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# A RELAXED QUASINORMALITY CONDITION AND THE BOUNDEDNESS OF DUAL AUGMENTED LAGRANGIAN SEQUENCES*

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**Abstract.** Global convergence of augmented Lagrangian methods to a first-order stationary point is well known to hold under considerably weak constraint qualifications. In particular, several constant rank-type conditions have been introduced for this purpose which turned out to be relevant also beyond this scope. In this paper we show that in fact under these conditions subsequences of approximate Lagrange multipliers associated with accumulation points generated by the algorithm remains bounded. This important stability property is associated with both the practical effectiveness of the algorithm and its computational complexity. In order to obtain this result we introduce a relaxed version of the quasinormality constraint qualification which adequately treats equality constraints by means of informative Lagrange multipliers, a topic that has been extensively studied.

**Key words.** relaxed quasinormality, augmented Lagrangian method, constraint qualification, informative Lagrange multipliers

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## 1. Introduction.

In this paper we are interested in the general smooth nonlinear programming problem with equality and inequality constraints. More specifically, we are interested in the properties of implementable algorithms for solving the problem. Most algorithms are primal-dual, in the sense that they build a sequence of primal iterates that hopefully converges to a solution, but they also build approximations of Lagrange multipliers (dual solutions) to help guide the algorithm towards a solution. These sequences play different roles in the analysis as, for instance, boundedness of the primal iterates may be guaranteed by adding large enough box constraints to the problem, while the dual solutions may be unbounded.

The most well-known approach for bounding the dual sequence generated by an algorithm is assuming the Mangasarian–Fromovitz constraint qualification (MFCQ) at the point of interest. This is equivalent to saying that the set of Lagrange multipliers at the point is bounded. However, this may be considered too stringent for practical purposes as, for instance, it does not allow redundancies in the problem formulation; MFCQ always fails when an equality constraint is replaced by two inequalities or when

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an equality constraint appears twice in the problem formulation. Of course, these situations may sometimes be prevented by preprocessing the problem, but this may not be possible or it may be very time consuming, especially when the optimization process appears in the middle of a more complicated application.

Thus, we are interested in bounding the dual sequence generated by the algorithm even when the primal sequence is converging to a point that fails to satisfy MFCQ. For instance, it has been shown in [19] that the popular interior point method IPOPT tends to find an unbounded dual sequence when MFCQ fails, hindering its practical performance, in contrast with other interior point methods [21].

In the context of augmented Lagrangian methods, several constraint qualifications have been used to show that the primal iterate converges to a stationary point [15]. Although in previous works it was not recognized whether dual sequences are bounded or not, an approximate KKT point is achieved. Boundedness of the dual sequence has been shown only recently in [3, 17] under the so-called quasinormality constraint qualification [20] (actually, only the subsequence associated with the primal accumulation point is bounded), which is weaker than MFCQ, the constant rank constraint qualification (CRCQ [22]), and the constant positive linear dependence condition (CPLD [28]). These constraint qualifications were used in the original global convergence analysis of the popular safeguarded augmented Lagrangian method Algencan [1]; see [9]. However, global convergence to a stationary point is known to hold under considerably weaker conditions. In [26], it has been shown that equality constraints should be treated differently in the formulation of CRCQ, giving rise to a relaxed variant of CRCQ (RCRCQ) which has essentially the same properties as the original formulation. This approach has been exploited in the definition of a relaxed variant of CPLD (RCPLD [7]), which later gave rise to the so-called constant rank of the subspace component constraint qualification (CRSC [8]), the weakest of the constant rank-type constraint qualifications. Besides global convergence of algorithms, several applications and extensions of these conditions have been discussed in the literature, for instance, concerning second-order necessary optimality conditions and a facial reduction procedure for removing redundancies in the problem formulation. We refer the reader to [4] and the references therein for a thorough discussion on this topic, specifically on the central role played by CRSC in this context.

Condition CRSC and the relaxed variants of CRCQ and CPLD are not related to quasinormality constraint qualification, thus boundedness of dual sequences is not known under these conditions. The purpose of this paper is to bridge the gap in terms of the global convergence of the safeguarded augmented Lagrangian method to a stationary point and the boundedness of the dual sequence associated with the primal accumulation point. In particular, we will define a relaxed variant of the quasinormality constraint qualification that is implied by CRSC and that still guarantees boundedness of the dual augmented Lagrangian sequences. The definition is inspired by the well-known notion of an informative Lagrange multiplier [13] and the relaxed variants of CRCQ and CPLD.

The quasinormality condition, introduced by Hestenes [20] and further studied and adapted to different contexts [11, 12, 18, 23, 25, 30], has found several other applications such as in exact penalty [13], computation of the value function [29], and error bound properties [27, 29]. Although these applications are out of the scope of this paper, we believe that our version of quasinormality adequately treats equality constraints in a similar fashion to the relaxed variants of CRCQ and CPLD, preserving its properties, so that we expect these applications to be extended under relaxed quasinormality.

This paper is organized as follows: Section 2 introduces the preliminary results and definitions. In section 3 we present the two main proofs that relaxed quasinormality implies boundedness of dual augmented Lagrangian subsequences, whereas it is implied by CRSC. In section 4 we present some more refined results in terms of feasibility of the primal sequence and a more general algorithm with a scaled criterion for solving the augmented Lagrangian subproblems. Section 5 presents some more refined comparisons with other constraint qualifications, and section 6 presents some concluding remarks.

*Notation.* We use $\mathbb{R}_+$ to denote the set of nonnegative real numbers. Given $z \in \mathbb{R}^r$, $z_+ \in \mathbb{R}^r_+$ is the vector whose $i$th coordinate is $\max\{0, z_i\}, i = 1, \ldots, r$. Given a function $q\colon \mathbb{R}^n \to \mathbb{R}^r$, $\nabla q(x)$ is the $n \times r$ matrix whose columns are the gradients $\nabla q_i(x)$, $i = 1, \ldots, r$, at a point $x \in \mathbb{R}^n$ (transposed Jacobian). We use $\|\cdot\|_2$ and $\|\cdot\|_\infty$ to denote the Euclidean norm and the sup-norm, respectively. When $g\colon \mathbb{R}^n \to \mathbb{R}^p$ and $x \in \mathbb{R}^n$ is such that $g(x) \le 0$, the set $\mathcal{A}(x) = \{j \in \{1, \ldots, p\} \mid g_j(x) = 0\}$ is the set of indices of active inequality constraints at $x$. Given real sequences $\{a_k\}$ and $\{b_k\}$, $a_k = o(b_k)$ means that there is a sequence $\{m_k\} \subset \mathbb{R}$, $m_k > 0$, converging to zero such that $|a_k| \le m_k |b_k|$ for all $k$. Given a tuple (an ordered finite set) $\mathcal{J} = (j_1, \ldots, j_\ell) \subseteq \{1, \ldots, r\}$ and $z \in \mathbb{R}^r$, we define $z_{\mathcal{J}} = (z_{j_1}, \ldots, z_{j_\ell}) \in \mathbb{R}^\ell$. The number of elements in a tuple $\mathcal{J}$ is denoted by $|\mathcal{J}|$.

