

PAPER

# Particle creation from the vacuum by an exponentially decreasing electric field

To cite this article: T C Adorno *et al* 2015 *Phys. Scr.* **90** 074005

View the [article online](#) for updates and enhancements.

## You may also like

- [Cosmological particle production: a review](#)  
L H Ford
- [Particle creation and particle number in an expanding universe](#)  
Leonard Parker
- [Quantum thermodynamics for a model of an expanding Universe](#)  
Nana Liu, John Goold, Ivette Fuentes et al.

# Particle creation from the vacuum by an exponentially decreasing electric field

T C Adorno<sup>1,5</sup>, S P Gavrilov<sup>1,2,3</sup> and D M Gitman<sup>1,2,4</sup>

<sup>1</sup>Institute of Physics, University of São Paulo, CP 66318, CEP 05315-970 São Paulo, SP, Brazil

<sup>2</sup>Department of Physics, Tomsk State University, 634050, Tomsk, Russia

<sup>3</sup>Department of General and Experimental Physics, Herzen State Pedagogical University of Russia, Moyka embankment 48, 191186 St. Petersburg, Russia

<sup>4</sup>P. N. Lebedev Physical Institute, 53 Leninskiy prospect, 119991, Moscow, Russia

E-mail: [tadornov@usp.br](mailto:tadornov@usp.br), [gavrilovsergeyp@yahoo.com](mailto:gavrilovsergeyp@yahoo.com) and [gitman@if.usp.br](mailto:gitman@if.usp.br)

Received 15 October 2014, revised 5 December 2014

Accepted for publication 15 December 2014

Published 8 June 2015



## Abstract

We analyze in detail the creation of fermions and bosons from a vacuum by an electric field that exponentially decreases in time. In our calculations, we use quantum electrodynamics (QED) and mainly consider the particle creation effect in a homogeneous electric field. To this end, we find complete sets of exact solutions of the  $d$ -dimensional Dirac equation in the exponentially decreasing electric field, and we use them to calculate all the characteristics of the effect, and specifically the total number of created particles and the probability that a vacuum will remain a vacuum. Note that the latter quantities were derived in the case under consideration for the first time. All possible asymptotic regimes are discussed in detail. In addition, switching on and switching off effects are studied.

Keywords: vacuum instability, quantum electrodynamics, external fields

## 1. Introduction

Particle creation from a vacuum by strong external electromagnetic fields is an important nonperturbative effect, the theoretical study of which has a long history, as seen in [1–7]. To be observable, the effect needs very strong electric fields in magnitudes compared with the Schwinger critical field,  $E_c = m^2 c^3 / e \hbar \simeq 1.3 \times 10^{16} \text{ V} \cdot \text{cm}^{-1}$ . However, recent progress in laser physics allows one to hope that the non-perturbative regime of pair production may be reached in the near future (see [8] for a review). Electron–hole pair creation from the vacuum also becomes observable in the laboratory effect in graphene physics, an area that is currently under intense development [9, 10]. In particular, this effect is crucial for understanding the conductivity of graphene, especially in the so-called nonlinear regime, as seen in [11]. Particle creation from the vacuum by external electric and gravitational backgrounds also plays an important role in cosmology and astrophysics [6].

Note that particle creation from the vacuum by external fields is a nonperturbative effect, and its calculation

essentially depends on the structure of the external fields. Sometimes calculations can be done in the framework of relativistic quantum mechanics, and sometimes using semiclassical and numerical methods (see [6, 8, 12] for a review). The vast majority of analytic works in quantum electrodynamics (QED) are based on the worldline and instanton formalisms, rather than on solving the Dirac equation (for example, see [13, 14] and references therein). In fact, in all these cases, the authors calculate the one-loop effective action, whose imaginary part is related to the probability that a vacuum will remain a vacuum. However, in those cases, when the semiclassical approximation does not work, the most convincing consideration is formulated in the framework of quantum field theory (QFT), in particular, in the framework of QED (see [3, 4, 7]). In the latter approach, nonperturbative calculations are based on the existence of exact solutions of the Dirac equation with the corresponding external electromagnetic field. In fact, until now, there have only been a few known exactly solvable cases for either time-dependent homogeneous or constant inhomogeneous electric fields. One of them is related to the constant uniform electric field [1]. There are more realistic examples that are related to the so-called adiabatic electric field  $E(t) = E \cosh^{-2}(t/\alpha)$

<sup>5</sup> Author to whom any correspondence should be addressed.

[15] (see also [16]) and the  $T$ -constant electric field [17–19], which corresponds to a constant electric field that turns on and off at definite time instants,  $t_1$  and  $t_2$ , ( $t_2 - t_1 = T$ ) being constant inside of the time interval,  $T$ . There is also the case related to a periodic alternating electric field [20], and the number of a constant inhomogeneous electric fields of the similar forms where time,  $t$ , is replaced by the spatial coordinate,  $x$ . To complete the picture, we note that exact solutions of the Dirac equation exist, with some electric fields satisfying more complicated symmetries, as with potentials given in the light-cone variables (see [21] and [22]). The existence of exactly solvable cases of particle creation is extremely important both for deep understanding of QFT in general and for studying quantum vacuum effects in the corresponding external fields.

In this article, we present a new, exactly solvable case of particle creation that corresponds to the so-called  $T$ -exponentially decreasing in time electric field, which switches on at the time instant  $t_1$ , switches off at the time instant  $t_2$  ( $t_2 - t_1 = T$ ), and within the time interval,  $T$  has the form  $E_x(t) = E e^{-k_0(t-t_1)}$ , where  $k_0$  and  $E$  are some positive constants. In particular, this field presents the example of an exponentially decaying electric field when  $t_2 \rightarrow \infty$ . Technically, this exactly solvable case differs from all the previously mentioned cases because of an asymmetrical asymptotic behavior of the external electric field. Consideration of such a case has an interesting physical motivation. The corresponding external electric field can be treated as one that is created by an external current, which switches on fast enough, and then slowly switches off (decreases) because of dissipation processes. One can demonstrate that under certain conditions, the main contribution to particle creation is due to the decreasing part of the electric field, whereas the contribution from the increasing part of the field is relatively small. The qualitative difference in the asymptotic behavior of the external electric field under consideration allows one to study the role of switching on and switching off for an electric field. From the beginning, we only consider general ( $d = D + 1$ )-dimensional Minkowski space-time, so we can use the case  $D = 3$  for describing high-energy effects, while the case  $D = 1, 2, 3$  could be adequate for condensed-matter problems. For completeness, the case of scalar particles is considered too.

