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# INTEGRO-DIFFERENTIAL EQUATIONS WITH LINEAR CONSTRAINTS AND DISCONTINUOUS SOLUTIONS

Uma Introdução

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Chaim Samuel Hönig

In this paper we give a survey of the main results we obtained in the theory of linear differential and integral equations and of equations of a more general type, the Volterra Stieltjes-integral equations. We prove the existence of a resolvent for these equations (§5). We consider also the case when the solutions satisfy a linear constraint, i.e., a generalized boundary condition; in this case we find a Green function for the solutions (§8). We relate our results to those found in the literature and we present also some results and extensions obtained by our students. At the end of this article we suggest several directions of research (§10). Let us mention that the results here presented may be applied to the theory of Optimal Control, to the study of Feedback Systems etc.

Our theory is developed in such a way that discontinuous solutions are allowed; these solutions appear in a natural way in Mathematics, Physics, Technology etc. Let us only recall that the components of an eletric field are discontinuous across the surface of an eletric conductor or across a surface that separates two different dieletrics.

In this survey we avoid, as much as possible, technical definitions and details. For the proofs we refer to our monograph [H] which contains a complete and detailed exposition of the whole theory. An abatract of our main results can be found in [$H\text{-}BAMS_2$].

§1. Given an n-order linear differential equations on an interval [a,b],

$$x^{(n)} + b_1 x^{(n-1)} + \dots + b_n x = \varphi,$$

we recall that if we define $y_1 = x$, $y_2 = x'$, ..., $y_n = x^{(n-1)}$ the n-order differential equation is transformed into a system of n first order linear differential equations

$$\begin{aligned} y_1' - y_2 &= 0 \\ &\vdots \\ y_{n-1}' - y_n &= 0 \\ y_n' + b_1 y_n + \dots + b_n y_1 &= \varphi . \end{aligned}$$

If the function x is real or complex valued, the same is true for the functions $y_1, y_2, \dots, y_n$ and if we consider the vector function

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

the first order system may be written in the form

$$(L') \qquad y' + B.y = g \; ,$$

where

$$B = \begin{bmatrix} 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 0 & -1 \\ b_n & b_{n-1} & b_{n-2} & \dots & b_2 & b_1 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \varphi \end{bmatrix};$$

here $y$ takes values in $\mathbb{R}^n$ or $\mathbb{C}^n$ and for every $t \in [a,b]$, $B(t)$ is a real or complex $n$-order square matrix, hence defines a linear mapping from $\mathbb{R}^n$ or $\mathbb{C}^n$ into itself.

More generally we will consider differential equations of the form (L'), where the functions $y$ and $g$ are defined in $[a,b]$ and take values in a Banach space $X$; we suppose that $B(t) \in L(X)$ (the space of linear continuous mappings from $X$ into itself) for every $t \in [a,b]$.

If $A: [a,b] \to L(X)$ is a primitive of $B$ (i.e., if we have $A(t) = A(a) + \int_a^t B(s)ds$) and $f$ a primitive of $g$, then (L') may be written in the form

$$\text{(L)} \quad y(t) - y(s) + \int_a^t dA(\sigma).y(\sigma) = f(t)-f(s) \text{ for all } s,t \in [a,b];$$

the integral is taken in the sense of Riemann-Stieltjes (see §3). Equation (L) however may be considered even if $A$ and $f$ are not primitives (see §5 and §6).

Analogously if we have a linear Volterra integral equation of the form

$$\text{(V)} \quad y(t) - y(t_o) + \int_{t_o}^t B(t,\sigma).y(\sigma)d\sigma = f(t) - f(t_o), \quad t \in [a,b]$$

and if we define $K(t,\sigma) = \int_{t_o}^{\sigma} B(t,s)ds$ then (V) takes the form

(K) $$y(t) - y(t_0) + \int_{t_0}^{t} d_\sigma K(t,\sigma).y(\sigma) = f(t) - f(t_0);$$

here too we suppose that $y$ and $f$ are defined in $[a,b]$ with values in the Banach space $X$ and that $K(t,\sigma) \in L(X)$; hence equation (L) is a particular case of equation (K); linear delay differential equations may also be reduced to the form (K) (see [H] p. 91, Example D) as well as other types of equations.

The main object of our theory is the study of (K); in §5 we specify the hypothesis made on $y$, $f$ and $K$.

§2. We will now define the type of discontinuous functions $y$ and $f$ we consider in the equations (L) and (K). Let $X$ and $Y$ denote Banach spaces. The discontinuous functions we will work with are the regulated ones, i.e., the functions that have only discontinuities of the first kind. More precisely, we say that a function $f: [a,b] \to X$ is regulated, and we write $f \in G([a,b],X)$, if for every $t \in [a,b[$ there exists $f(t_+) = \lim_{\varepsilon \downarrow 0} f(t+\varepsilon)$ and for every $t \in ]a,b]$ there exists $f(t_-)$.

2.1. $G([a,b],X)$ is a Banach space when endowed with the norm $\|f\| = \sup_{a \le t \le b} \|f(t)\|$ (see [H], I.3.6).

For the solutions of the differential and integral equations or, more generally, of equations of type (L) and (K), in most situations the value of a functions $f$ at a point $t$ does not matter but only matter the limits $f(t_+)$ and $f(t_-)$.

Therefore we also consider the quotient space $\tilde{G}([a,b],X)$ of $G([a,b],X)$ where we take as equivalent two functions $f$ and $g$ such that at every point $t \in [a,b[$ we have $f(t_+) = g(t_+)$ and at every $t \in ]a,b]$ we have $f(t_-) = g(t_-)$, or, equivalently, for all $s$, $t \in [a,b]$ we have $\int_s^t [f(\sigma)-g(\sigma)]d\sigma=0$.

