms of embedding of Y into E , depending on the properties of the basic sequence $(x_n)_n$ in E . For example, one can require $(x_n)_n$ to be a block sequence of the basis $(e_n)_n$ of E or a sequence of disjointly supported vectors in E .

The authors of [10] also stated that after a variation of the notion of embedding in the definition of tight basis, and consequently modifying the methods involved in the proof of the *third dichotomy*, the following result can be obtained:

THEOREM ([10, Theorem 3.16]). *Every Banach space with a basis contains a block subspace $E = [e_n]_n$ satisfying one of the following properties:*

- (1) *For any $[y_n]_n \leq E$, there is a sequence $(I_n)_n$ of successive intervals in \mathbb{N} such that for any $A \in [\mathbb{N}]^\infty$, $[y_n]_n$ does not embed into $[e_n : n \notin \bigcup_{i \in A} I_i]$ as a sequence of disjointly supported vectors or as a block sequence.*
- (2) *For any $[y_n]_n \leq E$, $(e_n)_n$ is equivalent to a sequence of disjointly supported vectors of $[y_n]_n$, or to a block sequence of $[y_n]_n$.*

Therefore, modifying the embedding we obtain a corresponding type of minimality and the associated dual type of tightness. In this work we define and study different types of minimality and the respective dual notions of tightness, in order to obtain new dichotomies between them. An additional attractive aspect of this point of view is to allow extending the techniques to the study of subsequences of a given basis, instead of subspaces of a given space.

Those ways of interpreting the embedding are coded in what we call an admissible system of blocks, which is a pair $(\mathcal{D}_E, \mathcal{A}_E)$ associated to a Banach space E with a fixed normalized basis $(e_n)_n$. Basically, a set \mathcal{D}_E of blocks (see Definition 3.2) for E is a set “containing” the possible bases of the block

subspaces one admits to consider. Meanwhile, an admissible set (see Definition 3.6) \mathcal{A}_E for E is the set of infinite sequences of vectors which are the images of the embeddings one wants to consider. Using this coding in the case of “being equivalent to a subsequence of $(e_n)_n$ ”, for example, \mathcal{D}_E would be the set whose elements are the vectors of the basis and \mathcal{A}_E is the set of all subsequences of E . The properties of sets of blocks and admissible families will be studied in Section 3.

This coding for embedding through admissible sets \mathcal{A}_E of vectors naturally leads us to define the notions of \mathcal{A}_E -minimality and \mathcal{A} -tightness, which depend on the pair $(\mathcal{D}_E, \mathcal{A}_E)$, as follows: given a set of blocks \mathcal{D}_E and an admissible set \mathcal{A}_E for E , we say that E is \mathcal{A}_E -minimal if for every block sequence $(x_n)_n \in (\mathcal{D}_E)^\omega$, there is a sequence $(y_n)_n \in \mathcal{A}_E \cap X^\omega$ equivalent to $(e_n)_n$. We say that $(e_n)_n$ is an \mathcal{A}_E -tight basis for E if for every Banach space Y there is an sequence $(I_i)_i$ of successive intervals such that for every infinite subset A of \mathbb{N} ,

$$(1) \quad Y \not\hookrightarrow^{\mathcal{A}} \left[e_n : n \notin \bigcup_{i \in A} I_i \right].$$

The study of \mathcal{A}_E -embeddings and \mathcal{A}_E -minimality is taken up in Sections 4, 5 and summarized in Section 6. Basic properties of \mathcal{A}_E -tight bases are studied in Section 7.

In this work, we generalize the methods of [10] to use admissible systems and we prove the main theorem of this work:

THEOREM 1.2. *Let E be a Banach space with a normalized basis $(e_n)_n$ and $(\mathcal{D}_E, \mathcal{A}_E)$ be an admissible system of blocks for E . Then E contains a \mathcal{D}_E -block subspace which is either \mathcal{A}_E -tight or \mathcal{A}_E -minimal.*

The authors of [10] stated that modifying the notion of embedding in the definition of tight basis, and consequently modifying the methods involved in the proof of the third dichotomy, one can deduce the following statement:

- *Every Banach space with a basis contains a block subspace $E = [e_n]_n$ such that either for any $[y_n]_n \leq E$, there is a sequence $(I_n)_n$ of successive intervals in \mathbb{N} such that for any $A \in [\mathbb{N}]^\infty$, $[y_n]_n$ does not embed into $[e_n : n \notin \bigcup_{i \in A} I_i]$ as a permutation of a block sequence; or for any $[y_n]_n \leq E$, $(e_n)_n$ is permutatively equivalent to a block sequence of $[y_n]_n$.*

But, as we see in Proposition 5.4 below, a basic sequence $(y_n)_n$ being embedded in $[e_n]_n = E$ as a permutation of $(e_n)_n$ is not an \mathcal{A}_E -embedding obtained from an admissible set for E , and this is fundamental for the proofs in this statement to work. We have no evidence that in this case the above paper is true, but it cannot be obtained just by modifying the embedding notion in the proof of the third dichotomy, as claimed in [10].

1.1. A sketch of the proof. The main tool used in [10] in order to prove the third dichotomy is the notion of *generalized asymptotic game* which is a generalization of the notion of infinite asymptotic game (see [17, 14]). A modification of the infinite asymptotic game was first defined by Ferenczi [6] to prove that a space saturated with subspaces with a Schauder basis, which embed into the closed linear span of any subsequence of their basis, must contain a minimal subspace. The work in [6] generalized the methods and the result of Pelczar [16]: a Banach space saturated with subsymmetric basic sequences contains a minimal subspace.

Let $X = [x_n]_n$ and $Y = [y_n]_n$ be two block subspaces of a Banach space E with a Schauder basis $(e_n)_n$. The *generalized asymptotic game* $H_{Y,X}$ with constant C is a game with infinite rounds between player I and player II where in the k th round, player I picks a natural number n_k and player II responds with a natural number m_k and a not necessarily normalized finitely supported vector u_k such that $\text{supp}(u_k) \subseteq \bigcup_{i=0}^k [n_i, m_i]$. The outcome of the game is a not necessarily block sequence $(u_n)_n$. Player II wins the game if $(y_n)_n$ is C -equivalent to $(u_n)_n$.

In order to prove Theorem 1.2, we follow the demonstration of the third dichotomy generalizing the arguments for the context of \mathcal{A}_E -minimality and \mathcal{A}_E -tightness, creating the notion of “admissible systems of blocks”. First, we shall adapt to \mathcal{D}_E -block subspaces two technical lemmas (8.1 and 8.2), whose original versions for block subspaces were proved in [10] and in [15], respectively. We define an \mathcal{A} -version of the generalized asymptotic game $H_{Y,X}^{\mathcal{A}}$ with constant C , depending on an admissible set \mathcal{A}_E , requiring that the outcome $(u_n)_n$ of the game be an element of $\mathcal{A}_E \cap X^\omega$. Again, the game $H_{Y,X}^{\mathcal{A}}$ with constant C is open for player I and so, by the determinacy of open Gale–Stewart games, is determined.

In Section 8 we prove technical lemmas by varying the methods of Ferenczi and Rosendal: we show that if E is in some way saturated by \mathcal{D}_E -block subspaces X and Y such that player I has a winning strategy for the game $H_{Y,X}^{\mathcal{A}}$ with constant C , then E has an \mathcal{A}_E -tight subspace.

Before the proof of our main theorem it is necessary to introduce two games for \mathcal{A}_E -minimality: the game $G_{Y,X}^{\mathcal{A}}$ with constant C and a version assuming that finitely many moves have been made in $G_{Y,X}^{\mathcal{A}}$. This will be done in Section 9. The main result in that section relates the existence of a winning strategy for player II in the game $H_{Y,X}^{\mathcal{A}}$ to the existence of a winning strategy for player II in the game $G_{Y,X}^{\mathcal{A}}$. Finally, after the proof of Theorem 1.2 in Section 10, some tight-minimal dichotomies are presented.

2. Preliminaries. If E is a Banach space then \mathbb{S}_E , \mathbb{B}_E and $\overline{\mathbb{B}}_E$ denote the unit sphere and the open and closed ball of E , respectively. For $\varepsilon > 0$ and

$x \in E$, $\mathbb{B}_E(x, \varepsilon)$ and $\overline{\mathbb{B}}_E(x, \varepsilon)$ denote the open and closed ball in E centered in x with radius ε .

Suppose that $(e_n)_n$ is a basis for E . We define the *support* of $x \in E$ (written $\text{supp}_E(x)$) in the basis $(e_n)_n$ as the set $\{n \in \mathbb{N} : e_n^*(x) \neq 0\}$, where e_k^* are the coordinate functionals defined by $x = \sum_{n=0}^{\infty} \lambda_n e_n \mapsto \lambda_k$ for $k \in \mathbb{N}$. The support of the zero vector of E is the empty set.

We say that a Banach space X is isomorphic to a Banach space Y with constant K (denoted as $Y \simeq_K X$) if there exists a one-to-one bounded linear operator T from X onto Y such that T^{-1} is bounded and $K \geq \|T\| \cdot \|T^{-1}\|$. We say that X contains a K -isomorphic copy of Y , or Y is K -embeddable in X (denoted as $Y \hookrightarrow_K X$), if $Y \simeq_K Z$ for some subspace Z of X . Finally, Y is *isomorphically embeddable*, or just *embeddable*, in X (in symbols $Y \hookrightarrow X$) if $Y \hookrightarrow_K X$ for some $K \geq 1$. In this case we say that X contains an *isomorphic copy*, or just a *copy*, of Y .

For $K \geq 1$, two basic sequences $(x_n)_n$ and $(y_n)_n$ are K -equivalent $((x_n)_n \sim_K (y_n)_n)$ if for all $k \in \mathbb{N}$ and every finite sequence $(a_i)_{i=0}^k$ of scalars we have

$$\frac{1}{K} \left\| \sum_{n=0}^k a_n x_n \right\| \leq \left\| \sum_{n=0}^k a_n y_n \right\| \leq K \left\| \sum_{n=0}^k a_n x_n \right\|.$$

Two basic sequences are *equivalent* if they are K -equivalent for some $K \geq 1$.

We shall use the following well known result:

PROPOSITION 2.1. *Let X be a Banach space with a basis $(x_n)_n$ with basis constant C and let $M \geq 1$. Then there is a constant $c \geq 1$, which depends on C and M , such that if $(z_n)_n$ and $(y_n)_n$ are normalized block bases of $(x_n)_n$ which differ only in M terms, then $(y_n)_n \sim_c (z_n)_n$.*

If A is a nonempty set, then $|A|$ denotes the cardinality of A , $\mathbb{P}(A)$ denotes the power set of A , and $[A]^{<\infty}$ and $[A]^\infty$ denote the set of finite subsets of A and the set of infinite subsets of A , respectively. Given $A, B \subset \mathbb{N}$ and assuming that $\max A$ and $\min B$ exist, we write $A < B$ to mean that $\max A < \min B$. When we refer to a sequence $(I_n)_n$ of *successive* finite subsets of \mathbb{N} , we mean that $I_n < I_{n+1}$ for every $n \in \mathbb{N}$. Also, when we refer to an *interval* I of natural numbers, we mean that $I = [a, b] \cap \mathbb{N}$ for some $0 \leq a < b$. Let us denote the set of nonempty finite sets of \mathbb{N} by FIN , that is, $\text{FIN} := [\mathbb{N}]^{<\infty} \setminus \{\emptyset\}$. We denote by FIN^ω the set of infinite sequences of non-empty finite subsets of \mathbb{N} .

We shall consider the Cantor space $2^\omega = \{0, 1\}^\omega$ with the product topology where $\{0, 1\}$ is endowed with the discrete topology. If $\mathfrak{s} = (s_i)_i \in 2^\omega$, define $\text{supp}(\mathfrak{s}) = \{i \in \mathbb{N} : s_i = 1\}$. Notice that $\mathbb{P}(\mathbb{N})$ can be identified with 2^ω using characteristic functions: if $A \in \mathbb{P}(\mathbb{N})$, then the characteristic function χ_A belongs to 2^ω and $A = \text{supp}(\chi_A)$. Thus, families of subsets of

\mathbb{N} will sometimes be viewed as families of sequences in \mathbb{N} . Therefore, any $\mathcal{F} \subseteq \mathbb{P}(\mathbb{N})$ can be seen as a topological subspace of 2^ω . A basic open subset of 2^ω determined by $\mathfrak{s} \in 2^\omega$ and $J \in [\mathbb{N}]^{<\omega}$ is given by

$$\mathcal{N}_{\mathfrak{s},J} := \{u = (u_n)_n \in 2^\omega : \forall n \in J (u_n = s_n)\}.$$

2.1. A law and Ramsey-like theorems. A *Polish space* is a separable completely metrizable topological space. In this subsection we shall recall some classical theorems. The next theorem is known as the *first topological 0-1 law*:

THEOREM 2.2 ([13, (8.46)]). *Let X be a Polish space, and G be a group of homeomorphisms of X with the following property: for any non-empty open subsets U and V of X , there is $g \in G$ such that $g(U) \cap V \neq \emptyset$. If $A \subseteq X$ has the Baire property and is G -invariant (i.e. $g(A) = A$ for every $g \in G$), then A is meager or comeager in X .*

The next theorem is known as *Galvin–Prikry’s Theorem*:

THEOREM 2.3 ([13, (19.11)]). *Let $[\mathbb{N}]^\infty = P_0 \cup \dots \cup P_{k-1}$, where each P_i is Borel and $k \in \mathbb{N}$. Then there are $H \in [\mathbb{N}]^\infty$ and $i < k$ with $[H]^\infty \subseteq P_i$.*

According to Ramsey theory’s nomenclature, a subset \mathcal{C} of $[\mathbb{N}]^\infty$ is *Ramsey* if there is some $H \in [\mathbb{N}]^\infty$ such that $[H]^\infty \subseteq \mathcal{C}$ or $[H]^\infty \subseteq [\mathbb{N}]^\infty \setminus \mathcal{C}$. So, Galvin–Prikry’s Theorem can be enunciated as follows: Borel sets of $[\mathbb{N}]^\infty$ are Ramsey. The next theorem, *Silver’s Theorem*, says that analytic subsets of $[\mathbb{N}]^\infty$ are completely Ramsey, which implies that analytic subsets of $[\mathbb{N}]^\infty$ are Ramsey (since all completely Ramsey subsets are Ramsey). We recommend [13] for more information about these definitions and proofs.

THEOREM 2.4 ([13, (29.8)]). *Analytic subsets of $[\mathbb{N}]^\infty$ are completely Ramsey.*

3. Admissible sets and families. Along this section suppose E is a Banach space with a Schauder basis $(e_n)_n$. Set $\mathcal{B}_E := \{e_n : n \in \mathbb{N}\}$ and $\mathcal{B}_E^\pm := \{e_n : n \in \mathbb{N}\} \cup \{-e_n : n \in \mathbb{N}\}$. Let \mathbf{F}_E be a countable subfield of \mathbb{R} containing the rationals such that for all $\sum_{i=0}^n \lambda_i e_i$ with $n \in \mathbb{N}$ and $(\lambda_i)_{i=0}^n \in (\mathbf{F}_E)^{n+1}$, the norm $\|\sum_{i=0}^n \lambda_i e_i\|$ is in \mathbf{F}_E . We denote by \mathbb{D}_E the countable set of nonzero not necessarily normalized finite \mathbf{F}_E -linear combinations of $(e_n)_n$.

3.1. Definitions and notations

DEFINITION 3.1. Let $(x_n)_n$ be a sequence of successive finitely supported vectors of E . For $X = [x_n]_n$, define $*_X : (\mathbb{D}_E \cap X)^\omega \times (\mathbb{D}_E)^\omega \rightarrow (\mathbb{D}_E)^\omega$ as follows: if $v = (v_n)_n \in (\mathbb{D}_E)^\omega$ and $u = (u_n)_n \in (\mathbb{D}_E \cap X)^\omega$ are such that for each $n \in \mathbb{N}$,

$$u_n = \sum_{i \in \text{supp}_X(u_n)} \lambda_i^n x_i,$$

then $u *_X v$ is the sequence $(w_n)_n$ where for each $n \in \mathbb{N}$,

$$w_n = \sum_{i \in \text{supp}_X(u_n)} \lambda_i^n v_i.$$

Notice that the set $\mathbb{D}_E \cap X$ could be empty. In our work we shall take subspaces generated by vectors in \mathbb{D}_E , so this will not occur.

DEFINITION 3.2. We define a *set of blocks* for the space E to be a set \mathcal{D}_E satisfying the following conditions:

- (a) $\mathcal{D}_E \subseteq \mathbb{D}_E$.
- (b) $\{e_n : n \in \mathbb{N}\} \subseteq \mathcal{D}_E$.
- (c) If $u \in \mathcal{D}_E$, then $u/\|u\| \in \mathcal{D}_E$.
- (d) For all $(u_n)_n, (v_n)_n \in (\mathcal{D}_E)^\omega$, we have $(u_n)_n *_E (v_n)_n \in (\mathcal{D}_E)^\omega$.
- (e) Let $(x_i)_{i=0}^n \in (\mathcal{D}_E)^{n+1}$ with $x_i < x_{i+1}$ for every $0 \leq i < n$, and $X = [x_i]_{i \leq n}$. If $u \in \mathcal{D}_E$ is such that

$$u = \sum_{i=0}^n \lambda_i x_i,$$

then

$$v = \sum_{i=0}^n \lambda_i e_i \in \mathcal{D}_E.$$

We say that a vector u is a \mathcal{D}_E -block if $u \in \mathcal{D}_E$.

EXAMPLE 3.3. $\mathcal{B}_E, \mathcal{B}_E^\pm$ and \mathbb{D}_E are sets of blocks for E .

DEFINITION 3.4. Let $D \subseteq \mathbb{D}_E$ be an infinite subset such that D^ω contains a block basis of $(e_n)_n$.

- (i) We say that $(y_n)_n \in E^\omega$ is a *D-block sequence* if $(y_n)_n$ is a block basis of $(e_n)_n$ and for each $n \in \mathbb{N}$ we have $y_n \in D$.
- (ii) A subspace Y is a *D-block subspace* if it is the closed subspace spanned by a *D-block sequence* $(y_n)_n$.

Without loss of generality we shall suppose that a *D-block subspace* is always generated by a normalized *D-block sequence*.

DEFINITION 3.5. Let \mathcal{D}_E be a set of blocks for E . Let X be a \mathcal{D}_E -block subspace.

- (i) We define $\mathbb{D}_X := \mathbb{D}_E \cap X$.
- (ii) We denote $\mathcal{D}_X := \mathcal{D}_E \cap X$.
- (iii) We denote by $bb_{\mathcal{D}}(E)$ the set of normalized \mathcal{D}_E -block sequences of E , i.e.

$$bb_{\mathcal{D}}(E) := \{(x_n)_n \in (\mathcal{D}_E)^\omega : (x_n)_n \text{ is a } \mathcal{D}_E\text{-block sequence of } E \\ \& \forall n \in \mathbb{N} (\|x_n\| = 1)\}.$$

- (iv) We denote by $bb_{\mathcal{D}}(X)$ the set of normalized \mathcal{D}_X -block sequences of E , i.e.

$$bb_{\mathcal{D}}(X) := \{(y_n)_n \in (\mathcal{D}_X)^\omega : (y_n)_n \text{ is a } \mathcal{D}_X\text{-block sequence of } E \\ \& \forall n \in \mathbb{N} (\|x_n\| = 1)\}.$$

If \mathcal{D}_E is a set of blocks for E and X is a \mathcal{D}_E -block subspace, then we sometimes identify an element $(y_n)_n$ of $bb_{\mathcal{D}}(X)$ with the \mathcal{D}_E -block subspace it generates.