Note that the differential mean number of particles created by a kind of exponentially decaying electric field was calculated previously in [23] in the framework of some semiclassical considerations, and in [24] using the Dirac-Heisenberg-Wigner function. However, the authors of the latter work did not present any analysis of how their results depend on the problem parameters in the case of a strong field; in fact, they studied the weak field limit only.

As was already said, in our calculations, we use the general theory of [3, 4] and consider the particle creation effect in a homogeneous electric field [18] (see the appendix for some basic elements). To this end, we find complete sets of exact solutions of the Dirac and Klein-Gordon equations in the  $T$ -exponentially decreasing electric field, and we use them

to calculate all the characteristics of the effect, and specifically the differential mean number of particles created, total number of created particles, and the probability that a vacuum will remain a vacuum. Note that the latter quantities were derived in the case under consideration for the first time. Using these solutions, we analyze particle creation in the case of the exponentially decaying electric field. All possible asymptotic regimes are discussed in detail. In addition, switching-on and switching-off effects are studied.

## 2. Exponentially decreasing electric field

We consider the Dirac equation<sup>6</sup> in ( $d = D + 1$ )-dimensional Minkowski space, with an external electromagnetic field given by potentials  $A_\mu(x)$ ,

$$(\gamma^\mu \hat{P}_\mu - m)\psi(x) = 0, \quad \hat{P}_\mu = \hat{p}_\mu - qA_\mu(x), \quad \hat{p}_\mu = i\partial_\mu. \quad (2.1)$$

Here,  $\psi(x)$  is a  $2^{[d/2]}$ -component spinor ( $[d/2]$  stands for the integer part of  $d/2$ ),  $m$  is the particle mass,  $q$  is the particle charge (for the electron  $q = -e$ , with  $e > 0$  being the absolute value of the electron charge),  $x = (x^\mu) = (x^0, \mathbf{x})$ ,  $\mathbf{x} = (x^i)$ ,  $x^0 = t$ ; the Greek and Latin indexes assume values  $\mu = 0, 1, \dots, D$  and  $i = 1, \dots, D$ , respectively, and  $\gamma$ -matrices satisfy the standard anticommutation relations:

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \quad \eta_{\mu\nu} = \text{diag}(1, -1, \dots, -1).$$

Using the Ansatz  $\psi(x) = (\gamma^\mu \hat{P}_\mu + m)\phi(x)$ , one finds that the spinor,  $\phi(x)$ , satisfies the following equation:

$$\left( \hat{P}^2 - m^2 - \frac{q}{2} \sigma^{\mu\nu} F_{\mu\nu} \right) \phi(x) = 0, \quad \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu], \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.2)$$

In what follows, we consider the so-called  $T$ -exponentially decreasing electric field with a constant direction along the  $x$  axis (see figure 1). This field switches on at  $t_1$  and switches off at  $t_2$ , being nonzero within the time interval,  $T = t_2 - t_1 > 0$ , and zero outside of it,

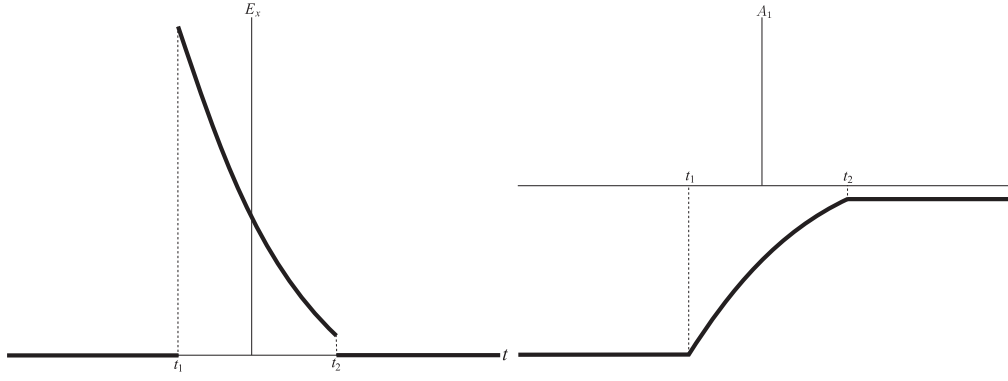
$$E_x(t) = E \begin{cases} 0, & t \in \text{I} = (-\infty, t_1) \\ e^{-k_0(t-t_1)}, & t \in \text{II} = [t_1, t_2], \quad k_0 > 0. \\ 0, & t \in \text{III} = (t_2, +\infty) \end{cases} \quad (2.3)$$

We choose the corresponding potentials as  $A^\mu(t) = \delta_1^\mu A_x(t)$  with only one nonzero component,

$$A_x(t) = \frac{E}{k_0} \begin{cases} 1, & t \in \text{I} \\ e^{-k_0(t-t_1)}, & t \in \text{II}. \\ e^{-k_0 T}, & t \in \text{III} \end{cases} \quad (2.4)$$

We admit that the switching off can occur in the remote future such that  $t_2$  can be infinite,  $t_2 = +\infty$ , under the condition that  $t_1$  remains finite.

<sup>6</sup> From this section and in what follows, we consider the system of units,  $\hbar = c = 1$ , and the fine structure constant is  $\alpha = e^2$ .



**Figure 1.** The exponentially decreasing electric field and its potential.

Solving equation (2.2), we will use a set of constant orthonormalized spinors,  $v_{s,\sigma}$ ,

$$v_{s,\sigma}^\dagger v_{s',\sigma'} = \delta_{s,s'} \delta_{\sigma,\sigma'}, \quad v v^\dagger = \mathbb{I},$$

with  $s = \pm 1$ , and  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{[d/2]-1})$ ,  $\sigma_j = \pm 1$ , such that  $\gamma^0 \gamma^1 v_{s,\sigma} = s v_{s,\sigma}$ . For  $d \geq 4$ , the indices,  $\sigma_j$ , describe the spin polarization, which is not coupled to the electric field, and together with the additional index,  $s$ , they provide a suitable parametrization of the solutions. Note that for  $d = 2, 3$  there is only one spin degree of freedom, and the spinors are labeled either by  $s = +1$  or by  $s = -1$ . Solutions of equation (2.2) with the potential given by equation (2.4) can be found in the form

$$\phi_{\mathbf{p},s,\sigma}(x) = \varphi_{\mathbf{p},s}(t) e^{i\mathbf{p} \cdot \mathbf{x}} v_{s,\sigma}, \quad (2.5)$$

where scalar functions,  $\varphi_{\mathbf{p},s}(t)$ , satisfy the following second-order differential equation

$$\left[ \frac{d^2}{dt^2} + [p_x - qA_x(t)]^2 + \mathbf{p}_\perp^2 + m^2 + i s q E_x(t) \right] \varphi_{\mathbf{p},s}(t) = 0, \quad (2.6)$$

where  $\mathbf{p}_\perp$  is the transversal particle momentum,  $\mathbf{p}_\perp = (0, p^2, \dots, p^D)$ .

