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Recursive Estimation for Discrete-Time Markovian Jump Singular Systems with Random State Delays ⋆

Elizandra K. Odorico * **Kaio D. T. Rocha** * **Marco H. Terra** *

* *São Carlos School of Engineering, University of São Paulo, São Carlos, Brazil. (e-mails: elizandra.odorico@usp.br, kaiodouglas@usp.br and terra@sc.usp.br).*

Abstract: This paper deals with the estimation problem for discrete-time Markovian jump singular systems with known random state delays. By application of the lifting method and modeling of time-varying delay as a mode-dependent Markov chain, the system with random delay is converted to an augmented delay-free Markovian jump singular system. A recursive predictive estimator is obtained by means of the weighted least-squares estimation approach. The proposed solution is given in terms of Riccati equations presented in a square matrix framework. The effectiveness of the proposed estimator is demonstrated with a numerical example.

Keywords: Singular systems, time-delay, discrete-time systems, recursive estimation, Markov models, Riccati equations

1. INTRODUCTION

Singular systems are a generalized form of state-space systems widely applied in economic systems, communication networks, chemical processes, biological systems, and many others (Xu and Lam, 2006). On the other hand, when the dynamics of a singular system changes abruptly, Markovian jump singular systems (MJSS) have the advantage of better representing such systems, for instance, in manufacturing systems, networked control systems, robotics, power systems and aircraft control, see, e.g., Wang et al. (2015) and references therein.

Additionally, time-delay systems are present in a vast range of applications, such as networked systems (Lai and Lu, 2013), problems of information transfer delay and risk of data loss (Zhang et al., 2013), the spread of epidemics (Zuo et al., 2015), and in unmanned aerial vehicle systems (Li et al., 2016). In such systems, the future dynamics depends on both current and past values of the state and/or the control input. Therefore, the delay may considerably affect the characteristics of the plant, changing its dynamics and increasing the difficulty in stability analysis.

The class of Markovian jump singular systems with delay (MJSSD) is one that encompasses all three aforementioned types of systems. However, it is possible to transform this kind of system into an augmented delay-free system by using a classic technique called *lifting method* or *augmented approach*, see, e.g., Sun and Chen (2012) and Fridman (2014). Furthermore, the randomly varying aspect of the delay can be modeled as a Markov chain with known transition probability matrix (Ma and Boukas, 2009; Gong and Zeng, 2014), independent from the one that governs the changing system dynamics. Each mode of this chain thus corresponds to an amount of time-delay, which is restricted to a given interval. In this way, the original MJSSD can be treated as a delay-free MJSS.

In the literature, some of the approaches that can be found to design estimators for Markovian jump singular systems with time-varying delay are H_∞ filtering (Ding et al., 2014) and l_2–l_∞ filtering (Wu et al., 2011). In addition, many results on H_∞ filtering for MJSSD stand out in networked systems applications (Lai and Lu, 2014), in which the system mode is transmitted through an unreliable network. These results are established in terms of linear matrix inequalities (LMIs). To the best of the authors knowledge, recursive estimation approaches for MJSSD have not been addressed in the literature.

The aim of this paper is to propose a recursive predictive estimator (RPE) for Markovian jump singular systems with random state delays. Moreover, we consider the more general discrete-time singular system in rectangular form and with correlated state and measurement noises. The methodology consists in converting the original MJSSD to a delay-free augmented MJSS. An RPE is thus deduced for the MJSS, based on Ishihara et al. (2010). The solution is obtained using the weighted least-squares estimation approach, which combines penalty functions and weighted least-squares. The recursiveness of the proposed estimator is established through Riccati equations presented in a square matrix framework, which makes it suitable for use in online applications. The solution resembles the standard Kalman filter for singular systems in its predictor form, in the sense that it extends the one proposed by Nikoukhah et al. (1992).

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Notations: Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^n$ the set of n-dimensional vectors with elements in $\mathbb{R}$ and $\mathbb{R}^{m\times n}$ the set of $m \times n$ real matrices. I_n is the identity matrix of dimensions $n \times n$. For a real matrix P, $P \succ 0$ ($P \succeq 0$) stands for a symmetric positive (semi)definite matrix. The superscript T denotes matrix transposition. $\mathbb{E}\{x\}$ is the expected value of x. The weighted squared Euclidean norm of x is denoted by $\|x\|_P^2 = x^T P x$.

2. PROBLEM FORMULATION

Consider a class of discrete-time Markovian jump singular systems with random state delay described by

$$\begin{aligned} &E_{\theta(k+1),k+1}x_{k+1} = F_{\theta(k),k}x_k + F_{d,\theta(k),k}x_{k-d_k} + G_{\theta(k),k}\nu_k, \\ &y_k = H_{\theta(k),k}x_k + H_{d,\theta(k),k}x_{k-d_k} + K_{\theta(k),k}\nu_k, \; k \in [0,\; N] \\ &x_k = \varphi_0(k), \;\; k \in [-d_{max},\; 0], \end{aligned} \tag{1}$$