We endow $(\mathcal{D}_E)^\omega$ with the product topology obtained by considering \mathcal{D}_E with the discrete topology; then $(\mathcal{D}_E)^\omega$ is a Polish space. Also, the set $(\mathbb{N} \times \mathbb{N} \times \mathcal{D}_E)^\omega$ with its natural product topology is Polish. The set $bb_{\mathcal{D}}(E)$ is a nonempty closed subspace of $(\mathcal{D}_E)^\omega$, so it is Polish.

DEFINITION 3.6. Let \mathcal{D}_E be a set of blocks for E . We say that a set \mathcal{A}_E is *admissible* for E if it satisfies the following conditions:

- (a) \mathcal{A}_E is a closed subset of $(\mathcal{D}_E)^\omega$.
- (b) \mathcal{A}_E contains all the \mathcal{D}_E -block sequences.
- (c) For every $(y_n)_n \in \mathcal{A}_E$ and every \mathcal{D}_E -block subspace $X = [x_n]_n$, if $(u_n)_n \in (\mathcal{D}_X)^\omega$, then

$$(u_n)_n \in \mathcal{A}_E \iff (u_n)_n *_X (y_n)_n \in \mathcal{A}_E.$$

- (d) Let $(y_n)_n$ be a \mathcal{D}_E -block sequence and $Y = [y_n]_n$. For every $(u_n)_n \in \mathcal{A}_E$ and $k \in \mathbb{N}$, there is $(v_n)_n \in Y^\omega$ such that $(u_0, \dots, u_k, v_0, v_1, \dots) \in \mathcal{A}_E$.

DEFINITION 3.7. Let \mathcal{D}_E be a set of blocks for E , \mathcal{A}_E an *admissible set* for E , and X be a \mathcal{D}_E -block subspace.

- (i) Set $\mathcal{A}_X := \mathcal{A}_E \cap X^\omega$.
- (ii) We denote by $[\mathcal{A}_X]$ the set of *initial parts* of \mathcal{A}_X , that is,

$$[\mathcal{A}_X] := \bigcup_{n \in \mathbb{N}} \{(u_0, u_1, \dots, u_n) \in (\mathcal{D}_X)^{n+1} : \exists (w_i)_i \in \mathcal{A}_X (w_i = u_i \text{ for } 0 \leq i \leq n)\}.$$

REMARK 3.8. (i) Notice that an admissible set depends on the set of blocks that has been chosen for E .

(ii) Since $(\mathcal{D}_E)^i$ is a discrete topological space, the set $[\mathcal{A}_E] \cap (\mathcal{D}_E)^i$ is a clopen subset of $(\mathcal{D}_E)^i$ for every $i \geq 1$.

(iii) If X and Y are \mathcal{D}_E -block subspaces such that $Y \subseteq X$, then $\mathcal{A}_Y \subseteq \mathcal{A}_X$.

DEFINITION 3.9. Let \mathcal{D}_E be a set of blocks for E and \mathcal{A}_E be an admissible set for E . We say that $(\mathcal{D}_E, \mathcal{A}_E)$ is an *admissible system of blocks* for E if for every \mathcal{D}_E -block subspace X of E , for every sequence $(\delta_n)_n$ with $0 < \delta_n < 1$, and $K \geq 1$, there is a collection $(A_n)_n$ of nonempty subsets of \mathcal{D}_X with the following properties:

- (a) For each n and each $d \in [\mathbb{N}]^{<\infty}$ such that there is $w \in \mathcal{D}_X$ with $\text{supp}_X(w) = d$, there are finitely many vectors $u \in A_n$ with $\text{supp}_X(u) = d$.
- (b) For every sequence $(w_i)_i \in \mathcal{A}_X$ satisfying $1/K \leq \|w_i\| \leq K$, for every i , there is $(u_i)_i \in \mathcal{A}_X$ such that for each n we have
 - (b.1) $u_n \in A_n$,
 - (b.2) $\text{supp}_X(u_n) \subseteq \text{supp}_X(w_n)$,
 - (b.3) $\|w_n - u_n\| < \delta_n$.

3.2. Properties of admissible sets

PROPOSITION 3.10. *Let \mathcal{D}_E be a set of blocks and \mathcal{A}_E be an admissible set for E . Then the following are equivalent:*

- (i) *For every $(y_n)_n \in \mathcal{A}_E$ and every \mathcal{D}_E -block subspace $X = [x_n]_n$, if $(u_n)_n \in (\mathcal{D}_X)^\omega$ then*

$$(u_n)_n \in \mathcal{A}_E \iff (u_n)_n *_X (y_n)_n \in \mathcal{A}_E.$$
- (ii) *For all $(y_n)_n, (z_n)_n \in \mathcal{A}_E$, if $(w_n)_n \in (\mathcal{D}_E)^\omega$ then*

$$(w_n)_n *_E (y_n)_n \in \mathcal{A}_E \iff (w_n)_n *_E (z_n)_n \in \mathcal{A}_E.$$
- (iii) *For all $(y_n)_n, (z_n)_n \in \mathcal{A}_E$ and every \mathcal{D}_E -block subspace $X = [x_n]_n$, if $(u_n)_n \in (\mathcal{D}_X)^\omega$ then*

$$(u_n)_n *_X (y_n)_n \in \mathcal{A}_E \iff (u_n)_n *_X (z_n)_n \in \mathcal{A}_E.$$

Proof. This follows directly from Definitions 3.2 and 3.6. ■

We can easily prove that a set of blocks has the following heredity properties:

PROPOSITION 3.11. *Let \mathcal{D}_E be a set of blocks and \mathcal{A}_E be an admissible set for E . If $X = [x_n]_n$ is a \mathcal{D}_E -block subspace, then the following hold:*

- (i) $\mathcal{D}_X \subseteq \mathbb{D}_X$.
- (ii) $\{x_n : n \in \mathbb{N}\} \subseteq \mathcal{D}_X$.
- (iii) If $u \in \mathcal{D}_X$, then $u/\|u\| \in \mathcal{D}_X$.
- (iv) For every $(u_n)_n \in (\mathcal{D}_X)^\omega$ and $(v_n)_n \in (\mathcal{D}_E)^\omega$, we have $(u_n)_n *_X (v_n)_n \in (\mathcal{D}_E)^\omega$. In particular, if $(v_n)_n \in (\mathcal{D}_X)^\omega$, then $(u_n)_n *_X (v_n)_n \in (\mathcal{D}_X)^\omega$.
- (v) Let $(y_i)_{i=0}^n \in (\mathcal{D}_X)^{n+1}$ with $y_i < y_{i+1}$ for every $0 \leq i < n$, and $Y = [y_i]_{i \leq n}$. If $u \in \mathcal{D}_X$ is such that

$$u = \sum_{i=0}^n \lambda_i y_i,$$

then

$$v = \sum_{i=0}^n \lambda_i x_i \in \mathcal{D}_X.$$

Proof. This follows directly from the definition of a set of blocks. ■

The last heredity property is also valid for an admissible set:

PROPOSITION 3.12. *Let \mathcal{D}_E be a set of blocks and \mathcal{A}_E be an admissible set for E . Let $X = [x_n]_n$ be a \mathcal{D}_E -block subspace. The set \mathcal{A}_X has the following properties:*

- (i) \mathcal{A}_X is a closed subset of $(\mathcal{D}_X)^\omega$.
- (ii) Any block basis $(y_n)_n$ in $(\mathcal{D}_X)^\omega$ belongs to \mathcal{A}_X .
- (iii) For every $(v_n)_n \in \mathcal{A}_X$ and every \mathcal{D}_X -block subspace $Y = [y_n]_n$, if $(u_n)_n \in (\mathcal{D}_Y)^\omega$, then

$$(u_n)_n \in \mathcal{A}_X \iff (u_n)_n *_Y (v_n)_n \in \mathcal{A}_X.$$
- (iv) Let $Y = [y_n]_n$ be a \mathcal{D}_X -block subspace. For every $(u_n)_n \in \mathcal{A}_X$ and $k \in \mathbb{N}$, there is $(v_n)_n \in Y^\omega$ such that $(u_0, \dots, u_k, v_0, v_1, \dots) \in \mathcal{A}_X$.

Proof. This follows directly from Definition 3.6. ■

Notice that if X is a \mathcal{D}_E -block subspace, then using Proposition 3.12(ii) we conclude that $[\mathcal{A}_X]$ is infinite. If $(\mathcal{D}_E, \mathcal{A}_E)$ is an admissible system of blocks for E and X is a \mathcal{D}_E -block subspace, then as a consequence of Propositions 3.11 and 3.12, the pair $(\mathcal{D}_X, \mathcal{A}_X)$ can be thought of as an “admissible subsystem of blocks” relative to X . The “relativization” of the condition given in Definition 3.9 to X is clearly true: For every \mathcal{D}_X -block subspace Y of X , for every sequence $(\delta_n)_n$ with $0 < \delta_n < 1$, and $K \geq 1$, there is a collection $(A_n)_n$ of nonempty subsets of \mathcal{D}_Y with the following properties:

- (a) For each n and each $d \in [\mathbb{N}]^{<\omega}$ such that there is $w \in \mathcal{D}_Y$ with $\text{supp}_Y(w) = d$, there are finitely many vectors $u \in A_n$ with $\text{supp}_Y(u) = d$.
- (b) For every sequence $(w_i)_i \in \mathcal{A}_Y$ satisfying $1/K \leq \min_i \|w_i\| \leq \sup_i \|w_i\| \leq K$, there is $(u_i)_i \in \mathcal{A}_Y$ such that for each n ,
 - (b.1) $u_n \in A_n$,
 - (b.2) $\text{supp}_Y(u_n) \subseteq \text{supp}_Y(w_n)$,
 - (b.3) $\|w_n - u_n\| < \delta_n$.

PROPOSITION 3.13. *Let \mathcal{D}_E be a set of blocks for E , and \mathcal{A}_E be an admissible set for E .*

- (i) *Let X be a \mathcal{D}_E -block subspace. If $(u_i)_i \in (\mathcal{D}_X)^\omega$ is such that for every $n \in \mathbb{N}$ the finite sequence $(u_i)_{i=0}^n$ is in $[\mathcal{A}_E]$, then $(u_i)_i \in \mathcal{A}_X$.*
- (ii) *If $X = [x_n]_n$ and $Y = [y_n]_n$ are \mathcal{D}_E -block subspaces such that $(x_n)_n \sim (y_n)_n$, and $T : X \rightarrow Y$ is the linear map such that $\forall n \in \mathbb{N} (T(x_n) = y_n)$, then $T(\mathcal{A}_X) = \mathcal{A}_Y$.*
- (iii) *If X is a \mathcal{D}_E -block subspace, then*

$$[\mathcal{A}_X] = [\mathcal{A}_E] \cap \bigcup_{i \geq 1} X^i.$$

Proof. (i) Let X and $u = (u_i)_i \in (\mathcal{D}_X)^\omega$ be as in the hypothesis. For each $n \in \mathbb{N}$ let $v^n = (v_i^n)_i \in \mathcal{A}_E$ be such that $u_i = v_i^n$ for every $0 \leq i \leq n$. Without loss of generality, we can suppose each v^n is in \mathcal{A}_X (using Definition 3.6(d)) we can find a sequence in \mathcal{A}_X which coincides with v^n in the first n coordinates). Thus, $v_j^n = u_j$ for every $n \geq j$. This means that for each $j \in \mathbb{N}$ we have $(v_j^n) \rightarrow u_j$ in \mathcal{D}_X as $n \rightarrow \infty$. Therefore, $v^n \rightarrow u$ in $(\mathcal{D}_X)^\omega$. Proposition 3.12(i) yields $u \in \mathcal{A}_X$.

(ii) Let $X = [x_n]_n$ and $Y = [y_n]_n$ be \mathcal{D}_E -block subspaces of E , and let $T : X \rightarrow Y$ be as in the hypothesis. Notice that by Definition 3.6(b), $(x_n)_n, (y_n)_n, (e_n)_n \in \mathcal{A}_E$.

Let $(u_n)_n \in \mathcal{A}_X$ with

$$u_n = \sum_{i \in \text{supp}_X(u_n)} \lambda_i^n x_i$$

for each $n \in \mathbb{N}$. We want to show that $(T(u_n))_n \in \mathcal{A}_Y$. Indeed, $(T(u_n))_n = (u_n)_n *_{\mathcal{D}_X} (y_n)_n$, so by Definition 3.6(c), $(T(u_n))_n \in \mathcal{A}_E \cap Y^\omega = \mathcal{A}_Y$.

On the other hand, let $(v_n)_n \in \mathcal{A}_Y$ with

$$v_n = \sum_{i \in \text{supp}_Y(v_n)} \alpha_i^n y_i$$

for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ set

$$u_n = \sum_{i \in \text{supp}_Y(v_n)} \alpha_i^n x_i.$$

Clearly, $T(u_n) = v_n$ for every n and $(u_n)_n = (v_n)_n *_{\mathcal{D}_Y} (x_n)_n$. By Definition 3.6(c), $(u_n)_n \in \mathcal{A}_E \cap X^\omega = \mathcal{A}_X$.

(iii) Let $X = [x_n]_n$ be a \mathcal{D}_E -block subspace. Since $\mathcal{A}_E \cap X^\omega = \mathcal{A}_X$, it follows that

$$[\mathcal{A}_X] \subseteq [\mathcal{A}_E] \cap \bigcup_{i \geq 1} X^i.$$

Suppose that $(u_i)_{i=0}^n \in [\mathcal{A}_E] \cap X^{n+1}$ for some $n \in \mathbb{N}$. By Definition 3.6(d), there is $(u_i)_{i=n+1}^\infty \in X^\omega$ such that $u = (u_0, \dots, u_n, u_{n+1}, \dots) \in \mathcal{A}_X$. Then $(u_i)_{i=0}^n \in [\mathcal{A}_X]$. ■

3.3. Admissible families

DEFINITION 3.14. We define $\otimes : \mathbb{P}(\omega)^\omega \times \mathbb{P}(\omega)^\omega \rightarrow \mathbb{P}(\omega)^\omega$ as follows: given $U = (U_i)_i$ and $V = (V_i)_i$ in $\mathbb{P}(\omega)^\omega$, we define $U \otimes V = (W_i)_i$ by $W_i = \bigcup_{j \in U_i} V_j$ for every $i \in \mathbb{N}$.

DEFINITION 3.15.

- (i) We denote by bb the set of sequences of successive nonempty finite subsets of \mathbb{N} , that is,

$$bb := \{(U_i)_i \in \text{FIN}^\omega : \forall i \in \mathbb{N} (U_i < U_{i+1})\}.$$

- (ii) We denote by db the set of sequences of nonempty finite subsets of \mathbb{N} whose elements are mutually disjoint:

$$db(\mathbb{N}) := \{(U_i)_i \in \text{FIN}^\omega : \forall i \neq j (U_i \cap U_j = \emptyset)\}.$$

REMARK 3.16. (i) The operation \otimes is internal on each of FIN^ω , bb and db .

- (ii) If $U = (U_i)_i$ and $V = (V_i)_i$ in $\mathbb{P}(\omega)^\omega$ and $U \otimes V = (W_i)_i$, then

$$\bigcup_{i \in \mathbb{N}} W_i \subseteq \bigcup_{i \in \mathbb{N}} V_i.$$

- (iii) $e := (\{i\})_i$ is a neutral element for \otimes , that is, if $U \in \mathbb{P}(\omega)^\omega$, then $U \otimes e = e \otimes U = U$.

We shall consider FIN^ω as a topological subspace of $(2^\omega)^\omega$, where $(2^\omega)^\omega$ is endowed with the product topology which results from considering 2^ω as the Cantor space with its topology. The following proposition follows directly from the definition of the operations $*_X$ and \otimes .

PROPOSITION 3.17. *Let \mathcal{D}_E be a set of blocks for E . Let X be a \mathcal{D}_E -block subspace of E . Suppose $(u_n)_n \in (\mathcal{D}_X)^\omega$ and $(v_n)_n \in (\mathcal{D}_E)^\omega$. If $(w_n)_n = (u_n)_n *_X (v_n)_n$, then*

$$(2) \quad (\text{supp}_E(w_n))_n = (\text{supp}_X(u_n))_n \otimes (\text{supp}_E(v_n))_n.$$

Also, if $(v_n)_n$ is a basic sequence, then for each n ,

$$(3) \quad \text{supp}_{[v_i]_i}(w_n) = \text{supp}_X(u_n).$$

DEFINITION 3.18. We say that a nonempty subset $\mathfrak{B} \subseteq \text{FIN}^\omega$ is an *admissible family* if the following conditions are satisfied:

- (a) \mathfrak{B} is a closed subset of FIN^ω .
 (b) $bb \subseteq \mathfrak{B}$.
 (c) For every $(U_i)_i, (V_i)_i \in \mathfrak{B}$ and every $(W_i)_i \in \text{FIN}^\omega$, we have
 (4) $(W_i)_i \otimes (U_i)_i \in \mathfrak{B} \iff (W_i)_i \otimes (V_i)_i \in \mathfrak{B}$.
 (d) For every $(U_i)_i, (V_i)_i \in \mathfrak{B}$ and $n \in \mathbb{N}$, there is a subsequence $(\{t_i\})_i$ of e such that

$$(U_0, U_1, \dots, U_n, W_0, W_1, \dots) \in \mathfrak{B},$$

where $(W_i)_i = (t_i)_i \otimes (V_i)_i$.

REMARK 3.19. (i) If \mathfrak{B} is an admissible set, condition (b) implies that the neutral element e belongs to \mathfrak{B} .

(ii) It is easy to see that condition (c) in Definition 3.18 is equivalent to the following: for every $(V_i)_i \in \mathfrak{B}$ and every $(W_i)_i \in \text{FIN}^\omega$,

$$(5) \quad (W_i)_i \in \mathfrak{B} \iff (W_i)_i \otimes (V_i)_i \in \mathfrak{B}.$$

PROPOSITION 3.20. *The sets FIN^ω , bb and db are admissible families.*

Proof. It is clear that FIN^ω is an admissible set. Both bb and $db(\mathbb{N})$ satisfy condition (a) of Definition 3.18 as a consequence of the topology we are considering on FIN^ω . Conditions (b)–(d) in Definition 3.18 are consequences of the properties of \otimes and the fact that sequences of the type $(\{m+i\})_i$ are in bb , and therefore in $db(\mathbb{N})$. ■

PROPOSITION 3.21. *The set*

per := $\{(U_i)_i \in \text{FIN}^\omega : \exists \pi \text{ a permutation of } \mathbb{N}, \forall i \in \mathbb{N} (U_{\pi(i)} < U_{\pi(i+1)})\}$
is not an admissible family.

Proof. Consider

$$U := (\{0, 1\}, \{2\}, \{3\}, \dots) \quad \text{and} \quad V := (\{0\}, \{2\}, \{1\}, \{3\}, \{4\}, \dots),$$

both in $\text{per}(\mathbb{N})$. Notice that $U = U \otimes e$ and V belong to $\text{per}(\mathbb{N})$, but

$$U \otimes V = (\{0, 2\}, \{1\}, \{3\}, \{4\}, \dots)$$

does not. Thus, $\text{per}(\mathbb{N})$ fails to satisfy condition (c) in Definition 3.18. ■

The next definition establishes that an admissible family determines an admissible set for E .

