Anal Math., **51** (1) (2025), 211–224
https://doi.org/10.1007/s10476-025-00070-z
First published online February 24, 2025

THE BISHOP–PHELPS–BOLLOBÁS PROPERTY FOR OPERATORS DEFINED ON c_0-SUM OF EUCLIDEAN SPACES

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(Received August 18, 2023; revised July 25, 2024; accepted August 30, 2024)

Abstract. The main purpose of this paper is to study the Bishop–Phelps–Bollobás property for operators on c_0-sum of Euclidean spaces. We show that the pair $(c_0(\bigoplus_{k=1}^{\infty} \ell_2^k), Y)$ has the Bishop–Phelps–Bollobás property for operators (shortly BPBp for operators) whenever Y is a uniformly convex Banach space.

1. Introduction

In 1961, Bishop and Phelps [6] proved that, for any Banach space, the subset of norm attaining functionals is dense in the topological dual space. This result is known as Bishop–Phelps theorem. These authors posed the problem of possible extensions of such a result to operators. In 1963, Lindenstrauss [15], started the study of extensions of Bishop–Phelps theorem for operators. In full generality there is no parallel version of Bishop–Phelps theorem for operators. Motivated by this result, there has been an effort of many authors to study some geometric conditions of the Banach spaces X and Y in order to get the Bishop–Phelps theorem for operators. In 1970, Bollobás [7], proved a "quantitative" version of the Bishop–Phelps theorem, which stated that every norm one functional and its almost norming points can be approximated by a norm attaining functional and its norm attaining point. The result is known nowadays as the Bishop–Phelps–Bollobás theorem. In 2008, Acosta, Aron, Garcia and Maestre [3] dealt with "quantitative" versions of the Bishop–Phelps theorem for operators. They defined a

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Key words and phrases: Banach space, Bishop–Phelps–Bollobás theorem, norm attaining operator.

Mathematics Subject Classification: primary 46B04, secondary 46B07, 46B20.

new notion for a pair of Banach spaces, which is called the Bishop–Phelps–Bollobás property for operators, and provided many notable results. We recommend the surveys [2] and [12] on the recent progress concerning the Bishop–Phelps–Bollobás property.

Many references in the field have appeared, among others, [1,3–5,8–11, 13,14].

In [3] the authors showed that (ℓ_∞^n, Y) satisfies the Bishop–Phelps–Bollobás property for operators for every $n \in \mathbb{N}$, whenever Y is a uniformly convex Banach space. They also raised the question if (c_0, Y) satisfies the BPBp for operators, whenever Y is a uniformly convex Banach space. In this sense, S. K. Kim [14], answered the question in a positive way. More generally, G. Choi and S. K. Kim [11] proved that $(c_0(\bigoplus_{k=1}^\infty X), Y)$ has BPBp for operators, if X is uniformly convex Banach space and Y is $\mathbb{C}$-uniformly convex Banach space. Every uniformly convex complex space is $\mathbb{C}$-uniformly convex but the converse is not true.

The purpose of this paper is to show that $\big(c_0\big(\bigoplus_{k=1}^\infty \ell_2^k\big), Y\big)$ satisfies the BPBp for operators whenever Y is a uniformly convex Banach space. In this sense, we obtained a different result than the one in [11], since the c_0-sum is composed with different spaces. Notice that the Banach space $c_0\big(\bigoplus_{k=1}^\infty \ell_2^k\big)$, known as c_0-sum of the Euclidean n-spaces, is not isometric to c_0. The importance of such a space is due to the fact that C. Stegall, in [16], showed that $\ell_\infty\big(\bigoplus_{k=1}^\infty \ell_2^k\big)$ does not have the Dunford–Pettis property, but its predual, $\ell_1\big(\bigoplus_{k=1}^\infty \ell_2^k\big)$ does.

Each $x \in c_0\big(\bigoplus_{k=1}^\infty \ell_2^k\big)$ can be represented by $x = \sum_{n=1}^\infty \sum_{k\in I(n)} x_k e_k$, where for every $n \in \mathbb{N}$, $I(n) = \{l \in \mathbb{N} : s(n-1)+1 \le l \le s(n)\}$ with $s\colon \mathbb{N}_0 \to \mathbb{N}_0$ the auxiliar function defined by $s(n) = 0$ if $n = 0$ and $s(n) = 1+2+\cdots+n$ if $n \neq 0$, and (e_j) is the standard basis of $c_0\big(\bigoplus_{k=1}^\infty \ell_2^k\big)$. The norm of x is given by the formula $\|x\| := \sup_{n\in\mathbb{N}}\big(\sum_{k\in I(n)}|x_k|^2\big)^{1/2}$.

2. Results

It will be convenient to recall the following notation. Let X and Y be Banach spaces (over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We denote by S_X, B_X, X^* and $\mathcal{L}(X, Y)$, the unit sphere, the closed unit ball, the topological dual space of X and the space of all bounded linear operators from X into Y, respectively. An operator $T \in \mathcal{L}(X, Y)$ is said to attain its norm at $x_0 \in S_X$, if $\|T\| = \|T(x_0)\|$. Now, we recall a few definitions.