with

$$\begin{bmatrix} G_{\theta(k),k} \\ K_{\theta(k),k} \end{bmatrix} = \begin{bmatrix} G_{w,\theta(k),k} & G_{v,\theta(k),k} \\ K_{w,\theta(k),k} & K_{v,\theta(k),k} \end{bmatrix} \text{ and } \nu_k = \begin{bmatrix} w_k \\ v_k \end{bmatrix},$$

in which $x_k \in \mathbb{R}^n$ is the state at time instant k, $x_{k-d_k} \in \mathbb{R}^n$ is the state delayed by d_k samples, $y_k \in \mathbb{R}^r$ is the measured output, and the vectors $w_k \in \mathbb{R}^p$ and $v_k \in \mathbb{R}^q$ are correlated zero-mean Gaussian noise signals with known covariance matrices

$$\mathbb{E}\left\{ \begin{bmatrix} w_k \\ v_k \end{bmatrix} \begin{bmatrix} w_j \\ v_j \end{bmatrix}^T \right\} = \begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} \delta_{kj},$$

where $\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ otherwise, $Q_k \in \mathbb{R}^{p\times p}$, $R_k \in \mathbb{R}^{q\times q}$ and $S_k \in \mathbb{R}^{p\times q}$. The rectangular matrices $E_{\theta(k+1),k+1}$, $F_{\theta(k),k}$, $F_{d,\theta(k),k} \in \mathbb{R}^{m\times n}$, $G_{w,\theta(k),k} \in \mathbb{R}^{m\times p}$, $G_{v,\theta(k),k} \in \mathbb{R}^{m\times q}$, $H_{\theta(k),k}$, $H_{d,\theta(k),k} \in \mathbb{R}^{r\times n}$, $K_{w,\theta(k),k} \in \mathbb{R}^{r\times p}$ and $K_{v,\theta(k),k} \in \mathbb{R}^{r\times q}$ are known and $\varphi_0(k)$ denotes the initial condition for $k = -d_{max}, -d_{max}+1, \ldots, 0$. The delay d_k is assumed to be known and to vary randomly between two bounds, i.e.,

$$0 \le d_{min} \le d_k \le d_{max},$$

with known constant positive integers d_{min} and d_{max} representing the minimum and maximum delay bounds, respectively. In the time-invariant delay case, we have $d_{min} = d_{max}$.

The switching of System (1) operating modes occurs according to an underlying finite state discrete-time Markov chain $\theta(k) \in \mathcal{S} = \{1, \ldots, s\}$ associated with a transition probability matrix $\mathbb{P} = [p_{ij}] \in \mathbb{R}^{s\times s}$, $i, j \in \mathcal{S}$ and entries satisfying the conditions

$$\begin{aligned} &\Pr[\theta(k+1) = j \,|\, \theta(k) = i] = p_{ij}, \; \Pr[\theta(0) = i] = \pi_{i,0}, \\ &\sum_{j=1}^{s} p_{ij} = 1 \text{ and } 0 \le p_{ij} \le 1. \end{aligned} \tag{2}$$

Our goal is to design a state estimator for MJSSD (1), assuming that the output y_k, jump variable $\theta(k)$ and the delay d_k are available at each time instant k. Since the state sequence $\{x_k\}$ associated with MJSSD (1) is not perfectly observed, the problem consists in calculating the optimal predicted state estimate $\hat{x}_{k+1|k}$ for the state x_{k+1} of the system based on the information set $\mathbf{Y}_k$ available up to step k,

$$\mathbf{Y}_k = \{y_0, \ldots, y_k, \theta(0), \ldots, \theta(k), d_0, \ldots, d_k\}.$$

As discussed in Ishihara et al. (2010), a deterministic interpretation of estimation problems is admitted, such that it becomes a data fitting problem in which w_k and v_k are understood as fitting errors. We then associate with MJSSD (1) the N-stage cost function

$$J_N = \|x_0 - \hat{x}_{0|-1}\|^2_{P^{-1}_{0|-1}} + \sum_{k=0}^{N} \|\nu_k\|^2_{\Omega_k^{-1}}, \tag{3}$$

where $\Omega_k := \begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} \succ 0$ is a weighting matrix regarding the fitting errors ν_k, $\hat{x}_{0|-1}$ is an initial estimate of x_0 and $P_{0|-1} \succ 0$ is a weighting matrix related to the initial estimation error $x_0 - \hat{x}_{0|-1}$.

One approach to treat a time-delay system consists in the augmentation of the system state vector by application of the *lifting method* (Sun and Chen, 2012; Fridman, 2014), such that a correspondent delay-free system is obtained. Based on this technique, the MJSSD (1) can therefore be rewritten as the delay-free augmented MJSS

$$\begin{aligned} \mathcal{E}_{\bar{\theta}(k+1),k+1} z_{k+1} &= \mathcal{F}_{\bar{\theta}(k),k} z_k + \mathcal{G}_{\bar{\theta}(k),k}\nu_k, \\ y_k &= \mathcal{H}_{\bar{\theta}(k),k} z_k + \mathcal{K}_{\bar{\theta}(k),k}\nu_k, \; k \in [0,\; N], \end{aligned} \tag{4}$$