We define

$$G_-([a,b],X) = \{f \in G([a,b],X) \mid f(a)=0,\ f(t_-)=f(t) \text{ for } t \in ]a,b]\}.$$

<u>2.2.</u> <u>The subspace</u> $G_-([a,b],X)$ <u>of</u> $G([a,b],X)$ <u>is isometric to the Banach space</u> $\tilde{G}([a,b],X)$ (see [H], I.3.13).

In general it is simpler to work with $G_-([a,b],X)$ then with $\tilde{G}([a,b],X)$.

§3. We will now give a precise meaning to the integrals in the equations (L) and (K) and in the representation theorem (Theorem 4.1) for linear constraints (§7). We need the following definitions:

A <u>division</u> of the interval $[a,b]$ is a finite sequence $d$: $t_o = a < t_1 < \ldots < t_n = b$; we write $|d| = n$, $\Delta d = \sup_{1\leq i\leq n} |t_i - t_{i-1}|$ and denote by $D$ the set of all divisions of $[a,b]$. For $d_1, d_2 \in D$ we write $d_1 \leq d_2$ or $d_2 \geq d_1$ if every point of $d_1$ is in $d_2$. Given a topological space $E$ and a function $d \in D \longmapsto x_d \in E$ we write $x = \lim_{d \in D} x_d$, where $x \in E$, if for every neighborhood $V$ of $x$ there exists $d_V \in D$ such that $d \geq d_V$ implies $x_d \in V$.

For $\alpha: [a,b] \to L(X,Y)$ and $f: [a,b] \to X$ we define the usual Riemann-Stieltjes integral

$$\int_a^b d\alpha(t).f(t) = \lim_{\Delta d \to 0} \sum_{i=1}^{|d|} [\alpha(t_i)-\alpha(t_{i-1})].f(\xi_i)$$

where $\xi_i \in [t_{i-1},t_i]$ (see [H], [G], [D]) and the <u>interior</u> or Dushnik integral

$$\int_a^b \cdot d\alpha(t).f(t) = \lim_{d \in D} \sum_{i=1}^{|d|} [\alpha(t_i)-\alpha(t_{i-1})].f(\xi_i^{\cdot})$$

where $\xi_i^{\cdot} \in ]t_{i-1},t_i[$ (see [H], [K], [H-ti] p. 96) whenever these limits exist.

<u>3.1. The existence of the first integral implies the existence of the second one (and their equality</u> - see [H], I.1.1).

Reciprocally

<u>3.2. The existence of the second integral implies the existence of the first one and their equality, if</u> $\alpha$ <u>and</u> $f$ <u>are bounded and have no common point of discontinuity, for instance, if one of them is continuous</u> (see [H], I.1.2).

For the first integral we have the integration by parts formula:

<u>3.3. There exists</u> $\int_a^b d\alpha(t).f(t)$ <u>if and only if there exists</u> $\int_a^b \alpha(t).df(t)$ <u>and then we have</u>

$$\int_a^b d\alpha(t).f(t) + \int_a^b \alpha(t).df(t) = \alpha(b).f(b)-\alpha(a).f(a)$$

(see [H], I.1.3).

For the interior integral the integration by parts formula is not valid (see [H], I.4.21 and I.4.22) and this fact turns its use more delicate.

Given a normed space $E$ and $\beta: [a,b] \to E$ we define the variation of $\beta$ (in $[a,b]$) by

$$V[\beta] = V_{[a,b]}[\beta] = \sup_{d \in D} V_d[\beta]$$

where $V_d[\beta] = \sum_{i=1}^{|d|} \|\beta(t_i) - \beta(t_{i-1})\|$.

If $V[\beta] < \infty$ we say that $\beta$ is a function of bounded variation and we write $\beta \in BV([a,b],E)$. For $E = \mathbb{R}$ the functions of bounded variation are exactly the functions that are difference of two monotonic ones.

Given $\alpha: [a,b] \to L(X,Y)$ we define the semivariation of $\alpha$ (in $[a,b]$) by

$$SV[\alpha] = SV_{[a,b]}[\alpha] = \sup_{d \in D} SV_d[\alpha]$$

where $SV_d[\alpha] = \sup\{\| \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot x_i \| \mid x_i \in X, \|x_i\| \leq 1\}$.

If $SV[\alpha] < \infty$ we say that $\alpha$ is a function of bounded semivariation and we write $\alpha \in SV([a,b],L(X,Y))$. If we have furthermore $\alpha(a) = 0$ we write $\alpha \in SV_o([a,b],L(X,Y))$.

3.4. $SV_o([a,b],L(X,Y))$ is a Banach space when endowed with the norm $SV[\alpha]$ (see [H-IME], I.3.3 or [H], I.5.1).

It is not difficult to see that

3.5. $SV([a,b],L(X,Y)) = BV([a,b],X')$ (see [H], p. 23).

3.6. We have $BV([a,b],L(X,Y)) \subset SV([a,b],L(X,Y))$ and these spaces are different if (and only if) $\dim Y = \infty$.

3.7. If $\alpha \in SV([a,b],L(X,Y))$, for every $g \in C([a,b],X)$ there exists $F_\alpha[g] = \int_a^b d\alpha(t).g(t)$ and $\|F_\alpha[g]\| \leq SV[\alpha]\|g\|$, hence $F_\alpha \in L[C([a,b],X),Y]$ (see [H], I.4.6).