## 2. Preliminaries.

We consider the nonlinear programming problem

$$\text{(P)} \qquad \min f(x) \quad \text{s.t.} \quad h(x) = 0, \quad g(x) \le 0,$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$, $h\colon \mathbb{R}^n \to \mathbb{R}^m$, and $g\colon \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable functions. We denote by

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^p \mu_j g_j(x)$$

the Lagrangian function associated with (P). A constraint qualification is any condition on the description of the feasible set of (P) such that whenever $\bar{x} \in \mathbb{R}^n$ is a local minimizer of (P), there exist so-called Lagrange multipliers $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p_+$ such that $\nabla L(\bar{x}, \lambda, \mu) = 0$ with $\mu_i = 0$ for all $i \notin \mathcal{A}(\bar{x})$, where the derivative is taken with respect to $x$. In other words, it must be the case that $-\nabla f(\bar{x}) \in \mathcal{K}(\bar{x}; \bar{x})$, where

$$\mathcal{K}(x; \bar{x}) = \left\{ \sum_{i=1}^m \lambda_i \nabla h_i(x) + \sum_{j \in \mathcal{A}(\bar{x})} \mu_j \nabla g_j(x) \,\middle|\, \mu_j \ge 0 \ \ \forall j \in \mathcal{A}(\bar{x}) \right\}.$$

When $x = \bar{x}$, we may write $\mathcal{K}(\bar{x}; \bar{x}) = \mathcal{K}(\bar{x})$, which is the polar of the linearized tangent cone at $\bar{x}$. It is easy to see that the set of Lagrange multipliers at $\bar{x}$ is bounded if and only if the gradients of equalities and active inequalities are *positively linearly independent* (that is, MFCQ holds):

$$\sum_{i=1}^m \lambda_i \nabla h_i(\bar{x}) + \sum_{j \in \mathcal{A}(\bar{x})} \mu_j \nabla g_j(\bar{x}) = 0,\ \mu \ge 0 \ \text{ implies } \ \lambda = 0,\ \mu_j = 0 \ \ \forall j \in \mathcal{A}(\bar{x}).$$

In this paper we consider the (safeguarded) augmented Lagrangian method described in [2, 15], whose implementation is known as ALGENCAN.[1] Given the *penalty*

[1] Freely available at www.ime.usp.br/∼egbirgin/tango.

*parameter* $\rho > 0$ and the *projected multipliers* $\bar{\lambda} \in \mathbb{R}^m$, $\bar{\mu} \in \mathbb{R}^p_+$, let us consider the Powell–Hestenes–Rockafellar (PHR) augmented Lagrangian function

$$L_{\rho,\bar{\lambda},\bar{\mu}}(x) = f(x) + \frac{\rho}{2}\left[\left\|h(x) + \frac{\bar{\lambda}}{\rho}\right\|_2^2 + \left\|\left(g(x) + \frac{\bar{\mu}}{\rho}\right)_+\right\|_2^2\right].$$

In ALGENCAN, the iterate $x^k$ is obtained by minimizing $L_{\rho_k,\bar{\lambda}^k,\bar{\mu}^k}(x)$ for fixed $\rho_k$, $\bar{\lambda}^k$ and $\bar{\mu}^k$. The projected multipliers sequences $\{\bar{\lambda}^k\}$ and $\{\bar{\mu}^k\}$ are computed within a predefined box (*safeguards*).

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**Algorithm 2.1.** Safeguarded augmented Lagrangian method.

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The parameters are $\tau \in [0,1)$, $\gamma > 1$, $-\infty < \lambda_{\min} \le \lambda_{\max} < \infty$, $0 \le \mu_{\max} < \infty$, and $\rho_1 > 0$. Let $\bar{\lambda}^1 \in [\lambda_{\min}, \lambda_{\max}]^m$, $\bar{\mu}^1 \in [0, \mu_{\max}]^p$, and a sequence $\{\varepsilon_k\} \subset \mathbb{R}_+$ such that $\lim_{k\to\infty} \varepsilon_k = 0$. Initialize $k \leftarrow 1$.

**Step 1** (Solving the subproblem): Compute an approximate stationary point $x^k$ of $L_{\rho_k,\bar{\lambda}^k,\bar{\mu}^k}(x)$, that is, $x^k$ satisfying $\|\nabla L_{\rho_k,\bar{\lambda}^k,\bar{\mu}^k}(x^k)\|_\infty \le \varepsilon_k$.

**Step 2** (Updating the penalty parameter): Compute

$$V^k = \min\left\{-g(x^k), \frac{\bar{\mu}^k}{\rho_k}\right\}.$$

If $k = 1$ or $\max\{\|h(x^k)\|_\infty, \|V^k\|_\infty\} \le \tau \max\{\|h(x^{k-1})\|_\infty, \|V^{k-1}\|_\infty\}$, set $\rho_{k+1} = \rho_k$. Otherwise, take $\rho_{k+1} \ge \gamma\rho_k$.

**Step 3** (New projected multipliers): Choose $\bar{\lambda}^{k+1} \in [\lambda_{\min}, \lambda_{\max}]^m$ and $\bar{\mu}^{k+1} \in [0, \mu_{\max}]^p$.

**Step 4**: Set $k \leftarrow k+1$ and go to Step 1.

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The dual (approximate Lagrange multipliers) sequences generated by Algorithm 2.1 are defined as

$$\lambda^k = \bar{\lambda}^k + \rho_k h(x^k) \quad \text{and} \quad \mu^k = \left[\bar{\mu}^k + \rho_k g(x^k)\right]_+, k \ge 1. \tag{2.1}$$

The global convergence of Algorithm 2.1 was established and improved over several works; see, for example, [2, 3, 5, 6]. Let us recall here one of the main conditions used in this analysis.

DEFINITION 2.1 (CRSC [8]). *A feasible point $\bar{x}$ for* (P) *satisfies the* constant rank of the subspace component condition (CRSC) *if the rank of the gradients*

$$\nabla h_i(x),\ i = 1,\dots,m, \qquad \nabla g_j(x),\ j \in \mathcal{A}_-(\bar{x}),$$

*remains constant for all $x$ in a neighborhood of $\bar{x}$, where*

$$\mathcal{A}_-(\bar{x}) = \{j \in \mathcal{A}(\bar{x}) \mid -\nabla g_j(\bar{x}) \in \mathcal{K}(\bar{x})\}. \tag{2.2}$$

This condition improves several others [7, 22, 26, 28] by considering a single set of constraints to have the constant rank property. Notice that when MFCQ holds, CRSC also holds with $\mathcal{A}_-(\bar{x}) = \emptyset$. Several applications of this condition have been found, and we refer the interested reader to [4] and the references therein. In particular, under

CRSC, the constraints indexed in the set $\mathcal{A}_-(\bar{x})$ behave locally as equality constraints in the description of the feasible set; this procedure is known as *facial reduction* [16] in the context of conic programming. Now let $\bar{x}$ be a limit point of a sequence $\{x^k\}$ generated by Algorithm 2.1. If $\bar{x}$ is feasible and satisfies CRSC, then it was shown in [8] that $\bar{x}$ is a stationary point of (P); however, no information has been provided with respect to the dual sequences (2.1). In order to provide such information, one relies on the quasinormality constraint qualification, which is also weaker than MFCQ, but it is independent of CRSC. The definition is as follows.

DEFINITION 2.2 (quasinormality [20]). *A feasible point $\bar{x}$ of* (P) *satisfies the* quasinormality (QN) *constraint qualification if there is no $(\lambda,\mu)\in\mathbb{R}^m\times\mathbb{R}^p_+$ such that*

1. $\nabla h(\bar{x})\lambda+\nabla g(\bar{x})\mu=0$;
2. $(\lambda,\mu)\neq 0$;
3. *defining the index sets*

$$I_{\neq}=\{i \mid \lambda_i\neq 0\} \quad \text{and} \quad J_+=\{j \mid \mu_j>0\}, \tag{2.3}$$

*there is a sequence $\{x^k\}$ converging to $\bar{x}$ such that, for all $k$, $\lambda_i h_i(x^k)>0$ for all $i\in I_{\neq}$, and $g_j(x^k)>0$ for all $j\in J_+$.*

We start by recalling the fact that CRSC and QN are independent conditions.