which, in turn, has its operating modes governed by a second finite state discrete-time Markov chain $\bar{\theta}(k) \in \mathcal{T} = \{1, \ldots, t\}$, $t = d_{max} - d_{min} + 1$, associated with a transition probability matrix $\bar{\mathbb{P}} = [\bar{p}_{ij}] \in \mathbb{R}^{t\times t}$, $i, j \in \mathcal{T}$ with entries analogously satisfying the conditions established in (2). The augmented system state z_k and initial condition z_0 vectors are given by

$$z_k = \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-d_{max}+1} \\ x_{k-d_{max}} \end{bmatrix} \text{ and } z_0 = \begin{bmatrix} \varphi_0(0) \\ \varphi_0(-1) \\ \vdots \\ \varphi_0(-d_{max}+1) \\ \varphi_0(-d_{max}) \end{bmatrix}.$$

The augmented matrices $\mathcal{E}_{\bar{\theta}(k+1),k+1}$, $\mathcal{F}_{\bar{\theta}(k),k} \in \mathbb{R}^{m_d\times n_d}$, $\mathcal{G}_{\bar{\theta}(k),k} \in \mathbb{R}^{m_d\times(p+q)}$, $\mathcal{H}_{\bar{\theta}(k),k} \in \mathbb{R}^{r\times n_d}$, and $\mathcal{K}_{\bar{\theta}(k),k} \in \mathbb{R}^{r\times(p+q)}$, with $m_d := (m + d_{max}\, n)$ and $n_d := (n + d_{max}\, n)$, are given by

$$\mathcal{E}_{\bar{\theta}(k+1),k+1} = \left[\begin{array}{c|ccc} E_{\theta(k+1),k+1} & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & I_{d_{max}n} & \\ 0 & & & \end{array}\right],$$

$$\mathcal{F}_{\bar{\theta}(k),k} = \left[\begin{array}{ccccc|c} F_{\theta(k),k} & \overbrace{0 \cdots 0}^{\bar{\theta}(k)+d_{min}-2} & F_{d,\theta(k),k} & 0 & \cdots & 0 \\ \hline & & & & & 0 \\ & & I_{d_{max}n} & & & \vdots \\ & & & & & 0 \end{array}\right], \tag{5}$$

$$\mathcal{G}_{\bar{\theta}(k),k} = \begin{bmatrix} G_{\theta(k),k} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \;\; \mathcal{K}_{\bar{\theta}(k),k} = K_{\theta(k),k} \text{ and}$$

$$\mathcal{H}_{\bar{\theta}(k),k} = \left[H_{\theta(k),k} \;\; \overbrace{0 \cdots 0}^{\bar{\theta}(k)+d_{min}-2} \;\; H_{d,\theta(k),k} \;\; 0 \cdots 0\right],$$

where $\bar{\theta}(k) + d_{min} - 2$ represents the number of $(m \times n)$ null matrices between $F_{\theta(k),k}$ and $F_{d,\theta(k),k}$. When this quantity equals -1, then the first block of $\mathcal{F}_{\bar{\theta}(k),k}$ is $F_{\theta(k),k} + F_{d,\theta(k),k}$. An analogous observation is valid for the matrix $\mathcal{H}_{\bar{\theta}(k),k}$. Notice that the switching of original system matrices is still governed by the first Markov chain $\theta(k) \in \mathcal{S}$, whereas the second chain $\bar{\theta}(k) \in \mathcal{T}$ handles the time-delay amount variation by adjusting the structure of the augmented system matrices, considering that both jump variables are known at each instant k.

We now associate an N-stage quadratic cost function to the augmented delay-free MJSS (4), which is the augmented version of the cost function (3) associated with MJSSD (1), as follows:

$$\mathcal{J}_N = \|z_0 - \hat{z}_{0|-1}\|^2_{\mathcal{P}^{-1}_{0|-1}} + \sum_{k=0}^{N} \|\nu_k\|^2_{\Omega_k^{-1}}, \tag{6}$$

in which $\mathcal{P}_{0|-1} \succ 0$ is the weighting matrix of the initial estimation error $z_0 - \hat{z}_{0|-1}$ and $\Omega_k \succ 0$ is the same fitting error weighting matrix defined for (3).

The application of the lifting method allows us to develop an RPE for MJSSD (1) in terms of the equivalent augmented MJSS (4). In order achieve our goal, we consider the following constrained optimization problem in which the cost function (6) is minimized along the trajectory sequence z_k, for $k \in [0,\ N]$,

$$\min_{z_k,\, z_{k+1},\, \nu_k} \mathcal{J}_N$$
$$\text{s.t.} \begin{cases} \mathcal{E}_{\bar{\theta}_{k+1},k+1} z_{k+1} = \mathcal{F}_{\bar{\theta}(k),k} z_k + \mathcal{G}_{\bar{\theta}(k),k}\nu_k, \\ y_k = \mathcal{H}_{\bar{\theta}(k),k} z_k + \mathcal{K}_{\bar{\theta}(k),k}\nu_k,\ k \in [0,\ N]. \end{cases} \tag{7}$$

In the next sections, the solution of problem (7) is presented and the optimality of the RPE obtained is ensured by its equivalence with the standard Kalman filter in predictor form.