Definition 2.1 [3, Definition 1.1]. Let X and Y be real or complex Banach spaces. We say that the pair (X, Y) has the *Bishop–Phelps–Bollobás property for operators* (shortly BPBp for operators) if given $\varepsilon > 0$, there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t\to 0}\beta(t) = 0$ such that for all $T \in S_{\mathcal{L}(X,Y)}$, if

$x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ that satisfy the conditions

$$\|Su_0\| = 1, \quad \|u_0 - x_0\| < \beta(\varepsilon) \quad \text{and} \quad \|S - T\| < \varepsilon.$$

A Banach space X is *uniformly convex* if for every $\varepsilon > 0$ there is a $0 < \delta < 1$ such that for all $x, y \in B_X$ with $\|\frac{x+y}{2}\| > 1 - \delta$, we have $\|x - y\| < \varepsilon$. In this case, the modulus of convexity is given by $\delta(\varepsilon) = \inf\bigl\{1 - \|\frac{x+y}{2}\| : x, y \in B_X,\ \|x - y\| \geq \varepsilon\bigr\}$. A Banach space X is *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_X$ and $x \neq y$. We remark that uniform convexity implies strict convexity, but the converse is not true.

We remark that in the following results we will use similar techniques as can be found in [1,3,5,11,14]. We decided to include the proof of these results for the sake of completeness.

LEMMA 2.2. *Let Y be a strictly convex Banach space and $T : c_0\bigl(\bigoplus_{k=1}^{\infty} \ell_2^k\bigr) \to Y$ a bounded linear operator. If $\|T(x)\| = \|T\|$ for some norm one vector $x = \sum_{n=1}^{\infty} \sum_{k\in I(n)} x_k e_k$, then*

$$T(e_j) = 0 \quad \textit{for all } j \in I(k) \textit{ if } \sum_{i\in I(k)} |x_i|^2 < 1.$$

PROOF. We can assume that $\|T\| > 0$, otherwise nothing is to be proven. Let $x = \sum_{n=1}^{\infty} \sum_{k\in I(n)} x_k e_k$ be an element of $S_{c_0(\bigoplus_{k=1}^{\infty} \ell_2^k)}$ such that $\|T(x)\| = \|T\|$. By the definition of c_0-sum, there is $k_0 \in \mathbb{N}$ such that $\sum_{i\in\ I(k_0)} |x_i|^2 < 1$, where $I(k_0) = \{j, \dots, l\}$. Now, we argue by a contradiction, and assume that $T(e_j) \neq 0$. Let $v := \bigl(x_j \pm \bigl(1 - \bigl(\sum_{i\in I(k_0)} |x_i|^2\bigr)^{1/2}\bigr), x_{j+1}, \dots, x_l\bigr) \in \ell_2^{k_0}$. Then,

$$\begin{aligned}\|v\|_2 &= \left\| \left(x_j \pm \left(1 - \Bigl(\sum_{i\in I(k_0)} |x_i|^2 \Bigr)^{1/2} \right), x_{j+1}, \dots, x_l \right) \right\|_2 \\ &\leq \bigl\|(x_j, x_{j+1}, \dots, x_l)\bigr\|_2 + \left\| \left(1 - \Bigl(\sum_{i\in I(k_0)} |x_i|^2 \Bigr)^{1/2}, 0, \dots, 0 \right) \right\|_2 \\ &= \Bigl(\sum_{i\in I(k_0)} |x_i|^2 \Bigr)^{1/2} + 1 - \Bigl(\sum_{i\in I(k_0)} |x_i|^2 \Bigr)^{1/2} = 1.\end{aligned}$$

This implies that

$$\left\| x \pm \left(1 - \Bigl(\sum_{i\in I(k_0)} |x_i|^2 \Bigr)^{1/2} \right) e_j \right\| = \sup_{n\in\mathbb{N}} \left\{ \Bigl(\sum_{i\in I(n)\setminus I(k_0)} |x_i|^2 \Bigr)^{1/2}, \|v\|_2 \right\} \leq 1.$$

By assumption $\|T(x)\| = \|T\|$, we have naturally $\|T(2x)\| = 2\|T\|$,

$$2\|T\| \le \Big\|T\Big(x + \Big(1 - \Big(\sum_{i\in I(k_0)} |x_i|^2\Big)^{1/2}\Big)e_j\Big)\Big\|$$

$$+ \Big\|T\Big(x - \Big(1 - \Big(\sum_{i\in I(k_0)} |x_i|^2\Big)^{1/2}\Big)e_j\Big)\Big\| \le 2\|T\|.$$

So, $\big\|T\big(x \pm \big(1 - \big(\sum_{i\in I(k_0)}|x_i|^2\big)^{1/2}\big)e_j\big)\big\| = \|T\|$, and

$$\frac{T(x \pm (1 - (\sum_{i\in I(k_0)}|x_i|^2)^{1/2})e_j)}{\|T\|} \in S_Y.$$

Finally,

$$\left\| \frac{\frac{T(x+(1-(\sum_{i\in I(k_0)}|x_i|^2)^{1/2})e_j)}{\|T\|} + \frac{T(x-(1-(\sum_{i\in I(k_0)}|x_i|^2)^{1/2})e_j)}{\|T\|}}{2} \right\| = \left\|\frac{2T(x)}{2\|T\|}\right\| = 1.$$

Since Y is strictly convex we get that $T(e_j) = 0$. This is a contradiction, as we are assuming that $T(e_j) \neq 0$. By similar argument one shows that $T(e_i) = 0$ for each $i \in I(k_0)$. □

Considering the real case, when Y is strictly convex, we prove that if $\big(c_0\big(\bigoplus_{k=1}^\infty \ell_2^k\big), Y\big)$ satisfies BPBp for operators then Y is uniformly convex.