*Example* 2.3. Consider the constraint set defined by $h_1(x)=x$ and $h_2(x)=x^2$ at $\bar{x}=0\in\mathbb{R}$. QN does not hold since we can take $\lambda_1=0$ and $\lambda_2=1$ together with the sequence $x^k=1/k$, where we have that $\lambda_1\nabla h_1(\bar{x})+\lambda_2\nabla h_2(\bar{x})=0$ with $\lambda_2 h_2(x^k)>0$ for all $k$. On the other hand, CRSC holds since the set $\{\nabla h_1(x),\nabla h_2(x)\}$ has full (constant) rank for all $x$ nearby $\bar{x}$. The reverse situation occurs with the constraint set defined by $g_1(x)=-x_1$ and $g_2(x)=x_1-x_2^2$ at $\bar{x}=(0,0)\in\mathbb{R}^2$. The set $\mathcal{K}(\bar{x})$ is equal to $\mathbb{R}\times\{0\}$, where $\mathcal{A}_-(\bar{x})=\{1,2\}$ but the rank of $\{\nabla g_1(x),\nabla g_2(x)\}$ increases from 1 at $\bar{x}$ to 2 for $x$ nearby $\bar{x}$ with $x_2\neq 0$. Thus CRSC fails. To see that QN holds, notice that $\mu_1\nabla g_1(\bar{x})+\mu_2\nabla g_2(\bar{x})=0$ with $0\neq\mu\geq 0$ implies that $\mu_1=\mu_2>0$. However, if $g_1(x)>0$ for some $x$, it must be the case that $g_2(x)<0$. Thus, no sequence satisfying item 3 in Definition 2.2 exists and QN holds.

Under quasinormality, it was proved in [3, 19] that if $\bar{x}$ is a feasible limit point of a sequence $\{x^k\}$ generated by Algorithm 2.1, that is, $\lim_{k\in K}x^k=\bar{x}$ for an infinite set of indexes $K$, then the sequences of approximate Lagrange multipliers $\{\lambda^k\}_{k\in K}$ and $\{\mu^k\}_{k\in K}$ as defined in (2.1) are bounded. In fact, by Step 1 of the algorithm, a simple computation gives $\nabla L(x^k,\lambda^k,\mu^k)\to 0$. Thus, assuming that the dual sequences are not both bounded, one arrives at a pair $(\lambda,\mu)\in\mathbb{R}^m\times\mathbb{R}^p_+$ such that items 1 and 2 of Definition 2.2 are satisfied. Now, by (2.1) it is easy to see that when $\lambda_i^k\to+\infty$ then, since $\{\bar{\lambda}^k\}$ is bounded, it must be the case that $\rho_k\to+\infty$ and $h_i(x^k)>0$ for sufficiently large $k$. A similar analysis holds when $\lambda_i^k\to-\infty$ or $\mu_j^k\to+\infty$ so that the sign condition given by item 3 is also satisfied. Therefore, under quasinormality, no such sequence exists and the dual sequences must be bounded.

An alternative motivation for the definition of quasinormality [20] comes from an enhanced Fritz John theorem, where it is shown that around a local minimizer there exists a sequence that violates the constraints in a particular way. This inspired several different definitions of a Lagrange multiplier with additional requirements concerning constraint violation. The most general of these results is the following.

THEOREM 2.4 ([13, Proposition 2.1]). *Let $\bar{x}$ be a local minimizer of* (P). *Then there is $(\sigma,\lambda,\mu)\in\mathbb{R}_+\times\mathbb{R}^m\times\mathbb{R}^p_+$ such that*

1. $\sigma\nabla f(\bar{x})+\nabla h(\bar{x})\lambda+\nabla g(\bar{x})\mu=0$;
2. $(\sigma,\lambda,\mu)\neq 0$;
3. *if* $I_{\neq}\cup J_+\neq\emptyset$, *where* $I_{\neq}$ *and* $J_+$ *are defined in* (2.3), *then there is a sequence* $\{x^k\}$ *converging to* $\bar{x}$ *such that, for all* $k$,
   (a) $\lambda_i h_i(x^k)>0$ *for all* $i\in I_{\neq}$, *and* $g_j(x^k)>0$ *for all* $j\in J_+$;
   (b) $|h_i(x^k)|=o(w(x^k))$ *for all* $i\notin I_{\neq}$ *and* $g_j(x^k)_+=o(w(x^k))$ *for all* $j\notin J_+$, *where*

$$w(x^k)=\min\left\{\min_{i\in I_{\neq}}|h_i(x^k)|,\ \min_{j\in J_+}g_j(x^k)_+\right\}. \tag{2.4}$$

It is easy to see that item 3(a) of the above theorem implies the usual complementary slackness $\mu_j g_j(\bar{x})=0$ for all $j=1,\dots,p$. Notice that Theorem 2.4 implies that QN is a constraint qualification, since Definition 2.2 prevents the existence of a sequence satisfying items 1, 2, and 3(a) of Theorem 2.4 when $\sigma=0$. Thus, at a local minimizer, QN implies that there exists a Lagrange multiplier $(\lambda,\mu)$ with the additional constraint violation given by items 3(a–b) of Theorem 2.4. This has been called an *informative* Lagrange multiplier in [13]. It was shown in [10] that this additional dual information is not relevant for distinguishing a primal solution; that is, a feasible point for (P) admits an informative Lagrange multiplier if and only if it admits a standard Lagrange multiplier. However, this additional dual information will be crucial in our analysis. We start by noticing that it is clear that Theorem 2.4 suggests a weaker definition of QN by incorporating also item 3(b) as follows.

DEFINITION 2.5. *A feasible point* $\bar{x}$ *for* (P) *satisfies the* relaxed quasinormality *(RQN) condition if there is no* $(\lambda,\mu)\in\mathbb{R}^m\times\mathbb{R}^p_+$ *such that*

1. $\nabla h(\bar{x})\lambda+\nabla g(\bar{x})\mu=0$;
2. $(\lambda,\mu)\neq 0$;
3. *there is a sequence* $\{x^k\}$ *converging to* $\bar{x}$ *such that, for all* $k$,
   (a) $\lambda_i h_i(x^k)>0$ *for all* $i\in I_{\neq}$ *and* $g_j(x^k)>0$ *for all* $j\in J_+$;
   (b) $|h_i(x^k)|=o(w(x^k))$ *for all* $i\notin I_{\neq}$ *and* $g_j(x^k)_+=o(w(x^k))$ *for all* $j\notin J_+$, *where* $I_{\neq}$ *and* $J_+$ *are defined as in* (2.3) *and* $w(x^k)$ *as in* (2.4).

This definition has not been exploited yet in the literature, and it will be the main focus in this paper. It is clearly a constraint qualification since it implies that it must be the case that $\sigma>0$ in Theorem 2.4. Recalling Example 2.3 with the constraint set defined by $h_1(x)=x$ and $h_2(x)=x^2$ at $\bar{x}=0\in\mathbb{R}$, notice that the sequence $x^k=1/k$ with $\lambda_1=0$ and $\lambda_2=1$ fails to comply with item 3(b) in Definition 2.5. Indeed, it is not the case that $|h_1(x^k)|=o(|h_2(x^k)|)$ because $\frac{|x^k|}{(x^k)^2}\not\to 0$. Actually, this is the case for any sequence $x^k\to\bar{x}, x^k\neq\bar{x}$, and since items 1–2 of Definition 2.5 imply that $\lambda_1=0$ and $\lambda_2\neq 0$, this shows that RQN holds.

It turns out that RQN will provide an adequate way of dealing with equality constraints, similarly to how it is done in the relaxed variants of CRCQ and CPLD. Namely, while QN is implied by CRCQ and CPLD, it is independent of the weaker variants RCRCQ, RCPLD, and CRSC. We will show that CRSC (and all other constant rank-type constraint qualifications) strictly implies RQN. This will give rise to a new stability property under constant rank-type constraint qualifications, as we will show that RQN will be enough for providing boundedness of the dual augmented Lagrangian sequences associated with primal accumulation points.