3. WEIGHTED LEAST-SQUARES ESTIMATION

This section presents the approach that supports the solution of the proposed optimization problem (7). It consists in solving a constrained weighted least-squares problem by transforming it into an unconstrained problem via the *penalty function method* (Luenberger and Ye, 2008; Bazaraa et al., 2006).

Consider the problem of obtaining an optimal estimate $\hat{z}$ of a state vector $z \in \mathbb{R}^r$ from an observation vector $y \in \mathbb{R}^l$ related by the dynamic system

$$y = Cw + Dz,$$

with known matrices $C \in \mathbb{R}^{l\times p}$ and $D \in \mathbb{R}^{l\times r}$, and a zero-mean Gaussian noise vector $w \in \mathbb{R}^p$ with covariance matrix $\mathbb{E}\{ww^T\} = R \succ 0$. As presented in Nikoukhah et al. (1992), the optimal estimate $\hat{z}$ of z is obtained by solving the following deterministic minimization problem

$$\min_{z,w} F(w) = \|w\|^2_{R^{-1}}$$
$$\text{s.t.} \quad y = Cw + Dz. \tag{8}$$

The constrained optimization problem (8) can be transformed into an equivalent unconstrained problem by application of the penalty function method, such that the constraint is inserted into the objective function via the *penalty parameter* $\mu > 0$, which penalizes violations of this constraint. Thus, for each μ, problem (8) can be treated by the usual unconstrained weighted least-squares problem

$$\min_{x_\mu} \mathcal{F}(x_\mu) = (Ax_\mu - b)^T W_\mu (Ax_\mu - b), \tag{9}$$

where

$$x_\mu = \begin{bmatrix} w \\ z \end{bmatrix},\ A = \begin{bmatrix} I_p & 0 \\ C & D \end{bmatrix},\ b = \begin{bmatrix} 0 \\ y \end{bmatrix} \text{ and } W_\mu = \begin{bmatrix} R^{-1} & 0 \\ 0 & \mu I_l \end{bmatrix}.$$

The problem is then solved iteratively, i.e., for each $\mu > 0$ of a monotonically increasing sequence $\{\mu_k\}$, an optimal solution $\hat{x}_\mu$ to (9) is obtained. According to Luenberger and Ye (2008), the optimal solution of the original constrained problem (8) is attained when $\mu \to +\infty$.

The optimization problem (9) is solved in Kailath et al. (2000) and the solution is rewritten in Ishihara et al. (2010) as a square matrix framework, according to Lemma 1.

Lemma 1. Consider the optimization problems (8) and (9). Suppose that W_μ is positive definite. Then, there exist unique optimal solutions $\hat{x}$ to (9) and $\hat{z}$ to (8), obtained according to:

(i) For each $\mu > 0$, the optimal estimate $\hat{x}_\mu$ and the weighting matrix P_μ of the estimation error $x_\mu - \hat{x}_\mu$ for problem (9) are given by

$$[\hat{x}_\mu\ \ P_\mu] = \begin{bmatrix} 0 \\ I \end{bmatrix}^T \begin{bmatrix} W_\mu^{-1} & A \\ A^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} b & 0 \\ 0 & -I \end{bmatrix}. \tag{10}$$

(ii) If the matrix $[\,C\ D\,]$ has full row rank, then we can consider $\mu \to +\infty$, such that the optimal estimate $\hat{z}$ and the weighting matrix P of the estimation error $z - \hat{z}$ for problem (8) are given by

$$[\hat{z}\ \ P] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_r \end{bmatrix}^T \begin{bmatrix} R^{-1} & 0 & I_p & 0 \\ 0 & 0 & C & D \\ I_p & C^T & 0 & 0 \\ 0 & D^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ y & 0 \\ 0 & 0 \\ 0 & -I_r \end{bmatrix}. \tag{11}$$

4. RECURSIVE PREDICTIVE ESTIMATION FOR MJSSD

At this point, we are ready to find a solution to the proposed optimization problem (7), so that a recursive predictive estimator for MJSSD (1) is obtained. The constrained optimization problem (7) is defined upon an N-stage objective function and a common technique to deal with such functions is *dynamic programming*, more specifically the forward version, which is suitable for state estimation (Larson and Peschon, 1966). Based on this approach, the cost function (6) is divided into several one-step cost functions. Thereby, at each step k we consider the problem

$$\min_{\mathcal{Z}_{k+1},\psi_k} \|\psi_k\|^2_{\mathscr{P}_k^{-1}}$$
$$\text{s.t.} \quad \mathcal{Y}_k = \mathcal{C}_k\psi_k + \mathcal{D}_k\mathcal{Z}_{k+1}, \tag{12}$$

where $\mathcal{Z}_{k+1} = z_{k+1}$,

$$\mathscr{P}_k = \begin{bmatrix} \mathcal{P}_{k|k-1} & 0 \\ 0 & \Omega_k \end{bmatrix},\ \psi_k = \begin{bmatrix} z_k - \hat{z}_{k|k-1} \\ \nu_k \end{bmatrix}, \mathcal{D}_k = \begin{bmatrix} -\mathcal{E}_{\bar{\theta}(k+1),k+1} \\ 0 \end{bmatrix},$$

$$\mathcal{Y}_k = \begin{bmatrix} -\mathcal{F}_{\bar{\theta}(k),k}\hat{z}_{k|k-1} \\ -\mathcal{H}_{\bar{\theta}(k),k}\hat{z}_{k|k-1} + y_k \end{bmatrix} \text{ and } \mathcal{C}_k = \begin{bmatrix} \mathcal{F}_{\bar{\theta}(k),k} & \mathcal{G}_{\bar{\theta}(k),k} \\ \mathcal{H}_{\bar{\theta}(k),k} & \mathcal{K}_{\bar{\theta}(k),k} \end{bmatrix}.$$