Theorem 2.3. *Let X be the real Banach space $c_0\big(\bigoplus_{k=1}^\infty \ell_2^k\big)$ and let Y be a strictly convex real Banach space. If (X, Y) has the BPBp for operators, then Y is uniformly convex.*

Proof. Suppose that Y is not a uniformly convex Banach space. Then there exist $\varepsilon > 0$ and sequences $(y_k), (z_k) \subset S_Y$ such that

$$\lim_{k\to\infty} \left\|\frac{y_k + z_k}{2}\right\| = 1 \quad \text{and} \quad \|y_k - z_k\| > \varepsilon \quad \text{for all } k. \tag{2.1}$$

For each positive integer $i \in \mathbb{N}$, we define $T_i \colon X \longrightarrow Y$ by

$$T_i(x) = \Big(\frac{x_1 + x_2}{2}\Big)y_i + \Big(\frac{x_1 - x_2}{2}\Big)z_i, \quad x = (x_k) \in X.$$

For each $i \in \mathbb{N}$ and each $x \in S_X$ we have that

$$\|T_i(x)\| \le \frac{1}{2}(|x_1 + x_2| + |x_1 - x_2|) \le 1.$$

As $\|T_i(e_1+e_2)\|=1$, it follows that $\|T_i\|=1$, for each $i\in\mathbb{N}$. We observe that, for each $i\in\mathbb{N}$, $\|T_i(e_1)\|=\left\|\frac{y_i+z_i}{2}\right\|$, thus $\|T_i(e_1)\|$ converges to 1 when $i\to\infty$. This fact, together with the hypothesis that (X,Y) has the BPBp for operators, guarantees that there are $\eta(\varepsilon)>0$, $\beta(\varepsilon)>0$ with $\lim_{t\to 0}\beta(t)=0$, $i_0\in\mathbb{N}$ such that $\|T_{i_0}(e_1)\|>1-\eta(\frac{\varepsilon}{2})$, an operator $R\in S_{\mathcal{L}(X,Y)}$ and a point $u\in S_X$ such that

$$\|R(u)\|=1,\quad \|u-e_1\|<\beta\Big(\frac{\varepsilon}{2}\Big)<1,\quad \|R-T_{i_0}\|<\frac{\varepsilon}{2}. \tag{2.2}$$

Then $\big(\sum_{i\in I(k)}|u_i|^2\big)^{1/2}<1$ for all $k\in\mathbb{N}\setminus\{1\}$, and by Lemma 2.2,

$$R(e_k)=0\quad\text{for all } k\in\mathbb{N}\setminus\{1\}.$$

Therefore, $R(e_1)=R(e_1+e_2)=R(e_1-e_2)$. This implies that

$$\begin{aligned}\|y_{i_0}-z_{i_0}\|&=\|T_{i_0}(e_1+e_2)-T_{i_0}(e_1-e_2)\|\\&=\|T_{i_0}(e_1+e_2)-R(e_1+e_2)+R(e_1-e_2)-T_{i_0}(e_1-e_2)\|\\&\le\|T_{i_0}-R\|\|e_1+e_2\|+\|R-T_{i_0}\|\|e_1-e_2\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

This is a contradiction, so Y is a uniformly convex Banach space. □

We need the next lemma to show the main result. In order to state it, let us recall that for $A\subset\mathbb{N}$ (resp. $A\subset\{1,\dots,n\}$) and $X=c_0\big(\bigoplus_{k=1}^{\infty}\ell_2^k\big)$ (resp. $X=\ell_\infty\big(\bigoplus_{k=1}^{n}\ell_2^k\big)$), $P_A\colon X\to X$ is a projection on the components in A.

Lemma 2.4. *Let $F\subset\mathbb{N}$ and $A=\bigcup_{i\in F}I(i)$. Suppose that $0<\varepsilon<1$ and Y is a uniformly convex Banach space with modulus of convexity $\delta(\varepsilon)$. If $T\in S_{\mathcal{L}(c_0(\bigoplus_{k=1}^{\infty}\ell_2^k),Y)}$ satisfy that $\|TP_A\|>1-\delta(\varepsilon)$, then*

$$\|T(I-P_A)\|\le\varepsilon.$$

Analogously, if $T\in S_{\mathcal{L}(\ell_\infty(\bigoplus_{k=1}^{n}\ell_2^k),Y)}$ and $A\subset\{1,\dots,n\}$ satisfy $\|TP_A\|>1-\delta(\varepsilon)$, then $\|T(I-P_A)\|\le\varepsilon$.

Proof. Let $0<\varepsilon<1$ and $T\in S_{\mathcal{L}(c_0(\bigoplus_{k=1}^{\infty}\ell_2^k),Y)}$ an operator such that $\|TP_A\|>1-\delta(\varepsilon)$. Then there exists an $x\in S_{c_0(\bigoplus_{k=1}^{\infty}\ell_2^k)}\cap P_A\big(c_0\big(\bigoplus_{k=1}^{\infty}\ell_2^k\big)\big)$ such that $\|TP_A(x)\|>1-\delta(\varepsilon)$. Fix an element $y=\sum_{n=1}^{\infty}\sum_{k\in I(n)}y_ke_k\in B_{c_0(\bigoplus_{k=1}^{\infty}\ell_2^k)}$ with $\operatorname{supp}y\subset\mathbb{N}\setminus A$, then

$$\|x\pm y\|=\sup_{j\in\mathbb{N}}\bigg\{\bigg(\sum_{i\in I(j),j\in F}|x_i|^2\bigg)^{1/2},\bigg(\sum_{i\in I(j),j\in\mathbb{N}\setminus F}|y_i|^2\bigg)^{1/2}\bigg\}\le 1.$$