3.8. If $\alpha \in SV([a,b],L(X,Y))$, for every $f \in G([a,b],X)$ there exists $F_\alpha[f] = \int_a^b \cdot\, d\alpha(t).f(t)$ and $\|F_\alpha[f]\| \leq SV[\alpha]\|f\|$, hence $F_\alpha \in L[G([a,b],X),Y]$ (see [H], I.4.12).

§4. We recall that the Riesz representation theorem for linear continuous functionals on $C([a,b])$ says that for every $F \in C([a,b])'$ there exists $\alpha \in BV([a,b])$ such that $F = F_\alpha$, i.e., such that for every $\varphi \in C([a,b])$ we have

$$F[\varphi] = \int_a^b \varphi(t)\,d\alpha(t).$$

The importance of the interior integral and of the functions of bounded semivariation lies in the fact that they allow to extend the Riesz representation theorem to the elements of $L[G_-([a,b],X),Y]$:

Theorem 4.1. The mapping

$$\alpha \in SV_o([a,b],L(X,Y)) \longmapsto F_\alpha \in L[G_-([a,b],X),Y]$$

is an isometry (i.e., $\|F_\alpha\| = SV[\alpha]$) of the first Banach space (see 3.4) onto the second one, where for $f \in G([a,b],X)$ we define $F_\alpha[f] = \int_a^b \cdot\, d\alpha(t).f(t)$; for every $t \in [a,b]$ and $x \in X$ we have $\alpha(t).x = F_\alpha[\chi_{]a,t]}x]$, where $\chi_A$ denotes the characteristic function of the set $A$: $\chi_A(s) = 1$ if $s \in A$ and $\chi_A(s) = 0$ if $s \notin A$. (See [H], I.5.1)

See [H], p. 39 to 41 for examples.

More generally, it is not difficult to give a representation theorem for the elements of $L[G([a,b],X),Y]$ (see [H], I.5.6). These theorems extend one of Kaltenborn, [K], proved in the numerical case (i.e., for $X = Y = \mathbb{R}$).

Using the theorem above and other results (specially a generalization of a theorem of Helly: [H], I.5.8) we obtain integral representations for linear continuous mappings between many function spaces, for instance, for the elements of $L[G\_([a,b],X),Z]$ where $Z = G([c,d],Y)$, $C(K,Y)$ etc. (where $K$ is a compact topological space).

Example. Given a function $A: K \times [a,b] \to L(X,Y)$ we write $A \in C^\sigma SV_o^u(K \times [a,b],L(X,Y))$ if $A$ has the following properties:

$(C^\sigma)$: for every $s \in [a,b]$ and $x \in X$ the function $A_s x: t \in K \longmapsto A(t,s)x \in Y$ is continuous.

$(SV_o^u)$: for every $t \in K$ we have $A^t \in SV_o([a,b],L(X,Y))$

and $SV^u[A] = \sup_{t\in K} SV[A^t] < \infty$.

Here, and later on, given a function $A: T \times S \to U$ we write $A^t(s) = A_s(t) = A(t,s)$ for $t \in T$ and $s \in S$.

We have then the following

<u>Theorem 4.2.</u> <u>The mapping</u>

$$A \in C^\sigma SV_o^u(K \times [a,b], L(X,Y)) \longmapsto F_A \in L[G_-([a,b],X), C(K,Y)]$$

<u>is an isometry</u> (i.e., $\|F_A\| = SV^u[A]$) <u>of the first Banach space onto the second, where for every</u> $f \in G_-([a,b],X)$ <u>and</u> $t \in K$ <u>we define</u>

$$F_A[f](t) = \int_a^b d_\sigma A(t,\sigma).f(\sigma);$$

<u>for</u> $(t,\sigma) \in K \times [a,b]$ <u>and</u> $x \in X$ <u>we have</u> $A(t,\sigma)x = F_A[X_{]a,\sigma]}x](t)$ (see [H], I.5.11 and remark 9).

§5. Let us now go back to the study of the Volterra Stieltjes-integral equation

(K) $\qquad y(t) - y(t_o) + \int_{t_o}^t d_\sigma K(t,\sigma).y(\sigma) = f(t) - f(t_o).$

We suppose that $y, f \in G([a,b],X)$ and that the function $K: [a,b] \times [a,b] \to L(X)$ satisfies the properties

(G) For every $s \in [a,b]$ we have $K_s \in G([a,b],L(X))$.

($SV^{uo}$) $\lim_{\varepsilon \downarrow 0} \sup\{SV_{[s-\varepsilon,s+\varepsilon]}[K^t] \mid s,t \in [a,b]\} = 0$

and we write then $K \in G^{uo} = G^{uo}([a,b] \times [a,b], L(X))$. (G) expresses that $K$ is regulated as a function of the first variable.

5.1. $(SV^{uo})$ implies that $K$ is continuous as a function of the second variable ([H], III.1.2).

5.2. $G^{uo}$ is a Banach space when endowed with the norm $|||K||| = \|K\| + SV^u[K]$ where $\|K\| = \sup\{\|K(t,s)\| \mid s, t \in [a,b]\}$ and $SV^u[K] = \sup\{SV[K^t] \mid a \leq t \leq b\}$ (see [H], III.1.4 and III.1.14).

5.3. If $K \in G^{uo}$ then for every $y \in G([a,b],X)$ the function

$$t \in [a,b] \longmapsto \int_{t_o}^{t} d_\sigma K(t,\sigma).y(\sigma) \in X$$

is regulated (see [H], III.1.3).

Hence the equation (K) is well defined.