The optimization problem (12) is equivalent to the one in (8), therefore a solution can be calculated via the weighted least-squares estimation procedure presented in Lemma 1.

We then perform the following identifications between problems (12) and (8):

$$w \leftarrow \psi_k, \quad z \leftarrow \mathcal{Z}_{k+1}, \quad R \leftarrow \mathcal{P}_k,$$
$$C \leftarrow \mathcal{C}_k, \quad D \leftarrow \mathcal{D}_k \text{ and } y \leftarrow \mathcal{Y}_k.$$

Lemma 2. Consider the optimization problem (12) with $\mu > 0$ fixed. The optimal predicted estimate $\hat{z}_{k+1|k}$ and the weighting matrix $\mathcal{P}_{k+1|k}$ of the estimation error $z_{k+1} - \hat{z}_{k+1|k}$ are given by the square matrix framework (13), for all $k \in [0,\ N]$.

Remark 1. The solution obtained in Lemma 2 is optimal when $\mu \rightarrow \infty$, consequently, $\mu^{-1}I \rightarrow 0$. In this case, it should be remarked that the matrix block $\begin{bmatrix} \mathcal{F}_{\bar{\theta}(k),k} & \mathcal{G}_{\bar{\theta}(k),k} & -\mathcal{E}_{\bar{\theta}(k+1),k+1} \\ \mathcal{H}_{\bar{\theta}(k),k} & \mathcal{K}_{\bar{\theta}(k),k} & 0 \end{bmatrix}$ must have full row rank with $\mathcal{E}_{\bar{\theta}(k+1),k+1}$ full column rank, such that the invertibility of the central matrix in (13) is guaranteed.

Algorithm 1 summarizes the proposed recursive predictive estimator for MJSSD.

Algorithm 1 Recursive Predictive Estimator for MJSSD

Model: Assume MJSSD (1).

Initialization: Set μ, N, $\hat{z}_{0|-1}$ and $\mathcal{P}_{0|-1} \succ 0$.

For each $k = 0, \ldots, N$:
- Define matrices $\mathcal{E}_{\bar{\theta}(k+1),k+1}$, $\mathcal{F}_{\bar{\theta}(k),k}$, $\mathcal{G}_{\bar{\theta}(k),k}$, $\mathcal{H}_{\bar{\theta}(k),k}$ and $\mathcal{K}_{\bar{\theta}(k),k}$, according to (5).
- Calculate $\hat{z}_{k+1|k}$ and $\mathcal{P}_{k+1|k}$ via (13).
- Extract $\hat{x}_{k+1|k}$ from augmented state vector $\hat{z}_{k+1|k}$.

The next result shows that the solution in Lemma 2, presented in terms of a square matrix framework, can be reduced to an equivalent recursive Riccati equation form. The necessary and sufficient conditions for the existence of a solution for the proposed recursive predictive estimation problem are given based on the standard Kalman filter for singular systems in its predictor form (see, e.g., Nikoukhah et al. (1992)).

Theorem 1. Consider the optimization problem (12). The optimal recursive algebraic solution, for each step k and each $\mu > 0$, is given by

$$\mathcal{P}_{k+1|k} = \Big(\mathcal{E}^T_{\bar{\theta}(k+1),k+1}\Big[\overline{\mathcal{W}}_k - \mathcal{F}_{\bar{\theta}(k),k}\mathcal{P}_{k|k-1}\mathcal{H}^T_{\bar{\theta}(k),k}\Big(\overline{\mathcal{R}}_k + \mathcal{H}_{\bar{\theta}(k),k}\mathcal{P}_{k|k-1}\mathcal{H}^T_{\bar{\theta}(k),k}\Big)^{-1}\mathcal{H}_{\bar{\theta}(k),k}\mathcal{P}_{k|k-1}\mathcal{F}^T_{\bar{\theta}(k),k}\Big]^{-1}\mathcal{E}_{\bar{\theta}(k+1),k+1}\Big)^{-1}$$

with $\overline{\mathcal{W}}_k = \mathcal{W}_k - \overline{\mathcal{G}}_{\bar{\theta}(k),k} - \overline{\mathcal{F}}_{\bar{\theta}(k),k}$,

$$\mathcal{W}_k = \mu^{-1}I + \mathcal{F}_{\bar{\theta}(k),k}\mathcal{P}_{k|k-1}\mathcal{F}^T_{\bar{\theta}(k),k} + \mathcal{G}_{\bar{\theta}(k),k}\Omega_k\mathcal{G}^T_{\bar{\theta}(k),k},$$