Thus, in what follows, we are going to deal with two complete sets of solutions of the Dirac equation (2.1) of the following structure

$$\begin{aligned} \pm \psi_n(x) &= (\gamma^\mu \hat{p}_\mu + m) \pm \phi_{\mathbf{p},s,\sigma}(x), \\ \pm \psi_n(x) &= (\gamma^\mu \hat{p}_\mu + m) \pm \phi_{\mathbf{p},s,\sigma}(x), \end{aligned} \quad (2.7)$$

where spinors  $\pm \phi_{\mathbf{p},s,\sigma}(x)$  and  $\pm \phi_{\mathbf{p},s,\sigma}(x)$  are given by equation (2.5) with solutions  $\zeta \varphi_{\mathbf{p},s}(t)$  and  $\xi \varphi_{\mathbf{p},s}(t)$ , respectively, satisfying equation (2.6), with initial or final conditions that are specified in what follows. Here, we use  $n = (\mathbf{p}, \sigma)$  to denote a complete set of quantum numbers of the Dirac spinor for cases  $s = +1$  or  $s = -1$ . Note that for  $d \geq 4$ , the Dirac spinors given by the choice of  $s = +1$  in (2.7) are linearly dependent, with the spinors given by the choice of  $s = -1$ . Thus, one can form physically equivalent complete sets of the Dirac spinors for both choices of parametrization. The algebra

of the  $\gamma$ -matrices has two inequivalent representations in  $d = 3$  dimensions; the representations given by  $s = +1$  and  $s = -1$  are associated with different fermion species.

Note that a formal reduction to the spinless case that corresponds to the use of the Klein-Gordon equation instead of the Dirac equation can be done by setting  $s = 0$  in (2.6) and  $v_{s,\{r\}} = 1$  in (2.5). In this case,  $n = (\mathbf{p})$ .

In the first region I and in the third region III, the electric field is absent and equation (2.6) has plane wave solutions  $\zeta \varphi_{\mathbf{p},s}(t)$  and  $\xi \varphi_{\mathbf{p},s}(t)$ , respectively, with an additional quantum number  $\zeta = \pm$ , which satisfies simple dispersion relations

$$\text{I: } \zeta \varphi_{\mathbf{p},s}(t) \sim e^{-i\zeta p_0(t_1)t}, \quad \text{III: } \xi \varphi_{\mathbf{p},s}(t) \sim e^{-i\xi p_0(t_2)t},$$

$$p_0(t) = \sqrt{\left( p'_x - \frac{|qE|}{k_0} e^{-k_0(t-t_1)} \right)^2 + \mathbf{p}_\perp^2 + m^2}, \quad (2.8)$$

where  $p'_x = \kappa p_x$  and  $\kappa = \text{sgn}(qE)$ . Here, the quantum numbers,  $\zeta$ , label particle/antiparticle states such that positive ( $\zeta = +$ )/negative ( $\zeta = -$ ) values define particle/antiparticle states, respectively.

In the second region II, it is convenient to introduce a new variable,  $\eta$ , and represent the functions  $\varphi_{\mathbf{p},s}$  as

$$\begin{aligned} \eta &= \frac{2i|qE|}{k_0^2} e^{-k_0(t-t_1)}, \quad \varphi_{\mathbf{p},s}(t) = e^{-\eta/2} \eta^\nu \chi_{\mathbf{p},s}(\eta), \\ \nu &= i \frac{\omega_0}{k_0}, \quad \omega_0 = \sqrt{\mathbf{p}^2 + m^2}. \end{aligned} \quad (2.9)$$

Then the functions  $\chi_{\mathbf{p},s}(\eta)$  satisfy the confluent hypergeometric equation [25],

$$\left[ \eta \frac{d^2}{d\eta^2} + (c - \eta) \frac{d}{d\eta} - a \right] \chi_{\mathbf{p},s}(\eta) = 0,$$

with parameters

$$c = 1 + 2\nu, \quad a = \frac{1}{2}(1 - \kappa s) - \frac{ip'_x}{k_0} + \nu. \quad (2.10)$$

The complete set of solutions for this equation is formed by

two linearly independent confluent hypergeometric functions:

$$\Phi(a, c; \eta) \text{ and } \eta^{1-c} \Phi(a - c + 1, 2 - c; \eta),$$

where

$$\Phi(a, c; \eta) = 1 + \frac{a}{c} \frac{\eta}{1!} + \frac{a(a+1)}{c(c+1)} \frac{\eta^2}{2!} + \dots$$

Thus, one can find the general solution of equation (2.6) in time region II as the following linear superposition

$$\begin{aligned} \varphi(t) &= a_2 \varphi_1(t) + a_1 \varphi_2(t), \\ \varphi_1(t) &= e^{-\eta/2} \eta^\nu \Phi(a, c; \eta), \\ \varphi_2(t) &= e^{-\eta/2} \eta^{-\nu} \Phi(a - c + 1, 2 - c; \eta), \end{aligned} \quad (2.11)$$

where constants  $a_1$  and  $a_2$  are fixed by initial conditions.