In order to prove 5.3, as well as the formulas of Dirichlet and the theorem of Bray (see §8) we made in chapters I and II of [H] a carefull analysis of the properties of the interior integral (dependence on parameters ([H], I.5.9) and on the endpoints of the interval ([H], I.4.14); inversion of repeated integrals ([H], II.1.1, II.1.10) etc.). This analysis is particularly delicate since we can not use integration by parts and because a function of bounded semivariation, in general; is not even measurable (in the sense of Bochner-Lebesgue). For instance, by Theorem 4.1 the identical automorphism of $G_-([a,b])$ is represented by the function

$$\alpha : t \in [a,b] \mapsto \chi_{]a,t]} \in G_-([a,b])$$

i.e., for every $\varphi \in G_-([a,b])$ we have $\varphi = \int_a^b \cdot\, d\alpha(t).\varphi(t)$. This function $\alpha$ is not measurable since for $\varepsilon > 0$ there exists no compact $K_\varepsilon \subset [a,b]$ such that $\complement K_\varepsilon$ has Lebesgue measure $< \varepsilon$ and such that the restriction of $\alpha$ to $K_\varepsilon$ is a continuous function; this is obvious since for $a \leq s < t \leq b$ we have $\|\alpha(t) - \alpha(s)\| = \|\chi_{]s,t]}\| = 1$.

In equation (K) we may replace $K(t,\sigma)$ by $K(t,\sigma) - K(t,t)$, i.e., we may suppose that $K(t,t) = 0$ (see [H], p. 88, remark 4); in this case we write $K \in G_o^{uo}$. Analogously we write $U \in G_I^{uo}$ if we have $U \in G^{uo}$ and $U(t,t) = I_X$ (the identical automorphism of $X$).

The next theorem is fundamental for the resolution of the equation (K).

**Theorem 5.4.** Given $K \in G_o^{uo}$ we have

I. There is one and only one element $R \in G_I^{uo}$, the resolvent of (K), such that

$$(R^*) \qquad R(t,s) = I_X - \int_s^t d_\sigma K(t,\sigma).R(\sigma,s) \quad \text{for all} \quad t,\ s \in [a,b]$$

II. For every $f \in G([a,b],X)$ and $x \in X$ the system

$$(K) \qquad y(t)-y(t_o)+\int_{t_o}^t d_\sigma K(t,\sigma).y(\sigma)=f(t)-f(t_o), \quad t \in [a,b]$$

$$y(t_o) = x$$

has one and only one solution $y \in G([a,b],X)$; this solution is given by

$$(\rho) \qquad Y(t) = R(t,t_o)x + \int_{t_o}^{t} R(t,\sigma)df(\sigma)$$

and depends continuously on $x$, $f$ and $K$.

III. We have

$$(R_*) \qquad R(t,s) = I_X + \int_s^t R(t,\sigma).d_\sigma K(\sigma,s) \quad \text{for all } s,t\varepsilon[a,b].$$

IV. The mapping that to every $K \varepsilon G_o^{uo}$ associates its resolvent $R \varepsilon G_I^{uo}$ is a bicontinuous (non linear) bijection from $G_o^{uo}$ onto $G_I^{uo}$. (see [H], III.1.5).

Theorem 5.4 generalizes results of Mac-Nerney proved for the equation (L) (see the comments made after Theorem 6.3). An analogous theorem was proved for equation (K) by Hinton, [Hi], under the hypothesis of bounded variation for $K$ (instead of bounded semivariation) but allowing certain types of discontinuity in the second variable (see [Hi], the definition of the class $F$ on p. 318, and his Theorem 3.1). The techniques used in the proofs in [$M_1$] and [Hi] are completely different from ours and do not extend to the case of functions of bounded semivariation.

In Theorem 5.4 it is not difficult to prove the existence of $R$ satisfying $(R^*)$; the really difficult part however is the proof that $R \varepsilon G^{uo}$ and this is necessary for the integrals in $(\rho)$ and $(R_*)$ to be defined, for the proof that $y$ given by $(\rho)$ satisfies (K) and in the proof of IV.

The analogous of Theorem 5.4 is true if we replace $G^{uo}$ by its subspace $E^{uo}$ of continuous functions (see [H], III.1.27), by its subspace $E^{co}$ of functions K that satisfy

$$(SV^c) \qquad \lim_{t\to t_1} SV[K^t - K^{t_1}] = 0 \quad \text{for every } t_1 \in [a,b],$$

(see [H], III.1.30), by the corresponding spaces $GBV^{uo}$, $CBV^{uo}$, $CBV^{co}$ of functions of bounded variation (see [H], p. 114 remark 9) etc.

§6. We will now particularize Theorem 5.4 to the equation (L), i.e., we suppose that $K(t,\sigma) = A(\sigma)$ (more precisely, $K(t,\sigma) = A(\sigma) - A(t)$, since $K(t,t) = 0$); if we take then equation (K) for $t$ and $s$, by subtraction we obtain

$$(L) \qquad y(t) - y(s) + \int_s^t dA(\sigma)\cdot y(\sigma) = f(t) - f(s) \quad \text{for all } s,t \in [a,b].$$

Property $(SV^{uo})$ for K implies that $A: [a,b] \to L(X)$ satisfies

$$(SV^o) \qquad \lim_{\delta\downarrow 0} SV_{[s-\delta,s+\delta]}[A] = 0 \quad \text{for every } s \in [a,b].$$

<u>6.1.</u> <u>If</u> $A: [a,b] \to L(X,Y)$ <u>satisfies</u> $(SV^o)$, <u>then</u> A <u>is a continuous function of bounded semivariation.</u>

We ignore if reciprocally every continuous function $A \in SV([a,b], L(X,Y))$ satisfies $(SV^o)$; this is true if Y is reflexive or, more generally, weakly sequentially complete.