## 3. Main results.

Our first main result concerning RQN is the fact that it subsumes all constant rank-type constraint qualifications. That is, we will show that

CRSC implies RQN. Clearly, the implication is strict due to the second constraint set defined in Example 2.3, where CRSC fails and QN (thus RQN) holds. We will make use of the following lemma, which is a consequence of the inverse function theorem.

LEMMA 3.1 ([9, Lemma 3.2]). *Let $\bar{x} \in \mathbb{R}^n$, let $\mathcal{V} \subseteq \mathbb{R}^n$ be an open neighbourhood of $\bar{x}$, and let $F\colon \mathcal{V} \to \mathbb{R}$. Suppose that*

$$\nabla F(\bar{x}) = \sum_{j=1}^{r} \alpha_j \nabla c_j(\bar{x})$$

*for some $C^1$ function $c\colon \mathcal{V} \to \mathbb{R}^r$ such that $\{\nabla c_j(\bar{x})\}_{j\in\{1,\dots,r\}}$ is linearly independent. Also, suppose that $\nabla F(x)$ is a linear combination of $\nabla c_j(x)$, $j = 1,\dots,r$ for all $x \in \mathcal{V}$. Then there exists an open neighborhood $\mathcal{U} \subseteq \mathbb{R}^r$ of $c(\bar{x})$ and a $C^1$ function $\varphi\colon \mathcal{U} \to \mathbb{R}$ such that $c(x) \in \mathcal{U}$ and $F(x) = \varphi(c(x))$ for all $x \in \mathcal{V}$, and*

$$\alpha_j = \left[\nabla\varphi(c(\bar{x}))\right]_j, \quad j = 1,\dots,r.$$

THEOREM 3.2. *CRSC implies RQN.*

*Proof.* Suppose by contradiction that the feasible point $\bar{x}$ for (P) satisfies CRSC but not RQN. Then there exist $(\lambda,\mu)$ and $\{x^k\}$ satisfying items 1, 2, and 3 of Definition 2.5, that is,

1. $\nabla h(\bar{x})\lambda + \nabla g(\bar{x})\mu = 0$;
2. $(\lambda,\mu) \neq 0$;
3. $\lim_k x^k = \bar{x}$ and, for all $k$,
   (a) $\lambda_i h_i(x^k) > 0$ for all $i \in I_{\neq}$, and $g_j(x^k) > 0$ for all $j \in J_+$;
   (b) $|h_i(x^k)| = o(w(x^k))$ for all $i \notin I_{\neq}$ and $g_j(x^k)_+ = o(w(x^k))$ for all $j \notin J_+$, where $I_{\neq}$ and $J_+$ are defined as in (2.3) and $w(x^k)$ as in (2.4).

First, we affirm that $\mu_j = 0$ for all $j \notin \mathcal{A}_-(\bar{x})$, or equivalently, $J_+ \subseteq \mathcal{A}_-(\bar{x})$. In fact, if $\mu_j > 0$ then $\nabla h(\bar{x})\lambda + \nabla g(\bar{x})\mu = 0$ implies

$$-\nabla g_j(\bar{x}) = \frac{1}{\mu_j}\left[\sum_{i=1}^{m} \lambda_i \nabla h_i(\bar{x}) + \sum_{\ell\neq j} \mu_\ell \nabla g_\ell(\bar{x})\right] \in \mathcal{K}(\bar{x}),$$

which in turn implies $j \in \mathcal{A}_-(\bar{x})$. Hence, item 1 takes the form

$$\sum_{i=1}^{m} \lambda_i \nabla h_i(\bar{x}) + \sum_{j\in\mathcal{A}_-(\bar{x})} \mu_j \nabla g_j(\bar{x}) = 0. \tag{3.1}$$

Let $\mathcal{I} \subseteq \{1,\dots,m\}$ and $\mathcal{J} \subseteq \mathcal{A}_-(\bar{x})$ be such that $\{\nabla h_i(\bar{x})\}_{i\in\mathcal{I}} \cup \{\nabla g_j(\bar{x})\}_{j\in\mathcal{J}}$ is a basis for span$\{\nabla h(\bar{x}), \nabla g_{\mathcal{A}_-(\bar{x})}(\bar{x})\}$, the subspace generated by $\{\nabla h_i(\bar{x})\}_{i=1}^m \cup \{\nabla g_j(\bar{x})\}_{j\in\mathcal{A}_-(\bar{x})}$. The CRSC condition guarantees that the rank of the gradients of the equality constraints and inequality constraints with indices $\mathcal{A}_-(\bar{x})$ remains constant in a neighborhood of $\bar{x}$, so

$$\operatorname{span}\{\nabla h_{\mathcal{I}}(x), \nabla g_{\mathcal{J}}(x)\} = \operatorname{span}\{\nabla h(x), \nabla g_{\mathcal{A}_-(\bar{x})}(x)\} \quad \forall x \text{ close to } \bar{x}. \tag{3.2}$$

Now, let us define

$$F(x) = -\sum_{i\in\{1,\dots,m\}\setminus\mathcal{I}} \lambda_i h_i(x) - \sum_{j\in\mathcal{A}_-(\bar{x})\setminus\mathcal{J}} \mu_j g_j(x) \tag{3.3}$$

and the $C^1$ function $c:\mathbb{R}^n \to \mathbb{R}^{|\mathcal{I}|} \times \mathbb{R}^{|\mathcal{J}|}$ given by $c(x) = (h_{\mathcal{I}}(x), g_{\mathcal{J}}(x))$. From (3.2), $\nabla F(x) \in \operatorname{span}\{\nabla h_{\mathcal{I}}(x), \nabla g_{\mathcal{J}}(x)\}$ for all $x$ near $\bar{x}$. In particular, from (3.1) and (3.3) we have

$$\nabla F(\bar{x}) = \left[ -\sum_{i=1}^{m} \lambda_i \nabla h_i(\bar{x}) - \sum_{j \in \mathcal{A}_-(\bar{x})} \mu_j \nabla g_j(\bar{x}) \right] + \sum_{i \in \mathcal{I}} \lambda_i \nabla h_i(\bar{x}) + \sum_{j \in \mathcal{J}} \mu_j \nabla g_j(\bar{x})$$
$$= \sum_{i \in \mathcal{I}} \lambda_i \nabla h_i(\bar{x}) + \sum_{j \in \mathcal{J}} \mu_j \nabla g_j(\bar{x}).$$

So, we can apply Lemma 3.1 to obtain a $C^1$ function $\varphi:\mathbb{R}^{|\mathcal{I}|} \times \mathbb{R}^{|\mathcal{J}|} \to \mathbb{R}$ such that

$$F(x) = \varphi(c(x)) = \varphi(h_{\mathcal{I}}(x), g_{\mathcal{J}}(x)) \ \forall x \text{ near } \bar{x} \quad \text{and} \quad (\lambda_{\mathcal{I}}, \mu_{\mathcal{J}}) = \nabla \varphi(h_{\mathcal{I}}(\bar{x}), g_{\mathcal{J}}(\bar{x})). \tag{3.4}$$

Note that $(h_{\mathcal{I}}(\bar{x}), g_{\mathcal{J}}(\bar{x})) = 0$, $\varphi(0) = F(\bar{x}) = 0$, and then the Taylor expansion of $\varphi$ around the origin gives

$$\varphi(z) = \nabla \varphi(h_{\mathcal{I}}(\bar{x}), g_{\mathcal{J}}(\bar{x}))^T z + o(\|z\|_\infty).$$

Taking $z = (h_{\mathcal{I}}(x^k), g_{\mathcal{J}}(x^k)_+)$, which converges to $(h_{\mathcal{I}}(\bar{x}), g_{\mathcal{J}}(\bar{x}))$ as $\bar{x}$ is feasible and $\mathcal{J} \subseteq \mathcal{A}(\bar{x})$, the above expression together with (3.4) implies

$$F(x^k) = \sum_{i \in \mathcal{I}} \lambda_i h_i(x^k) + \sum_{j \in \mathcal{J}} \mu_j g_j(x^k)_+ + w_k \tag{3.5}$$

for all $k$ large enough, where

$$w_k = o(\|(h_{\mathcal{I}}(x^k), g_{\mathcal{J}}(x^k)_+)\|_\infty).$$

From (3.3), (3.5), and the fact that $g_j(x^k) > 0$ when $j \in J_+$ by item 3(a), we have

$$\begin{aligned} 0 &= \sum_{i=1}^{m} \lambda_i h_i(x^k) + \sum_{j \in \mathcal{J}} \mu_j g_j(x^k)_+ + \sum_{j \in \mathcal{A}_-(\bar{x}) \setminus \mathcal{J}} \mu_j g_j(x^k) + w_k \\ &= \sum_{i \in I_{\neq}} \lambda_i h_i(x^k) + \sum_{j \in J_+} \mu_j g_j(x^k)_+ + w_k \end{aligned} \tag{3.6}$$

for all $k$ large enough.