Taking into account expressions (2.8) and (2.11), one can construct orthonormalized solutions for the complete time interval in the following form

$$\begin{aligned} \zeta \varphi_{\mathbf{p},s}(t) &= \begin{cases} g(-|\zeta) - C e^{ip_0(t_1)(t-t_1)} + g(+|\zeta) + C e^{-ip_0(t_1)(t-t_1)}, & t \in \text{I} \\ a_2^\zeta \varphi_1(t) + a_1^\zeta \varphi_2(t), & t \in \text{II}, \\ \zeta C e^{-i\zeta p_0(t_2)(t-t_2)}, & t \in \text{III} \end{cases} \end{aligned} \quad (2.12)$$

where constants  $\zeta C$  and  $\zeta C$  are defined by normalization conditions for the Dirac spinors (A.4),

$$\begin{aligned} \zeta C &= [2Vp_0(t_1)p_\zeta(t_1)]^{-1/2}, \\ \zeta C &= [2Vp_0(t_2)p_\zeta(t_2)]^{-1/2}, \\ p_\zeta(t) &= p_0(t) - \zeta s [p_x - qA_x(t)], \end{aligned} \quad (2.13)$$

and  $p_0(t)$  is given by equation (2.8). Note that notation  $g(\zeta|\zeta)$  corresponds to definition (A.6) from the appendix. Coefficients  $a_1^\zeta$ ,  $a_2^\zeta$ ,  $g(-|\zeta)$ , and  $g(+|\zeta)$  are specified by the following gluing conditions:

$$\begin{aligned} {}^+ \varphi_{\mathbf{p},s}(t_k - 0) &= {}^+ \varphi_{\mathbf{p},s}(t_k + 0), \\ \partial_t {}^+ \varphi_{\mathbf{p},s}(t) \Big|_{t=t_k-0} &= \partial_t {}^+ \varphi_{\mathbf{p},s}(t) \Big|_{t=t_k+0}, \\ k &= 1, 2. \end{aligned} \quad (2.14)$$

One can see from the consideration given in the appendix that the probability of a vacuum to remain a vacuum, the probability of a particle scattering, a pair creation, and a pair annihilation can be expressed via the differential mean number of particles created from vacuum  $N_n$ , given by equation (A.9). It follows that one can describe a vacuum instability for the case under consideration using the quantity

$$N_{\mathbf{p},\sigma} = |g(-|^\sigma)|^2 \quad (2.15)$$

only. Then it is enough to consider only the case  $\zeta = +$  in

equation (2.12). Using conditions (2.14), we obtain

$$a_1^+ = -\frac{i^+ C p_0(t_2)}{W} f_1(t_2), \quad a_2^+ = \frac{i^+ C p_0(t_2)}{W} f_2(t_2),$$

where  $W$  is the corresponding Wronskian of the solutions [25],

$$W = \varphi_1(t) \frac{d}{dt} \varphi_2(t) - \varphi_2(t) \frac{d}{dt} \varphi_1(t) = 2i\omega_0,$$

and

$$f_{1,2}(t) = \left[ 1 + \frac{ik_0 \eta}{p_0(t)} \frac{d}{d\eta} \right] \varphi_{1,2}(t). \quad (2.16)$$

We finally find that the coefficient  $g(-|^\sigma)$  takes the form

$$\begin{aligned} g(-|^\sigma) &= \frac{1}{4\omega_0} \sqrt{\frac{p_0(t_2)p_-(t_1)p_0(t_1)}{p_+(t_2)}} [f_1(t_1)f_2(t_2) \\ &\quad - f_2(t_1)f_1(t_2)]. \end{aligned} \quad (2.17)$$

One can demonstrate that in the case of a sufficiently long duration of the exponential electric field, when  $t_2 \rightarrow +\infty$  and  $p_0(t_1)(t_2 - t_1) \gg 1$ , the differential mean numbers, given by expression (2.17), coincide in the leading-order term approximation with the result obtained in [24].

Taking into account that the normalization constants,  $\zeta C$  and  $\zeta C$ , for the scalar case are

$$\zeta C = [2Vp_0(t_1)]^{-1/2}, \quad \zeta C = [2Vp_0(t_2)]^{-1/2},$$

we find that in this case, the coefficient  $g(-|^\sigma)$  has the following form

$$\begin{aligned} g(-|^\sigma) &= \frac{1}{4\omega_0} \sqrt{p_0(t_2)p_0(t_1)} [f_1(t_1)f_2(t_2) \\ &\quad - f_2(t_1)f_1(t_2)], \end{aligned} \quad (2.18)$$

where  $f_{1,2}(t)$  are given by equation (2.16) at  $s=0$ . The differential mean number of created scalar particles is expressed via  $g(-|^\sigma)$  (2.18) as  $N_{\mathbf{p}} = |g(-|^\sigma)|^2$ .

Expression (2.17) does not depend on spin polarization parameters,  $\sigma_j$ . That is why all the probabilities and the mean number do not depend on  $\sigma_j$ , so that the total (summed over all  $\sigma_j$ ) probabilities and the mean number are  $J_{(d)}$  times greater than the corresponding differential quantities. Here,  $J_{(d)} = 2^{[\frac{d}{2}]-1}$  is the number of spin degrees of freedom. For example, the total number of particles created with a given momentum,  $\mathbf{p}$ , is

$$\sum_{\sigma} N_{\mathbf{p},\sigma} = J_{(d)} |g(-|^\sigma)|^2. \quad (2.19)$$

To get the total number,  $N$ , of created particles, one has to sum over the spin projections, using equation (2.19), and then over the momenta. The latter sum can be easily transformed

into an integral,

$$N = \sum_{\mathbf{p}} \sum_{\sigma} N_{\mathbf{p},\sigma} = \frac{VJ_{(d)}}{(2\pi)^{d-1}} \int d\mathbf{p} |g(-|\mathbf{p}|)|^2, \quad (2.20)$$

where  $V$  is  $(d-1)$ -dimensional spatial volume.

The expression above depends essentially on the time interval of the field duration,  $T = t_2 - t_1$ . Then the effect of pair creation depends on two dimensionless parameters,  $k_0 T$  and  $|qE|/k_0^2$ . For  $k_0$  fixed, the first allows one to analyze the characteristics with respect to the time duration,  $T$ , of the electric field, while the second also provides information on the maximum magnitude of the field,  $|E|$ , for  $k_0$  fixed.

### 3. Exponentially decaying strong field

#### 3.1. Differential quantities

Let us consider the exponentially decaying electric field given by equation (2.4), with

$$\frac{|qE|}{k_0^2} e^{-k_0 T} \ll 1; \quad (3.1)$$

when its initial magnitude is sufficiently large,

$$\frac{|qE|}{k_0^2} \gg K_f, \quad K_f \gg \max\left(\frac{\omega_0}{k_0}, 1\right), \quad (3.2)$$

where  $K_f$  is a given number. We stress that condition (3.2) corresponds to the most interesting case of a strong electric field where a perturbative consideration is not applicable.