PROPOSITION 3.22. *Let \mathfrak{B} be an admissible family. Let \mathcal{D}_E be a set of blocks for E . Define*

$$(6) \quad \mathfrak{B}(\mathcal{D}_E) = \{(u_i)_i \in (\mathcal{D}_E)^\omega : (\text{supp}_E(u_i))_i \in \mathfrak{B}\}.$$

Then $\mathfrak{B}(\mathcal{D}_E)$ is an admissible set for E .

Proof. Set $\mathcal{A} := \mathfrak{B}(\mathcal{D}_E)$. Let us check each condition of Definition 3.6.

(a) Suppose $v := (v_i)_i \in \overline{\mathcal{A}} \subseteq (\mathcal{D}_E)^\omega$ and let $(u_i)_i \in \mathcal{A}^\omega$ converge to v . If for each i , $u_i = (u_j^i)_j$, then $u_j^i \rightarrow v_j$ in $(\mathcal{D}_E)^\omega$ as $i \rightarrow \infty$, for every $j \in \mathbb{N}$. Thus, for each $j \in \mathbb{N}$ there is $N_j > 0$ such that $u_j^i = v_j$ (in particular $\text{supp}_E(u_j^i) = \text{supp}_E(v_j)$) for every $i > N_j$. This means that for each $j \in \mathbb{N}$,

$$(7) \quad \text{supp}_E(u_j^i) \xrightarrow{i \rightarrow \infty} \text{supp}_E(v_j) \quad \text{in } \text{FIN}.$$

For each $i \in \mathbb{N}$, $u_i \in \mathcal{A} \Rightarrow U_i := (\text{supp}_E(u_j^i))_j \in \mathfrak{B}$. Note that (7) shows that $(U_i)_i$ converges to $(\text{supp}_E(v_j))_j \in \text{FIN}^\omega$. Since \mathfrak{B} is closed in FIN^ω , we have $(\text{supp}_E(v_j))_j \in \mathfrak{B}$. By the definition of \mathcal{A} , this means that $v \in \mathcal{A}$.

(b) Let $(y_n)_n$ be a sequence of successive blocks, that is, we have $\forall n \in \mathbb{N} (y_n \in \mathcal{D} \text{ \& } y_n < y_{n+1})$. Then $(\text{supp}_E(y_i))_i \in bb(\mathbb{N})$. By Definition 3.18(b), $bb(\mathbb{N}) \subseteq \mathfrak{B}$, so $(y_n)_n \in \mathcal{A}$.

(c) Let $(y_n)_n \in \mathcal{A}$ and $X = [x_n]_n$ be a \mathcal{D}_E -block subspace. Suppose $(u_n)_n \in (\mathcal{D}_X)^\omega$, where for each $n \in \mathbb{N}$,

$$u_n = \sum_{i \in \text{supp}_X(u_n)} \lambda_i^n x_i.$$

We want to see that

$$(8) \quad (u_n)_n \in \mathcal{A} \iff (v_n)_n := (u_n)_n *_X (y_n)_n \in \mathcal{A}.$$

Observe that $(u_n)_n \in (\mathcal{D}_X)^\omega$ and, due to Proposition 3.11(iv), we know that $(u_n)_n *_X (y_n)_n \in (\mathcal{D}_E)^\omega$.

By Proposition 3.17,

$$(9) \quad (\text{supp}_E(v_n))_n = (\text{supp}_X(u_n))_n \circledast (\text{supp}_E(y_n))_n.$$

As a consequence of this equality, the definition of \mathcal{A} and Definition 3.18(c), we obtain

$$\begin{aligned} (u_n)_n \in \mathcal{A} &\iff (\text{supp}_E(u_n))_n \in \mathfrak{B} \\ &\iff (\text{supp}_X(u_n))_n \circledast (\text{supp}_E(x_n))_n \in \mathfrak{B} \\ &\iff (\text{supp}_X(u_n))_n \circledast (\text{supp}_E(y_n))_n \in \mathfrak{B} \\ &\iff (\text{supp}_E(v_n))_n \in \mathfrak{B} \\ &\iff (v_n)_n \in \mathcal{A}. \end{aligned}$$

(d) Let $(y_n)_n$ a \mathcal{D}_E -block sequence and $Y = [y_n]_n$. By using item (b), we have $(\text{supp}_E(y_n))_n \in \mathfrak{B}$. Let $(u_i)_i \in \mathcal{A}$, so $(\text{supp}_E(u_i))_i \in \mathfrak{B}$. By Definition 3.18(d) there is $(\{a_i\})_i \in \text{bb}(\mathbb{N})$ such that

$$(10) \quad (\text{supp}_E(u_0), \text{supp}_E(u_1), \dots, \text{supp}_E(u_n), B_0, B_1, \dots) \in \mathfrak{B},$$

where $(B_i)_i = (\{a_i\})_i \circledast (\text{supp}_E(y_i))_i$. For each $i \in \mathbb{N}$, let $z_i = y_{a_i}$. It is clear that $(z_i)_i \in (\mathcal{D}_Y)^\omega$ and $\text{supp}_E(z_i) = B_i$ for every $i \in \mathbb{N}$. Then, by (10),

$$(u_0, \dots, u_n, z_0, z_1, \dots) \in \mathcal{A}. \blacksquare$$

Under the hypothesis of Proposition 3.22, we shall refer to the resulting set $\mathfrak{B}(\mathcal{D}_E)$ as the admissible set for E determined by the admissible family \mathfrak{B} .

PROPOSITION 3.23. *Let \mathfrak{B} be an admissible family. Let \mathcal{D}_E be a set of blocks for E . Let X be a \mathcal{D}_E -block subspace of E . If $\mathcal{A}_E = \mathfrak{B}(\mathcal{D}_E)$, then $\mathcal{A}_X = \mathfrak{B}(\mathcal{D}_X)$.*

Proof. This follows from the facts that $\mathcal{A}_X = \mathcal{A}_E \cap X^\omega$, $\mathcal{D}_X = \mathcal{D}_E \cap X$, and for every $(u_n)_n \in (\mathcal{D}_X)^\omega$,

$$(11) \quad (\text{supp}_E(u_n))_n \in \mathfrak{B} \iff (\text{supp}_X(u_n))_n \in \mathfrak{B}.$$

And this last fact follows from Proposition 3.17. \blacksquare

From Proposition 3.22 we obtain immediately:

PROPOSITION 3.24. *Let \mathcal{D}_E be a set of blocks for E . The following sets are admissible for E :*

- (i) *The set $(\mathcal{D}_E)^\omega$ of infinite sequences of \mathcal{D}_E -blocks.*
- (ii) *The set $bb(\mathcal{D}_E)$ of \mathcal{D}_E -block sequences of E .*
- (iii) *The set $db(\mathcal{D}_E)$ of infinite sequences of pairwise disjointly supported \mathcal{D}_E -blocks.*

4. Embeddings and minimality. In this section we shall use the previous sets of blocks and admissible sets to code different kinds of embeddings. Doing this we shall be able to associate to each embedding a notion of tightness and of minimality, which in some cases coincide with notions studied previously, for example in [9]. To simplify the notations we shall fix a Banach space E with a normalized basis $(e_n)_n$.

DEFINITION 4.1. Let \mathcal{D}_E be a set of blocks for E and \mathcal{A}_E an admissible set for E . Suppose that X is a \mathcal{D}_E -block subspace. Let Y be a Banach space with a normalized basis $(y_n)_n$ and suppose $K \geq 1$.

- (i) We shall say that Y \mathcal{A}_X -embeds in X with constant K (in symbols $Y \xrightarrow{A}_K X$) if there is some sequence $(u_n)_n \in \mathcal{A}_X$ of blocks such that $(u_n)_n \sim_K (y_n)_n$.
- (ii) We say that Y \mathcal{A}_X -embeds in X (in symbols $Y \xrightarrow{A} X$) if $Y \xrightarrow{A}_K X$ for some constant $K \geq 1$.

A number of natural properties follow directly from Definition 4.1. For example, the definition guarantees that if Y is a \mathcal{D}_X -block subspace of X and $Z \xrightarrow{A} Y$, then $Z \xrightarrow{A} X$ as well.

DEFINITION 4.2. Let E be a Banach space with a normalized basis $(e_n)_n$. Let \mathcal{D}_E be a set of blocks for E , and \mathcal{A}_E an admissible set for E . Suppose that X is a \mathcal{D}_E -block subspace. We say that X is \mathcal{A}_E -minimal if $X \xrightarrow{A} Y$ for every \mathcal{D}_X -block subspace Y .

The following proposition establishes that the property of being \mathcal{A}_E -minimal is hereditary under taking \mathcal{D}_E -subspaces.

PROPOSITION 4.3. *Let E be a Banach space with a normalized basis $(e_n)_n$. Let \mathcal{D}_E be a set of blocks for E , and \mathcal{A}_E be an admissible set for E . Suppose that X is a \mathcal{D}_E -block subspace which is \mathcal{A}_E -minimal. If Y is a \mathcal{D}_X -block subspace of X , then Y is \mathcal{A}_E -minimal.*

Proof. Let $Y = [y_n]_n$ be a \mathcal{D}_X -block subspace of X . Let $Z = [z_n]_n$ be a \mathcal{D}_Y -block subspace of Y (so it is also a \mathcal{D}_X -block subspace of $(x_n)_n$). We want to see that $Y \xrightarrow{A} Z$.

By the \mathcal{A}_E -minimality of X , we have $X \xrightarrow{A} Z$, thus there is $(u_n)_n \in \mathcal{A}_Z \subseteq \mathcal{A}_X$ such that $(x_n)_n \sim (u_n)_n$. By Proposition 3.12(iii) we have

$$(y_n)_n \in \mathcal{A}_X \implies (w_n)_n := (y_n)_n *_X (u_n)_n \in \mathcal{A}_X \cap Z = \mathcal{A}_Z.$$

Thus, $(w_n)_n$ is a block basis of the basic sequence $(u_n)_n$ of \mathcal{D}_Z -blocks (it is not necessarily a block sequence of X because $(u_n)_n$ need not be a block sequence). Since $(u_n)_n \sim (x_n)_n$ and each w_n has the same scalars in its expansion as y_n , we have $(y_n)_n \sim (w_n)_n$. So, $Y \xrightarrow{A} Z$. ■

REMARK 4.4. In the context of Proposition 3.24, for a fixed set \mathcal{D}_E of blocks, we have

$$bb(\mathcal{D}_E)\text{-minimality} \implies db(\mathcal{D}_E)\text{-minimality} \implies (\mathcal{D}_E)^\omega\text{-minimality}.$$

5. Interpretations for the set of blocks. Depending on the set of blocks $\mathcal{D}_E \subseteq \mathbb{D}_E$ we have chosen for the Banach space E , it is possible to give different interpretations for the admissible set considered. In this subsection we shall explore various sets of blocks and analyze the admissible sets obtained in Proposition 3.24 in each context.

5.1. Blocks as nonzero \mathbf{F} -linear combinations. We start with the biggest set of blocks possible. Consider the set of blocks \mathbb{D}_E , that is, the set of all nonzero finitely supported \mathbf{F}_E -linear combinations of the basis $(e_n)_n$. This set of blocks coincides with the blocks used by A. Pelczar [16] and also by V. Ferenczi and Ch. Rosendal [10].

In this context, a \mathbb{D}_E -block sequence is a block basis whose elements are nonzero finitely supported \mathbf{F}_E -linear combinations and

$$bb_{\mathbb{D}}(E) = \{(x_n)_n \in (\mathbb{D}_E)^\omega : \forall n \in \mathbb{N} (x_n < x_{n+1} \text{ \& } \|x_n\| = 1)\}.$$

REMARK 5.1. Any normalized finitely supported basic sequence $(y_n)_n$ in $E = [e_n]_n$ is equivalent to $(z_n)_n \in (\mathbb{D}_E)^\omega$ with $\text{supp}_E(z_n) = \text{supp}_E(y_n)$ for every n . This is a consequence of the density of \mathbb{D}_E in E and the principle of small perturbations.

The proof of the following proposition is an adaptation of the beginning of the proof of [10, Lemma 3.7].

PROPOSITION 5.2. *Suppose that we are considering \mathbb{D}_E the set of blocks for E and that \mathcal{A}_E is the admissible set for E determined by an admissible family \mathfrak{B} , i.e., $\mathcal{A}_E = \mathfrak{B}(\mathbb{D}_E)$. Then $(\mathbb{D}_E, \mathcal{A}_E)$ is an admissible system of blocks for E .*

Proof. Let $X = [x_n]_n$ be a \mathbb{D}_E -block subspace, and $(\delta_n)_n$ with $0 < \delta_n < 1$ and $K \geq 1$. We are going to construct for each $n \in \mathbb{N}$ sets D_n of not necessarily normalized \mathbb{D}_X -blocks with the following properties:

- (1) For each $d \in [\mathbb{N}]^{<\infty}$, there are a finite number of vectors $u \in D_n$ such that $\text{supp}_X(u) = d$.
- (2) If w is a \mathbb{D}_X -block vector with norm in $[1/K, K]$, then there is some $u \in D_n$ with the same support in X as w such that $\|w - u\| < \delta_n$.

Before the proof of the existence of such sets D_n , let us show why this is sufficient: Let $(v_i)_i \in \mathcal{A}_X$ satisfy $1/K \leq \|v_i\| \leq K$ for every $i \in \mathbb{N}$. Since $(v_i)_i \in \mathcal{A}_X$ and \mathcal{A}_E is the admissible set for E determined by an admissible family \mathfrak{B} , it follows that

$$(12) \quad (\text{supp}_X(v_i))_i \in \mathfrak{B}.$$

Using (2), for each i there is $w_i \in D_i$ with $\|w_i - v_i\| < \delta_i$ and $\text{supp}_X(w_i) = \text{supp}_X(v_i)$, so by (12) $(\text{supp}_X(w_i))_i \in \mathfrak{B}$, which means that $(w_i)_i \in \mathcal{A}_X$. Therefore, $(\mathbb{D}_E, \mathcal{A}_E)$ is an admissible system of blocks for E .

Let us prove that such sets D_n exist. Set $n \in \mathbb{N}$. We proceed by induction. If $d \in [\mathbb{N}]^1$, then since the closed K -ball of $[x_i]_{i \in d}$ is totally bounded and \mathbb{D}_E is dense in E , it is possible to find a finite $U_d = \{u_1^d, \dots, u_{m(d)}^d\} \subset \overline{\mathbb{B}}_K([x_i]_{i \in d}) \cap \mathbb{D}_E$ such that if $w \in [x_i]_{i \in d}$ and $1/K \leq \|w\| \leq K$, then there is some $j \leq m(d)$ with $\|w - u_j^d\| < \delta_n$.

Suppose we have found for every $d \in [\mathbb{N}]^{<m}$ such vectors $U_d = \{u_1^d, \dots, u_{m(d)}^d\} \subset \overline{\mathbb{B}}_K([x_i]_{i \in d}) \cap \mathbb{D}_E$ with the desired property. Let $d \in [\mathbb{N}]^m$. Then as the closed K -ball in

$$[x_i]_{i \in d} \setminus \bigcup_{d' \subset d} [x_i]_{i \in d'}$$

is again totally bounded and \mathbb{D}_E is dense in E , there is $U_d = \{u_1^d, \dots, u_{m(d)}^d\} \subset \overline{\mathbb{B}}([x_i]_{i \in d}) \cap \mathbb{D}_E$ such that if $w \in [x_i]_{i \in d}$, $1/K \leq \|w\| \leq K$ and $\text{supp}(w) = d$, then there is some $j \leq m(d)$ such that $\|w - u_j^d\| < \delta_n$. Finally, set

$$D_n = \bigcup_{d \in [\mathbb{N}]^{<\infty}} U_d. \blacksquare$$

As an immediate consequence of Propositions 3.24 and 5.2, we deduce:

COROLLARY 5.3. *The pairs $(\mathbb{D}_E, \mathbb{D}_E^\omega)$, $(\mathbb{D}_E, db(\mathbb{D}_E))$ and $(\mathbb{D}_E, bb(\mathbb{D}_E))$ are admissible systems of blocks for E .*

Notice that it is a fact frequently used, for example in [16, 10], that after a perturbation argument, a $(\mathbb{D}_E)^\omega$ -embedding is “equivalent” to the usual isomorphic embedding, i.e. if $X = [x_n]_n$ is a \mathbb{D}_E -block subspace, then $Y \xrightarrow{A} X \Leftrightarrow Y \hookrightarrow X$, when $\mathcal{A}_E = (\mathbb{D}_E)^\omega$. Furthermore, if $Y \hookrightarrow_K X$ for some $K \geq 1$, then for any $\varepsilon > 0$ we have $Y \xrightarrow{A}_{K+\varepsilon} X$.

As proved in Proposition 3.21, the family *per* is not admissible for FIN^ω . So, Proposition 3.22 cannot be used to determine whether the set of sequences of blocks that are a permutation of a block basis is an admissible

set for the Banach space E . In the next proposition we actually prove that such a set is not admissible for E .

PROPOSITION 5.4. *The set*

$$\text{per}(\mathbb{D}_E) := \{(x_n)_n \in (\mathbb{D}_E)^\omega : (\text{supp}_E(x_n))_n \in \text{per}\}$$

is not admissible for E .

Proof. Let

$$(z_n)_n = (e_0, e_2, e_1, e_3, e_4, \dots) \quad \text{and} \quad (w_n)_n = (e_0 + e_1, e_2, e_3, e_4, \dots)$$

Both $(z_n)_n$ and $(w_n)_n$ are permutations of \mathbb{D}_E -block sequences but $(w_n)_n *_E (z_n)_n = (e_0 + e_2, e_1, e_3, \dots)$ is not. So, condition (c) in Definition 3.6 is not satisfied. ■

5.2. Blocks as vectors of the basis. The smallest set of blocks we can consider is the set for which the blocks are exclusively the vectors of the basis \mathcal{B}_E . Notice that in this case all blocks are normalized. In this context a \mathcal{B}_E -block sequence is a subsequence of the basis, and a sequence of disjointly supported blocks is a sequence of different elements of the basis (not necessarily in increasing order).

PROPOSITION 5.5. *Let \mathfrak{B} be an admissible family. Then $(\mathcal{B}_E, \mathfrak{B}(\mathcal{B}_E))$ is an admissible system of blocks for E .*

Proof. This follows directly from the fact that for each $n \in \mathbb{N}$ only one \mathcal{B}_E -block has support $\{n\}$. In this case, the conditions asked in Definition 3.9 are trivial. What we are saying is that for the case of embedding, minimality or tightness by sequences, it is not necessary to perturb the vectors along the proofs. ■

As a corollary, using the admissible families bb and db (Proposition 3.24), we obtain:

COROLLARY 5.6. *The pairs $(\mathcal{B}_E, bb(\mathcal{B}_E))$ and $(\mathcal{B}_E, db(\mathcal{B}_E))$ are admissible system of blocks for E , corresponding to the admissible sets of subsequences of $(e_n)_n$ and of pairwise distinct elements of $(e_n)_n$, respectively.*

REMARK 5.7. Let \mathcal{D}_E be a set of blocks for the Banach space E , and \mathcal{A}_E be an admissible set determined by an admissible family. Notice that Proposition 5.5 is true in the case where for each $d \in [\mathbb{N}]^{<\omega}$ such that there is $w \in \mathcal{D}_E$ with $\text{supp}_E(w) = d$, the set $\{u \in \mathcal{D}_E : d = \text{supp}_E(u)\}$ is finite. Under this hypothesis a pair $(\mathcal{D}_E, \mathcal{A}_E)$ is an admissible system of blocks for E .