We have

$$w_k = o(\|(\ h(x^k)\ ,\ g_{\mathcal{J}}(x^k)_+\ ,\ g_{\mathcal{A}_-(\bar{x}) \setminus \mathcal{J}}(x^k)_+\ )\|_\infty)$$

since the term in parentheses is greater than or equal to $\|(h_{\mathcal{I}}(x^k), g_{\mathcal{J}}(x^k)_+)\|_\infty$. By item 3(b), each sequence $\{|h_i(x^k)|\}$ and $\{g_j(x^k)_+\}$ such that $i \in I_{\neq}$ and $j \in J_+$ asymptotically bounds above all such sequences with indices outside $I_{\neq}$ or $J_+$, which allows us to write

$$w_k = o(\|(\ h_{I_{\neq}}(x^k)\ ,\ g_{J_+}(x^k)\ )\|_\infty)$$

(remember that $g_j(x^k) > 0$ when $j \in J_+$). So, dividing (3.6) by $\|(h_{I_{\neq}}(x^k), g_{J_+}(x^k))\|_\infty$ and passing to the limit on a subsequence if necessary, we obtain

$$(h^*, g^*) = \lim_k \frac{(\ h_{I_{\neq}}(x^k)\ ,\ g_{J_+}(x^k)_+)}{\|(h_{I_{\neq}}(x^k), g_{J_+}(x^k))\|_\infty} \neq 0$$

satisfying

$$\sum_{i\in I_{\neq}} \lambda_i h_i^* + \sum_{j\in J_+} \mu_j g_j^* = 0.$$

But this is impossible since by item 3(a), $\lambda_i h_i^* \geq 0$, $i \in I_{\neq}$, $\mu_i g_i^* \geq 0$, $j \in J_+$, and thus the left-hand side of the above expression is positive. We then conclude that RQN holds at $\bar{x}$. ∎

We now show that, similar to what is known about QN, the dual sequences generated by Algorithm 2.1 are bounded under RQN.

THEOREM 3.3. *Let* $\{x^k\}$ *be a sequence generated by Algorithm* 2.1 *and* $\bar{x}$ *a feasible limit point of it, let us say,* $\lim_{k\in K} x^k = \bar{x}$. *If* $\bar{x}$ *satisfies RQN, then the associated dual subsequences* $\{\lambda^k\}_{k\in K}$ *and* $\{\mu^k\}_{k\in K}$ *given in* (2.1) *are bounded. In particular, all limit points of these sequences are Lagrange multipliers associated with* $\bar{x}$.

*Proof.* Let $\{\rho_k\}, \{\bar{\lambda}^k\}$, and $\{\bar{\mu}^k\}$ be corresponding sequences produced by Algorithm 2.1 and suppose that $\{M_k := \|(1, \lambda^k, \mu^k)\|_\infty\}$ is unbounded. By (2.1), we have $\rho_k \to \infty$ and then $\mu_i^k = 0$ for all $i \notin \mathcal{A}(\bar{x})$. So, dividing the expression $\nabla L(x^k, \lambda^k, \mu^k) \to 0$ as provided by Step 1 by $M_k$ and taking the limit over $K$, we arrive at

$$\sum_{i=1}^{m} \lambda_i \nabla h_i(\bar{x}) + \sum_{j\in\mathcal{A}(\bar{x})} \mu_j \nabla g_j(\bar{x}) = 0,$$

with $(\lambda, \mu) \neq 0$ and

$$\lambda_i = \lim_{k\in K} \frac{\bar{\lambda}_i^k + \rho_k h_i(x^k)}{M_k}, \quad \mu_j = \lim_{k\in K} \frac{\left[\bar{\mu}_j^k + \rho_k g_j(x^k)\right]_+}{M_k}$$

for all $i, j$. If $\lambda_i \neq 0$, then we can extract a subsequence so that $h_i(x^k)$ always has the same sign of $\lambda_i$ (the same is valid for $\mu_j$). Thus, passing to a subsequence if necessary, we can suppose without loss of generality that

$$\forall k \in K, \quad \lambda_i h_i(x^k) > 0 \text{ if } \lambda_i \neq 0 \quad \text{and} \quad g_j(x^k) > 0 \text{ if } \mu_j > 0. \tag{3.7}$$

Therefore, if all entries of $(\lambda, \mu)$ are nonzero, then item 3(b) of Definition 2.5 holds trivially with $\{x^k\}_{k\in K}$, contradicting the validity of RQN at $\bar{x}$.

Now, suppose that $\lambda_i \neq 0$ and $\lambda_\ell = 0$. Clearly $\lim_{k\in K} |\rho_k h_i(x^k)| = \infty$ and, by (3.7), $h_i(x^k) \neq 0$ for all $k \in K$. For each $k \in K$, let us define

$$A_k := \frac{|\bar{\lambda}_\ell^k + \rho_k h_\ell(x^k)|}{|\rho_k h_i(x^k)|} = \left| \frac{\bar{\lambda}_\ell^k}{\rho_k h_i(x^k)} + \frac{h_\ell(x^k)}{h_i(x^k)} \right|.$$

We affirm that $\liminf_{k\in K} A_k = 0$. In fact, if $A_k \geq \varepsilon > 0$ for all $k \in K$ large enough, we would have $|\bar{\lambda}_\ell^k + \rho_k h_\ell(x^k)| \geq \varepsilon |\rho_k h_i(x^k)|$ for all $k \in K$ large enough and therefore

$$0 < \varepsilon|\lambda_i| = \lim_{k\in K} \varepsilon \left| \frac{\bar{\lambda}_i^k + \rho_k h_i(x^k)}{M_k} \right| = \lim_{k\in K} \varepsilon \frac{|\rho_k h_i(x^k)|}{M_k} \leq \lim_{k\in K} \frac{|\bar{\lambda}_\ell^k + \rho_k h_\ell(x^k)|}{M_k} = |\lambda_\ell| = 0,$$

a contradiction. Hence, there is an infinite set of indices $K_1 \subseteq K$ such that

$$\lim_{k\in K_1} \frac{|h_\ell(x^k)|}{|h_i(x^k)|} = \lim_{k\in K_1} A_k = 0.$$

A similar argument is valid changing $\lambda_i \neq 0$ to $\mu_j > 0$, $\lambda_\ell = 0$ to $\mu_\ell = 0$, and/or $|h_\ell(x^k)|$ to $g_\ell(x^k)_+$. Thus, applying it successively we obtain an infinite set $K_* \subseteq \cdots \subseteq K_1 \subseteq K$ such that

$$(3.8) \qquad \lim_{k \in K_*} \frac{|h_\ell(x^k)|}{w(x^k)} = 0 \ \text{ if } \ \lambda_\ell = 0 \quad \text{and} \quad \lim_{k \in K_*} \frac{g_\ell(x^k)_+}{w(x^k)} = 0 \ \text{ if } \ \mu_\ell = 0,$$

where $w(x^k)$ is as in (2.4). Finally, from (3.7) and (3.8) we conclude that RQN at $\bar{x}$ is violated using the sequence $\{x^k\}_{k \in K_*}$, and the proof is complete. □

Theorem 3.3 was known only under QN [3, 19]. The following example shows that when RQN fails, the dual subsequences generated by Algorithm 2.1 corresponding to a convergent primal subsequence may in fact be unbounded.