We fix now a point $\bar{0} \in [a,b]$ and write $A \in A_{\bar{0}}$ if $A: [a,b] \to L(X)$ satisfies $(SV^o)$ and if $A(\bar{0}) = 0$.

6.2. $\mathcal{A}_{\bar{0}}$ is a Banach space when endowed with the norm $A \mapsto SV[A]$.

For functions $R: [a,b] \times [a,b] \to L(X)$ let $(SV_{uo})$, $(SV_o)$ and $(SV_c)$ denote the analogous for the first variable, of the properties $(SV^{uo})$, $(SV^o)$ and $(SV^c)$ in the second variable. We say that $R$ is harmonic, or an harmonic operator, we write $R \in \mathcal{H} = \mathcal{H}([a,b] \times [a,b], L(X))$, if $R$ satisfies $(SV^{uo})$, $(SV^c)$, $(SV_{uo})$, $(SV_c)$ and

(o) $R(t,t) = I_X$, $R(t,\sigma).R(\sigma,s) = R(t,s)$ for all $t,s,\sigma \in [a,b]$.

We denote by $\mathcal{H}^{co}$ the set $\mathcal{H}$ with the topology induced by $G^{uo}$ (see 5.2); analogously we define $\mathcal{H}_{co}$.

Theorem 6.3. A. Given $A \in \mathcal{A}_{\bar{0}}$ we have

I. There is one and only one $R \in \mathcal{H}$, the resolvent of (L) or $A$, such that

$$R(t,s) = R(\zeta,s) - \int_{\zeta}^{t} dA(\sigma).R(\sigma,s) \quad \text{for all} \quad s,\zeta,t \in [a,b].$$

II. For every $f \in G([a,b],X)$ and $x \in X$ the equation (L) has one and only one solution $y \in G([a,b],X)$ such that $y(s) = x$; this solution is given by

$$y(t) = R(t,s)x + \int_{s}^{t} R(t,\sigma)df(\sigma)$$

and depends continuously on $x$, $f$ and $A$.

III. We have

$$R(t,s) = R(t,\sigma) + \int_{\sigma}^{s} R(t,\zeta).dA(\zeta) \quad \underline{\text{for all}}\ s,\sigma,t \in [a,b].$$

IV. <u>We have</u>

$$A(t) = \int_{t}^{\overline{0}} d_{\sigma} R(\sigma,s).R(s,\sigma) \quad \underline{\text{for any}} \quad s \in [a,b].$$

B. <u>If</u> $R: [a,b] \times [a,b] \to L(X)$ <u>satisfies</u> (o) <u>and</u> $(SV_o)$ <u>then</u> $R \in H$ <u>and</u> R <u>is the resolvent of</u> A <u>given in</u> IV.

C. <u>On</u> $H$ <u>the topologies of</u> $H^{co}$ <u>and</u> $H_{co}$ <u>coincide and the mapping that to every</u> $A \in A_{\overline{0}}$ <u>associates its resolvent</u> $R \in H$ <u>is a bicontinuous (nonlinear) bijection from</u> $A_{\overline{0}}$ <u>onto</u> $H$. (See [H], III.2.1, III.2.2 and III.2.3).

Parts A and B of Theorem 6.3 extend to the case of bounded semivariation, the Theorems 3.1 and 3.3 of Mac-Nerney, $[M_1]$, formulated for the case where A is a continuous function of bounded variation; see also [W].

§7. We will now look for solutions y of (K) that satisfy an equation

(F) $$F[y] = c$$

where $F \in L[G([a,b],X),Y]$; (F) is called a <u>linear constraint</u>.

Examples of linear constraints:

1. Initial conditions: we take Y = X and $\zeta \in [a,b]$; $F[y] = y(\zeta)$.

2. Boundary conditions: we take Y = X and

$F[y] = \alpha.y(a) + \beta.y(b)$ where $\alpha, \beta \in L(X)$.

3. Periodicity conditions: Theorem 5.4 extends to open intervals $]a,b[$ (see [H], p. 114 remark 10); we take $]a,b[ = \mathbb{R}$, the locally convex space $Y = G(\mathbb{R},X)$ and we give $p > 0$ (the period); we define $F[y](t) = y(t+p)-y(t)$, $t \in \mathbb{R}$.

4. Left discontinuity: we take $Y = X^2$, $\zeta \in ]a,b]$ and $F[y] = (y(\zeta), y(\zeta_-))$.

5. Multiple point condition or the Nicoletti problem: we give $t_1, \ldots, t_m \in [a,b]$ and $A_1, \ldots, A_m \in L(X,Y)$; $F[y] = \sum_{j=1}^{m} A_j.y(t_j)$.

6. Integral conditions: we give $\alpha \in SV([a,b], L(X,Y))$ and $F[y] = \int_a^b \cdot\, d\alpha(t).y(t)$.

7. Interface conditions: we give $\zeta \in ]a,b[$ and $A_-$, $A$, $A_+ \in L(X,Y)$; $F[y](t) = A_-.y(\zeta_-) + A.y(\zeta) + A_+.y(\zeta_+)$.

8. Integral equations: we give $Y = C(K,Z)$ and $A \in C^\sigma SV_o^u(K \times [a,b], L(X,Z))$ (see Theorem 4.2); $F[y](t) = \int_a^b \cdot\, d_\sigma A(t,\sigma).y(\sigma)$, $t \in K$.