DEFINITION 5.8. Let Y be a Banach space with a normalized basis $(y_n)_n$. We write $(y_n)_n \xrightarrow{s} (e_n)_n$ to denote that $(y_n)_n$ is equivalent to a subsequence of $(e_n)_n$.

Note that $(y_n)_n \xrightarrow{s} (e_n)_n$ if and only if $Y \xrightarrow{A} E$ where the set of blocks is \mathcal{B}_E and $\mathcal{A}_E = bb(\mathcal{B}_E)$ is the admissible set for E . Therefore this definition fits into the general context of our paper.

5.3. Blocks as signed elements of the basis. Additionally, we shall study the case of the set of blocks \mathcal{B}_E^\pm for E , where we recall that $x \in \mathcal{B}_E^\pm$ if and only if $x = \varepsilon e_k$ for some $k \in \mathbb{N}$ and some sign $\varepsilon \in \{-1, 1\}$.

Since for each $n \in \mathbb{N}$ only two vectors e_n and $-e_n$ in \mathcal{B}_E have $\{n\}$ as support, from Remark 5.7 we have immediately:

PROPOSITION 5.9. *Let \mathfrak{B} be an admissible family. Then $(\mathcal{B}_E^\pm, \mathfrak{B}(\mathcal{B}_E^\pm))$ is an admissible system of blocks for E .*

DEFINITION 5.10. We say that $(x_n)_n$ is a *signed subsequence* of $(e_n)_n$ if $(x_n)_n \in bb(\mathcal{B}_E^\pm) := \{(\varepsilon_i e_{n_i})_i : (n_i)_i \in \mathbb{N}^\omega \text{ is increasing \& } (\varepsilon_i)_i \in \{-1, 1\}^\omega\}$. The sequence $(x_n)_n$ is a *signed permutation of a subsequence* of $(e_n)_n$ if $(x_n)_n \in db(\mathcal{B}_E^\pm) := \{(\varepsilon_i e_{n_i})_i : (n_i)_i \in \mathbb{N}^\omega \text{ are mutually distinct \& } (\varepsilon_i)_i \in \{-1, 1\}^\omega\}$.

From Propositions 3.22 and 5.9, we have:

COROLLARY 5.11. *The pairs $(\mathcal{B}_E^\pm, bb(\mathcal{B}_E^\pm))$ and $(\mathcal{B}_E^\pm, db(\mathcal{B}_E^\pm))$ are admissible systems of blocks for E , associated to the admissible sets of signed subsequences of $(e_n)_n$ and signed permutations of subsequences of $(e_n)_n$, respectively.*

6. Summary of types of minimality. We can summarize the interpretation of each embedding as follows: Let Y be a Banach space with normalized basis $(y_n)_n$. Suppose that we are considering the set of blocks \mathcal{D}_E to be \mathcal{B}_E , \mathcal{B}_E^\pm or \mathbb{D}_E , and \mathcal{A}_E the admissible set determined by any of the admissible families FIN^ω , bb or db . To say that $Y \xrightarrow{A} E$ means in each case that the basis $(y_n)_n$ is equivalent to a sequence $(x_n)_n$ in E^ω which satisfies the respective condition we have represented in Table 1.

Notice that since $(y_n)_n$ is a basic sequence, in the trivial cases when the admissible family is FIN^ω , $(x_n)_n$ must also be basic, so in particular $x_n \neq x_m$ for $n \neq m$. For that reason, the first and third rows of the \mathcal{B}_E and \mathcal{B}_E^\pm columns are the same.

We can summarize the notions of \mathcal{A}_E -minimality which follow from each non-trivial \mathcal{A}_E -embedding notion given in Table 1. For this we first give a few simple definitions.

In [8] a basis $(e_n)_n$ for a Banach space E was defined to be *block equivalence minimal* if any block sequence has a further block sequence equivalent to $(e_n)_n$.

Table 1. \mathcal{A} -embeddings for an admissible set determined by an admissible family \mathfrak{B}

$\mathfrak{B} \backslash \mathcal{D}_E$	\mathcal{B}_E	\mathcal{B}_E^\pm	\mathbb{D}_E
FIN^ω	$(x_n)_n$ is a permutation of a subsequence of $(e_n)_n$	$(x_n)_n$ is a permutation of a signed subsequence	$(x_n)_n$ is a sequence of finitely supported vectors of \mathbb{D}_E
bb	$(x_n)_n$ is a subsequence of $(e_n)_n$	$(x_n)_n$ is a signed subsequence of $(e_n)_n$	$(x_n)_n$ is a \mathbb{D}_E -block sequence
db	$(x_n)_n$ is a permutation of a subsequence of $(e_n)_n$	$(x_n)_n$ is a permutation of a signed subsequence	$(x_n)_n$ is a sequence of disjointly finitely supported vectors of \mathbb{D}_E

Recall that two basic sequences $(x_n)_n$ and $(y_n)_n$ are said to be *permutatively equivalent* if $(x_n)_n \sim (y_{\sigma(n)})_n$ for some permutation σ of the integers. Similarly:

DEFINITION 6.1. Let $(x_n)_n$ and $(y_n)_n$ be two basic sequences. We say that $(x_n)_n$ is *signed equivalent* to $(y_n)_n$ if there is some $(\epsilon_n)_n \in \{-1, 1\}^\omega$ such that $(x_n)_n \sim (\epsilon_n y_n)_n$. We say that $(x_n)_n$ is *signed permutatively equivalent* to $(y_n)_n$ if it is permutatively equivalent to $(\epsilon_n y_n)_n$ for some $(\epsilon_n)_n \in \{-1, 1\}^\omega$.

Recall that a basic sequence is *spreading* when it is equivalent to all its subsequences. Similarly:

DEFINITION 6.2. We say that the basic sequence $(e_n)_n$ is *signed* (resp. *permutatively*, *signed permutatively*) *spreading* if $(e_n)_n$ is signed equivalent (resp. permutatively equivalent, signed permutatively equivalent) to all its subsequences.

In [8] it was proved that, as a consequence of the Galvin–Prikry Theorem, if a basis $(e_n)_n$ has the property that every subsequence has a further subsequence equivalent to $(e_n)_n$, then $(e_n)_n$ is spreading. Adapting the proof of this fact, we shall use Silver’s Theorem to prove the natural form of minimalities in the specific case of subsequences. All notions of minimality are summarized in the next proposition.

PROPOSITION 6.3. *Let E be a Banach space with a normalized basis $(e_n)_n$.*

- *Consider the set of blocks \mathcal{B}_E for E , and $X = [x_n]_n$ where $(x_n)_n$ is a subsequence of $(e_n)_n$. We have:*
 - (i) *X is $bb(\mathcal{B}_E)$ -minimal if and only if $(x_n)_n$ is spreading.*
 - (ii) *X is $db(\mathcal{B}_E)$ -minimal if and only if $(x_n)_n$ is permutatively spreading.*
- *Consider the set of blocks \mathcal{B}_E^\pm for E , and $X = [x_n]_n$ where $(x_n)_n$ is a signed subsequence of $(e_n)_n$. We have:*

- (iii) X is $bb(\mathcal{B}_E^\pm)$ -minimal if and only if $(x_n)_n$ is signed spreading.
- (iv) X is $db(\mathcal{B}_E^\pm)$ -minimal if and only if $(x_n)_n$ is signed permutatively spreading.
- Consider the set of blocks \mathbb{D}_E for E , and $X = [x_n]_n$ a \mathbb{D}_E -block subspace of E . We have:
 - (v) X is $bb(\mathbb{D}_E)$ -minimal if and only if $(x_n)_n$ is block equivalence minimal.
 - (vi) X is $db(\mathbb{D}_E)$ -minimal if and only if for every \mathbb{D}_X -block sequence $(y_n)_n$ of $(x_n)_n$ there is a sequence $(z_n)_n$ of disjointly supported blocks of $Y = [y_n]_n$ such that $(z_n)_n \sim (x_n)_n$.
 - (vii) X is $(\mathbb{D}_E)^\omega$ -minimal if and only if X is minimal.

Proof. (i) First, suppose that the set of blocks for E is \mathcal{B}_E , and $X = [x_n]_n$ is a \mathcal{B}_E -block subspace of E , i.e. $(x_n)_n$ is a subsequence of $(e_n)_n$. By Definition 4.2, X is $bb_{\mathcal{B}}(E)$ -minimal if and only if for every subsequence $(y_n)_n$ of $(x_n)_n$, there is a further subsequence $(y_{n_k})_k$ equivalent to $(x_n)_n$, which implies that the sequence $(x_n)_n$ is spreading (see [9]).

Now, consider the set of blocks \mathbb{D}_E for E , and suppose that $X = [x_n]_n$ is a \mathbb{D}_E -block subspace of E .

(vii) As noticed in Section 5.1, $Y \xrightarrow{\mathcal{A}} X \Leftrightarrow Y \hookrightarrow X$ when $\mathcal{A}_E = (\mathbb{D}_E)^\omega$. So, the conclusion is clear.

(v) X is $bb(\mathbb{D}_E)$ -minimal if and only if for every $\mathbb{D}(X)$ -block sequence there is a further $\mathbb{D}(X)$ -block sequence equivalent to $(x_n)_n$. Therefore, $(x_n)_n$ is a block equivalence minimal basis.

(vi) simply follows from Definition 4.2.

It remains to prove (ii)–(iv), which are consequences of the facts that if every subsequence of a basis $(e_n)_n$ admits a subsequence which is (ii) permutatively equivalent, or (iii) signed equivalent, or (iv) signed permutatively equivalent to $(e_n)_n$, then $(e_n)_n$ must be respectively (ii) permutatively equivalent, (iii) signed equivalent, or (iv) signed permutatively equivalent to all its subsequences. We shall only prove (iv), leaving the very similar and easier proofs of (ii) and (iii) as exercises. We follow the proof of a similar lemma from [8], with the difference that we shall use the fact that analytic sets in $[\mathbb{N}]^\omega$ are Ramsey.

Let $\mathcal{C} \subseteq [\mathbb{N}]^\omega$ be such that

$$\{n_k : k \in \mathbb{N}\} \in \mathcal{C} \iff (e_{n_k})_k \text{ is signed permutatively equivalent to } (e_k)_k.$$

Consider the Polish space $[\mathbb{N}]^\omega \times \{-1, 1\}^\omega \times \text{Bij}(\omega)$, where $\{-1, 1\}^\omega$ is equipped with the usual topology, and the Polish topology on the set $\text{Bij}(\omega)$ of bijections of ω is induced by its inclusion in ω^ω . Then \mathcal{C} can be expressed as follows:

$$\begin{aligned}
\mathcal{C} &= \left\{ \{n_k : k \in \mathbb{N}\} \in [\mathbb{N}]^\infty : \exists (\delta_k)_k \in \{-1, 1\}^\omega, \exists \sigma \in \text{Bij}(\omega) \right. \\
&\quad \left. ((\delta_k e_{n_k})_k \sim (e_{\sigma(k)})_k) \right\} \\
&= \left\{ \{n_k : k \in \mathbb{N}\} \in [\mathbb{N}]^\infty : \exists (\delta_k)_k \in \{-1, 1\}^\omega, \exists \sigma \in \text{Bij}(\omega) \right. \\
&\quad \left. (\{n_k : k \in \mathbb{N}\}, (\delta_k)_k, \sigma) \in \mathcal{B} \right\} \\
&= \text{proj}_{[\mathbb{N}]^\infty}(\mathcal{B}),
\end{aligned}$$

where

$$\mathcal{B} = \bigcup_{C \geq 1} \bigcap_{k \geq 0} \left\{ (\{n_i : i \in \mathbb{N}\}, (\delta_i)_i, \sigma) \in [\mathbb{N}]^\infty \times \{-1, 1\}^\omega \times \text{Bij}(\omega) \right. \\
\left. ((\delta_i e_{n_i})_{i=0}^k \sim_C (e_{\sigma(i)})_{i=0}^k) \right\}.$$

Since \mathcal{B} a countable union of countable intersections of open sets in $[\mathbb{N}]^\infty \times \{-1, 1\}^\omega \times \text{Bij}(\omega)$, it is a Borel subset of $[\mathbb{N}]^\infty \times \{-1, 1\}^\omega \times \text{Bij}(\omega)$. Therefore, \mathcal{C} is analytic in $[\mathbb{N}]^\infty$. By Silver's Theorem, there is some $H \in [\mathbb{N}]^\infty$ such that either $[H]^\infty \subseteq \mathcal{C}$ or $[H]^\infty \subseteq [\mathbb{N}]^\infty \setminus \mathcal{C}$.

If $[H]^\infty \subseteq [\mathbb{N}]^\infty \setminus \mathcal{C}$ then the sequence $(e_n)_{n \in H}$ has the property that none of its subsequences is signed permutatively equivalent to $(e_n)_n$, which is a contradiction. On the contrary, if $[H]^\infty \subseteq \mathcal{C}$, then $(e_n)_{n \in H}$ is signed permutatively equivalent to all its subsequences, and so is $(e_n)_n$ because it is signed permutatively equivalent to $(e_n)_{n \in H}$. ■

Let us end this section with a few obvious implications:

$$\begin{array}{ccc}
(e_n)_n \text{ is spreading} & \implies & (e_n)_n \text{ is signed spreading} \\
\Downarrow & & \Downarrow \\
(e_n)_n \text{ is permutatively spreading} & \implies & (e_n)_n \text{ is signed permutatively spreading}
\end{array}$$

and also

$$(e_n)_n \text{ is block equivalence minimal} \implies (e_n)_n \text{ is } db(\mathbb{D}_E)\text{-minimal} \implies E \text{ is minimal.}$$

The canonical basis of c_0 and ℓ_p , with $1 \leq p < \infty$, is, in each case, block equivalence minimal. In [1] it was proved that the canonical basis of the Schlumprecht space \mathcal{S} is block equivalence minimal. In [6] it was observed that \mathbf{T}^* has no “block minimal” block subspaces, and so in particular does not have block equivalence minimal block subspaces. It may actually be seen that \mathbf{T}^* contains no block subspace with (vi). Here is a sketch of proof: \mathbf{T}^* is “strongly asymptotically ℓ_∞ ” (see [4, 5]), which means that n normalized disjointly supported vectors supported far enough on the basis are equivalent to the natural basis of ℓ_∞^n ; on the other hand, a standard diagonalization argument (see e.g. Lemma 8.1) shows that a block subspace with (vi) must have a further block subspace $(x_n)_n$ with the uniform version of (vi), i.e. in any block sequence $(y_n)_n$, the existence of disjointly supported blocks K -equivalent to (x_n) for some fixed K ; the conjunction of the two implies that $(x_n)_n$ must be K -equivalent to the unit basis of c_0 , contradicting the fact that

\mathbf{T}^* does not contain a copy of c_0 . In conclusion, \mathbf{T}^* satisfies (vii) and does not satisfy the minimality conditions of (v) (or (vi)) in Proposition 6.3. We do not know of spaces satisfying the condition of (vi) but not of (v).

7. Results on \mathcal{A} -tightness

7.1. Notions of tightness

DEFINITION 7.1. Let E be a Banach space with a normalized basis $(e_n)_n$. Let \mathcal{D}_E be a set of blocks for E and \mathcal{A}_E an admissible set for E . Suppose that $X = [x_n]_n$ is a \mathcal{D}_E -block subspace. We say that a Banach space Y with Schauder basis is \mathcal{A}_E -tight in the basis $(x_n)_n$ if there is a sequence $(I_i)_i$ of successive intervals such that for every $A \in [\mathbb{N}]^\infty$,

$$(13) \quad Y \xrightarrow{\mathcal{A}} \left[x_n : n \notin \bigcup_{i \in A} I_i \right].$$

DEFINITION 7.2. Let E be a Banach space with a normalized basis $(e_n)_n$. Let \mathcal{D}_E be a set of blocks for E , and \mathcal{A}_E an admissible set for E . Suppose that $X = [x_n]_n$ is a \mathcal{D}_E -block subspace. The basis $(x_n)_n$ is \mathcal{A}_E -tight if every \mathcal{D}_X -block subspace Y of X is \mathcal{A}_E -tight in the basis $(x_n)_n$. The \mathcal{D}_E -block subspace X is \mathcal{A}_E -tight if the basis $(x_n)_n$ is \mathcal{A}_E -tight.

REMARK 7.3. Let E , \mathcal{D}_E and \mathcal{A}_E be as in Definition 7.2. Then E is \mathcal{A}_E -tight if and only if every \mathcal{D}_E -block subspace X is \mathcal{A}_E -tight in $(e_n)_n$.

The following result extends Proposition 3.1 of Ferenczi–Godefroy [7] for the original notion of tightness. To prove it we use [7, Corollary 2.4] where the following characterization of meager and comeager subsets of the Cantor space is given. Let B be a subset of 2^ω closed under supersets.

- (i) B is meager if and only if there exists a sequence $(I_i)_i$ of successive intervals in \mathbb{N} such that

$$\mathbf{u} \in B \implies \{n \in \omega : \text{supp}(\mathbf{u}) \cap I_n = \emptyset\} \text{ is finite.}$$

- (ii) B is comeager if and only if there exists a sequence $(I_i)_i$ of successive intervals in \mathbb{N} such that

$$\{n \in \omega : I_n \subseteq \text{supp}(\mathbf{u})\} \text{ is infinite} \implies \mathbf{u} \in B.$$

PROPOSITION 7.4. Let E be a Banach space with a normalized basis $(e_n)_n$. Let \mathcal{D}_E be a set of blocks for E , and \mathcal{A}_E an admissible set for E . Suppose that $X = [x_n]_n$ is a \mathcal{D}_E -block subspace and Y is a \mathcal{D}_X -block subspace of X . Then Y is \mathcal{A}_X -tight in the basis $(x_n)_n$ if and only if the set

$$(14) \quad E_{Y,X}^{\mathcal{A}} := \{\mathbf{u} \in 2^\omega : Y \xrightarrow{\mathcal{A}} [x_n : n \in \text{supp}(\mathbf{u})]\}$$

is meager in 2^ω .

Proof. If Y is \mathcal{A}_E -tight in $(x_n)_n$, then there are intervals $I_0 < I_1 < \dots$ such that for any $A \in [\mathbb{N}]^\infty$,

$$(15) \quad Y \xrightarrow{\mathcal{A}} \left[x_n : n \notin \bigcup_{i \in A} I_i \right].$$

Let $\mathbf{u} \in E_{Y,X}^{\mathcal{A}}$ (clearly $\text{supp}(\mathbf{u}) \in [\mathbb{N}]^\infty$) and suppose for contradiction that $A_{\mathbf{u}} = \{i \in \mathbb{N} : I_i \cap \text{supp}(\mathbf{u}) = \emptyset\}$ is infinite. We have

$$\text{supp}(\mathbf{u}) \subseteq \mathbb{N} \setminus \bigcup_{i \in A_{\mathbf{u}}} I_i.$$

By the observation after Definition 4.1, we obtain

$$Y \xrightarrow{\mathcal{A}} [x_n : n \in \text{supp}(\mathbf{u})] \Rightarrow Y \xrightarrow{\mathcal{A}} \left[x_n : n \notin \bigcup_{i \in A_{\mathbf{u}}} I_i \right],$$

contradicting (15). Therefore $A_{\mathbf{u}}$ is finite and, by [7, Corollary 2.4], $E_Y^{\mathcal{A}}$ is meager in 2^ω .