This implies that $\|T(x \pm y)\| \leq 1$ for every $y \in B_{c_0(\bigoplus_{k=1}^{\infty} \ell_2^k)}$ with $\operatorname{supp} y \subset \mathbb{N} \setminus A$. Notice that, for every $z \in B_{c_0(\bigoplus_{k=1}^{\infty} \ell_2^k)}$, the support of the vector $(I - P_A)(z)$ is a subset of $\mathbb{N} \setminus A$ and then, $\|T(x) \pm T(I - P_A)(z)\| \leq 1$. Moreover,

$$\left\| \frac{T(x + (I - P_A)(z)) + T(x - (I - P_A)(z))}{2} \right\| = \|TP_A(x)\| > 1 - \delta(\varepsilon).$$

As Y is uniformly convex Banach space, we conclude that

$$\|T(x + (I - P_A)(z)) - T(x - (I - P_A)(z))\| < \varepsilon.$$

That is, $\|T(I - P_A)(z)\| < \frac{\varepsilon}{2} < \varepsilon$, whenever $z \in B_X$. This shows that $\|T(I - P_A)\| < \varepsilon$. $\square$

In [3] the authors proved that the pair (X, Y) has the BPBp for operators when X and Y are both finite-dimensional. The next proposition claims that the pair (X, Y) has the BPBp for a specific finite dimensional space X and any uniformly convex space Y. We remark that Proposition 2.6 is similar to [11, Theorem 2.4] and can be proven in a completely analogously way, but we give our version for reader's convenience. We need the next lemma to prove Proposition 2.6. We omit the proof because is just modifications of [11, Lemma 2.3].

Lemma 2.5 [11, Lemma 2.3]. *Let Y be a Banach space and $0 < \eta < 1$ be given. Assume that $T \in S_{\mathcal{L}(\ell_\infty(\bigoplus_{k=1}^{n} \ell_2^k), Y)}$, $y^* \in S_{Y^*}$ and $x \in S_{\ell_\infty(\bigoplus_{k=1}^{n} \ell_2^k)}$ satisfy the estimate $y^*(Tx) = \|Tx\| > 1 - \eta$. Then for all $0 < \eta' < 1$, the sets*

$$N = \left\{ k \in \{1, \dots, n\} : \sum_{j \in I(k)} |(T^* y^*)(j)| \neq 0 \right\}$$

and

$$A = \left\{ k \in N : \operatorname{Re} \sum_{j \in I(k)} (T^* y^*)(j) x(j) > (1 - \eta') \sum_{j \in I(k)} |(T^* y^*)(j)| \right\}$$

satisfy the estimate $\sum_{k \in A} \sum_{j \in I(k)} |(T^* y^*)(j)| > 1 - \frac{\eta}{\eta'}$. In particular,

$$\operatorname{Re} \sum_{k \in A} \sum_{j \in I(k)} (T^* y^*)(j) x(j) > \left(1 - \frac{\eta}{\eta'}\right)(1 - \eta').$$

To prove the next proposition, we recall that $NA(X, Y)$ is the subset of $\mathcal{L}(X, Y)$ of all norm attaining operators between X and Y.

Proposition 2.6 [11, Theorem 2.4]. *If Y is a uniformly convex Banach space with modulus of convexity $\delta(\varepsilon)$, then $\big(\ell_\infty(\bigoplus_{k=1}^n \ell_2^k), Y\big)$ has the Bishop–Phelps–Bollobás property for operators.*

Proof. Let $0<\varepsilon<1$. We define $\eta(\varepsilon) = \min\big\{\frac{\varepsilon}{16}, \delta(\frac{\varepsilon}{16}), \delta_1(\frac{\varepsilon}{2}), \dots, \delta_n(\frac{\varepsilon}{2})\big\}$, where $\delta_k(\varepsilon)$ is the modulus of convexity of the spaces ℓ_2^k, for all $k = 1, \dots, n$. Let $T \in S_{\mathcal{L}(\ell_\infty(\bigoplus_{k=1}^n \ell_2^k), Y)}$ and $x_0 = \sum_{k=1}^n \sum_{j\in I(k)} x_0(j)e_j \in S_{\ell_\infty(\bigoplus_{k=1}^n \ell_2^k)}$ such that

$$\|T(x_0)\| > 1 - \frac{\eta(\varepsilon)^6}{64}.$$

Choose a $y_0^* \in S_{Y^*}$ such that $y_0^*(T(x_0)) = \|T(x_0)\|$ and define the subsets

$$N = \Big\{k \in \{1, \dots, n\} : \sum_{j\in I(k)} |(T^* y_0^*)(j)| \neq 0\Big\}$$

and

$$A = \Big\{k \in N : \operatorname{Re} \sum_{j\in I(k)} (T^* y_0^*)(j) x_0(j) > \Big(1 - \frac{\eta(\varepsilon)^3}{8}\Big) \sum_{j\in I(k)} |(T^* y_0^*)(j)|\Big\}.$$

According to Lemma 2.5,

$$\|TP_A\| \geq \|TP_A(x_0)\| \geq |y_0^*(TP_A(x_0))| = |T^* y_0^*(P_A(x_0))| \geq \operatorname{Re} T^* y_0^*(P_A(x_0))$$

$$= \operatorname{Re} \sum_{k\in A} \sum_{j\in I(k)} (T^* y_0^*)(j) x_0(j) > \left(1 - \frac{\frac{\eta(\varepsilon)^6}{64}}{\frac{\eta(\varepsilon)^3}{8}}\right)\Big(1 - \frac{\eta(\varepsilon)^3}{8}\Big) > 1 - \delta\Big(\frac{\varepsilon}{16}\Big).$$