§8. We consider the system (K), (F) and we suppose that $K \in E_o^{uo}$ (see the end of §5); hence the resolvent $R$ of (K) satisfies $R \in E_I^{uo}$ and is therefore a continuous function. For every $s \in [a,b]$ we define $F[R_s] \in L(X,Y)$ by $F[R_s]x = F[R_s x]$, where $x \in X$ ($R_s x$ denotes the function $t \in [a,b] \longmapsto R(t,s)x \in X$). By Theorem 4.1, to $F \in L[G([a,b],X),Y]$ corresponds a function

$\alpha \in SV_o([a,b],L(X,Y))$ such that for every $g \in C([a,b],X)$ or $g \in C([a,b],L(X))$ we have $F[g] = \int_a^b d\alpha(t).g(t)$ (see 3.2); hence

<u>8.1.</u> <u>For every</u> $s \in [a,b]$ <u>we have</u>

$$F[R_s] = \int_a^b d\alpha(t).R(t,s) \in L(X,Y_o)$$

<u>where</u> $Y_o = F[K^{-1}(0)] = \{F[y] \in Y | K[y] = 0,\ y \in C([a,b],X)\}$ <u>and</u> $K[y] = f$ <u>is an abridged way of writing</u> (K).

We define $J(s) = F[R_s]$, $s \in [a,b]$.

<u>8.2.</u> $J: [a,b] \to L(X,Y)$ <u>satisfies</u> $(SV^o)$ (see [H], III.3.3).

<u>8.3.</u> <u>In the particular instance of the equation</u> (L), J <u>satisfies the adjoint equation</u>

$$J(t) - J(s) - \int_s^t J(\sigma).dA(\sigma) = 0, \quad s,\ t \in [a,b]$$

(see [H], III.3 5).

It is not difficult to prove the

<u>Theorem 8.4.</u> <u>The following properties are equivalent:</u>

(i) $y = 0$ <u>is the only solution of the system</u> $K[y] = 0$, $F[y] = 0$.

(ii) <u>For every</u> $c \in Y_o$ <u>the system</u> $K[y] = 0$, $F[y] = c$ <u>has one and only one solution.</u>

(iii) <u>The mapping</u> $y \in K^{-1}(0) \longmapsto F[y] \in Y_o$ <u>is bijective.</u>

(iv) $J(t_o): X \to Y_o$ <u>is bijective (continuous but not bicontinuous in general)</u> (see [H], III.3.6).

<u>From now on we suppose that the equivalent properties of Theorem 8.4 are satisfied by the system</u> (K), (F).

<u>8.5.</u> <u>The following properties are equivalent:</u>

(a) <u>The solution</u> $y_c$ <u>of</u> $K[y] = 0$, $F[y] = c$ <u>depends continuously on</u> $c \in Y_o$.

(b) <u>The operator</u> $J(t_o)^{-1}: Y_o \to X$ <u>is continuous</u>.

(c) $Y_o$ <u>is closed in</u> $Y$.

(see [H], III.3.30).

We will now look for a Green function of the system (K), (F), i.e., a function

$$G: [a,b] \times [a,b] \to L(X,Y)$$

such that the solution of $K[y] = g$, $F[y] = 0$, where $g \in C([a,b],X)$, is given by

$$y(t) = \int_a^b G(t,s).dg(s) \; .$$

We will proceed in a heuristic way to find the Green function and see the properties we need in order to justify our procedure.

Let $y$ be the solution of the system $K[y] = g$, $F[y] = 0$; by ($\rho$) of Theorem 5.4 we have

$$y(t) = R(t,t_o).y(t_o) + \int_{t_o}^{t} R(t,\sigma)dg(\sigma)$$

and if we apply $F$ to this equality we get

$$0 = J(t_o)y(t_o) + F\left[\int_{t_o}^{t} R(t,\sigma)dg(\sigma)\right]$$

i.e., $y(t_o) = J(t_o)^{-1}F\left[\int_{t}^{t_o} R(t,\sigma)dg(\sigma)\right]$; if we replace this in the equation for $y$ we obtain

$$y(t)= R(t,t_o).J(t_o)^{-1}.F\left[\int_{\zeta}^{t_o} R(\zeta,\sigma)dg(\sigma)\right] + \int_{t_o}^{t} R(t,\sigma)dg(\sigma) .$$

We write $\bar{J}(t) = R(t,t_o).J(t_o)^{-1} \in L(Y_o,X)$ and take the decomposition $\int_{\zeta}^{t_o} = \int_{\zeta}^{b} - \int_{t_o}^{b}$ :

$$y(t)=\bar{J}(t)\{F\left[\int_{\zeta}^{b} R(\zeta,\sigma)dg(\sigma)\right] - F\left[\int_{t_o}^{b} R(\zeta,\sigma)dg(\sigma)\right]\}+\int_{t_o}^{t} R(t,\sigma)dg(\sigma) =$$

$$= \bar{J}(t)\{\int_a^b d\alpha(\zeta)\left[\int_{\zeta}^{b} R(\zeta,\sigma)dg(\sigma)\right] - \int_a^b d\alpha(\zeta)\left[\int_{t_o}^{b} R(\zeta,\sigma)dg(\sigma)\right]\}+\int_{t_o}^{b} R(t,\sigma)dg(\sigma).$$

At this point in order to proceed, the formula of Dirichlet and the Theorem of Bray proved in chapter II of [H] (II.1.10, II.1.1 and II.2.4) are essential; they assure, respectively, that we have

$$\int_a^b d\alpha(\zeta)\left[\int_{\zeta}^{b} R(\zeta,\sigma)dg(\sigma)\right] = \int_a^b\left[\int_a^{\sigma} d\alpha(\zeta).R(\zeta,\sigma)\right]dg(\sigma)$$

$$\int_a^b d\alpha(\zeta)\left[\int_{t_o}^{b} R(\zeta,\sigma)dg(\sigma)\right] = \int_{t_o}^{b}\left[\int_a^{b} d\alpha(\zeta).R(\zeta,\sigma)\right]dg(\sigma)$$

and by the definition of $J(\sigma)$ the last integral is equal to $\int_{t_o}^{b} J(\sigma)dg(\sigma)$; hence we have

$$y(t) = \bar{J}(t)\{\int_a^b\left[\int_a^{\sigma} d\alpha(\zeta).R(\zeta,\sigma)\right]dg(\sigma) - \int_{t_o}^{b} J(\sigma)dg(\sigma)\}+\int_{t_o}^{t} R(t,\sigma)dg(\sigma).$$