In this case, using the asymptotics of the confluent hypergeometric functions [25], we first find from expression (2.17) that the differential mean numbers of created fermions are:

$$N_{\mathbf{p},\sigma} \simeq e^{-\frac{\pi}{k_0}(\omega_0 - p'_x)} \frac{\sinh\left[\pi(\omega_0 + p'_x)/k_0\right]}{\sinh(2\pi\omega_0/k_0)}. \quad (3.3)$$

We note that this case is not analyzed in [24], the only case when  $|qE| \rightarrow 0$  is considered there. Under the same condition, the differential mean numbers of created scalar bosons follow from equation (2.18). They are

$$N_{\mathbf{p}} \simeq e^{-\frac{\pi}{k_0}(\omega_0 - p'_x)} \frac{\cosh\left[\pi(\omega_0 + p'_x)/k_0\right]}{\sinh(2\pi\omega_0/k_0)}. \quad (3.4)$$

Note that if the kinetic energy of final particles is big enough,  $\frac{|qE|}{k_0\omega_0} \ll 1$ , the problem can be considered perturbatively. In this case, the weak time-dependent external field violates the vacuum very little, and the corresponding pair creation can be neglected in comparison with the main contribution given by equations (3.3) and (3.4), which is formed in the momentum range (3.2).

The difference in distributions (3.3) and (3.4) that is stipulated by the statistics is maximal for the fast varying field

when  $\omega_0/k_0 \ll 1$ . Then

$$N_{\mathbf{p},\sigma} \simeq \frac{1}{2} \left(1 + \frac{p'_x}{\omega_0}\right), \quad N_{\mathbf{p}} \simeq \frac{k_0}{2\pi\omega_0}. \quad (3.5)$$

In the spinless case, the mean numbers,  $N_{\mathbf{p}}$ , given by equation (3.5) grow without limits. This is an indication of a big back-reaction effect. Thus, we can suppose that for scalar QED, the concept of the external field is limited by the condition  $2\pi m/k_0 \gtrsim 1$ . At the same time, in the case of spinor QED, the mean number,  $N_{\mathbf{p},\sigma}$ , given by equation (3.5), is limited by  $N_{\mathbf{p},\sigma} \leq 1$ . This allows us to study fermion creation for all possible parameters given by equation (3.2) by using the external field concept.

It follows from equations (3.3) and (3.4) that for a large negative longitudinal momentum,

$$p'_x < 0, \quad |p'_x|/k_0 > K_x, \quad (3.6)$$

where  $K_x \gg 1$  is a given number, the mean number of created boson and fermion pairs is exponentially small.

In what follows, we show that the main contribution to the total number of created fermions is due to the sufficiently large positive longitudinal momenta,  $p'_x$ , from the range

$$p'_x/k_0 > K_x, \quad (3.7)$$

where it is assumed that  $K_f \gtrsim K_x$ . In this range, it follows from equations (3.3) and (3.4) that

$$N_{\mathbf{p},\sigma} = N_{\mathbf{p}}^{\text{as}}, \quad N_{\mathbf{p}}^{\text{as}} \simeq e^{-\frac{2\pi}{k_0}(\omega_0 - p'_x)} \quad (3.8)$$

both for fermions and bosons, taking into account that for bosons,  $N_{\mathbf{p}} = N_{\mathbf{p}}^{\text{as}}$ . We see that  $N_{\mathbf{p}}^{\text{as}} \leq 1$ . Note that equation (3.8) holds true for any transversal energy,  $\sqrt{m^2 + \mathbf{p}_\perp^2}$ . In particular, if  $(p'_x)^2 \gg m^2 + \mathbf{p}_\perp^2$ , distribution (3.8) can be approximated as

$$N_{\mathbf{p}}^{\text{as}} \simeq \exp\left(-\pi \frac{m^2 + \mathbf{p}_\perp^2}{k_0 p'_x}\right), \quad (3.9)$$

such that  $N_{\mathbf{p}}^{\text{as}} \rightarrow 1$  as  $p'_x/k_0 \rightarrow \infty$ . If  $(p'_x)^2 \lesssim m^2 + \mathbf{p}_\perp^2$ , then distribution (3.8) can be approximated as

$$N_{\mathbf{p}}^{\text{as}} \lesssim \exp\left[-\frac{2\pi}{k_0}(\sqrt{2} - 1)\sqrt{m^2 + \mathbf{p}_\perp^2}\right]. \quad (3.10)$$

We see that this expression is exponentially small in the momentum range  $\sqrt{m^2 + \mathbf{p}_\perp^2}/k_0 \gtrsim p'_x/k_0 > K_x$ .

The above analysis shows that maximum contribution to the differential number of created fermions is provided by large, positive longitudinal momenta,  $p'_x$ , given by expression (3.9), with a relatively small transversal momentum,  $|\mathbf{p}_\perp|$ . Thus, taking the inequality (3.2) into account, we can conclude that the essential contribution to the total number of created fermions is due to the longitudinal momenta,  $p'_x$ , from



the wide uniform range

$$K_x < p'_x/k_0 < \frac{|qE|}{k_0^2} - K_f, \quad (3.11)$$

where

$$|qE|/k_0^2 \gg K_x, \quad p'_x \gg m. \quad (3.12)$$

Note that the contribution to the total number of created particles from the relatively narrow momentum range of the width  $K_x$  is finite and of the order  $K_x$  if  $N_p \lesssim 1$ . For example, we can use this estimation for the total number of created fermions in the finite range of  $p'_x$  that is restricted by the inequality

$$-K_x < p'_x/k_0 < K_x. \quad (3.13)$$

This contribution is much less than the contribution from a very wide range (3.11). The same is true for bosons when  $2\pi m/k_0 \gtrsim 1$ . That is why the contribution to the total number of created fermions in the range (3.11) is the main contribution. The main contribution to the total number of created bosons is due to the range (3.11) for the slowly decaying electric field when  $2\pi m/k_0 \gtrsim 1$ .

Note that if  $(\omega_0 - p'_x)/k_0 \gg 1$ , Wentzel–Kramers–Brillouin approximation holds true for  $N_{p,\sigma}$ , given by equation (3.8). In this domain, expression (3.8) coincides exactly with an estimation obtained previously in [23] using the semiclassical consideration, while our approximation (3.8) is valid for any value of  $(\omega_0 - p'_x)/k_0$  and our exact results, given by equations (3.3) and (3.4), are quite different from the semiclassical ones.