Then, Lemma 2.4 implies that $\|TP_A - T\| < \frac{\varepsilon}{16}$. Now, let

$$\tilde{x}_0 := P_A\Bigg(\sum_{k=1}^n \frac{\sum_{j\in I(k)} x_0(j) e_j}{\big(\sum_{j\in I(k)} |x_0(j)|^2\big)^{1/2}}\Bigg).$$

Then

$$\|T(\tilde{x}_0)\| \geq |y_0^*(T\tilde{x}_0)| = |(T^* y_0^*(\tilde{x}_0))| \geq \operatorname{Re} \sum_{j\in A} \sum_{j\in I(k)} (T^* y_0^*)(j) \tilde{x}_0(j)$$

$$> \sum_{j\in A} \frac{1}{\big(\sum_{j\in I(k)} |x_0(j)|^2\big)^{1/2}} \Big(1 - \frac{\eta(\varepsilon)^3}{8}\Big) \sum_{j\in I(k)} |(T^* y_0^*)(j)|$$

$$\geq \Big(1 - \frac{\eta(\varepsilon)^3}{8}\Big) \sum_{j\in A} \sum_{j\in I(k)} |(T^* y_0^*)(j)| > \Big(1 - \frac{\eta(\varepsilon)^3}{8}\Big)\Big(1 - \frac{\eta(\varepsilon)^3}{8}\Big) > 1 - \frac{\eta(\varepsilon)^3}{4},$$

and

$$\Big(\sum_{j\in I(k)}|\tilde{x}_0(j)-x_0(j)|^2\Big)^{1/2}=\Big\|\frac{\sum_{j\in I(k)}x_0(j)e_j}{\big(\sum_{j\in I(k)}|x_0(j)|^2\big)^{1/2}}-\sum_{j\in I(k)}x_0(j)e_j\Big\|_2$$
$$=\Big|1-\Big(\sum_{j\in I(k)}|x_0(j)|^2\Big)^{1/2}\Big|<\frac{\eta(\varepsilon)^3}{8}\quad\text{for all } k\in A.$$

Choose $y_1^*\in S_Y^*$ such that $y_1^*(T(\tilde{x}_0))=\|T(\tilde{x}_0)\|$. Let $R\colon P_A\big(\ell_\infty\big(\bigoplus_{k=1}^n\ell_2^k\big)\big)\to Y$ be the linear bounded operator defined by

$$R(z)=TP_A(z)+\eta(\varepsilon)y_1^*(TP_A(z))\,\frac{T(\tilde{x}_0)}{\|T(\tilde{x}_0)\|}.$$

As $P_A\big(\ell_\infty\big(\bigoplus_{k=1}^n\ell_2^k\big)\big)$ is finite dimensional,

$$\overline{NA\Big(P_A\Big(\ell_\infty\Big(\bigoplus_{k=1}^n\ell_2^k\Big)\Big),Y\Big)}=\mathcal{L}\Big(P_A\Big(\ell_\infty\Big(\bigoplus_{k=1}^n\ell_2^k\Big)\Big),Y\Big).$$

Then, there is $Q\in NA\big(P_A\big(\ell_\infty\big(\bigoplus_{k=1}^n\ell_2^k\big)\big),Y\big)$ such that $\|Q-R\|<\frac{\eta(\varepsilon)^3}{4}$. That is, there exist $w_0\in S_{P_A(\ell_\infty(\bigoplus_{k=1}^n\ell_2^k))}$ such that $\|Q(w_0)\|=\|Q\|$, $\|Q\|=\|R\|$ and $\|Q-R\|<\frac{\eta(\varepsilon)^3}{4}$. Furthermore, we obtain the following estimate for $\|R(\tilde{x}_0)\|$

$$\|R(\tilde{x}_0)\|=\Big\|TP_A(\tilde{x}_0)+\eta(\varepsilon)y_1^*(TP_A(\tilde{x}_0))\frac{T(\tilde{x}_0)}{\|T(\tilde{x}_0)\|}\Big\|$$
$$=\Big\|T(\tilde{x}_0)+\eta(\varepsilon)y_1^*(T(\tilde{x}_0))\frac{T(\tilde{x}_0)}{\|T(\tilde{x}_0)\|}\Big\|$$
$$=\|T(\tilde{x}_0)+\eta(\varepsilon)\|T(\tilde{x}_0)\|\frac{T(\tilde{x}_0)}{\|T(\tilde{x}_0)\|}\|=\|T(\tilde{x}_0)\|(1+\eta(\varepsilon))$$
$$>\Big(1-\frac{\eta(\varepsilon)^3}{4}\Big)(1+\eta(\varepsilon))=1-\frac{\eta(\varepsilon)^3}{4}+\eta(\varepsilon)\Big(1-\frac{\eta(\varepsilon)^3}{4}\Big)$$

and

$$\|R(\tilde{x}_0)\|\le\|R\|=\|Q\|=\|Q(w_0)\|$$
$$\le\|Q(w_0)-R(w_0)\|+\|R(w_0)\|<\frac{\eta(\varepsilon)^3}{4}+1+\eta(\varepsilon)|y_1^*(T(w_0))|.$$

By composing with an isometry on $P_A\big(\ell_\infty\big(\bigoplus_{k=1}^n \ell_2^k\big)\big)$ if necessary, we may assume that $|y_1^*(T(w_0))| = \operatorname{Re} y_1^*(T(w_0))$. Combining the estimates obtained, we see that $\operatorname{Re} y_1^*(T(w_0)) > 1-\eta(\varepsilon)^2$. Thus,

$$\begin{aligned}\operatorname{Re} y_1^*\Big(T\Big(\frac{w_0+\tilde{x_0}}{2}\Big)\Big) &> \frac{1}{2}\Big(1-\eta(\varepsilon)^2+1-\frac{\eta(\varepsilon)^3}{4}\Big)\\ &= 1-\frac{\eta(\varepsilon)^2+\frac{\eta(\varepsilon)^3}{4}}{2} \geq 1-\eta(\varepsilon)^2.\end{aligned}$$