For the converse, suppose that $E_Y^{\mathcal{A}}$ is meager in 2^ω . By [7, Corollary 2.4], there are subsets $I_0 < I_1 < \dots$ such that if $\mathbf{u} \in E_Y^{\mathcal{A}}$, then $\{i \in \mathbb{N} : I_i \cap \text{supp}(\mathbf{u}) = \emptyset\}$ is finite. If there is $A \in [\mathbb{N}]^\infty$ such that $Y \xrightarrow{\mathcal{A}} [x_n : n \notin \bigcup_{i \in A} I_i]$, then take $v = \mathbb{N} \setminus \bigcup_{i \in A} I_i$. Clearly $\chi_v \in E_Y^{\mathcal{A}}$ and $\{i \in \mathbb{N} : I_i \cap v = \emptyset\}$ is infinite, which contradicts $E_Y^{\mathcal{A}}$ being meager in 2^ω . ■

The following lemma uses the same scheme as in [7] to prove that the set $E_Y = \{u \subseteq \omega : Y \hookrightarrow [x_n : n \in u]\}$ is meager or comeager.

LEMMA 7.5. *Let E be a Banach space with a normalized basis $(e_n)_n$. Let \mathcal{D}_E be a set of blocks for E , and \mathcal{A}_E an admissible set for E . Suppose that $X = [x_n]_n$ is a \mathcal{D}_E -block subspace and Y is a \mathcal{D}_X -block subspace of X . Then $E_{Y,X}^{\mathcal{A}}$ defined in (14) is either meager or comeager in 2^ω .*

Proof. As stated in [7, Example 2.2], the relation E'_0 defined on $\mathbb{P}(\omega)$ by $uE'_0v \iff \exists n \geq 0 ((u \cap [n, \infty) = v \cap [n, \infty)) \ \& \ (|u \cap [0, n-1]| = |v \cap [0, n-1]|))$ is an equivalence relation and its equivalence classes are the orbits of the group G'_0 of permutations of \mathbb{N} with finite support. Once we see $\mathbb{P}(\omega)$ as the Cantor space, it is Polish and clearly for any nonempty open subsets U and V of $\mathbb{P}(\omega)$, there is $g \in G'_0$ such that $g(U) \cap V \neq \emptyset$. We want to use the first topological 0-1 law (Theorem 2.2) to conclude that $E_{Y,X}^{\mathcal{A}}$ is meager or comeager in 2^ω , or more specifically we have to prove:

- (i) $E_{Y,X}^{\mathcal{A}}$ has the Baire property.
- (ii) $E_{Y,X}^{\mathcal{A}}$ is G'_0 -invariant.

To prove (i) we shall see that $E_{Y,X}^{\mathcal{A}}$ is an analytic subset of 2^ω (see [13, Theorem 21.6]). Notice that we can write $E_{Y,X}^{\mathcal{A}}$ as the projection on the first

coordinate of the set $B := \bigcup_{k \in \omega} B_k$, where for each $k \in \omega$,

$$B_k := \left\{ (u, (w_n)_n) \in 2^\omega \times (\mathcal{D}_X)^\omega : \right. \\ \left. (y_n)_n \sim_k (w_n)_n \ \& \ (w_n)_n \in [x_i : i \in u] \ \& \ (w_n)_n \in \mathcal{A}_X \right\}.$$

Each B_k is Borel in $2^\omega \times (\mathcal{D}_X)^\omega$ since the relation of two sequences being equivalent is closed and \mathcal{A}_X is a closed subset of $(\mathcal{D}_X)^\omega$.

In order to prove (ii) we shall see that $E_{Y,X}^A$ is E'_0 -saturated (this is sufficient because the orbits of the group G'_0 coincide with the equivalence classes of the relation E'_0), that is,

$$E_{Y,X}^A = (E_{Y,X}^A)^{E'_0} := \left\{ v \subseteq \omega : \exists u \in E_{Y,X}^A \ (u E'_0 v) \right\}.$$

Clearly, $E_{Y,X}^A \subseteq (E_{Y,X}^A)^{E'_0}$. Take $v \in E_{Y,X}^A$ and let $u \in E_{Y,X}^A$ be such that $u E'_0 v$. Notice that there is M such that u and v only differ on M elements and $(x_n)_{n \in u}$ and $(x_n)_{n \in v}$ are \mathcal{D}_X -block sequences. So, by Proposition 2.1, there is $K \geq 1$ such that $(x_n)_{n \in u} \sim_K (x_n)_{n \in v}$. Let T be a K -isomorphism from $X_u := [x_n]_{n \in u}$ to $X_v := [x_n]_{n \in v}$. From Proposition 3.13(ii) we know that $T[\mathcal{A}_{X_u}] = \mathcal{A}_{X_v}$. Therefore,

$$\begin{aligned} u \in E_{Y,X}^A &\implies \exists (z_n)_n \in \mathcal{A}_{X_u} ((y_n)_n \sim (z_n)_n) \\ &\implies (y_n)_n \sim (T(z_n))_n \text{ and } (T(z_n))_n \in \mathcal{A}_{X_v} \\ &\implies v \in E_{Y,X}^A. \blacksquare \end{aligned}$$

PROPOSITION 7.6. *Let E be a Banach space with a normalized basis $(e_n)_n$. Let \mathcal{D}_E be a set of blocks for E , and \mathcal{A}_E an admissible set for E . Suppose that $X = [x_n]_n$ is a \mathcal{D}_E -block subspace, Y is a \mathcal{D}_X -block subspace of X , and Z is a \mathcal{D}_Y -block subspace. If Z is \mathcal{A}_E -tight in X , then Z is \mathcal{A}_E -tight in Y .*

Proof. Set

$$E_{Z,X}^A := \{u \subseteq \omega : Z \overset{A}{\hookrightarrow} [x_n : n \in u]\}, \quad E_{Z,Y}^A := \{u \subseteq \omega : Z \overset{A}{\hookrightarrow} [y_n : n \in u]\}.$$

By hypothesis, $E_{Z,X}^A$ is meager in $\mathbb{P}(\omega)$ after identification of $\mathbb{P}(\omega)$ with 2^ω . Using Lemma 7.5, $E_{Z,Y}^A$ is meager or comeager. If it were meager, by Proposition 7.4 the demonstration ends. Suppose that $E_{Z,Y}^A$ is comeager in $\mathbb{P}(\omega)$. By [7, Corollary 2.4], there are sequences $(I_i)_i$ and $(J_i)_i$ of successive intervals such that

$$(16) \quad u \in E_{Z,X}^A \implies \{n \in \omega : u \cap I_n = \emptyset\} \text{ is finite,}$$

$$(17) \quad \{n \in \omega : J_n \subseteq v\} \text{ is infinite} \implies v \in E_{Z,Y}^A.$$

Let $A \in [\mathbb{N}]^\omega$ be such that

$$\left\{ k \in \mathbb{N} : \left(\bigcup_{n \in A} \bigcup_{i \in J_n} \text{supp}_X(y_i) \right) \cap I_k = \emptyset \right\}$$

is infinite. Such an A exists because each I_i and each J_i is finite and each y_i is finitely supported. Let $v = \bigcup_{n \in A} J_n$. Then by (17), we have $v \in E_{Z,Y}^A$. If $u = \bigcup_{k \in v} \text{supp}_X(y_k)$, then

$$Z \xrightarrow{A} [y_n : n \in v] \implies Z \xrightarrow{A} [x_n : n \in u].$$

This implication follows from the observation after Definition 4.1. Therefore, $u \in E_{Z,X}^A$ but it is disjoint from infinitely many intervals I_k , contradicting (16). ■

COROLLARY 7.7. *Let E be a Banach space with a normalized basis $(e_n)_n$. Let \mathcal{D}_E be a set of blocks for E , and \mathcal{A}_E an admissible set for E . Suppose that $X = [x_n]_n$ is an \mathcal{A}_E -tight \mathcal{D}_E -block subspace. Then any \mathcal{D}_E -block sequence $(y_n)_n$ of $(x_n)_n$ is an \mathcal{A} -tight basis.*

Proof. Let Z be a \mathcal{D}_Y -block subspace of Y . Since Z is a \mathcal{D}_X -block subspace of X and Z is \mathcal{A}_E -tight in X , by Proposition 7.6, Z is \mathcal{A}_E -tight in Y . ■

THEOREM 7.8. *Let E be a Banach space with a normalized basis $(e_n)_n$. Let \mathcal{D}_E be a set of blocks for E , and \mathcal{A}_E an admissible set for E . If $X = [x_n]_n$ is an \mathcal{A}_E -tight \mathcal{D}_E -block subspace, then it contains no \mathcal{A}_E -minimal \mathcal{D}_X -block subspaces.*

Proof. Towards a contradiction, suppose $Y = [y_n]_n$ is an \mathcal{A}_E -minimal \mathcal{D}_X -block subspace of X . Let $Z = [z_n]_n$ be a \mathcal{D}_Y -block subspace of Y , so Z is \mathcal{A}_E -tight in X . By Proposition 7.6, Z is \mathcal{A}_E -tight in Y , so

$$E_{Z,Y}^A = \{u \subseteq \omega : Z \xrightarrow{A} [y_n : n \in u]\}$$

must be meager in $\mathbb{P}(\omega)$.

We shall see that $E_{Z,Y}^A$ coincides with the set of all characteristic functions of infinite subsets of \mathbb{N} , which is comeager, leading to a contradiction. Suppose $v \subseteq \omega$ is infinite. Then by the \mathcal{A}_E -minimality of Y ,

$$Y \xrightarrow{A} [y_n : n \in v],$$

so there is $(u_n)_n \in \mathcal{A}_E \cap [y_n : n \in v]$ such that $(y_n)_n \sim (u_n)_n$.

We know that $(y_n)_n, (u_n)_n, (z_n)_n \in \mathcal{A}_Y$ and by Proposition 3.10(iii) we have

$$(z_n)_n *_Y (y_n)_n = (z_n)_n \in \mathcal{A}_Y \implies (w_n)_n := (z_n)_n *_Y (u_n)_n \in \mathcal{A}_Y.$$

Thus, $(w_n)_n$ is a \mathcal{D}_Y -block sequence of the basic sequence $(u_n)_n$ (it is not necessarily a block sequence of X because $(u_n)_n$ is not necessarily a block sequence). Also, each w_n has the same scalars in its expansion as z_n . Since $(u_n)_n \sim (y_n)_n$, we have $(z_n)_n \sim (w_n)_n$ and also we already know that $(w_n)_n \in \mathcal{A}_E \cap [y_n : n \in v]$. So, $Z \xrightarrow{A} [y_n : n \in v]$, which means that $v \in E_{Z,Y}^A$. We have just proved that $[\mathbb{N}]^\infty$ is contained in $E_{Z,Y}^A$, so they coincide. ■

From Definition 7.1 we obtain the following observation.

PROPOSITION 7.9. *Let E be a Banach space with a normalized basis $(e_n)_n$. Consider \mathbb{D}_E as the set of blocks for E and set $\mathcal{A}_E = (\mathbb{D}_E)^\omega$. A \mathbb{D}_E -block basis $(x_n)_n$ is \mathcal{A}_X -tight if and only if $(x_n)_n$ is tight (in the usual sense).*

In particular:

COROLLARY 7.10 ([10, Proposition 3.3]). *If E is a Banach space with a normalized tight basis $(e_n)_n$, then E has no minimal subspaces.*

As an exercise, the reader may write out the other forms of tightness associated to the set \mathcal{D}_E of blocks \mathcal{B}_E , \mathcal{B}_E^\pm or \mathbb{D}_E respectively, and to the choices $bb(\mathcal{D}_E)$, $db(\mathcal{D}_E)$ and \mathcal{D}_E^ω . All these forms of tightness will be made explicit in the final section when we list all dichotomies between minimality and tightness associated to each case.

8. Games for tightness. In this section the objective is to represent forms of tightness in terms of certain infinite games, as in [10]. Let $(x_n)_n$ and $(y_n)_n$ be two sequences of successive and finitely supported vectors of E . Let $Y = [y_n]_n$ and $X = [x_n]_n$. We write $Y \leq^* X$ if there is some $N \geq 1$ such that $y_n \in X$, for every $n \geq N$. First we need two preliminary lemmas. Lemma 8.1 is a modification of [10, Lemma 2.2] and Lemma 8.2 is a modification of [15, Lemma 2.1]. In both original cases the result was proved for usual block subspaces. We extend those results to \mathcal{D}_E -block subspaces.

PROPOSITION 8.1. *Let E be a Banach space and \mathcal{D}_E be a set of blocks for E . Suppose $X = [x_n^0]_n$ is a \mathcal{D}_E -block subspace and $[x_n^1]_n \geq [x_n^2]_n \geq \dots$ is a decreasing sequence of \mathcal{D}_X -block subspaces. Then there exists a \mathcal{D}_X -block sequence $(y_n)_n$ such that $(y_n)_n$ is \sqrt{K} -equivalent to a \mathcal{D}_X -block sequence of $[x_n^K]_n$ for every $K \geq 1$.*

Proof. Let C be the basis constant of $(x_n^0)_n$. For $M > 0$, consider the constant $c(M, C)$ that exists by Proposition 2.1 applied to X .

For each $K \geq 1$, let M_K be the greatest non-negative integer such that

$$(18) \quad c(M_K, C) \leq \sqrt{K}.$$

Using a diagonal argument, we can find an increasing sequence $(l_i)_i$ of natural numbers and a \mathcal{D}_X -block sequence $(y_n)_n$ with the property that for each K there is some $i \leq M_K$ such that $x_{i-1}^K < y_i$ and $(y_m)_{m \geq i}$ is a \mathcal{D}_X -block sequence of $[x_n^K : n \geq i]$. Therefore, $(y_n)_n$ differs in $i - 1$ terms from the block sequence $(x_0^K, x_1^K, \dots, x_{i-1}^K, y_i, y_{i+1}, \dots)$. Therefore, such sequences are $c(M_K, C)$ -equivalent and by (18) they are \sqrt{K} -equivalent. ■

LEMMA 8.2. *Let E be a Banach space and \mathcal{D}_E a set of blocks for E . Suppose that X is a \mathcal{D}_E -block subspace. Let N be a countable set and let $\mu : \text{bb}_{\mathcal{D}}(X) \rightarrow \mathbb{P}(N)$ satisfy one of the following monotonicity conditions:*

$$V \leq^* W \implies \mu(V) \subseteq \mu(W)$$

or

$$V \leq^* W \implies \mu(V) \supseteq \mu(W).$$

Then there exists a “stabilizing” \mathcal{D}_X -block subspace $V_0 \leq E$, i.e. a \mathcal{D}_X -block subspace such that $\mu(V) = \mu(V_0)$ for all $V \leq^ V_0$.*

Proof. If μ is increasing, suppose for contradiction that for every \mathcal{D}_X -block subspace W , there is $V \leq^* W$ such that $\mu(V) \subsetneq \mu(W)$. It is possible to construct a transfinite sequence $(W_\gamma)_{\gamma < \omega_1}$ of \mathcal{D}_X -block subspaces such that if $\gamma < \eta < \omega_1$, then $W_\eta \leq^* W_\gamma$ and $\mu(W_\eta) \subsetneq \mu(W_\gamma)$.

The sequence $(\mu(W_\eta))_{\eta < \omega_1}$ obtained is an uncountable strongly decreasing chain (with respect to inclusion) of subsets of N , which contradicts N being countable. If μ is decreasing, the result follows analogously. ■

We now define asymptotic games in same vein as in [10], with a careful choice of the sets of blocks in which the players are allowed to choose their moves.

DEFINITION 8.3. Let E be a Banach space with a normalized basis $(e_n)_n$, \mathcal{D}_E be a set of blocks for E , and \mathcal{A}_E be an admissible set for E . Let $X = [x_n]_n$ be a \mathcal{D}_E -block subspace, and let Y be a Banach space with a normalized basis $(y_n)_n$. Suppose $C \geq 1$. We define the asymptotic game $H_{Y,X}^A$ with constant C between players I and II taking turns as follows: I plays a natural number n_i , and II plays a natural number m_i and a not necessarily normalized \mathcal{D}_X -block vector $u_i \in X[n_0, m_0] + \dots + X[n_i, m_i]$, where $X[k, m] := [x_n : k \leq n \leq m] \cap \mathcal{D}_X$ for $k \leq m$ natural numbers. Diagrammatically,

$$\begin{array}{llll} \text{I} & n_0 & n_1 & \dots \\ \text{II} & m_0, u_0 & m_1, u_1 & \dots \end{array}$$

The sequence $(u_n)_n$ is the outcome of the game and we say that II wins the game $H_{Y,X}^A$ with constant C if $(u_n)_n \sim_C (y_n)_n$ and $(u_n)_n \in \mathcal{A}_X$.

The game $H_{Y,X}^A$ with constant C is determined since it is equivalent to a Gale–Stewart game, which is open for player I; we shall say that the game $H_{Y,X}^A$ with constant C is open for player I. Notice that if II has a winning strategy for the game $H_{Y,X}^A$ with constant C , then for any sequence $(I_i)_i$ of successive intervals we have $Y \xrightarrow[A]{C} (X, I_i)$. Therefore, if II has a winning strategy for $H_{Y,X}^A$ with constant C then Y is not \mathcal{A}_E -tight in X .

The following definition is similar to the one used in [10].

DEFINITION 8.4. Let E be a Banach space with a normalized basis $(e_n)_n$, \mathcal{D}_E be a set of blocks, and \mathcal{A}_E an admissible set for E . Let $X = [x_n]_n$ be a \mathcal{D}_E -block subspace, Y be a Banach space and $(I_i)_i$ be a sequence of successive nonempty intervals of natural numbers.

(i) Let K be a positive constant. We write

$$Y \xrightarrow{A}_K (X, I_i)$$

if there is $A \in [\mathbb{N}]^\infty$ containing 0 such that $Y \xrightarrow{A}_K [x_n : n \notin \bigcup_{i \in A} I_i]$.

(ii) We write

$$Y \xrightarrow{A} (X, I_i)$$

if there is $A \in [\mathbb{N}]^\infty$ such that $Y \xrightarrow{A} [x_n : n \notin \bigcup_{i \in A} I_i]$.

REMARK 8.5. Notice that under the hypothesis of Definition 8.4, if there is some $A \in [\mathbb{N}]^\infty$ such that $Y \xrightarrow{A} [x_n : n \notin \bigcup_{i \in A} I_i]$ and $0 \notin A$, then there is some $B \in [\mathbb{N}]^\infty$ containing 0 such that $Y \xrightarrow{A} [x_n : n \notin \bigcup_{i \in B} I_i]$.

In the original paper of Ferenczi–Rosendal, special attention is given to the (Borel, continuous, ...) dependence of the sequence I_j of intervals associated to a subspace Y in the definition of tightness. This has application to classification of the isomorphism relations between subspaces and the so-called “ergodic space” problem [9], as in [10, Theorem 7.3]. In the present paper we are not considering these aspects, which allows us to simplify certain parts of the proof – there is no reference to a Borel or continuous map defining those intervals as in the notion of continuous tightness [10, p. 165]. On the other hand, although the general scheme of the proof is the same, special attention has to be given to the roles of the set of blocks and of the type of embeddings to generalize the tight-minimal dichotomy from [10]. Approximation properties work similarly, but diagonalization properties must be ensured, as well as the topological properties (closed, open) of the outcomes, and this requires a careful definition of the infinite games at hand.