### 3.2. Total quantities

The obtained distribution,  $N_n = |g(-)|^2$ , plays the role of a cut-off factor in the integral over momenta (2.20) for the total number of created particles (for bosons  $J_{(d)} = 1$ ). However, for bosons, this result is valid only if the electric field decays slowly enough,  $2\pi m/k_0 \gtrsim 1$ . Then the total number of created particles can be represented by its main contribution in the range (3.11), as follows:

$$N \approx \frac{VJ_{(d)}}{(2\pi)^{d-1}} \int_{p'_x^{\min}}^{p'_x^{\max}} dp'_x N_{p'_x}^{\text{as}}, \quad N_{p'_x}^{\text{as}} = \int d\mathbf{p}_\perp N_{\mathbf{p}}^{\text{as}}, \quad (3.14)$$

where  $N_{\mathbf{p}}^{\text{as}}$  is given by equation (3.9) and

$$p'_x^{\max} = \frac{|qE|}{k_0} - K_f k_0, \quad p'_x^{\min} = K_x k_0. \quad (3.15)$$

Integrating over  $\mathbf{p}_\perp$  and taking into account that  $p'_x \gg m$ , we find that the total number of created particles with a given longitudinal momentum reads

$$N_{p'_x}^{\text{as}} \approx (k_0 p'_x)^{d/2-1} \exp\left(-\frac{\pi m^2}{k_0 p'_x}\right). \quad (3.16)$$

Using equation (3.16), we represent the integral (3.14) in the form

$$N \approx \frac{VJ_{(d)}}{(2\pi)^{d-1}} (k_0)^{d/2-1} Y^{(1)}, \quad (3.17)$$

where  $Y^{(1)}$  is the particular case of the integral

$$Y^{(k)} = \int_{p'_x^{\min}}^{p'_x^{\max}} dp'_x (p'_x)^{d/2-1} \times \exp\left(-k \frac{\pi m^2}{k_0 p'_x}\right), \quad k = 1, 2, \dots \quad (3.18)$$

Taking into account that  $|qE|/k_0^2 \gg K_f \gtrsim K_x$ , we obtain that the integral (3.18) is independent of the given numbers,  $K_f$  and  $K_x$ , in the leading-order term approximation. If  $m \neq 0$ , then the integral (3.18) in this approximation can be expressed via the incomplete gamma function as

$$Y^{(k)} \approx \left(\frac{k_0}{\pi m^2 k}\right)^{d/2} \Gamma\left(-\frac{d}{2}, k \frac{\pi m^2}{|qE|}\right), \quad k = 1, 2, \dots \quad (3.19)$$

Note that the representation (3.19) is suitable when the electric field is weak enough,  $k\pi m^2/|qE| \gg 1$ . In this case, one can use the following asymptotics of the incomplete gamma function,

$$\Gamma\left(-\frac{d}{2}, k \frac{\pi m^2}{|qE|}\right) \approx \exp\left(-k \frac{\pi m^2}{|qE|}\right) \left(k \frac{\pi m^2}{|qE|}\right)^{-d/2-1}. \quad (3.20)$$

For the case of a strong field, when  $k\pi m^2/|qE| \ll 1$ , where the case of massless fermions is also included, we find in the leading-order term approximation that

$$Y^{(k)} \approx \frac{2}{d} \left(\frac{|qE|}{k_0}\right)^{d/2}. \quad (3.21)$$

Then the total number of particles created from the vacuum is

$$N^{\text{strong}} \approx \frac{VJ_{(d)}}{(2\pi)^{d-1}} \frac{2(|qE|)^{d/2}}{k_0 d}. \quad (3.22)$$

Finally, taking into account the above results, we can represent the probability of a vacuum to remain a vacuum, defined by equation (A.11), as

$$P_v \approx \exp\left\{-\frac{VJ_{(d)}}{(2\pi)^{d-1}} \sum_{k=0}^{\infty} \frac{(-1)^{(1-\kappa)k/2}}{(k+1)^{d/2}} (k_0)^{d/2-1} Y^{(k+1)}\right\}, \quad (3.23)$$

where  $Y^{(k+1)}$  is given by the integral (3.18) and can be represented in the leading term approximation with the help of equations (3.19) and (3.21), respectively. For the strong field case, we find that the probability,  $P_v$ , is determined by the total number of created particles

$$P_v^{\text{strong}} = \exp\{-\mu N^{\text{strong}}\}, \quad \mu = -\sum_{k=0}^{\infty} \frac{(-1)^{(1-\kappa)k/2}}{(k+1)^{d/2}}. \quad (3.24)$$

One can see that the dependence of the total number of particles created from the vacuum by the strong exponential field on the field magnitude and space-time dimensions

mimics the case of particle creation by a strong  $T$ -constant electric field,  $E$  (see [18]), for big  $T$  and with the identification  $T = 2(k_0 d)^{-1}$ . This is due to the effect of saturation for the distribution,  $N_n \rightarrow 1$ , in the wide uniform range of initial longitudinal momentum, where there is a big increment of the kinetic momentum,  $|qE|/k_0$  and  $|qE| T$ , for both cases, respectively.

Let us consider two strong  $T$ -exponential electric fields of the same magnitude,  $E$ , but with distinct parameters,  $k_0^{(I)}$  and  $k_0^{(II)} \ll k_0^{(I)}$ . Let them create from a vacuum the total number of particles,  $N^{(I)}$  and  $N^{(II)}$ , respectively. One can see from equation (3.22) that  $N^{(II)} \gg N^{(I)}$ —that is, the electric field of a longer, more effective duration creates many more pairs. The total number of out-particles created from in-vacuum due to a decreasing exponential field is the same as the total number of particles created from a vacuum due to an increasing exponential field, provided that the modulus of potential difference is the same in both cases. Thus, we can consider  $N^{(I)}$  as the total number of particles created from a vacuum due to the increasing field. We see that if  $k_0^{(II)} \ll k_0^{(I)}$ , the main contribution to particle creation by an external electric field that switches on fast enough and then slowly decreases is due to its decreasing part, whereas the contribution from the increasing part of the field is relatively small. In particular, the exponentially decaying electric field can be treated as if it is created by an external current that switches on fast enough and then slowly switches off because of dissipation processes. Thus, we see that the exponentially decaying electric field under consideration allows one to study the role of the switching-on and switching-off processes.

## Acknowledgments

TCA acknowledges the support of FAPESP under contract 2013/00840-9. SPG thanks FAPESP for support and University of São Paulo for its hospitality. DMG is grateful to Brazilian foundations FAPESP and CNPq for permanent support. The work of SPG and DMG is also partially supported by the Tomsk State University Competitiveness Improvement Program and by grant from the Tomsk State University Academic D. I. Mendelev fund program.