*Example* 3.4. Consider the problem

$$\min f(x) = x_1^2 + x_2^2 \quad \text{s.t.} \quad x \in \Omega,$$

where

$$\Omega = \{x \in \mathbb{R}^2 \mid g_1(x) = x_1^3 - x_2 \leq 0,\ g_2(x) = x_1^3 + x_2 \leq 0,\ g_3(x) = -x_1 \leq 0\},$$

and its feasible point $\bar{x} = (0, 0)$. We affirm that this point can be reached by Algorithm 2.1 with unbounded associated multiplier sequences (2.1). In fact, let us consider the sequence $x^k = (1/\rho_k^a, 0)$, $\rho_k > 0$, where $a \in (1/5, 1/3)$ is a constant. For each $k$, take any $\bar{\mu}_1^k = \bar{\mu}_2^k \geq 0$ and $\bar{\mu}_3^k = 0$. The multiplier estimates (2.1) with these sequences are

$$\mu_2^k = \mu_1^k = \bar{\mu}_1^k + \rho_k^{1-3a}, \quad \mu_3^k = 0.$$

We have

$$\nabla L_{\rho_k, \bar{\mu}^k}(x^k) = \begin{bmatrix} \dfrac{2}{\rho_k^a} \\ 0 \end{bmatrix} + \left(\bar{\mu}_1^k + \rho_k^{1-3a}\right) \begin{bmatrix} \dfrac{3}{\rho_k^{2a}} \\ -1 \end{bmatrix} + \left(\bar{\mu}_1^k + \rho_k^{1-3a}\right) \begin{bmatrix} \dfrac{3}{\rho_k^{2a}} \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{2}{\rho_k^a} + \dfrac{6\bar{\mu}_1^k}{\rho_k^{2a}} + \dfrac{6}{\rho_k^{5a-1}} \\ 0 \end{bmatrix}$$

and

$$V_2^k = V_1^k = \min\left\{-\frac{1}{\rho_k^{3a}}, \frac{\bar{\mu}_1^k}{\rho_k}\right\} = -\frac{1}{\rho_k^{3a}}, \quad V_3^k = \min\left\{\frac{1}{\rho_k^a}, \frac{\bar{\mu}_3^k}{\rho_k}\right\} = 0.$$

If $\{\rho_k\}$ remained constant for all $k \geq k_0$, then we would have $\|V^k\|_\infty = 1/\rho_k^{3a} > \tau/\rho_{k-1}^{3a} = \tau\|V^{k-1}\|_\infty$ for any $\tau \in [0, 1)$, $k \geq k_0$, contradicting Step 2. On the contrary, $\rho_k \to \infty$ implies $\nabla L_{\rho_k, \bar{\mu}^k}(x^k) \to 0$ since $a > 1/5$. Therefore the sequence $(x^k, \rho_k, \bar{\mu}^k)$ with $\rho_k \to \infty$ can be generated by Algorithm 2.1. In this case $\mu_2^k = \mu_1^k \to \infty$ since $a < 1/3$.

To see that RQN fails at $\bar{x}$, just note that $\mu = (1, 1, 0)$ and $x^k = (1/k, 0)$ fulfill items 1, 2, and 3 of Definition 2.5.

Notice that $\bar{x} = (0, 0)$ in Example 3.4 is a KKT point that satisfies a weak constraint qualification called *constant positive generators* (CPG), as we will see in section 5. Actually, we will show that RQN and CPG are independent conditions.

**4. Extensions.** The result we presented related to the boundedness of dual augmented Lagrangian sequences (Theorem 3.3) assumes that the limit point $\bar{x}$ is feasible. This is not a serious drawback since the algorithm tends to find feasible limit points, when they exist, as their limit points are stationary to the problem of minimizing $\|h(x)\|_2^2 + \|g(x)_+\|_2^2$ [1]. These points are feasible when the gradients of equality constraints and violated or active inequality constraints are positively linearly independent, what is known as extended MFCQ [24]. However no feasibility result is known under a condition weaker than extended MFCQ. Let us show that the boundedness of the dual sequences is enough for ensuring feasibility, and that this is obtained by an extension of RQN to infeasible points. The definition is exactly the same as Definition 2.5, but it is simply not assumed that the point is feasible; in particular, it reduces to RQN when the point is feasible. We opted to present a simpler version of this result in Theorem 3.3 for clarity of exposition, but in fact this theorem is a particular case of the result we prove next.

DEFINITION 4.1. *A point $\bar{x} \in \mathbb{R}^n$, not necessarily feasible, satisfies the* extended RQN *condition if there is no $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p_+$ and sequence $x^k \to \bar{x}$ such that items* 1, 2, 3*(a) of Definition* 2.5 *hold and*

(b′) *$|h_i(x^k)| = o(w(x^k))$ for all $i \notin I_{\neq}$ with $h_i(\bar{x}) = 0$ and $g_j(x^k)_+ = o(w(x^k))$ for all $j \notin J_+$ with $g_j(\bar{x}) \le 0$, where $I_{\neq}$ and $J_+$ are defined as in* (2.3) *and $w(x^k)$ as in* (2.4).

THEOREM 4.2. *Let $\{x^k\}$ be a sequence generated by Algorithm* 2.1 *and $\bar{x}$ a limit point of it, let us say, $\lim_{k \in K} x^k = \bar{x}$. If $\bar{x}$ satisfies extended RQN, then $\bar{x}$ is feasible and the associated dual subsequences $\{\lambda^k\}_{k \in K}$ and $\{\mu^k\}_{k \in K}$ given in* (2.1) *are bounded. In particular, all limit points of these sequences are Lagrange multipliers associated with $\bar{x}$.*

*Proof.* Let $\{\rho_k\}, \{\bar{\lambda}^k\}$, and $\{\bar{\mu}^k\}$ be corresponding sequences produced by Algorithm 2.1.

If $\{\rho_k\}$ is bounded, it remains constant after a certain iteration as it is nondecreasing from Step 2. Thus, the test in Step 2 of Algorithm 2.1 ensures that $\max\{\|h(x^k)\|_\infty, \|V^k\|_\infty\} \to 0$, which implies that $\bar{x}$ is feasible. As a consequence, the multiplier sequences given by (2.1), namely

$$\lambda^k = \bar{\lambda}^k + \rho_k h(x^k) \quad \text{and} \quad \mu^k = \left[\bar{\mu}^k + \rho_k g(x^k)\right]_+, \tag{4.1}$$

are bounded.

If $\rho_k \to \infty$ and $\{M_k = \|(1, \lambda^k, \mu^k)\|_\infty\}_{k \in K}$ is bounded, then by (4.1) and the boundedness of $\{(\bar{\lambda}^k, \bar{\mu}^k)\}$ we also conclude that $\bar{x}$ is feasible.

From now on, we suppose that $\rho_k \to \infty$ and $\{M_k\}_{k \in K}$ is unbounded. By (4.1), we have $\mu_i^k = 0$ for all $i$ with $g_i(\bar{x}) < 0$ and $k \in K$ large enough. So, dividing $\nabla L(x^k, \lambda^k, \mu^k) \to 0$ by $M_k$ and taking the limit over $K$, we obtain $(\lambda, \mu) \neq 0$ such that

$$\lambda_i = \lim_{k \in K} \frac{\bar{\lambda}_i^k + \rho_k h_i(x^k)}{M_k}, \quad \mu_j = \lim_{k \in K} \frac{\left[\bar{\mu}_j^k + \rho_k g_j(x^k)\right]_+}{M_k} \tag{4.2}$$

for all $i$, $j$, and satisfying $\nabla h(\bar{x})\lambda + \nabla g(\bar{x})\mu = 0$.