LEMMA 8.6. *Let E be a Banach space with a normalized basis $(e_n)_n$, and $(\mathcal{D}_E, \mathcal{A}_E)$ be an admissible system of blocks for E . Suppose that $X = [x_n]_n$ is a \mathcal{D}_E -block subspace and that K and ε are positive constants such that for every \mathcal{D}_X -block subspace Y of X there is a winning strategy for player I in the game $H_{Y,X}^A$ with constant $K + \varepsilon$. Then for every \mathcal{D}_X -block subspace Y there exists a sequence $(I_j)_j$ of successive intervals such that $Y \xrightarrow{A}_K (X, I_j)$.*

Proof. We divide the proof into six steps:

STEP 1. By hypothesis, for each \mathcal{D}_X -block subspace Y of X there is a winning strategy σ_Y for player I in the game $H_{Y,X}^A$ with constant $K + \varepsilon$.

STEP 2. Let $C \geq 1$ be the basis constant of $(x_n)_n$. Let $\rho = 1 + \frac{\varepsilon}{K}$. Now, let $0 < \theta < 1$ be such that $(1 + \theta)(1 - \theta)^{-1} = \rho$. Take a sequence $\Delta = (\delta_n)_n$ of positive numbers such that $2CK^2 \sum_{n \in \mathbb{N}} \delta_n = \theta$.

Let $(w_n)_n$ be a KC -basic sequence of not necessarily normalized blocks with $1/K \leq \|w_i\| \leq K$ for any $i \in \mathbb{N}$. If $(u_n)_n$ is such that $\forall i \in \mathbb{N}$ ($\|w_i - u_i\| < \delta_i$), then

$$2KC \sum_{n \in \mathbb{N}} \frac{\|w_n - u_n\|}{\|w_n\|} = 2CK^2 \sum_{n \in \mathbb{N}} \delta_n = \theta < 1.$$

Thus, $(u_n)_n \sim_\rho (w_n)_n$.

STEP 3. We shall obtain some collection $\{D_n : n \in \mathbb{N}\}$ of sets of vectors which will be used in Step 4 to assist in the construction of a strategy for player I. Since $(\mathcal{D}_E, \mathcal{A}_E)$ is an admissible system of blocks for E , we infer that for X , the sequence $(\delta_n)_n$ and K , there is a collection $(D_n)_n$ of nonempty sets of vectors of \mathcal{D}_X such that

(C-1) For each n and for each $d \in [\mathbb{N}]^{<\infty}$ such that there is $w \in \mathcal{A}_X$ with $\text{supp}_X(w) = d$, there are a finite number of vectors $u \in D_n$ such that $\text{supp}_X(u) = d$.

(C-2) For every sequence $(w_i)_i \in \mathcal{A}_X$ satisfying $1/K \leq \min_i \|w_i\| \leq \sup_i \|w_i\| \leq K$, for each n there is $u_n \in D_n$ such that

$$(C-2.1) \quad \text{supp}_X(u_n) \subseteq \text{supp}_X(w_n),$$

$$(C-2.2) \quad \|w_n - u_n\| < \delta_n,$$

$$(C-2.3) \quad (u_i)_i \in \mathcal{A}_X.$$

STEP 4. Suppose now that Y is a \mathcal{D}_X -block subspace with normalized \mathcal{D}_X -block basis $(y_n)_n$. Suppose that $p = (n_0, u_0, m_0, \dots, n_i, u_i, m_i)$, with $u_j \in D_j$ for $j \leq i$ is a legal position in the game $H_{Y,X}^A$ in which I has played according to σ_Y .

$$\begin{array}{lllll} \text{I} & n_0 & n_1 & \dots & n_i \\ \text{II} & u_0, m_0 & u_1, m_1 & \dots & m_i, u_i \end{array}$$

We write $p < k$ if $n_j, u_j, m_j < k$ for all $j \leq i$. Since II is playing in $\prod_{j \leq i} D_j$, using condition (C-1), for every k there are only a finite number of such legal positions p which satisfy $p < k$. So, for every $k \in \mathbb{N}$ the following maximum exists:

$$(19) \quad \alpha(k) := \max \{k, \max \{\sigma_Y(p) : p < k\}\}.$$

We set $I_k = [k, \alpha(k)]$. The intervals in $(I_k)_k$ are not necessarily disjoint, but it is possible to extract a subsequence of successive intervals, with I_0 as first element.

STEP 5. To prove that $Y \xrightarrow{A}_K (X, I_j)$ we shall show that for every A in $[\mathbb{N}]^\infty$ containing 0, $Y \xrightarrow{A}_K [x_n : n \notin \bigcup_{k \in A} I_k]$.

For contradiction, suppose there is $A \in [\mathbb{N}]^\infty$ containing 0 and a sequence of blocks $(w_n)_n \in \mathcal{A}_X \cap [x_n : n \notin \bigcup_{k \in A} I_k]$ such that

$$(20) \quad (y_n)_n \sim_K (w_n)_n.$$

Recall that since $(y_n)_n$ is normalized, $1/K \leq \|w_n\| \leq K$ for all $n \in \mathbb{N}$.

By Step 3, condition (C-2), we can find for each i a block $u_i \in D_i$ such that $\|w_i - u_i\| < \delta_i$, $\text{supp}_X(u_i) \subseteq \text{supp}_X(w_i)$, $(u_n)_n \in \mathcal{A}_X$, and

$$(21) \quad (u_n)_n \sim_\rho (w_n)_n.$$

By (21), $(u_n)_n \sim_{K\rho} (y_n)_n$. Considering that $\rho = 1 + \varepsilon/K$, we conclude that $(u_n)_n \sim_{K+\varepsilon} (y_n)_n$.

STEP 6. Finally, we will construct a play \vec{p} in the game $H_{Y,X}^A$ with constant $K + \varepsilon$, where player I will follow his winning strategy and the outcome will be the sequence $(u_n)_n$. This means that player I wins the game, leading to a contradiction. In order to do that, we define natural numbers n_i, m_i and $a_i \in A$ as follows:

Let $a_0 = 0$ and $n_0 = \sigma_Y(\emptyset) = \alpha(0)$. Then, by definition of I_k , $I_0 = [0, \alpha(0)] = [0, n_0]$. Find $a_1 \in A$ such that $n_0, u_0, a_0 < a_1$ and set $m_0 = a_1 - 1$. Then $p_0 = (n_0, m_0, u_0)$ is a legal position in $H_{Y,X}^A$ in which I has played according to his winning strategy σ_Y . Since $w_0 \in X[n_0, m_0]$ and $\text{supp}_X(u_0) \subseteq \text{supp}_X(w_0)$, we have $u_0 \in X[n_0, m_0]$.

Now, as $p_0 < a_1$, by the definition of α , if $n_1 = \sigma_Y(n_0, m_0, u_0)$, we obtain $n_1 \leq \alpha(a_1)$. Therefore, $]m_0, n_1[= [m_0 + 1, n_1 - 1] = [a_1, n_1 - 1] \subseteq [a_1, \alpha(a_1)] = I_{a_1}$.

Suppose by induction that $n_0, \dots, n_i, m_0, \dots, m_i$ and $a_0, \dots, a_i \in A$ have been defined. Since $[0, n_0[\subseteq I_0$ and $]m_j, n_j + 1[\subseteq I_{a_{j+1}}$ for all $j < i$, we have

$$u_i \in X[n_0, m_0] + X[n_1, m_1] + \dots + X[n_i, \infty[.$$

Find some $a_{i+1} \in A$ greater than $n_0, \dots, n_i, u_0, \dots, u_i$ and a_0, \dots, a_i and let $m_i = a_{i+1} - 1$. Then

$$u_i \in X[n_0, m_0] + X[n_1, m_1] + \dots + X[n_i, m_i].$$

Therefore $p_i = (n_0, m_0, u_0, \dots, n_i, m_i, u_i)$ is a legal position of the game $H_{Y,X}^A$ with constant $K + \varepsilon$ in which I has played according to σ_Y . Since $p_i < a_{i+1}$, we have

$$n_{i+1} = \sigma_Y(n_0, m_0, u_0, \dots, n_i, m_i, u_i) \leq \alpha(a_{i+1})$$

and

$$]m_i, n_{i+1}[= [m_i + 1, n_{i+1} - 1] = [a_{i+1}, n_{i+1} - 1] \subseteq [a_{i+1}, \alpha(a_{i+1})] = I_{a_{i+1}}.$$

Let \vec{p} be the legal run such that each p_i is a legal position for the game. Such a \vec{p} is the run we were looking for to produce a contradiction. ■

The following technical lemma gives us a criterion for passing from the existence of intervals depending on K for which Y is not \mathcal{A} -embedded in $(I_j^{(K)})$ with constant K , to the existence of intervals $(J_j)_j$ for which Y is not embedded for any constant K . It is similar to [10, Lemma 3.8].

LEMMA 8.7. *Let E be a Banach space with a normalized basis $(e_n)_n$, and $(\mathcal{D}_E, \mathcal{A}_E)$ be an admissible system of blocks for E . Suppose that $X = [x_n]_n$ is a \mathcal{D}_E -block subspace and Y is a Banach space with a normalized basis $(y_n)_n$. If for every constant K there are successive intervals $(I_n^{(K)})$ of natural numbers such that $Y \xrightarrow{\mathcal{A}}_K (X, I_j^{(K)})$, then there is a sequence $(J_j)_j$ of successive intervals such that $Y \xrightarrow{\mathcal{A}} (X, J_j)$.*

Proof. We will construct the intervals $(J_j)_j$ inductively. The idea is to find such a sequence satisfying the following:

- (i) For each $n \geq 0$, J_n contains one interval of each $(I_i^{(n)})_i$.
- (ii) For each $n \geq 1$, if $M = \min J_n - 1$ and $K = \lceil n \cdot c(M) \rceil$ (where $c(M)$ is the constant guaranteed by Proposition 2.1 for $(x_n)_n$), then $\max J_n > \max I_0^{(K)} + M$.

This can be done as follows: Take $J_0 = I_0^{(1)}$. Now suppose that we have defined J_0, \dots, J_n satisfying (i) and (ii). Let a be a natural number greater than $\max J_n$, put $M = a - 1$ and $K = \lceil (n + 1) \cdot c(M) \rceil$. Take $b > \max I_0^{(K)} + M$ such that there exists $j_i \in \mathbb{N}$ with $I_{j(i)}^{(i)} \subseteq [a, b]$ for all $i \in \{1, \dots, n + 1\}$ (this can be done because the intervals are finite and we are looking at just the first $n + 1$ sequences). Let $J_{n+1} := [a, b]$. By construction, J_{n+1} satisfies conditions (i) and (ii).

For contradiction, suppose that $A \in [\mathbb{N}]^\infty$ and for some integer N , we have

$$Y \xrightarrow{\mathcal{A}}_N \left[x_n : n \notin \bigcup_{i \in A} J_i \right].$$

This implies that there is a sequence $(w_n)_n$ of \mathcal{D}_X -blocks in $\mathcal{A}_X \cap [x_n : n \notin \bigcup_{i \in A} J_i]$ such that $(y_n)_n \sim_N (w_n)_n$. Pick $a \in A$ such that $a \geq N$ and set $M = \min J_a - 1$ and $K = \lceil a \cdot c(M) \rceil$. Define an isomorphic embedding T from

$$\left[x_n : n \notin \bigcup_{i \in A} J_i \right]$$

into

$$[x_n : \max I_0^{(K)} < n \leq \max J_a] + \left[x_n : n \notin \bigcup_{i \in A} J_i \text{ \& } n > \max J_a \right]$$

by setting

$$(22) \quad T(x_n) = \begin{cases} x_n & \text{if } n > \max J_a, \\ x_{\max I_0^{(K)} + n + 1} & \text{if } n \leq M. \end{cases}$$

Notice that T is an isomorphism between those two \mathcal{D}_X -block subspaces. So, by Proposition 3.13(ii), we have $(T(w_n))_n \in \mathcal{A}_X$.

Since T only changes at most M vectors from $(x_n)_n$, it is a $C(M)$ -embedding. Hence

$$(y_n)_n \sim_N (w_n)_n \sim_{C(M)} (T(w_n))_n,$$

and because $N \cdot c(M) \leq a \cdot c(M) \leq K$, we obtain

$$(23) \quad Y \xrightarrow{A}_K \left[x_n : n \notin \bigcup_{i \in A} J_i \text{ \& } n > \max J_a \right] + [x_n : \max I_0^{(K)} < n \leq \max J_a].$$

Now, since for each $n \geq 1$, J_n contains one interval of each $(I_i^{(n)})_i$, for any $l \in A$ such that $l \geq K$ there is $b(l) \in \mathbb{N}$ such that $I_{b(l)}^{(K)} \subseteq J_l$. Let $B = \{0\} \cup \{b(l) : l \in A, l \geq K\}$. Then

$$\begin{aligned} \text{id} : \left[x_n : n \notin \bigcup_{i \in A} J_i \text{ \& } n > \max J_a \right] + [x_n : \max I_0^{(K)} < n \leq \max J_a] \\ \rightarrow \left[x_n : n \notin \bigcup_{i \in B} I_i^{(K)} \right] \end{aligned}$$

is an isomorphism onto its image and by Proposition 3.13(ii) and (23) we have

$$Y \xrightarrow{A}_K \left[x_n : n \notin \bigcup_{i \in B} I_i^{(K)} \right],$$

which contradicts our initial hypothesis. ■

The next lemma uses a “diagonalization” argument to relate the fact that a space E is saturated with \mathcal{D}_E -block subspaces X such that for every $Y \leq X$, player I has a winning strategy for the game $H_{Y,X}^A$ for any constant K , with the existence of a \mathcal{A}_E -tight \mathcal{D}_E -block subspace X . It is similar to [10, Lemma 3.9], without the study of the Borel dependence of the intervals in the definition of tightness, and on the other hand, with attention to the types of blocks in the construction so that the diagonalization property still holds.

LEMMA 8.8. *Let E be a Banach space with a normalized basis $(e_n)_n$, and $(\mathcal{D}_E, \mathcal{A}_E)$ be an admissible system of blocks for E . Suppose that for every \mathcal{D}_E -block subspace Z and constant K there is a \mathcal{D}_Z -block subspace X such that for every \mathcal{D}_X -block subspace Y , player I has a winning strategy in the game $H_{Y,X}^{\mathcal{A}}$ with constant K . Then there is a \mathcal{D}_E -block subspace X which is \mathcal{A}_E -tight.*

Proof. The idea is to construct inductively a sequence $X_0 \geq X_1 \geq \dots$ of \mathcal{D}_E -block subspaces and corresponding sequences $(I_j^K)_j$ of successive intervals such that for all $V \leq X_K$, $V \xrightarrow{\mathcal{A}}_{K^2} (X_K, I_j^K)$. Once these are constructed, we will use Proposition 8.1 to obtain the desired \mathcal{D}_E -block subspace.

Consider $X_0 = E$ and let $\varepsilon > 0$. Assuming $X_0 \geq X_1 \geq \dots \geq X_n$ have been defined, and applying the hypothesis to X_n , there is a \mathcal{D}_{X_n} -block subspace $X_{n+1} \leq X_n$ such that for every \mathcal{D} -block subspace $Y \leq X_{n+1}$ and for all $\varepsilon > 0$, player I has a winning strategy in the game $H_{Y,X_{n+1}}^{\mathcal{A}}$ with constant $(n+1)^2 + \varepsilon$. By Lemma 8.6, for every $\mathcal{D}_{X_{n+1}}$ -block subspace $V \leq X_{n+1}$, there are intervals I_j for which $V \xrightarrow{\mathcal{A}}_{(n+1)^2} (X_{n+1}, I_j)$.

Applying Lemma 8.1 to the sequence

$$X_0 \geq \dots \geq X_K \geq \dots,$$

we find a \mathcal{D}_E -block subspace $X_\infty = [x_n^\infty]_n \leq X_0 = E$ such that for each $K \geq 1$ there is a \mathcal{D}_{X_K} -block sequence $(z_n^K)_n$ with $Z_K = [z_n^K]_n \leq X_K$ such that

$$(24) \quad (x_n^\infty)_n \sim_{\sqrt{K}} (z_n^K)_n.$$

Let $Y = [y_n]_n \leq X_\infty$ be a \mathcal{D}_E -block subspace of X_∞ . For each $K \geq 1$ there exists a \mathcal{D}_{Z_K} -block subspace $V_K = [v_n^K]_n$ (using the form of the isomorphism given in (24) and Proposition 3.11(ii)) such that

$$(25) \quad (y_n)_n \sim_{\sqrt{K}} (v_n^K)_n,$$

and for such V_K we may by construction find $(I_j^K)_j$ such that

$$(26) \quad V_K \xrightarrow{\mathcal{A}}_{K^2} (X_K, I_j^K).$$

CLAIM. *There are successive intervals $(J_j^K)_j$ such that*

$$(27) \quad V_K \xrightarrow{\mathcal{A}}_{K^2} (Z_K, J_j^K).$$

Proof of the Claim. Let $(n_j)_j$ and $(m_j)_j$ be increasing sequences in \mathbb{N} such that for each $j \in \mathbb{N}$,

- $n_j < m_j < n_{j+1}$,
- there is $k_j > 0$ with

$$\text{supp}_{X_K}(z_{n_j}^K) < I_{k_j}^K < \text{supp}_{X_K}(z_{m_j}^K).$$

Let $J_j^K = [n_j, m_j]$ for each $j \in \mathbb{N}$. Such sequences $(n_j)_j$ and $(m_j)_j$ exist because all I_j^K and $\text{supp}_{X_K}(z_j^K)$ are finite subsets. Notice that for each $A \in [\mathbb{N}]^\infty$ we have

$$(28) \quad \left[z_n^K : n \notin \bigcup_{j \in A} J_j^K \right] \subseteq \left[x_n^K : n \notin \bigcup_{j \in A} I_{k_j}^K \right].$$

Now, suppose that there is $B \in [\mathbb{N}]^\infty$ such that

$$V_K \xrightarrow{A}_{K^2} \left[z_n^K : n \notin \bigcup_{i \in B} J_i^K \right].$$

Then there is $(w_n)_n \in \mathcal{A}_E \cap [z_n^K : n \notin \bigcup_{i \in B} J_i^K]$ such that $(v_n^K)_n \sim_{K^2} (w_n)_n$. By (28), $(w_n)_n \in \mathcal{A}_E \cap [x_n^K : n \notin \bigcup_{j \in B} I_{k_j}^K]$, so

$$V_K \xrightarrow{A}_{K^2} [x_n^K : n \notin \bigcup_{j \in A} I_j^K],$$

where $A = \{k_j : j \in B\}$, which contradicts (26). ■

Now, we will show that $Y \xrightarrow{A}_K (X_\infty, J_j^K)$. Suppose that, on the contrary, $Y \not\xrightarrow{A}_K (X_\infty, J_j^K)$. Then there is $A \in [\mathbb{N}]^\infty$ with $0 \in A$ and a sequence $(w_n)_n \in \mathcal{A}_E \cap [x_n^K : n \notin \bigcup_{i \in A} J_i^K]$ such that

$$(29) \quad (y_n)_n \sim_K (w_n)_n.$$

Recall that Y and each Z_K are \mathcal{D}_E -block subspaces. By the isomorphism in (24) and using Proposition 3.13(ii), we can find $(u_n^K)_n \in Z_K^\omega$ (image of $(w_n)_n$ by that isomorphism) such that $(u_n^K)_n \in \mathcal{A}_E \cap [z_n^K : n \notin \bigcup_{i \in A} J_i^K]$ and

$$(30) \quad (u_n^K)_n \sim_{\sqrt{K}} (w_n)_n.$$

Then, using (25), (29) and (30), we obtain

$$(31) \quad (v_n^K)_n \sim_{\sqrt{K}} (y_n)_n \sim_K (w_n)_n \sim_{\sqrt{K}} (u_n^K)_n.$$

Thus, $(v_n^K)_n \sim_{K^2} (u_n^K)_n$, which means that

$$[v_n^K]_n = V_K \xrightarrow{A}_{K^2} \left[z_n^K : n \notin \bigcup_{i \in A} J_i^K \right].$$

This contradicts (27).