We cannot put $\bar{J}(t)$ under sign of integration since $\bar{J}(t)$ is only defined on $Y_o$ and under the sign of integration we have elements of $Y$; however if we make the additional hypothesis that $Y_\alpha = Y_o$, where

$$Y_\alpha = \{\int_a^b d\alpha(t).f(t) \mid f \in G([a,b],X)\}$$

then the Green function

$$G(t,s) = \bar{J}(t).\int_a^s d\alpha(\zeta).R(\zeta,s) - Y(s-t_o)\bar{J}(t).J(s) + [Y(s-t_o)+Y(s-t)]R(t,s)$$

is well defined and it is immediate that we have

$$y(t) = \int_a^b G(t,s)dg(s).$$

<u>8.6.</u> <u>For the equation</u> (L) <u>we have</u>

$$R(t,s) = J(t)^{-1}.J(s) \quad \underline{\text{and}} \quad \bar{J}(t) = J(t)^{-1}$$

(see [H], III.3.7).

Hence for the problem (L), (F) the Green function reduces to

$$G(t,s) = J(t)^{-1}\int_a^s d\alpha(\zeta).R(\zeta,s) + Y(s-t)R(t,s) .$$

The heuristic procedure above may be completely justified and generalized and leads us to the following

<u>Theorem 8.7.</u> <u>If</u> K <u>and</u> F <u>satisfy the equivalent hypothesis of Theorem 8.4 and if</u> $Y_\alpha = Y_o$ <u>we have</u>

A. <u>The system</u> $K[y] = g$, $F[y] = c$ <u>has a solution</u> $y \in C([a,b],X)$ <u>if and only if</u> $(g,c) \in C([a,b],X) \times Y_o$; <u>then</u>

*this solution is given by*

$$y(t) = \bar{J}(t)c + \int_a^b G(t,s)dg(s)$$

*and (for* $c$ *fixed) the mapping* $g \mapsto y$ *is continuous.*

B. *The system* $K[y] = f$, $F[y] = c$ *has a solution* $y \in G([a,b],X)$ *if and only if* $c - F[f] \in Y_o$; *then this solution is given by*

$$y(t) = f(t) + \bar{J}(t)[c-F(f)] - \int_a^b G(t,s)d_s\left[\int_{t_o}^s d_\sigma K(s,\sigma).f(\sigma)\right];$$

*the mapping* $f \in G([a,b],X) \mapsto G_K(f) \in C([a,b],X)$ *is continuous, where*

$$G_K(f)(t) = \int_a^b G(t,s)d_s\left[\int_{t_o}^s d_\sigma K(s,\sigma).f(\sigma)\right]$$

(see [H], III.3.28, III.3.30 and III.3.31).

The existence of a Green function satisfying A of Theorem 8.7 implies that $\alpha \in SV_o([a,b],L(X,Y_o))$; we do not know if it implies that $Y_\alpha = Y_o$.

The Green function is characterized in the following theorem.

**Theorem 8.8.** *The Green function*

$$G: [a,b] \times [a,b] \to L(X)$$

*has the following properties:*

$(G_o)$ $F[G_s] = 0$ *for every* $s \in [a,b]$;

$(G_1)$ $G_s(t)-G_s(t_o)+\int_{t_o}^t d_\sigma K(t,\sigma).G_s(\sigma)=[-Y(s-t)+Y(s-t_o)]\,I_X$;

$(G_2)$ $\tilde{G}^t(s) + \int_a^s \tilde{G}^t(\sigma).d_\sigma K(\sigma,s) = \bar{J}(t).\alpha(s)$

<u>where</u> $\tilde{G}(t,\sigma) = G(t,\sigma)+Y(\sigma-t)R(t,\sigma)+Y(\sigma-t_o)[\bar{J}(t).J(\sigma)-R(t,\sigma)]$;

$(G_3)$ <u>For every</u> $s \in [a,b]$, $G_s$ <u>is continuous for</u> $t \neq s$;

$(G_4)$ $\sup_{a \leq t \leq b} SV[G^t] < \infty$ <u>and</u> $G(t,b) = 0$, $G(t,a) = 0$ <u>for</u> $a < t \leq b$, $G(a,a) = -I_X$; (see [H], III.3.29).

$(G_1)$ says that $G_s$ satisfies $\frac{d}{dt}K[G_s] = \delta_{(s)}$, in the sense of the Theory of Distributions; by Theorem 8.7 $(G_o)$ and $(G_1)$ determine the Green function $G$.

In the case of the equation (L), $(G_2)$ reduces to

$(G_2')$ $G(t,s)+Y(s-t)R(t,s)-\int_a^s [G(t,\sigma)+Y(\sigma-t)R(t,\sigma)].dA(\sigma) =$

$$= J(t)^{-1}.\alpha(s).$$

We mention again that all results of §2 to §8 may be extended to open intervals $]a,b[$ and $Y$ a separated sequentially complete locally convex topological vector space (see [H]). For concrete examples see [H], p. 146 to 150.

§9. In the preceding §§ we presented our main results from [H]. Our students generalized many of these results and are studing other questions related to them.