$$\overline{\mathcal{G}}_{\bar{\theta}(k),k} = \mathcal{G}_{\bar{\theta}(k),k}\Omega_k\mathcal{K}^T_{\bar{\theta}(k),k}\mathcal{B}_k(\mathcal{H}_{\bar{\theta}(k),k}\mathcal{P}_{k|k-1}\mathcal{F}^T_{\bar{\theta}(k),k} + \mathcal{K}_{\bar{\theta}(k),k}\Omega_k\mathcal{G}^T_{\bar{\theta}(k),k}),$$

$$\overline{\mathcal{F}}_{\bar{\theta}(k),k} = \mathcal{F}_{\bar{\theta}(k),k}\mathcal{P}_{k|k-1}\mathcal{H}^T_{\bar{\theta}(k),k}\mathcal{B}_k\mathcal{K}_{\bar{\theta}(k),k}\Omega_k\mathcal{G}^T_{\bar{\theta}(k),k},$$

$$\mathcal{B}_k = (\overline{\mathcal{R}}_k + \mathcal{H}_{\bar{\theta}(k),k}\mathcal{P}_{k|k-1}\mathcal{H}^T_{\bar{\theta}(k),k})^{-1} \text{ and}$$

$$\overline{\mathcal{R}}_k = \mu^{-1}I + \mathcal{K}_{\bar{\theta}(k),k}\Omega_k\mathcal{K}_{\bar{\theta}(k),k}.$$

Proof. It follows from algebraic manipulation of the expression presented in (13).

5. NUMERICAL EXAMPLE

The numerical example treated in this paper is adapted from Ishihara et al. (2010). We consider MJSSD (1) with two operating modes and the following data:

$$E_{1,k+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_{2,k+1} = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.2 \\ 0 & 0 \end{bmatrix},\ F_{1,k} = \begin{bmatrix} 0.8 & 0.1 \\ 0.1 & 0.7 \\ 0 & 0 \end{bmatrix},$$

$$F_{2,k} = \begin{bmatrix} 0.9 & 0.2 \\ 0.2 & 0.8 \\ 0 & 0 \end{bmatrix}, F_{d,1,k} = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.2 \\ 0 & 0 \end{bmatrix}, F_{d,2,k} = \begin{bmatrix} 0.15 & -0.10 \\ -0.10 & -0.25 \\ 0 & 0 \end{bmatrix},$$

$$G_{w,1,k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},\ G_{w,2,k} = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \\ 1 & 0 \end{bmatrix},\ G_{v,1,k} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix},$$

$$G_{v,2,k} = \begin{bmatrix} 0.3 \\ 0.5 \\ 0.2 \end{bmatrix},\ H_{1,k} = \begin{bmatrix} 1 & 0.8 \end{bmatrix},\ H_{2,k} = \begin{bmatrix} 0.8 & 0.5 \end{bmatrix},$$

$$H_{d,1,k} = \begin{bmatrix} -0.3 & 0.2 \end{bmatrix},\ H_{d,2,k} = \begin{bmatrix} -0.1 & 0.3 \end{bmatrix},\ K_{w,1,k} = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix},$$

$$K_{w,2,k} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, K_{v,1,k} = 1 \text{ and } K_{v,2,k} = 1.2,\ \forall k \in [0,\ N].$$

The weighting matrices, defined in (3), are given by

$$P_{0|-1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ Q_k = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix},$$

$$R_k = 0.1 \text{ and } S_k = \begin{bmatrix} 0.01 \\ 0.001 \end{bmatrix},\ \forall k \in [0,\ N].$$

Two delay cases are considered. In the first, the time-delay varies in the interval $1 \leq d_k \leq 3$, whereas in the second, it varies in the $8 \leq d_k \leq 10$ range.

For both delay cases, the transition probability matrices of the two underlying Markov chains, $\theta(k)$ and $\bar{\theta}(k)$, are, respectively,

$$\mathbb{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \text{ and } \bar{\mathbb{P}} = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}.$$

In order to verify the effectiveness of Algorithm 1, which concerns the proposed RPE for MJSSD, we compare

$$\begin{bmatrix} \hat{z}_{k+1|k} & \mathcal{P}_{k+1|k} \end{bmatrix}_\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I_{n_d} \end{bmatrix}^T \begin{bmatrix} \mathcal{P}_{k|k-1} & 0 & 0 & 0 & I_{n_d} & 0 & 0 \\ 0 & \Omega_k & 0 & 0 & 0 & I_{(p+q)} & 0 \\ 0 & 0 & \mu^{-1}I_{m_d} & 0 & \mathcal{F}_{\bar{\theta}(k),k} & \mathcal{G}_{\bar{\theta}(k),k} & -\mathcal{E}_{\bar{\theta}(k+1),k+1} \\ 0 & 0 & 0 & \mu^{-1}I_r & \mathcal{H}_{\bar{\theta}(k),k} & \mathcal{K}_{\bar{\theta}(k),k} & 0 \\ I_{n_d} & 0 & \mathcal{F}^T_{\bar{\theta}(k),k} & \mathcal{H}^T_{\bar{\theta}(k),k} & 0 & 0 & 0 \\ 0 & I_{(p+q)} & \mathcal{G}^T_{\bar{\theta}(k),k} & \mathcal{K}^T_{\bar{\theta}(k),k} & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{E}^T_{\bar{\theta}(k+1),k+1} & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \hat{z}_{k|k-1} & 0 \\ 0 & 0 \\ 0 & 0 \\ y_k & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I_{n_d} \end{bmatrix} \quad (13)$$

its performance with that of a version which does not incorporate the presence of time-delay in its formulation, so-called estimator without delay (ED). Both estimators are applied to MJSSD (1).