## Appendix A. Pair creation in a homogeneous electric field

Following general consideration in [18], in this appendix we recall some basic elements of the generalized Furry representation [3, 4, 7], which is used to describe vacuum instability in a strong external time-dependent electric field.

For the particular case of a homogeneous electric field, we assume that the potential,  $A_1(t)$  ( $A_\mu(t) = 0$ ,  $\mu \neq 1$ ), is constant for  $t < t_1$  and for  $t > t_2$ . Therefore, the initial (at  $t < t_1$ ) and the final (at  $t > t_2$ ) vacua are vacuum states of in- and out-free particles that correspond to the constant effective

potentials  $A_1(t_1)$  and  $A_1(t_2)$ , respectively. During the time interval,  $t_2 - t_1 = T$ , the quantum Dirac field interacts with the time-dependent effective potential,  $A_1(t)$ . In the general case, the initial and final vacua are different. We introduce an initial set of creation and annihilation operators,  $a_n^\dagger(\text{in})$ ,  $a_n(\text{in})$ , of in-particles (electrons), and operators,  $b_n^\dagger(\text{in})$ ,  $b_n(\text{in})$ , of in-anti-particles (positrons), with the corresponding in-vacuum being  $|0, \text{in}\rangle$ . There is also a final set of creation and annihilation operators,  $a_n^\dagger(\text{out})$ ,  $a_n(\text{out})$ , of out-electrons and operators,  $b_n^\dagger(\text{out})$ ,  $b_n(\text{out})$ , of out-positrons, the corresponding out-vacuum, being  $|0, \text{out}\rangle$ . Thus, for any quantum number,  $n$ , we have

$$\begin{aligned} a_n(\text{in})|0, \text{in}\rangle &= b_n(\text{in})|0, \text{in}\rangle = 0, \\ a_n(\text{out})|0, \text{out}\rangle &= b_n(\text{out})|0, \text{out}\rangle = 0. \end{aligned} \quad (\text{A.1})$$

In both cases, we use  $n = (\mathbf{p}, \sigma)$  to denote complete sets of quantum numbers that describe both in- and out-particles and antiparticles. The in-operators and the out-operators obey the canonical anticommutation relations. The above in- and out-operators are defined by two decompositions of the quantum Dirac field,  $\Psi(x)$ , in the exact solutions of the Dirac equation,

$$\begin{aligned} \Psi(x) &= \sum_n \left[ a_n(\text{in}) \psi_n(x) + b_n^\dagger(\text{in}) \bar{\psi}_n(x) \right] \\ &= \sum_n \left[ a_n(\text{out}) \psi_n(x) + b_n^\dagger(\text{out}) \bar{\psi}_n(x) \right]. \end{aligned} \quad (\text{A.2})$$

Thus, the in-operators are associated with a complete orthonormal set of solutions,  $\{\zeta \psi_n(x)\}$  (we call it the in-set), of equation (2.1), where  $\zeta = +$  stays for electrons and  $\zeta = -$  for positrons. Their asymptotics at  $t < t_1$  are wave functions of free particles in the presence of a constant electric potential,  $A_1(t_1)$ . The out-operators are associated with another complete orthonormal out-set of solutions,  $\{\zeta \psi_n(x)\}$  of equation (2.1). Their asymptotics at  $t > t_2$  are wave functions of free particles in the presence of a constant electric potential,  $A_1(t_2)$ .

The inner product between two solutions,  $\psi(x)$  and  $\psi'(x)$ , of the Dirac equation on the  $t$ -const hyperplane,

$$(\psi, \psi') = \int \psi^\dagger(x) \psi'(x) d\mathbf{x}, \quad (\text{A.3})$$

is time-independent. Then, taking into account the structure (2.7) and initial or final forms of the functions  $\zeta \psi_n(x)$  and  $\zeta \psi'_n$ , respectively, one finds the orthonormality relations:

$$(\zeta \psi_n, \zeta' \psi'_n) = \delta_{n,n'} \delta_{\zeta,\zeta'}, \quad (\zeta \psi_n, \zeta' \psi'_n) = \delta_{nn'} \delta_{\zeta,\zeta'}. \quad (\text{A.4})$$

Here we apply the standard QFT volume regularization, assuming that all the processes are confined in a big  $D$  dimensional space box with the volume,  $V$ . In- and out-solutions with given quantum numbers,  $n$ , are related by linear transformations of the form

$$\begin{aligned} \zeta \psi_n(x) &= g \left( \begin{smallmatrix} \zeta \\ + \end{smallmatrix} \right) \psi_n(x) + g \left( \begin{smallmatrix} \zeta \\ - \end{smallmatrix} \right) \bar{\psi}_n(x), \\ \zeta \psi'_n(x) &= g \left( \begin{smallmatrix} + \\ \zeta \end{smallmatrix} \right) \psi_n(x) + g \left( \begin{smallmatrix} - \\ \zeta \end{smallmatrix} \right) \bar{\psi}_n(x), \end{aligned} \quad (\text{A.5})$$

where the coefficients,  $g$ , are defined via the inner products of



these sets,

$$(\zeta \psi_n^{\zeta'} \psi_n^{\zeta'}) = \delta_{n,n'} g(\zeta|\zeta'), \quad g(\zeta'|\zeta) = g(\zeta|\zeta')^*. \quad (\text{A.6})$$

These coefficients satisfy the unitarity relations, which follow from the orthonormality relations (A.4), and can be expressed in terms of two of them:  $g(+|+)$  and  $g(-|+)$ . However, even these coefficients are not completely independent,

$$|g(-|+)|^2 + |g(+|+)|^2 = 1. \quad (\text{A.7})$$

A linear canonical transformation (Bogolyubov transformation) between in- and out-operators that can be derived from equation (A.2) has the form

$$\begin{aligned} a_n(\text{out}) &= g(+|+)a_n(\text{in}) + g(+|-)b_n^\dagger(\text{in}), \\ b_n^\dagger(\text{out}) &= g(-|+)a_n(\text{in}) + g(-|-)b_n^\dagger(\text{in}). \end{aligned} \quad (\text{A.8})$$

Then one can see that all the information about electron-positron creation, annihilation, and scattering in an external field can be extracted from the coefficients,  $g(\zeta|\zeta')$ .