Now, we analyze feasibility of $\bar{x}$ with respect to each constraint in two cases: whether the corresponding multiplier is zero or not. If $\lambda_i = 0$ and $h_i(\bar{x}) \neq 0$, then

$$\lim_{k \in K} \frac{\rho_k}{M_k} = 0 \tag{4.3}$$

by (4.2) and the boundedness of $\{\bar{\lambda}_i^k\}$. As $(\lambda,\mu)\neq 0$, there is an index $\ell$ such that $\lambda_\ell \neq 0$ or $\mu_\ell > 0$. However, due to (4.3), we would have

$$\lambda_\ell = \lim_{k\in K}\left[\frac{\bar{\lambda}_\ell^k}{M_k} + \frac{\rho_k}{M_k} h_\ell(x^k)\right] = 0 \quad \text{and} \quad \mu_\ell = \lim_{k\in K}\left[\frac{\bar{\mu}_\ell^k}{M_k} + \frac{\rho_k}{M_k} g_\ell(x^k)\right]_+ = 0,$$

a contradiction. Therefore, $h_i(\bar{x}) = 0$ whenever $\lambda_i = 0$. We can prove that $g_j(\bar{x}) \leq 0$ if $\mu_j = 0$ analogously, since, in view of (4.2) and the boundedness of $\{\bar{\mu}_j^k\}$, $\mu_j = 0$ and $g_j(\bar{x}) > 0$ imply (4.3).

The case where $\lambda_i \neq 0$ or $\mu_j > 0$ follows the same steps as in the proof of Theorem 3.3, as we can conclude that this contradicts the validity of extended RQN at $\bar{x}$ independently of the feasibility of $\bar{x}$. This completes the proof. □

Theorems 3.3 and 4.2 can also be extended in a different direction by considering a more general variation of Algorithm 2.1. Namely, in Step 1, instead of computing $x^k$ such that $\|\nabla L_{\rho_k,\bar{\lambda}^k,\bar{\mu}^k}(x^k)\|_\infty \leq \varepsilon_k$ for a sequence $\varepsilon_k \to 0$, we do so for a sequence $\varepsilon_k$ such that $\varepsilon_k = o(M_k)$, where $M_k = \|(1,\lambda^k,\mu^k)\|_\infty$ as defined in (2.1). A sequence of this type is computed when one aims at achieving a scaled stopping criterion, so that the subproblems may be solved to a less stringent accuracy, improving the efficiency without hindering its robustness. See the discussion and numerical experiments in [6] and [14]. In other words, [14, Theorem 2.1] may be proved under extended RQN, which we state below.

THEOREM 4.3. *Let $\bar{x}$ be the limit of a subsequence $\{x^k\}_{k\in K}$ as generated by Algorithm* 2.1 *that satisfies extended RQN where the subproblem tolerance $\varepsilon_k$ is such that $\varepsilon_k = o(M_k)$, where $M_k = \|(1,\lambda^k,\mu^k)\|_\infty$ as defined in* (2.1)*. Then $\bar{x}$ is feasible and $\{M_k\}_{k\in K}$ is bounded. In particular, all limit points of these sequences are Lagrange multipliers associated with $\bar{x}$.*

*Proof.* The proof is essentially the same as the ones presented previously. In the proof of Theorem 3.3, notice that the first step is to divide $\|\nabla L(x^k,\lambda^k,\mu^k)\|_\infty \leq \varepsilon_k$ by $M_k$ and use that the right-hand side goes to zero. This remains the case under our assumptions. □

**5. Relationship between RQN and other known constraint qualifications.** In this section we analyze the relationship between RQN and other known constraint qualifications from the literature besides CRSC and QN. Next, we recall the *AL-regular constraint qualification* (or *AL-regularity* condition), which is associated with the sequences generated by Algorithm 2.1 [5]. To this end, we consider the function $\mathcal{K}^{\mathrm{AL}}\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by

$$\mathcal{K}^{\mathrm{AL}}(x,\rho) = \nabla h(x)[\rho h(x)] + \nabla g(x)[\rho g(x)]_+.$$

Let $\bar{x}$ be a feasible point for (P). The upper limit of $\mathcal{K}^{\mathrm{AL}}(x,\rho)$ as $x \to \bar{x}$ and $\rho \to \infty$ is the set

$$\limsup_{x\to\bar{x},\rho\to\infty} \mathcal{K}^{\mathrm{AL}}(x,\rho) = \left\{\bar{y} \in \mathbb{R}^n \mid \exists\{(x^k,y^k)\} \to (\bar{x},\bar{y}),\ \exists\{\rho_k\}\to\infty \text{ s.t. } y^k = \mathcal{K}^{\mathrm{AL}}(x^k,\rho_k)\ \forall k\right\}.$$

We have $\mathcal{K}(\bar{x}) \subseteq \limsup_{x\to\bar{x},\rho\to\infty} \mathcal{K}^{\mathrm{AL}}(x,\rho)$ [5], but the contrary inclusion is not always true. AL-regularity is exactly that.

DEFINITION 5.1. *A feasible $\bar{x}$ for* (P) *satisfies the* AL-regularity *condition (or it is an* AL-regular *point) if*

$$\limsup_{x\to\bar{x},\rho\to\infty} \mathcal{K}^{AL}(x,\rho) \subseteq \mathcal{K}(\bar{x}).$$

The AL-regularity condition is in some sense the weakest possible property that guarantees that any feasible limit point of Algorithm 2.1 satisfies the KKT conditions.

THEOREM 5.2. *Let $\bar{x}$ be a feasible limit point of a sequence generated by Algorithm* 2.1*. If $\bar{x}$ satisfies the AL-regularity condition, then $\bar{x}$ satisfies the KKT conditions. Conversely, if, for every objective function that attains a constrained local minimum at $\bar{x}$, the KKT conditions are satisfied, then $\bar{x}$ is an AL-regular point.*

*Proof.* The statement follows from [5, Theorems 1 and 6]. ∎

The above theorem implies that, indeed, AL-regularity is a constraint qualification since every local minimizer is the limit point of a sequence generated by Algorithm 2.1 [5, Theorem 2] (actually, it implies Abadie's constraint qualification [5, Theorem 9]). This fact also gives an alternative proof that RQN is a constraint qualification.

THEOREM 5.3. *RQN implies AL-regularity.*

*Proof.* The statement follows directly from Theorems 3.3 and 5.2, as the boundedness of the multiplier sequence associated with a primal accumulation point of Algorithm 2.1 implies the validity of the KKT conditions. ∎

Another constraint qualification of interest is the constant positive generators (CPG), which we recall next. Let us consider the set

$$\mathcal{K}(x;\bar{x},\mathcal{I},\mathcal{J}) = \left\{ \sum_{i\in\mathcal{I}} \lambda_i \nabla h_i(x) + \sum_{j\in\mathcal{J}} \mu_j \nabla g_j(x) + \sum_{j\in\mathcal{A}_+(\bar{x})} \mu_j \nabla g_j(x) \;\middle|\; \mu_j \geq 0 \quad \forall j \in \mathcal{A}_+(\bar{x}) \right\},$$

where $\mathcal{I} \subseteq \{1,\ldots,m\}$, $\mathcal{J} \subseteq \mathcal{A}_-(\bar{x})$, $\mathcal{A}_+(x) = \mathcal{A}(x)\backslash\mathcal{A}_-(x)$, and $\mathcal{A}_-(x)$ is given in (2.2). In this set, inequality constraints with indices in $\mathcal{J}$ are treated as equalities in the sense that their associated multipliers are free of sign.