Now, we define the subset

$$B=\Big\{k\in A : \operatorname{Re}\sum_{j\in I(k)}(T^*y_1^*)(j)\Big(\frac{w_0(j)+\tilde{x_0}(j)}{2}\Big)>(1-\eta(\varepsilon))\sum_{j\in I(k)}|(T^*y_1^*)(j)|\Big\}.$$

Applying Lemma 2.5 we see that

$$\begin{aligned}\|TP_B\| &\geq \Big\|TP_B\Big(\frac{w_0+\tilde{x}_0}{2}\Big)\Big\| \geq \Big|y_1^*\Big(TP_B\Big(\frac{w_0+\tilde{x}_0}{2}\Big)\Big)\Big|\\ &= \Big|T^*y_1^*\Big(P_B\Big(\frac{w_0+\tilde{x}_0}{2}\Big)\Big)\Big| \geq \operatorname{Re}\, T^*y_1^*\Big(P_B\Big(\frac{w_0+\tilde{x}_0}{2}\Big)\Big)\\ &= \operatorname{Re}\sum_{j\in B}\sum_{j\in I(k)}(T^*y_1^*)(j)\Big(\frac{w_0(j)+\tilde{x_0}(j)}{2}\Big)\\ &> \Big(1-\frac{\eta(\varepsilon)^2}{\eta(\varepsilon)}\Big)(1-\eta(\varepsilon)) > 1-\delta\Big(\frac{\varepsilon}{16}\Big).\end{aligned}$$

Then, Lemma 2.4 implies that $\|TP_B-T\|<\frac{\varepsilon}{16}$. Further, for each $k\in B$ we have

$$\begin{aligned}1-\eta(\varepsilon) &< \operatorname{Re}\sum_{j\in I(k)}\frac{(T^*y_1^*)(j)}{\sum_{j\in I(k)}|(T^*y_0^*)(j)|}\Big(\frac{w_0(j)+\tilde{x}_0(j)}{2}\Big)\\ &\leq \Big\|\sum_{j\in I(k)}\frac{w_0(j)+\tilde{x}_0(j)}{2}e_j\Big\|_2.\end{aligned}$$

That is, $1-\delta_k(\frac{\varepsilon}{2}) < \big\|\sum_{j\in I(k)}\frac{w_0(j)+\tilde{x}_0(j)}{2}e_j\big\|_2$ for every $k\in B$. As ℓ_2^k is uniformly convex with modulus of convexity δ_k, then

$$\Big\|\sum_{j\in I(k)}(w_0(j)-\tilde{x}_0(j))e_j\Big\|_2<\frac{\varepsilon}{2}\quad\text{for every } k\in B.$$

Now, we define the linear bounded operator $\tilde{S}\colon P_A\big(\ell_\infty\big(\bigoplus_{k=1}^n \ell_2^k\big)\big) \to Y$ by

$$\tilde{S}(z) = QP_B(z) + Q(I - P_B)U(z),$$

where $U \in B_{\mathcal{L}(P_A(\ell_\infty(\bigoplus_{k=1}^n \ell_2^k)),P_A(\ell_\infty(\bigoplus_{k=1}^n \ell_2^k)))}$ is chosen such that

$$U\bigg(E_k\bigg(\sum_{j\in I(k)} \tilde{x}_0(j)e_j\bigg)\bigg) = E_k\bigg(\sum_{j\in I(k)} w_0(j)e_j\bigg)$$

for every $k \in A$ and $E_k\colon \ell_2^k \to P_A\big(\ell_\infty\big(\bigoplus_{k=1}^n \ell_2^k\big)\big)$ is the k^{th} injection map. Moreover, for each $z = \sum_{k\in A}\sum_{j\in I(k)} z_je_j \in S_{P_A(\ell_\infty(\bigoplus_{k=1}^n \ell_2^k))}$

$$\|\tilde{S}(z)\| = \bigg\|Q\bigg(\sum_{k\in B}\sum_{j\in I(k)} z_je_j + \sum_{k\in A\setminus B}\sum_{j\in I(k)} z_jU(e_j)\bigg)\bigg\| \le \|Q\|,$$

so, $\|\tilde{S}\|\le\|Q\|$. Let $S\colon \ell_\infty\big(\bigoplus_{k=1}^n \ell_2^k\big) \to Y$ be the canonical extension of $\frac{\tilde{S}}{\|\tilde{S}\|}$ and define

$$z_0 = \sum_{k\in B}\sum_{j\in I(k)} w_0(j)e_j + \sum_{k\in A\setminus B}\sum_{j\in I(k)} \tilde{x}_0(j)e_j + \sum_{k\in\{1,\dots,n\}\setminus A}\sum_{j\in I(k)} x_0(j)e_j \in S_{\ell_\infty(\bigoplus_{k=1}^n \ell_2^k)}.$$

Then

$$1 \ge \|S(z_0)\| = \frac{1}{\|\tilde{S}\|}\bigg\|\tilde{S}\bigg(\sum_{k\in B}\sum_{j\in I(k)} w_0(j)e_j + \sum_{k\in A\setminus B}\sum_{j\in I(k)} \tilde{x}_0(j)e_j\bigg)\bigg\|$$
$$\ge \frac{1}{\|Q\|}\bigg\|Q\bigg(\sum_{k\in B}\sum_{j\in I(k)} w_0(j)e_j + \sum_{k\in A\setminus B}\sum_{j\in I(k)} w_0(j)e_j\bigg)\bigg\| = \frac{\|Q(w_0)\|}{\|Q\|} = 1.$$