We have proved that, for every $Y \leq X_\infty$ and every $K \geq 1$, there is a sequence $(J_j^K)_j$ of successive intervals such that $Y \xrightarrow{A}_K (X_\infty, J_j^K)$. By Lemma 8.7 there exists a sequence $(L_i^Y)_i$ of successive intervals such that

$$Y \xrightarrow{A} (X_\infty, L_j^Y),$$

which finishes the proof. ■

9. Games for minimality

DEFINITION 9.1. Let E be a Banach space with a normalized basis $(e_n)_n$, \mathcal{D}_E be a set of blocks, and \mathcal{A}_E an admissible set for E . Suppose L and M are \mathcal{D}_E -block subspaces of a Banach space E and $C \geq 1$ a constant. We define the asymptotic game $G_{L,M}^A$ with constant C between players I and II taking turns as follows. In the $(i+1)$ th round, player I chooses a subspace $E_i \subseteq L$, spanned by a finite \mathcal{D}_L -block sequence, a not necessarily normalized \mathcal{D}_L -block $u_i \in E_0 + \dots + E_i$, and a natural number m_i . On the other hand, II plays for the first time an integer n_0 , and in all successive rounds II plays a subspace F_i spanned by a finite \mathcal{D}_M -block sequence, a not necessarily normalized \mathcal{D}_M -block vector $v_i \in F_0 + \dots + F_i$ and an integer n_{i+1} .

For a move to be legal we demand that $n_i \leq E_i$, $m_i \leq F_i$ and that for each play in the game, the chosen vectors u_i and v_i satisfy $(u_0, \dots, u_i) \in [\mathcal{A}_E]$ and $(v_0, \dots, v_i) \in [\mathcal{A}_E]$. We present the following diagram:

$$\begin{array}{llll}
 \text{I} & n_0 \leq E_0 \subseteq L & n_1 \leq E_1 \subseteq L & \dots \\
 & u_0 \in E_0, m_0 & u_1 \in E_0 + E_1, m_1 & \\
 & & (u_0, u_1) \in [\mathcal{A}_E] & \\
 \\
 \text{II} & n_0 & m_0 \leq F_0 \subseteq M & m_1 \leq F_1 \subseteq M \quad \dots \\
 & & v_0 \in F_0, n_1 & v_1 \in F_0 + F_1, n_2 \\
 & & & (v_0, v_1) \in [\mathcal{A}_E]
 \end{array}$$

The sequences $(u_i)_i$ and $(v_i)_i$ are the outcome of the games and we say that II wins the game $G_{L,M}^A$ with constant C if $(u_i)_i \sim_C (v_i)_i$.

In $G_{Y,X}^A$ with constant C , players I and II must choose \mathcal{A}_E -block subspaces and vectors in $[\mathcal{A}_E]$, in contrast to block subspaces and any block vectors as in the game $G_{Y,X}$ with constant C in [10]. Also, in the game $G_{Y,X}^A$ the outcomes $(u_i)_i$ and $(v_i)_i$ belong to \mathcal{A}_E , since for each $n \in \mathbb{N}$, we have $(u_i)_{i \leq n}, (v_i)_{i \leq n} \in [\mathcal{A}_E]$ and \mathcal{A}_E is closed in $(\mathcal{D}_E)^\omega$.

In addition, since the relation of two sequences being equivalent is closed, we know that if \vec{p} is a legal run in the game such that every finite stage of \vec{p} is a finite stage of a run where II wins the game $G_{Y,X}^A$ with constant C , then \vec{p} itself is a run where II wins $G_{Y,X}^A$ with constant C . In this sense we say that the winning condition is closed for player II. The next lemma relates the games $H_{Y,X}^A$ and $G_{Y,X}^A$ with the same constant.

LEMMA 9.2. Let E be a Banach space with a normalized basis $(e_n)_n$, \mathcal{D}_E be a set of blocks, and \mathcal{A}_E an admissible set for E . If X and Y are \mathcal{D}_E -block subspaces of E such that player II has a winning strategy in $H_{Y,X}^A$ with constant C , then II has a winning strategy in $G_{Y,X}^A$ with constant C .

Proof. Suppose that $X = [x_n]_n$ and $Y = [y_n]_n$ are \mathcal{D}_E -block subspaces. We shall exhibit the move of player II after i rounds in the game $G_{Y,X}^A$ with constant C , and we will prove that such moves determine a winning strategy for II in the game $G_{Y,X}^A$ with constant C . For each i (even $i = 0$), suppose player I has played i times, and we have the following stage in the game $G_{Y,X}^A$:

$$\begin{array}{ll}
 \text{I} & 0 \leq E_0 \subseteq Y \quad \dots \quad 0 \leq E_i \subseteq Y \\
 & u_0 \in E_0, m_0 \quad \quad \quad u_i \in E_0 + \dots + E_i, m_i \\
 & \quad \quad \quad (u_0, \dots, u_i) \in [\mathcal{A}_E] \\
 \text{II} & 0 \quad m_0 \leq F_0 \subseteq X \quad \dots \quad m_{i-1} \leq F_{i-1} \subseteq X \\
 & \quad v_0 \in F_0, 0 \quad \quad v_{i-1} \in F_0 + \dots + F_{i-1}, 0 \\
 & \quad \quad \quad (v_0, \dots, v_{i-1}) \in [\mathcal{A}_E]
 \end{array}$$

Notice that without loss of generality we can ask player II to play $n_j = 0$ for all j (which we may do so since then player I has more possibilities to play and makes the game more difficult for II). Let us write each block vector u_j as $\sum_{k=0}^{k_j} \lambda_k^j y_k$ for all $j \leq i$. We can assume that $k_{j-1} < k_j$ for all $j \leq i$.

Consider the following run in the game $H_{Y,X}^A$:

$$\begin{array}{ll}
 \text{I} & m_0 \quad \dots \quad m_0 \quad \dots \quad m_i \quad \dots \quad m_i \quad m_i \quad m_i \quad \dots \\
 \text{II} & p_0, w_0 \quad \dots \quad p_{k_0}, w_{k_0} \quad \dots \quad p_{k_{i-1}+1}, w_{k_{i-1}+1} \quad \dots \quad p_{k_i}, w_{k_i} \quad q_0, w'_0 \quad q_1, w'_1 \quad \dots
 \end{array}$$

where player I consecutively plays m_0 the first $(k_0 + 1)$ -times, then consecutively plays m_j for $(k_j - k_{j-1})$ -times, for any $j \in \{1, \dots, i\}$, and then he plays m_i constantly. Meanwhile, II moves according to her winning strategy for the game $H_{Y,X}^A$ with constant C , which, by using the properties of \mathcal{A} , guarantees that

$$w' := (w_0, \dots, w_{k_i}, w'_0, w'_1, \dots) \in \mathcal{A}_X = \mathcal{A}_E \cap X^\omega.$$

Since $(u_0, \dots, u_i) \in [\mathcal{A}_E]$, by Definition 3.6(d) there is $(t_n)_n \in Y^\omega$ such that $u' = (u_0, \dots, u_i, t_0, t_1, \dots) \in \mathcal{A}_Y = \mathcal{A}_E \cap Y^\omega$. Notice that $u' *_Y (y_n)_n = u' \in \mathcal{A}_E$ and $(y_n)_n \in \text{bb}_{\mathcal{D}}(E) \subseteq \mathcal{A}_E$, thus, using Definition 3.6(c), we have

$$v' := u' *_Y w' \in \mathcal{A}_E \cap X^\omega = \mathcal{A}_X.$$

If $v' = (v'_j)_j$, then it follows from the inductive construction that

- $v'_j = v_j$, for $j < i$,
- $v'_i = \sum_{k=0}^{k_i} \lambda_k^i w_k$,
- $(v'_0, \dots, v'_i) \in [\mathcal{A}_X]$.

Set $v_i := v'_i$ and

$$F_i = X[m_i, \max\{p_{k_{i-1}+1}, \dots, p_{k_i}\}].$$

Therefore, $(v_0, \dots, v_i) \in [\mathcal{A}_X]$ and $v_i \in F_0 + \dots + F_i$, with $m_i \leq F_i \subseteq X$. This means that $(F_i, v_i, 0)$ is a legal position for II to play in the game $G_{Y,X}^{\mathcal{A}}$ with constant C in the $(i+1)$ th round.

Suppose that we have continued with the game, where II has played by using the previous procedure in every round, and we have obtained the outcome: $(u_i)_i$ (played by I) and $(v_i)_i$ (played by II).

Using the closedness condition (i) in Proposition 3.13, $(u_i)_i$ and $(v_i)_i$ are in \mathcal{A}_E (each initial part is in $[\mathcal{A}_E]$). Since $(u_i)_i$ and $(v_i)_i$ are defined with the same coefficients over $(y_i)_i$ and $(w_i)_i$, respectively, we have $(u_i)_i \sim_C (v_i)_i$. Hence, we have showed the moves that II can make in each round to win the game. Consequently, II has a winning strategy in $G_{Y,X}^{\mathcal{A}}$ with constant C . ■

9.1. An auxiliary minimal game

DEFINITION 9.3. Let E be a Banach space with a normalized basis $(e_n)_n$, and \mathcal{D}_E be a set of blocks for E . We denote by \mathcal{F}_E the set of subspaces of E generated by a finite \mathcal{D}_E -block sequence.

DEFINITION 9.4. Let E be a Banach space with a normalized basis $(e_n)_n$, and \mathcal{D}_E be a set of blocks for E . A *state* s is a pair (a, b) with $a, b \in (\mathcal{D}_E \times \mathcal{F}_E)^{<\omega}$ such that if $a = (a_0, A_0, \dots, a_i, A_i)$ and $b = (b_0, B_0, \dots, b_j, B_j)$, then $j = i$ or $j = i - 1$. Denote by \mathbf{S}_E the (countable) set of states.

REMARK 9.5. Let E be a Banach space with a normalized basis $(e_n)_n$, \mathcal{D}_E be a set of blocks, and \mathcal{A}_E an admissible set for E . Take two \mathcal{D}_E -block subspaces M and L and $C \geq 1$. Consider the game $G_{L,M}^{\mathcal{A}}$ with constant C . If we forget the integers m'_i played by I and n_i played by II in such game, then the set \mathbf{S}_E contains the set of possible positions after a finite number of runs.

DEFINITION 9.6. Let E be a Banach space with a normalized basis $(e_n)_n$, \mathcal{D}_E be a set of blocks, and \mathcal{A}_E an admissible set for E . Let M and L be two \mathcal{D}_E -block subspaces and $C \geq 1$. We say that the state $s = ((a_0, A_0, \dots, a_i, A_i), (b_0, B_0, \dots, b_j, B_j)) \in \mathbf{S}_E$ is *valid for the game* $G_{L,M}^{\mathcal{A}}$ *with constant* C if the finite sequences $(a_0, \dots, a_i), (b_0, \dots, b_j)$ are in $[\mathcal{A}_E]$.

DEFINITION 9.7. Let E be a Banach space with a normalized basis $(e_n)_n$, \mathcal{D}_E be a set of blocks, and \mathcal{A}_E an admissible set for E . Let M and L be two \mathcal{D}_E -block subspaces and $C \geq 1$. Consider a valid state $s \in \mathbf{S}_E$ for the game $G_{L,M}^{\mathcal{A}}$ with constant C . We define $G_{L,M}^{\mathcal{A}}(s)$ to be the game $G_{L,M}^{\mathcal{A}}$ with constant C in which the vectors and finite subspaces in the state s have been played in the initial rounds. That is, if $s = (a, b)$ with $a = (a_0, A_0, \dots, a_i, A_i)$ and $b = (b_0, B_0, \dots, b_i, B_i)$ then the game $G_{L,M}^{\mathcal{A}}(s)$ goes as follows:

$$\begin{array}{ll}
\text{I} & n_{i+1} \leq E_{i+1} \subseteq L \quad \dots \\
& u_{i+1} \in A_0 + \dots + A_i + E_{i+1}, m_{i+1} \\
& ((a_0, \dots, a_i, u_{i+1}) \in [\mathcal{A}_E]) \\
\text{II} & n_{i+1} \quad m_{i+1} \leq F_{i+1} \subseteq M \quad \dots \\
& v_{i+1} \in B_0 + \dots + B_i + F_{i+1}, n_{i+1} \\
& ((b_0, \dots, b_i, v_{i+1}) \in [\mathcal{A}_E])
\end{array}$$

The outcome of the game is the pair of infinite sequences $(a_0, \dots, a_i, u_{i+1}, \dots)$ and $(b_0, \dots, b_i, v_{i+1}, \dots)$.

If $s = (a, b)$ with $a = (a_0, A_0, \dots, a_i, A_i)$ and $b = (b_0, B_0, \dots, b_{i-1}, B_{i-1})$ then the game $G_{L,M}^A(s)$ goes as follows:

$$\begin{array}{ll}
\text{I} & m_i \quad n_i \leq E_{i+1} \subseteq L \quad \dots \\
& u_{i+1} \in A_0 + \dots + A_i + E_{i+1}, m_{i+1} \\
& ((a_0, \dots, a_i, u_{i+1}) \in [\mathcal{A}_E]) \\
\text{II} & m_i \leq F_i \subseteq M \quad \dots \\
& v_i \in B_0 + \dots + B_i + F_i, n_i \\
& ((b_0, \dots, b_{i-1}, v_i) \in [\mathcal{A}_E])
\end{array}$$

The outcome of the game is the pair of infinite sequences $(a_0, \dots, a_i, u_{i+1}, \dots)$ and $(b_0, \dots, b_i, v_{i+1}, \dots)$. We say that player II wins the game $G_{L,M}^A(s)$ with constant C if $(a_0, \dots, a_i, u_{i+1}, \dots) \sim_C (b_0, \dots, b_i, v_{i+1}, \dots)$.

10. Tight-minimal dichotomies. Now we are ready to prove our main result, Theorem 1.2.

Proof of Theorem 1.2. We shall prove that if no \mathcal{D}_E -block subspace is \mathcal{A}_E -tight, then there is a \mathcal{D}_E -block subspace which is \mathcal{A}_E -minimal.

If E fails to have an \mathcal{A}_E -tight subspace then by Lemma 8.8 there are a \mathcal{D}_E -block subspace Z of E and a constant $C \geq 1$ such that for every \mathcal{D}_Z -block subspace X of Z there is a further \mathcal{D}_X -block subspace Y of X such that I has no winning strategy for the game $H_{Y,X}^A$ with constant C . If we prove that Z has a \mathcal{A}_E -minimal \mathcal{D}_E -block subspace, the proof will be complete. So, without loss of generality we can suppose that $Z = E$.

Summing up, we are supposing that for every \mathcal{D}_E -block subspace X there is a further \mathcal{D}_X -block subspace $Y \leq X$ such that player I has no winning strategy in the game $H_{Y,X}^A$ with constant C . Since the game $H_{Y,X}^A$ with constant C is determined, we can conclude that for any \mathcal{D}_E -block subspace X , there is a \mathcal{D}_X -block subspace Y such that II has a winning strategy in $H_{Y,X}^A$ with constant C .

Let $\tau : bb_{\mathcal{D}}(E) \rightarrow \mathbb{P}(\mathbf{S})$ be defined by

$$s \in \tau(M) \iff \exists L \text{ } \mathcal{D}_M\text{-block subspace such that player II has a winning strategy in } G_{L,M}^A(s) \text{ with constant } C.$$

First observe that the elements of $\tau(M)$ are valid states for $G_{L,M}^A$ and $\tau(M)$ is nonempty for each \mathcal{D}_E -block subspace $M \leq E$: we already saw that there is a \mathcal{D}_M -block subspace $L \leq M$ such that II has a winning strategy in $H_{L,M}^A$ with constant C , and by Lemma 9.2, II has a winning strategy in $G_{L,M}^A$ with constant C . Then it is possible to define a valid state $s = (a, b)$, with b being chosen following the winning strategy for II, such that player II has a winning strategy in $G_{L,M}^A(s)$ and $s \in \tau(M)$.

Consider now a \mathcal{D}_E -block subspace $M' \leq^* M$ and $s \in \tau(M')$. Then there is a $\mathcal{D}_{M'}$ -block subspace $L' \leq M'$ such that II has a winning strategy in $G_{L',M'}^A(s)$ with constant C . Since player I can always choose finite subspaces E_i in L' inside of M and choose integers n_i large enough to force player II to play in M' and inside of M (the game $G_{L',M'}^A$ is asymptotic, in the sense that it does not depend on the first coordinates), it follows that it is possible to find a \mathcal{D}_M -block subspace $L \leq M$ such that II has a winning strategy in $G_{L,M}^A(s)$ with constant C . Therefore, $s \in \tau(M)$, and we conclude that $\tau(M') \subseteq \tau(M)$.

By Lemma 8.2 there is a \mathcal{D}_E -block subspace $M_0 \leq E$ which is stabilizing for τ , i.e. $\tau(M_0) = \tau(M')$ for all \mathcal{D}_M -block subspaces $M' \leq^* M$.

Define $\rho : bb_{\mathcal{D}}(M_0) \rightarrow \mathbb{P}(\mathbf{S})$ by setting

$$s \in \rho(L) \iff \text{player II has a winning strategy in } G_{L,M_0}^A(s) \text{ with constant } C.$$

Notice that there is a \mathcal{D}_{M_0} -block subspace $L \leq M_0$ such that $\rho(L) \neq \emptyset$ (the same justification was used to show that $\tau(M) \neq \emptyset$ for every \mathcal{D}_E -block subspace $M \leq E$), so ρ is a nontrivial function. As before, let $L' \leq^* L$ be a \mathcal{D}_{M_0} -block subspace and $s \in \rho(L)$. If player II has a winning strategy in $G_{L,M_0}^A(s)$ then, by the asymptoticity of the game (same previous argument for τ), II has a winning strategy in $G_{L',M_0}^A(s)$, so $s \in \rho(L')$. Thus ρ is decreasing. We can apply Lemma 8.2 to ρ , to find a stabilizing \mathcal{D}_{M_0} -block subspace L_0 of M_0 for ρ . Additionally, we obtain

$$(32) \quad \rho(L_0) = \tau(L_0) = \tau(M_0).$$

Let us prove (32). Since $L_0 \leq M_0$ and M_0 stabilizes τ , we have $\tau(M_0) = \tau(L_0)$. If $s \in \rho(L_0)$, then player II has a winning strategy in $G_{L_0,M_0}^A(s)$, which means that $s \in \tau(M_0)$, so $\rho(L_0) \subseteq \tau(M_0)$. If $s \in \tau(M_0) = \tau(L_0)$, then there is some \mathcal{D}_{L_0} -block subspace $L' \leq L_0$ such that II has a winning strategy in $G_{L',L_0}^A(s)$. Since $L_0 \leq M_0$, in particular II has a winning strategy in $G_{L',M_0}^A(s)$ with constant C . Thus, $s \in \rho(L') = \rho(L_0)$ because L_0 is stabilizing for ρ .

CLAIM. For every \mathcal{D}_{L_0} -block subspace M , II has a winning strategy in the game $G_{L_0, M}^A$ with constant C .