All the routines were written and executed in the MATLAB® software, version 9.4 (R2018a). For the initialization step of Algorithm 1, we adopted

$$\mu = 10^{16},\ N = 1000,\ x_0 = \begin{bmatrix}10 & -10\end{bmatrix}^T \text{ and } \hat{x}_{0|-1} = \begin{bmatrix}12 & -12\end{bmatrix}^T.$$

Figures 1 and 2 show the average squared norm of the estimation error at each time step k for the first and second delay cases, respectively, considering $T = 1000$ Monte Carlo experiments. In addition, the noise signals w_k and v_k are randomly selected according to their covariance matrices for each step k. The system operating mode and the state delay d_k are selected according to their governing Markov chain transition probability matrices.

It is possible to observe that, for the two time-delay situations, the proposed RPE presented smaller estimation errors as well as faster convergence when compared with the ED. Additionally, in the second, more severe, delay case, the performance of both estimators are affected, nonetheless, the RPE still achieves a reasonable estimation error level relative to the ED, which exhibits greater performance degradation.

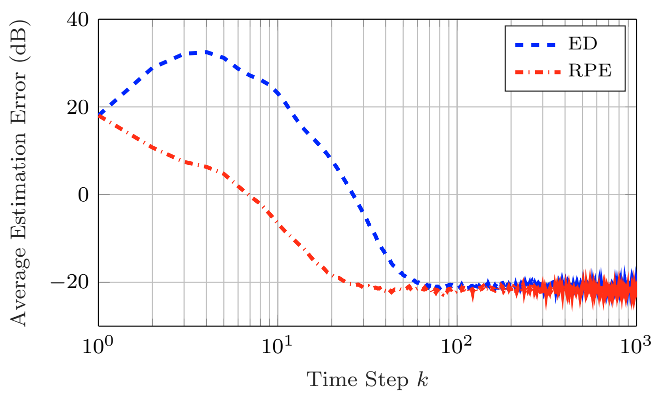

Fig. 1. Comparison of the average estimation errors when $1 \leq d_k \leq 3$.

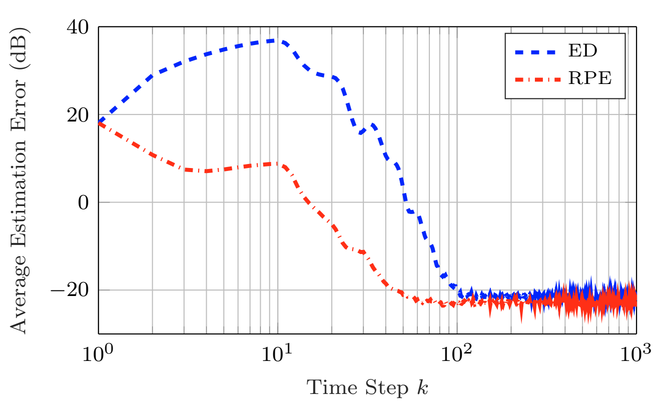

Fig. 2. Comparison of the average estimation errors when $8 \leq d_k \leq 10$.

Figures 3 and 4 present the real and estimated state variables $x_{1,k}$ and $x_{2,k}$ obtained using the RPE and the ED during the first 100 iterations of a single experiment for the two intervals of time-delay. Confirming the results previously shown in the error plots, the RPE can successfully track the real state variables in both delay cases.

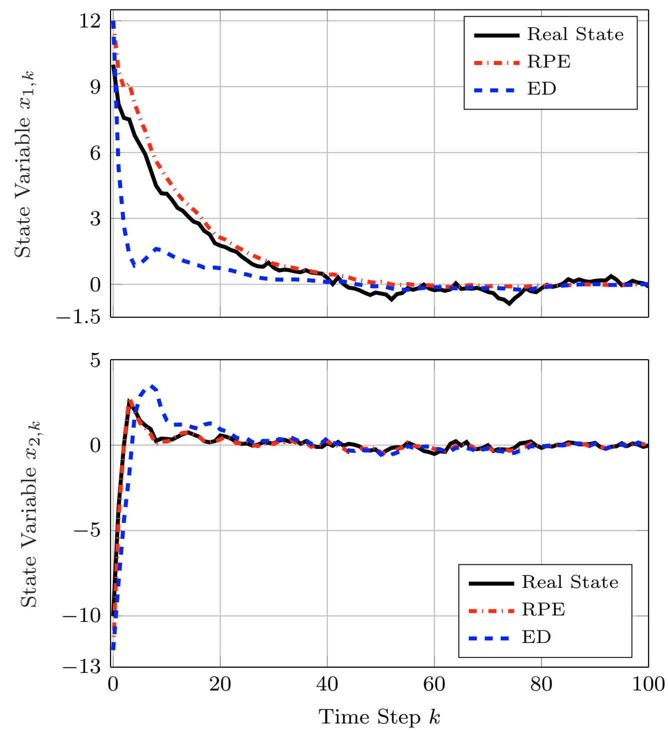

Fig. 3. Real and estimated state variables when $1 \leq d_k \leq 3$.