One of the most important quantities for the study of particle creation is the differential mean number of created particles, defined as the expectation value of an out number operator with respect to the in-vacuum,

$$N_n = \langle 0, \text{in} | a_n^\dagger(\text{out}) a_n(\text{out}) | 0, \text{in} \rangle = |g(-|+)|^2. \quad (\text{A.9})$$

It is equal to the mean number of particle-antiparticle pairs created. The total number of created particles is obtained by the summation over the quantum numbers,  $n$ ,

$$N = \sum_n N_n. \quad (\text{A.10})$$

The probability of a vacuum remaining a vacuum is defined as

$$P_v = |\langle 0, \text{out} | 0, \text{in} \rangle|^2 = \exp \left\{ \kappa \sum_n \ln(1 - \kappa N_n) \right\}, \quad (\text{A.11})$$

where  $\kappa = +1$  for fermions and  $\kappa = -1$  for bosons. The probability of the electron scattering,  $P(+|+)_{n,n'}$ , and the probability of a pair creation,  $P(-|+)_{n,n'}$  are, respectively,

$$\begin{aligned} P(+|+)_{n,n'} &= |\langle 0, \text{out} | a_n(\text{out}) a_n^\dagger(\text{in}) | 0, \text{in} \rangle|^2 \\ &= \delta_{n,n'} \frac{1}{1 - \kappa N_n} P_v, \\ P(-|+)_{n,n'} &= |\langle 0, \text{out} | b_n(\text{out}) a_n^\dagger(\text{in}) | 0, \text{in} \rangle|^2 \\ &= \delta_{n,n'} \frac{N_n}{1 - \kappa N_n} P_v. \end{aligned} \quad (\text{A.12})$$

The probabilities for a positron scattering and a pair annihilation are given by the same expressions,  $P(+|+)$  and  $P(-|+)$ , respectively.

## References

- [1] Schwinger J 1951 *Phys. Rev.* **82** 664
- [2] Nikishov A I 1969 *Zh. Eksp. Teor. Fiz.* **57** 1210  
Nikishov A I 1970 *Transl. Sov. Phys.—JETP* **30** 660  
Nikishov A I 1979 *Quantum Electrodynamics of Phenomena in Intense Fields* (Proc. P.N. Lebedev Phys. Inst. vol 111) (Moscow: Nauka) 153
- [3] Gitman D M 1977 *J. Phys. A: Math. Gen.* **10** 2007  
Fradkin E S and Gitman D M 1981 *Fortschr. Phys.* **29** 381
- [4] Fradkin E S, Gitman D M and Shvartsman S M 1991 *Quantum Electrodynamics with Unstable Vacuum* (Berlin: Springer)
- [5] Greiner W, Müller B and Rafelsky J 1985 *Quantum Electrodynamics of Strong Fields* (Berlin: Springer)
- [6] Birrell N D and Davies P C W 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)  
Grib A A, Mamaev S G and Mostepanenko V M 1994 *Vacuum Quantum Effects in Strong Fields* (St. Petersburg: Friedmann Laboratory Publishing)
- [7] Ruffini R, Vereshchagin G and Xue S 2010 *Phys. Rep.* **487** 1
- [8] Gavrilov S P, Gitman D M and Tomazelli J L 2008 *Nucl. Phys. B* **795** 645
- [9] Dunne G V 2009 *Eur. Phys. J. D* **55** 327  
Castro Neto A H et al 2009 *Rev. Mod. Phys.* **81** 109  
Peres N M R 2010 *Rev. Mod. Phys.* **82** 2673  
Vozmediano M A H, Katsnelson M I and Guinea F 2010 *Phys. Rep.* **496** 109
- [10] das Sarma D, Adam S, Hwang E H and Rossi E 2011 *Rev. Mod. Phys.* **83** 407
- [11] Gavrilov S P, Gitman D M and Yokomizo N 2012 *Phys. Rev. D* **86** 125022 (arXiv:1207.1749)
- [12] Dumlu C K and Dunne G V 2011 *Phys. Rev. D* **84** 125023
- [13] Dunne G V and Schubert C 2005 *Phys. Rev. D* **72** 105004  
Dunne G V, Gies H, Schubert C and Wang Q-H 2006 *Phys. Rev. D* **73** 065028
- [14] Ilderton A 2014 *J. High Energy Phys. JHEP09(2014)166*  
Schneider C and Schützhold R 2014 Dynamically assisted Sauter-Schwinger effect in inhomogeneous electric fields (arXiv:1407.3584)
- [15] Narozhnyi N B and Nikishov A I 1970 *Sov. J. Nucl. Phys. (USA)* **11** 596
- [16] Dunne G and Hall T 1998 *Phys. Rev. D* **58** 105022
- [17] Bagrov V G, Gitman D M and Shvartsman S M 1975 *Zh. Eksp. Teor. Fiz.* **68** 392  
Bagrov V G, Gitman D M and Shvartsman S M 1975 *Sov. Phys.—JETP* **41** 191
- [18] Gavrilov S P and Gitman D M 1996 *Phys. Rev. D* **53** 7162
- [19] Gavrilov S P and Gitman D M 2008 *Phys. Rev. Lett.* **101** 130403  
Gavrilov S P and Gitman D M 2008 *Phys. Rev. D* **78** 045017
- [20] Narozhnyi N B and Nikishov A I 1974 *Sov. Phys.—JETP* **38** 427  
Mostepanenko V M and Frolov V M 1974 *Sov. J. Nucl. Phys. (USA)* **19** 451
- [21] Tomaras T N, Tsamis N C and Woodard R P 2001 *J. High Energy Phys. JHEP11(2001)008*
- [22] Bagrov V G and Gitman D M 1990 *Exact Solutions of Relativistic Wave Equations* (Dordrecht: Kluwer)  
Bagrov V G and Gitman D M 2014 *Dirac Equation and its Solutions* (Boston, MA: de Gruyter)
- [23] Spokoinyi B L 1982 *Yad. Fiz.* **36** 474  
Spokoinyi B L 1982 *Sov. J. Nucl. Phys.* **36** 277 (English transl.)  
Spokoinyi B L 1982 *Phys. Lett. A* **88** 328
- [24] Bialynicki-Birula I, Rudnicki Ł and Wienczek A 2011 *Pair Creation By Time-Dependent Electric Fields: Analytic Solutions* arXiv:1108.2615
- [25] Erdelyi A et al (ed) 1953 *Higher Transcendental Functions (Bateman Manuscript Project vol 1)* (New York: McGraw-Hill)