For instance, in [S] Maria Ignez Souza extended Theorem 6.3 to the case where $A$ allows discontinuities; her results generalize theorems of Hildebrandt, [H-ie], proved under the restriction of bounded variation. M. I. de Souza is now trying to generalize Theorem 5.4 in the same way.

In [C], Carmen Silvia Cardassi proved that the existence of the derivatives $\frac{\partial K}{\partial t}$ or $\frac{\partial K}{\partial s}$ for K in several different senses, implies the existence of $\frac{\partial R}{\partial t}$ or $\frac{\partial R}{\partial s}$ with the corresponding meaning. The existence of $\frac{\partial K}{\partial t}$ and of f' implies then that (ρ) in Theorem 5.4 may be written as

$$y(t) = R(t,t_o)x+\int_{t_o}^{t} R(t,\sigma).f'(\sigma)d\sigma,$$

and that there exists

$$y'(t) = \frac{\partial R}{\partial t}(t,t_o)x + f'(t) + \int_{t_o}^{t} \frac{\partial R}{\partial t}(t,\sigma)f(\sigma)d\sigma.$$

The existence of $\frac{\partial K}{\partial s}$ implies the differentiability of y with respect to $t_o$:

$$\frac{\partial y}{\partial t_o}(t) = \frac{\partial R}{\partial t_o}(t,t_o)x-R(t,t_o).f'(t_o).$$

In [He] Sara Zisel Herscowicz studies classes of functions associated by the Riemann-Stieltjes integral (see §2 of [H-OP]) and in particular gives a complete proof of the Theorem 2.1 mentioned in [H-OP]. Galdino Cesar da Rocha Filho in [$R_2$] proves that there are other "homogeneous" classes (i.e., spaces of functions stable under certain groups of transformations of [a,b]) besides those given in Theorem 2.1 of [H-OP]. He does also a carefull analysis of associated classes.

In [H-DS] we prove the formulas of Dirichlet and of substitution under different hypothesis then those of chapter II of [H] and for different types of integrals. We also study the class of integrable functions that satisfy the Darboux criterium of integrability.

In [$R_1$] Galdino Cesar da Rocha Filho studies axiomatically different generalizations (in the sense of [H-ti]) of the Riemann-Stieltjes integral (see §12 of [H-OP]), in particular he proves that for these generalizations it is impossible to keep simultaneously all basic properties of the usual Riemann-Stieltjes integral.

Samy Elias Arbex is working in problems related to VI of §10. José Carlos Fernandez de Oliveira investigates certain nonlinear functional differential equations (see III of §10) and is relating them to questions in dynamical systems.

§10. In [H-OP] we gave some 80 open problems related to our work. The following problems have by now been partially or completely solved: (2.1), (2.2), (2.3), (6.5), (6.6), (7.1), (7.7), (7.8), (8.1) and (12.1).

We suggest now some directions of research we consider particularly interesting and promising:

I. Non linear equations. We consider the equation

$$y(t) - y(t_o) + \int_{t_o}^{t} d_\sigma K(t,\sigma).y(\sigma) = f[t,y(t)]-f[t_o,y(t_o)].$$

By Theorem 5.4 every solution of this equation is a solution of the Hammerstein equation

$$y(t) = R(t,t_o).y(t_o)+\int_{t_o}^{t} R(t,\sigma)d_\sigma f[\sigma,y(\sigma)].$$

If $y$ also has to satisfy a linear constraint (F) then by Theorem 8.7 we have to solve the equation

$$y(t) = \bar{J}(t)c + \int_a^b G(t,\sigma)\,d_\sigma f[\sigma,y(\sigma)].$$

For some partial results in the study of these equations see [H-IME], p. 121; see also [H-OP], problems (11.1) to (11.3).

II. Abstract differential equations and partial differential equations. See [H-OP], problems (11.6) and (11.7).

III. Functional differential equations. We consider equations of the type

$$(\bar{K}) \quad y(t)-y(s)+\int_a^b d_\sigma[K(t,\sigma)-K(s,\sigma)]\cdot y(\sigma) = f[t,y(t)]-f[s,y(s)],$$

eventually with a linear constraint. $(\bar{K})$ contains as particular instances the equations (K), (L), differential equations with deviating argument etc. See [H-OP], problems (9.2) to (9.5).

IV. Periodic solutions, bounded solutions, quasiperiodic solutions. In the case where $]a,b[ = \mathbb{R}$ it is obviously very important to find solutions with these properties, specially in the non linear case. For some results in this directions see [H-IME], p. 123; see also [H-OP], problems (8.4) to (8.9) and (9.1).

V. Non linear constraints. This is another important direction of research; we look for solutions $y$ of (L), (K) or

$(\overline{K})$ such that $\Phi[y] = c$ where $\Phi$ is a continuous (non linear) mapping from $G([a,b],X)$ into $Y$, satisfying adequate hypothesis.

VI. Extensions of existing results to a Banach space and linear constraints context. There are several groups of mathematicians working in questions related to those we exposed here. Mostly they work in $X = \mathbb{R}^n$ and even if they work in Banach spaces the linear constraints they consider are of a particular type, i.e., defined only by functions of bounded variation. It would be very interesting to extend their results to Banach spaces and to general linear constraints.

We mention specially the following groups:

1. The russian mathematicians that work at Tambov: Rachmatulina, Maximov, Azbelev and others. They work mainly on equations of type $(\overline{K})$; see [Ma] and the references given there.

2. In Czechoslovakia there are Kurzweil, T.rdý, Vejvoda, Schwabik and others; see [T] and the references given there.

3. In the United States there are Krall, Brown, Bryan, Green and others; see [B-K] and the references given there.

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