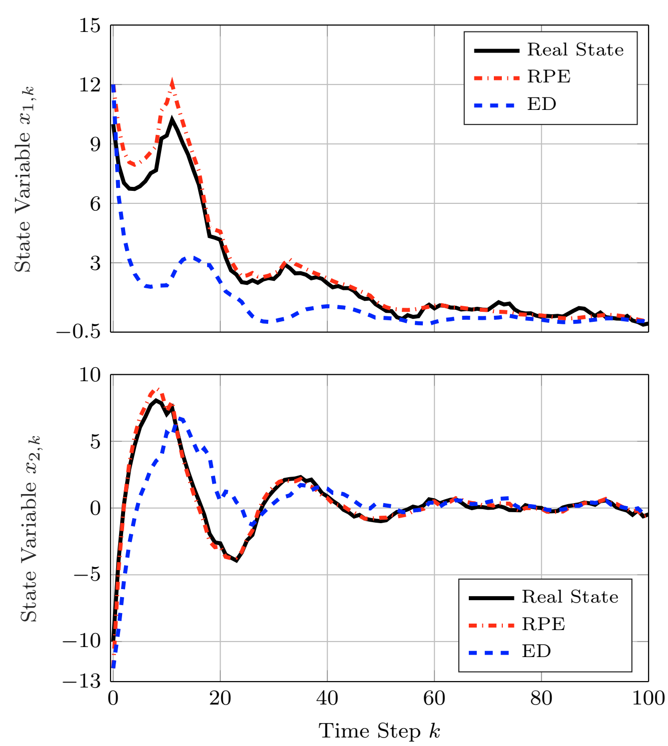

Fig. 4. Real and estimated state variables when $8 \leq d_k \leq 10$.

Finally, Fig. 5 illustrates the switching of operating modes of Markov chains $\theta(k)$ and $\bar{\theta}(k)$ corresponding to the experiment shown in Fig. 3. The switching occurs according to transition probability matrices $\mathbb{P}$ and $\bar{\mathbb{P}}$. Note that $\bar{\theta}(k)$ indicates the amount of delay d_k present in the system.

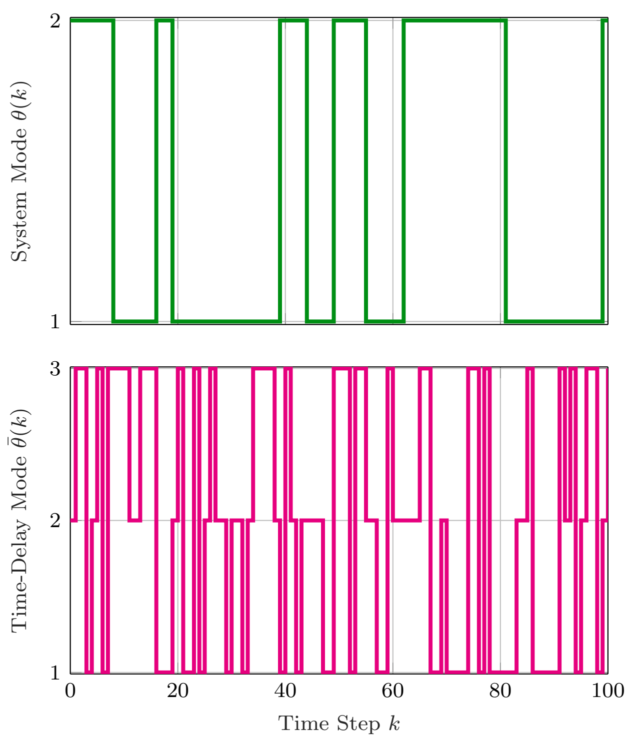

Fig. 5. Switching of system and time-delay operating modes for $1 \leq d_k \leq 3$.

6. CONCLUSION

The recursive predictive estimation problem for discrete-time Markovian jump singular systems with random state delay has been addressed in this paper. By application of the lifting method, the MJSSD was converted into an augmented delay-free MJSS. Then, a methodology was proposed to design a recursive estimator for this augmented MJSS, presented in terms of a square matrix framework, which is well suited for online applications.

The application of the proposed recursive predictive estimator to a numerical example demonstrated its effectiveness in comparison with an approach that does not incorporate the presence of delay in its formulation. Nevertheless, the lifting method leads to a higher state dimension, which may require a large computational effort. The development of alternative recursive methods that are able to soften the increase in the state dimension caused by the presence o time-delay will be considered in a future work.

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