Thus, S attains its norm at z_0. Furthermore, if $k \in B$ then

$$\bigg(\sum_{j\in I(k)} |z_0(j) - x_0(j)|^2\bigg)^{1/2} \le \bigg(\sum_{j\in I(k)} |w_0(j) - \tilde{x}_0(j)|^2\bigg)^{1/2} + \bigg(\sum_{j\in I(k)} |\tilde{x}_0(j) - x_0(j)|^2\bigg)^{1/2} < \frac{\varepsilon}{2} + \frac{\eta(\varepsilon)^3}{8}.$$

If $k \in A \setminus B$ then

$$\Big(\sum_{j\in I(k)} |z_0(j) - x_0(j)|^2\Big)^{1/2} = \Big(\sum_{j\in I(k)} |\tilde{x}_0(j) - x_0(j)|^2\Big)^{1/2} < \frac{\eta(\varepsilon)^3}{8};$$

and, if $k \in \{1,\dots,n\} \setminus A$ then $\big(\sum_{j\in I(k)}|z_0(j) - x_0(j)|^2\big)^{1/2} = 0$. Thus,

$$\|z_0 - x_0\| = \max\Big\{\Big(\sum_{j\in I(k)} |z_0(j) - x_0(j)|^2\Big)^{1/2} : k \in \{1,\dots,n\}\Big\}$$
$$< \frac{\varepsilon}{2} + \frac{\eta(\varepsilon)^3}{8} < \varepsilon.$$

Finally,

$$\|S - T\| \le \|S - TP_A\| + \|TP_A - T\| = \Big\|\frac{\tilde{S}}{\|\tilde{S}\|} - TP_A\Big\| + \|TP_A - T\|$$
$$\le \Big\|\frac{\tilde{S}}{\|\tilde{S}\|} - \tilde{S}\Big\| + \|\tilde{S} - Q\| + \|Q - R\| + \|R - TP_A\| + \|TP_A - T\|$$
$$< \big|1 - \|\tilde{S}\|\big| + \|Q(P_B - I) + Q(I - P_B)U\| + \frac{\eta(\varepsilon)^3}{4} + \eta(\varepsilon) + \frac{\varepsilon}{16}$$
$$\le \big|1 - \|R\|\big| + 2\|Q(I - P_B)\| + \frac{\eta(\varepsilon)^3}{4} + \eta(\varepsilon) + \frac{\varepsilon}{16}$$
$$< \frac{\varepsilon}{16} + \eta(\varepsilon) + 2\|TP_A - Q\| + 2\|TP_A(I - P_B)\| + \frac{\eta(\varepsilon)^3}{4} + \eta(\varepsilon) + \frac{\varepsilon}{16}$$
$$= \frac{\varepsilon}{16} + \eta(\varepsilon) + \frac{\varepsilon}{8} + 3\Big(\frac{\eta(\varepsilon)^3}{4} + \eta(\varepsilon)\Big) + \frac{\varepsilon}{16} < \varepsilon,$$

and the proof is complete. □

THEOREM 2.7. *If Y is a uniformly convex Banach space, then*

$$\Big(c_0\Big(\bigoplus_{k=1}^{\infty} \ell_2^k\Big), Y\Big)$$

has the Bishop–Phelps–Bollobás property for operators.

PROOF. Given $0 < \varepsilon < 1$, choose $\eta(\varepsilon) > 0$ the positive number in Proposition 2.6. Assume that $T \in S_{\mathcal{L}(c_0(\bigoplus_{k=1}^{\infty} \ell_2^k), Y)}$ and $x = \sum_{n=1}^{\infty} \sum_{k\in I(n)} x_k e_k \in S_{c_0(\bigoplus_{k=1}^{\infty} \ell_2^k)}$ satisfy $\|Tx\| > 1 - \eta(\varepsilon)^2$ and also $\|Tx\| > 1 - \delta(\varepsilon)$, where

$\delta(\varepsilon) > 0$ is the modulus of convexity of Y. Since c_{00} is a dense subspace of $c_0\big(\bigoplus_{k=1}^{\infty} \ell_2^k\big)$, we can choose a vector $u \in S_{c_0(\bigoplus_{k=1}^{\infty} \ell_2^k)}$ with finite support such that

$$\|T(u)\| > 1 - \eta(\varepsilon)^2, \quad \|T(u)\| > 1 - \delta(\varepsilon) \quad \text{and} \quad \|x - u\| < \varepsilon.$$

We define $n = \min\big\{k \in \mathbb{N} : \operatorname{supp} u \subset \bigcup_{j=1}^{k} I(j)\big\}$, and $A = \bigcup_{k=1}^{n} I(k)$. Thus,

$$\|TP_A\| \geq \|TP_A(u)\| = \|T(u)\| > 1 - \delta(\varepsilon) \quad \text{and} \quad \|TP_A\| > 1 - \eta(\varepsilon)^2.$$

According to the Lemma 2.4, $\|T(I - P_A)\| \leq \varepsilon$. Now, let $J \colon \ell_\infty\big(\bigoplus_{k=1}^{n} \ell_2^k\big) \to c_0\big(\bigoplus_{k=1}^{\infty} \ell_2^k\big)$ be the map defined by

$$J(w) = \begin{cases} w_j, & \text{if } j \in A, \\ 0, & \text{if } j \in \mathbb{N} \setminus A. \end{cases}$$