Proof of the claim. Fix a \mathcal{D}_{L_0} -block subspace M . The idea of the proof of this claim is to show inductively that for each valid state s from which player II has a winning strategy in $G_{L_0, M}^A(s)$ with constant C , there is another state s' which “extends” it such that player II has a winning strategy in $G_{L_0, M}^A(s')$. Then one uses the fact that the winning condition is closed for player II to justify that II has a winning strategy. This method was used by A. Pelczar [16] and we are using it in the same way that V. Ferenczi and Ch. Rosendal did in [10].

First, let us prove that $(\emptyset, \emptyset) \in \tau(L_0)$. We know that there is a \mathcal{D}_{L_0} -block subspace Y such that II has a winning strategy in H_{Y, L_0}^A with constant C . Lemma 9.2 implies that II has a winning strategy in G_{Y, L_0}^A with constant C , and, by definition of τ , this means that $(\emptyset, \emptyset) \in \tau(L_0)$. Now, we will show that:

- (i) For all valid states for the game $G_{L_0, M}^A(s)$,

$$s = ((u_0, E_0, \dots, u_i, E_i), (v_0, F_0, \dots, v_i, F_i)) \in \tau(L_0),$$

there is an n (which player II can play) such that for any subspace E spanned by a finite \mathcal{D}_{L_0} -block sequence of L_0 with support greater than n , and any $u \in E_0 + \dots + E_i + E$ such that $(u_0, \dots, u_i, u) \in [\mathcal{A}_E]$ (that is, any move that player I could do in his $(i+1)$ th round in $G_{L_0, M}^A(s)$, disregarding the integer m_{i+1}), we have

$$((u_0, E_0, \dots, u_i, E_i, u, E), (v_0, F_0, \dots, v_i, F_i)) \in \tau(L_0).$$

- (ii) For any $((u_0, E_0, \dots, u_{i+1}, E_{i+1}), (v_0, F_0, \dots, v_i, F_i)) \in \tau(L_0)$, and for all m , there is a subspace $F \geq m$ spanned by a finite \mathcal{D}_M -block sequence and $v \in F_0 + \dots + F_i + F$ with $(v_0, \dots, v_i, v) \in [\mathcal{A}_E]$ (which is a legal move that II can play) such that

$$((u_0, E_0, \dots, u_{i+1}, E_{i+1}), (v_0, F_0, \dots, v_i, F_i, v, F)) \in \tau(L_0).$$

This will be the case if both players have played $i+1$ rounds and player I has played in his $(i+1)$ th-move (E_{i+1}, u_{i+1}, m) , and it corresponds to player II making a legal move.

Let us prove statement (i). Suppose that

$$s = ((u_0, E_0, \dots, u_i, E_i), (v_0, F_0, \dots, v_i, F_i)) \in \tau(L_0).$$

By (32), II has a winning strategy in $G_{L_0, M_0}^A(s)$, which means that there is n such that for every subspace $n \leq E \subseteq L_0$ spanned by a finite \mathcal{D}_{L_0} -block sequence and $u \in E_0 + \dots + E_i + E$, II has a winning strategy in $G_{L_0, M_0}^A(s')$, where

$$s' = ((u_0, E_0, \dots, u_i, E_i, u, E), (v_0, F_0, \dots, v_i, F_i)).$$

So, $s' \in \rho(L_0) = \tau(L_0)$.

To prove (ii), suppose

$$((u_0, E_0, \dots, u_{i+1}, E_{i+1}), (v_0, F_0, \dots, v_i, F_i)) \in \tau(L_0)$$

and m is given. Then, as $M \leq L_0 \leq M_0$ and $\tau(M) = \tau(L_0)$, II has a winning strategy in $G_{L,M}^A(s)$ for some \mathcal{D}_M -block subspace $L \leq M$. Thus, there are $F \leq M$ with $m \leq F$ and $v \in F_0 + \dots + F_i + F$ such that II has a winning strategy in $G_{L,M}^A(s')$, where

$$s' = ((u_0, E_0, \dots, u_{i+1}, E_{i+1}), (v_0, F_0, \dots, v_i, F_i, v, F)).$$

So, $s' \in \tau(M) = \tau(L_0)$.

Starting with state $(\emptyset, \emptyset) \in \tau(L_0)$ and following those two steps inductively, we can obtain a sequence $(s_i)_i$ of states such that each $s_i \in \tau(L_0)$ is the initial part of $s_{i+1} \in \tau(L_0)$. We can define a strategy for player II as follows:

Since $(\emptyset, \emptyset) \in \tau(L_0)$, by (i) there is n_0 such that whenever $m_0, E_0 \leq L_0$ and $u_0 \in E_0$ such that $n_0 \leq E_0$ are played by I, we have

$$((u_0, E_0), \emptyset) \in \tau(L_0).$$

Let $\sigma((\emptyset, \emptyset)) = (n_0)$. Using (ii), there are $F_0 \leq M$ and $v_0 \in F_0$ such that

$$((u_0, E_0), (v_0, F_0)) \in \tau(L_0).$$

Again using (i), there is n_1 such that whatever $m_1, E_1 \leq L_0$ and $u_1 \in E_0 + E_1$ such that $n_1 \leq E_1$ are played by I, we have

$$((u_0, E_0, u_1, E_1), (v_0, F_0)) \in \tau(L_0).$$

Let $\sigma((E_0, u_0, m_0)) = (F_0, v_0, n_1)$. Following this process inductively, supposing that player I in the $(k+1)$ th round has played (E_k, u_k, m_k) , using (ii) there are $F_k \leq M$ and $v_k \in F_0 + \dots + F_k$ such that $m_k \leq F_n$ and

$$((u_0, E_0, \dots, u_k, E_k), (v_0, F_0, \dots, v_k, F_k)) \in \tau(L_0).$$

Using (i) there is n_{k+1} such that whatever $m_{k+1}, E_{k+1} \leq L_0$ and $u_{k+1} \in E_0 + \dots + E_{k+1}$ such that $n_{k+1} \leq E_{k+1}$ are played by I, we have

$$((u_0, E_0, \dots, u_{k+1}, E_{k+1}), (v_0, F_0, \dots, v_k, F_k)) \in \tau(L_0).$$

Let $\sigma((E_0, u_0, m_0, \dots, E_k, u_k, m_k)) = (F_k, v_k, n_{k+1})$. Then σ is a strategy for II to play in the game $G_{L_0,M}^A$ with constant C .

Let $\vec{p} = (n_0, E_0, u_0, m_0, F_0, v_0, n_1, \dots)$ be a legal run of $G_{L_0,M}^A$ where II follows the strategy σ . So, every finite stage

$$(n_0, E_0, u_0, m_0, F_0, v_0, n_1, \dots, E_i, u_i, m_i, F_i, v_i, n_{i+1})$$

of \vec{p} determines the state $s_i = ((u_0, E_0, \dots, u_i, E_i), (v_0, F_0, \dots, v_i, F_i)) \in \tau(L_0) = \rho(L_0)$ such that player II has a winning strategy in $G_{L_0,M_0}^A(s_i)$. By construction of σ , II actually plays in $M \leq L_0 \leq M_0$, so for every $i \in \mathbb{N}$, II has a winning strategy in $G_{L_0,M}^A(s_i)$.

Therefore, for every $i \in \mathbb{N}$, p_i is a finite stage of a legal run in $G_{L_0, M}^A$ with constant C where II wins. So, \vec{p} is a run in $G_{L_0, M}^A$ with constant C where II wins. Thus, σ is a winning strategy for II. ■

Returning to the proof of the theorem: for L_0 there is a \mathcal{D}_{L_0} -block subspace $Y = [y_n]_n$ such that II has a winning strategy in H_{Y, L_0}^A with constant C . We finish the proof by showing that for every \mathcal{D}_{L_0} -block subspace $M \leq L_0$, $Y \xrightarrow{A}_{C^2} M$.

Since II has a winning strategy for H_{Y, L_0}^A with constant C , player I can produce in $G_{L_0, M}^A$ a sequence $(u_i)_i \in \mathcal{A}_{L_0}$ such that $(u_i)_i \sim_C (x_i)_i$. That is, in each round of $G_{L_0, M}^A$, player I can choose the pair $(0, u_i)$, where each u_i is obtained by the moves of II in H_{Y, L_0}^A . By the Claim, II has a winning strategy in $G_{L_0, M}^A$ to produce $(v_i)_i \in \mathcal{A}_M$ such that $(u_i)_i \sim_C (v_i)_i$. By transitivity $(x_i)_i \sim_{C^2} (v_i)_i$, therefore $Y \xrightarrow{A}_{C^2} M$, which ends the proof. ■

REMARK 10.1. It is interesting to note that our theorem always provides us with a uniform version of \mathcal{A} -minimality, namely, there is a constant K such that Y \mathcal{A} -embeds with constant K into any \mathcal{D} -block subspace of Y . This fact was well-known for usual minimality, i.e. every minimal space must be K -minimal for some K .

10.1. Corollaries from the \mathcal{A} -tight-minimal dichotomy. As a corollary of Theorem 1.2 we obtain the third dichotomy of Ferenczi–Rosendal:

COROLLARY 10.2 (Third dichotomy, [10]). *Let E be a Banach space with a normalized basis $(e_n)_n$. Then E contains a tight block subspace or a minimal block subspace.*

Proof. In Theorem 1.2 consider the admissible system of blocks $(\mathbb{D}_E, (\mathbb{D}_E)^\omega)$. As already observed in Proposition 7.9, and in Proposition 6.3 for this particular admissible set, we obtain exactly our conclusion. ■

COROLLARY 10.3. *Let E be a Banach space with a normalized basis $(e_n)_n$. Then E contains a block subspace $X = [x_n]_n$ with one of the following properties:*

- (1) *For any $[y_n]_n \leq X$, there is a sequence $(I_n)_n$ of successive intervals in \mathbb{N} such that for any $A \in [\mathbb{N}]^\infty$, $[y_n]_n$ does not embed into $[x_n : n \notin \bigcup_{i \in A} I_i]$ as a block sequence.*
- (2) *$(x_n)_n$ is a block equivalence minimal basis.*

Proof. In Theorem 1.2 consider the admissible system of blocks $(\mathbb{D}_E, bb(\mathbb{D}_E))$ and apply Proposition 6.3(vi). ■

V. Ferenczi and Ch. Rosendal [10] also remarked that the case of block sequences in this theorem implies the main result of A. Pelczar [16] and an extension of it due to Ferenczi [6].

COROLLARY 10.4. *Let E be a Banach space with a normalized basis $(e_n)_n$. Then E contains a block subspace $X = [x_n]_n$ satisfying one of the following properties:*

- (1) *For any block basis $[y_n]_n$ of X , there is a sequence $(I_n)_n$ of successive intervals in \mathbb{N} such that for any $A \in [\mathbb{N}]^\infty$, $[y_n]_n$ does not embed into $[x_n : n \notin \bigcup_{i \in A} I_i]$, as a sequence of disjointly supported vectors.*
- (2) *For any block basis $[y_n]_n$ of X , $(x_n)_n$ is equivalent to a sequence of disjointly supported vectors of $[y_n]_n$.*

Proof. In Theorem 1.2 consider the admissible system of blocks $(\mathbb{D}_E, ds(\mathcal{D}_E))$ and apply Proposition 6.3(vii). ■

Notice that properties (1) and (2) in Corollary 10.3 and in Corollary 10.4 are incompatible (see Theorem 7.8). Corollaries 10.3 and 10.4 are stated as Theorem 3.16 in [10]. In its enunciation is also considered an embedding as a permutation of a block sequence. Nevertheless, as already seen in the proofs of this section, such an embedding corresponds to a nonadmissible set (see Proposition 5.4). So, the proofs we have presented do not work for the case of the embedding as a permutation of a block sequence, and we see no reason to think that the corresponding statement is true.

10.2. Corollaries from the \mathcal{A} -tight-minimal dichotomy: subsequences. We now pass to the case of subsequences, in which we shall see that Ramsey results allow one to reduce the number of relevant dichotomies.

COROLLARY 10.5. *For any basic sequence $(e_n)_n$ in a Banach space, there is $(x_n)_n \preceq (e_n)_n$ with one of the following properties:*

- (i) *For any $(y_n)_n \preceq (x_n)_n$ there is a sequence $(I_n)_n$ of successive intervals such that for every $A \in [\mathbb{N}]^\infty$, $(y_n)_n$ is permutatively equivalent to no subsequence of $(x_n)_n$ with indices in $\mathbb{N} \setminus \bigcup_{i \in A} I_i$.*
- (ii) *$(x_n)_n$ is spreading.*

Proof. In Theorem 1.2 consider the admissible system of blocks $(\mathcal{B}_E, db_{\mathcal{B}}(E))$. The result follows from item Proposition 6.3(ii), with “permutatively spreading” as a result of (ii). Additionally we use the fact that every permutatively spreading basis admits a spreading subsequence. This follows either from the techniques of [3], or from a proof similar to (and simpler than) that of the next corollary (Corollary 10.6). ■

COROLLARY 10.6. *For any basic sequence $(e_n)_n$ in a Banach space, there is a subsequence $(x_n)_n$ of $(e_n)_n$ with one of the following properties:*

- (i) *For any subsequence $(y_n)_n$ of $(x_n)_n$ there is a sequence $(I_n)_n$ of successive intervals such that for every $A \in [\mathbb{N}]^\infty$, $(y_n)_n$ is permutatively signed equivalent to no subsequence of $(x_n)_n$ with indices in $\mathbb{N} \setminus \bigcup_{i \in A} I_i$.*
- (ii) *$(x_n)_n$ is signed equivalent to a spreading sequence.*

Proof. In Theorem 1.2 consider the admissible system of blocks $(\mathcal{B}_E^\pm, db(\mathcal{B}_E^\pm))$. The result follows from Proposition 6.3(iv), with “ $(x_n)_n$ signed permutatively equivalent” as the conclusion of (ii). It remains to check that such an $(x_n)_n$ contains a subsequence which is sign equivalent to a spreading sequence. Let $(f_n)_n$ be a spreading model of $(x_n)_n$ (see for example [2]). From the hypothesis we know that for some constant C , for any n , there is a finite sequence (ε_n^k) of n signs and a linear order \leq_n on the integers such that $(\varepsilon_n^1 x_1, \dots, \varepsilon_n^n x_n) \sim_C (f_1, \dots, f_n)_{\leq_n}$, where the notation $(f_i)_{\leq}$ means that $\text{span}[f_i]$ is equipped with the norm $\|\sum_i \lambda_i f_i\|_{\leq} := \|\sum_i \lambda_i g_i\|$, if g_1, \dots, g_n is the \leq -increasing enumeration of f_1, \dots, f_n .

By compactness we find an infinite sequence $(\varepsilon_n)_n$ of signs and a linear order \leq on the integers such that $(\varepsilon_n x_n) \sim_C (f_n)_{\leq}$. By Ramsey’s theorem for sequences of length 2, we may find an infinite subset N of the integers such that \leq coincides either with the usual order on N , or with the reverse order. In the first case, $(\varepsilon_n x_n)_{n \in N}$ is C -equivalent to the spreading sequence $(f_n)_{n \in N}$ (or equivalently to $(f_n)_n$); in the second case, it is C -equivalent to the basic sequence (g_n) defined by $\|\sum_{i=1}^k \lambda_i g_i\| = \|\sum_{i=1}^k \lambda_{k-i} f_i\|$, which is also spreading. This completes the proof. ■

This last result is an interesting improvement on combinatorial results involving subsequences. Indeed, any basic sequence either contains a signed subsequence which is spreading, or satisfies a very strong form of tightness (involving changes of signs and permutations). On the other hand, the following seems to remain unknown.

QUESTION 10.7. *Let $(x_n)_n$ be a basic sequence such that all subspaces generated by subsequences of $(x_n)_n$ are isomorphic. Must $(x_n)_n$ contain a spreading subsequence?*

Acknowledgements. The first author was supported by the São Paulo Research Foundation (FAPESP), project 2017/18976-5. The second author was supported by the São Paulo Research Foundation (FAPESP), projects 2016/25574-8, 2022/04745-0 and 2023/12916-1, and by the National Council for Scientific and Technological Development (CNPq), grants 303731/2019-2 and 304194/2023-9.

References

- [1] G. Androulakis and Th. Schlumprecht, *The Banach space S is complementably minimal and subsequentially prime*, Studia Math. 156 (2003), 227–242.
- [2] S. Argyros and S. Todorcevic, *Ramsey Methods in Analysis*, Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, 2005.

- [3] J. Bourgain, P. G. Casazza, J. Lindenstrauss, and L. Tzafriri, *Banach spaces with a unique unconditional basis, up to permutation*, Mem. Amer. Math. Soc. 54 (1985), no. 322, iv+111 pp.
- [4] P. G. Casazza and E. Odell, *Tsirelson's space and minimal subspaces*, in: Texas Functional Analysis Seminar (1982-3), Longhorn Notes, Univ. of Texas, 1983, 61–72.
- [5] S. Dilworth, V. Ferenczi, D. Kutzarova, and E. Odell, *On strongly asymptotic ℓ_p spaces and minimality*, J. London Math. Soc. 75 (2007), 409–419.
- [6] V. Ferenczi, *Minimal subspaces and isomorphically homogeneous sequences in a Banach space*, Israel J. Math. 156 (2006), 125–140.
- [7] V. Ferenczi and G. Godefroy, *Tightness of Banach spaces and Baire category*, in: Topics in Functional and Harmonic Analysis, Theta, București, 2013, 43–55.
- [8] V. Ferenczi, A. Pelczar, and Ch. Rosendal, *On a question of Haskell P. Rosenthal concerning a characterization of c_0 and ℓ_p* , Bull. London Math. Soc. 36 (2004), 396–406.
- [9] V. Ferenczi and Ch. Rosendal, *Ergodic Banach spaces*, Adv. Math. 195 (2005), 259–282.
- [10] V. Ferenczi and Ch. Rosendal, *Banach spaces without minimal subspaces*, J. Funct. Anal. 257 (2009), 149–193.
- [11] W. T. Gowers, *A new dichotomy for Banach spaces*, Geom. Funct. Anal. 6 (1996), 1083–1093.
- [12] W. T. Gowers, *An infinite Ramsey theorem and some Banach-space dichotomies*, Ann. of Math. (2) 156 (2002), 797–833.
- [13] A. S. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math. 156, Springer, Berlin, 1995.
- [14] E. Odell and Th. Schlumprecht, *Trees and branches in Banach spaces*, Trans. Amer. Math. Soc. 354 (2002), 4085–4108.
- [15] A. M. Pelczar, *Remarks on Gowers' dichotomy*, in: Recent Progress in Functional Analysis (on the occasion of the 70th birthday of Manuel Valdivia, Valencia, 2000), Elsevier, Amsterdam, 2001, 201–213.
- [16] A. M. Pelczar, *Subsymmetric sequences and minimal spaces*, Proc. Amer. Math. Soc. 131 (2003), 765–771.
- [17] Ch. Rosendal, *Infinite asymptotic games*, Ann. Inst. Fourier (Grenoble) 59 (2009), 1359–1384.

Alejandra C. Cáceres-Rigo
 Departamento de Matemática
 Instituto de Matemática e Estatística
 Universidade de São Paulo
 05508-090 São Paulo SP, Brazil
 E-mail: caceres.rigo@gmail.com

Valentin Ferenczi
 Departamento de Matemática
 Instituto de Matemática e Estatística
 Universidade de São Paulo
 05508-090 São Paulo SP, Brazil
 and
 Équipe d'Analyse Fonctionnelle
 Institut de Mathématiques de Jussieu
 Sorbonne Université - UPMC
 75252 Paris, France
 E-mail: ferenczi@ime.usp.br