DEFINITION 5.4 (CPG [8]). *A feasible point $\bar{x}$ for* (P) *satisfies CPG if there are index sets $\mathcal{I} \subseteq \{1,\ldots,m\}$ and $\mathcal{J} \subseteq \mathcal{A}_-(\bar{x})$ such that the gradients within $\mathcal{K}(\bar{x};\bar{x},\mathcal{I},\mathcal{J})$ are positively linearly independent (that is, the unique way to $0 \in \mathcal{K}(\bar{x};\bar{x},\mathcal{I},\mathcal{J})$ is taking $(\lambda,\mu)=0$) and*

$$\mathcal{K}(x;\bar{x}) \subseteq \mathcal{K}(x;\bar{x},\mathcal{I},\mathcal{J})$$

*for all $x$ in a neighborhood of $\bar{x}$.*

It is known that CRSC implies CPG [8], which in turn implies AL-regularity [5]. However, CPG and RQN are independent of each other, as the next examples show.

*Example* 5.5 (CPG does not imply RQN). Let $\Omega = \{x \in \mathbb{R}^2 \mid g(x) \leq 0\}$, where [8]

$$g_1(x) = x_1^3 + x_2, \quad g_2(x) = x_1^3 - x_2, \quad g_3(x) = x_2^3, \quad g_4(x) = x_1$$

and $\bar{x} = (0,0) \in \Omega$, for which $\mathcal{A}(\bar{x}) = \{1,2,3,4\}$. It is easy to see that $\mathcal{A}_-(\bar{x}) = \{1,2,3\}$, so $\mathcal{A}_+(\bar{x}) = \{4\}$. We have $\nabla g_1(\bar{x}) = (0,1)$, $\nabla g_2(\bar{x}) = (0,-1)$, and $\nabla g_3(\bar{x}) = (0,0)$,

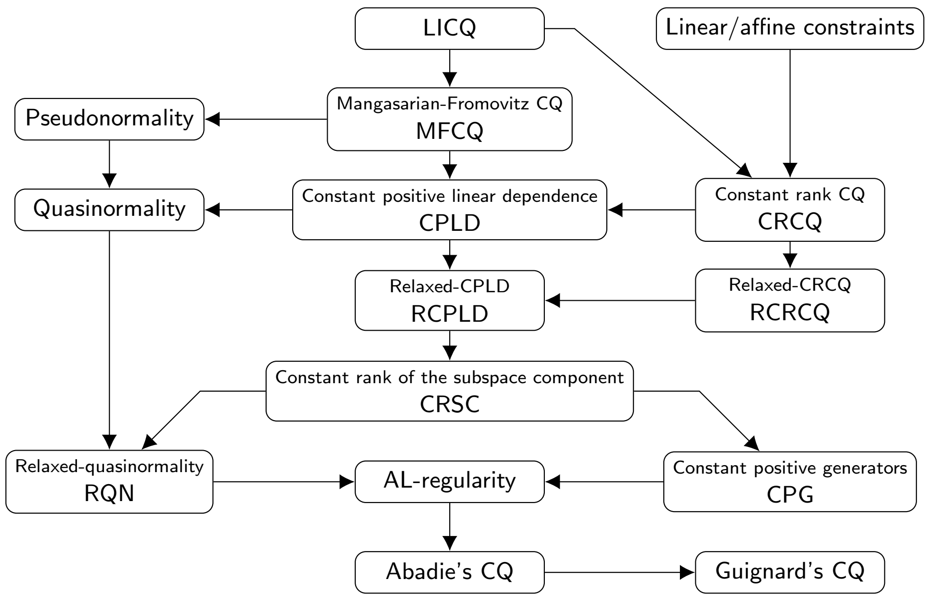


FIG. 1. *Diagram of constraint qualifications for nonlinear programming problems.*

which lead us to take $\mathcal{J}=\{1\}$ in Definition 5.4. Also, $\mathcal{K}(x;\bar{x})=\mathbb{R}_+\times\mathbb{R}=\mathcal{K}(x;\bar{x},\emptyset,\mathcal{J})$ for all $x$, and therefore CPG holds at $\bar{x}$.

To show that RQN does not hold at $\bar{x}$, it is enough to consider $\mu=(0,0,1,0)$ and $x^k=(-1/k^{1/3},1/k)$ for all $k\geq 1$. In fact, we have $\lim_k x^k=\bar{x}$, $\mu_1\nabla g_1(\bar{x})+\mu_2\nabla g_2(\bar{x})+\mu_3\nabla g_3(\bar{x})+\mu_4\nabla g_4(\bar{x})=0$ and, for all $k\geq 1$, $\mu_3 g_3(x^k)=1/k^3>0$ and $(g_1(x^k))_+=(g_2(x^k))_+=(g_4(x^k))_+=0=o(w(x^k))$.

*Example* 5.6 (RQN does not imply CPG). As in Example 2.3, consider the constraints $g_1(x)=-x_1\leq 0$, $g_2(x)=x_1-x_2^2\leq 0$, and the point $\bar{x}=(0,0)$. It was shown previously that QN holds at $\bar{x}$, so RQN also holds. On the other hand, CPG is not valid at $\bar{x}$. In fact, we have $\mathcal{A}_-(\bar{x})=\{1,2\}$. It cannot be $\mathcal{J}=\{1,2\}$ since $\nabla g_1(\bar{x})$ and $\nabla g_2(\bar{x})$ are not positively linearly independent. Furthermore, for any $\delta\neq 0$ we have $(1,-3\delta^2)\in\mathcal{K}((0,\delta);\bar{x})\backslash\mathcal{K}((0,\delta);\bar{x},\emptyset,\{1\})$ and $(-1,0)\in\mathcal{K}((0,\delta);\bar{x})\backslash\mathcal{K}((0,\delta);\bar{x},\emptyset,\{2\})$, so CPG does not hold at $\bar{x}$.

We summarize all the relations discussed in this section in Figure 1, which brings several known constraint qualifications from the literature not considered in this work. The reader is referred to [5] and references therein for details.

**6. Conclusions.** Weak constraint qualifications (in particular, those weaker than MFCQ and LICQ) have been largely used for several different purposes in nonlinear programming and more general optimization problems, namely for studying stability properties, error bound estimates, differentiability of the value function, global convergence of algorithms, among other applications. In particular, several studies have appeared related with constant rank-type constraint qualifications, which are the most well known of these conditions. On the other hand, the quasinormality constraint qualification has appeared in the context of enhanced Fritz John conditions connected with the notion of a more precise (enhanced) class of Lagrange multipliers.

In some sense, constant rank constraint qualifications introduced in recent years dictate that equality constraints should be treated differently than inequality constraints, with the exception of some inequalities that behave like equalities. In this paper we proposed a similar relaxation of the quasinormality condition, which turned out to be connected with the notion of informative Lagrange multipliers, where a Lagrange multiplier that vanishes must also somehow conform to a sign constraint with respect to how the constraints may be violated near the point of interest.

Concerning the global convergence properties of a safeguarded augmented Lagrangian method, several constraint qualifications have been used for this purpose, but only the strongest ones were known to provide boundedness of dual sequences. This property is particularly relevant for complexity analysis and for applications where a dual solution is actually sought (such as in energy pricing applications, among others). In this paper we showed that our relaxed quasinormality condition is enough to ensure this result, which implies that all constant rank-type constraint qualifications also inherit this property. This is particularly relevant due to the pivotal role played by the so-called constant rank of the subspace component condition (CRSC).

Other applications and extensions have been discussed, in particular connected with attaining a feasible limit point and the use of a scaled stopping criterion. We expect future research to expand the applicability of relaxed quasinormality to other areas where quasinormality was previously used.

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