Then,

$$\|J(w)\| = \max_{1 \leq k \leq n} \Big(\sum_{j \in I(k)} |w_j|^2 \Big)^{1/2} = \|w\|$$

for all $w \in \ell_\infty\big(\bigoplus_{k=1}^{n} \ell_2^k\big)$. Let $Q \colon \ell_\infty\big(\bigoplus_{k=1}^{n} \ell_2^k\big) \to Y$ be the bounded linear operator defined by $Q(w) = \frac{TP_A J}{\|TP_A J\|}(w)$ and the vector $z = (z_j) \in \ell_\infty\big(\bigoplus_{k=1}^{n} \ell_2^k\big)$ given by $z_j = u_j$, if $j \in \operatorname{supp} u$ and $z_j = 0$ if $j \in A \setminus \operatorname{supp} u$. It is easy to check that $\|Q\| = \|z\| = 1$. As $\|TP_A J\| \leq 1$, then

$$\|Q(z)\| = \left\| \frac{TP_A J}{\|TP_A J\|}(z) \right\| \geq \|TP_A(u)\| = \|T(u)\|, \tag{2.3}$$

and thus, $\|Q(z)\| > 1 - \eta(\varepsilon)$. By Proposition 2.6 the pair $\big(\ell_\infty\big(\bigoplus_{k=1}^{n} \ell_2^k\big), Y\big)$ has the BPBp for operators, then there are $\beta(\varepsilon) > 0$ with $\lim_{t \to 0} \beta(t) = 0$, $\tilde{R} \in S_{\mathcal{L}(\ell_\infty(\bigoplus_{k=1}^{n} \ell_2^k), Y)}$ and $\tilde{u} \in S_{\ell_\infty(\bigoplus_{k=1}^{n} \ell_2^k)}$, such that

$$\|\tilde{R}(\tilde{u})\| = 1, \quad \|z - \tilde{u}\| < \beta(\varepsilon) \quad \text{and} \quad \|\tilde{R} - Q\| < \varepsilon. \tag{2.4}$$

Let (e_j), (f_j) be the canonical basis of $c_0\big(\bigoplus_{k=1}^{\infty} \ell_2^k\big)$ and $\ell_\infty\big(\bigoplus_{k=1}^{n} \ell_2^k\big)$, respectively, and $R \colon c_0\big(\bigoplus_{k=1}^{\infty} \ell_2^k\big) \to Y$ be the bounded linear operator given by

$$R(y) = \sum_{k=1}^{\infty} \sum_{j \in I(k)} y_j R(e_j),$$

where

$$R(e_j) = \begin{cases} \tilde{R}(f_j), & \text{if } j \in A, \\ 0, & \text{if } j \in \mathbb{N} \setminus A. \end{cases}$$

Moreover, consider the vector $v = (v_j) \in c_0\big(\bigoplus_{k=1}^{\infty} \ell_2^k\big)$ defined by

$$v_j = \begin{cases} \tilde{u}_j, & \text{if } j \in A, \\ x_j, & \text{if } j \in \mathbb{N} \setminus A. \end{cases}$$

So $R \in S_{\mathcal{L}(c_0(\bigoplus_{k=1}^{\infty} \ell_2^k),Y)}$, $v \in S_{c_0(\bigoplus_{k=1}^{\infty} \ell_2^k)}$ and $\|R(v)\| = \|\tilde{R}(\tilde{u})\| = 1$.

It follows that R attains its norm at v. Next we will show that $\|R - T\| < \varepsilon$. We have

$$\begin{aligned} \|R - T\| &\le \left\| R - \frac{TP_A}{\|TP_A\|} \right\| + \left\| \frac{TP_A}{\|TP_A\|} - TP_A \right\| + \|TP_A - T\| \\ &= \left\| \tilde{R} - \frac{TP_AJ}{\|TP_AJ\|} \right\| + \left\| \frac{TP_A}{\|TP_A\|} - TP_A \right\| + \|TP_A - T\| \\ &= \left\| \tilde{R} - \frac{TP_AJ}{\|TP_AJ\|} \right\| + \|TP_A\| \left| \frac{1}{\|TP_A\|} - 1 \right| + \|TP_A - T\| \\ &= \|\tilde{R} - Q\| + \big|1 - \|TP_A\|\big| + \|TP_A - T\| < \varepsilon + 1 - 1 + \eta(\varepsilon)^2 + \varepsilon < 3\varepsilon. \end{aligned}$$

Finally, we show that the vectors x and v are close. Indeed,

$$\|v - x\| = \|P_A(v - x)\|$$

$$= \max_{1 \le k \le n} \Big(\sum_{j \in I(k)} |v_j - x_j|^2) \Big)^{1/2} = \max_{1 \le k \le n} \Big(\sum_{j \in I(k)} |\tilde{u}_j - x_j|^2) \Big)^{1/2}$$

$$\le \max_{1 \le k \le n} \Big(\sum_{j \in I(k)} |\tilde{u}_j - u_j|^2) \Big)^{1/2} + \max_{1 \le k \le n} \Big(\sum_{j \in I(k)} |u_j - x_j|^2) \Big)^{1/2}$$

$$\le \|\tilde{u} - z\| + \|u - x\| < \beta(\varepsilon) + \varepsilon,$$

where $\lim_{t \to 0}(\beta(t) + t) = 0$. Therefore $\big(c_0\big(\bigoplus_{k=1}^{\infty} \ell_2^k\big), Y\big)$ has the BPBp for operators. $\square$

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