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Self-duality and the Holomorphic Ansatz in Generalized BPS Skyrme Model

Luiz Agostinho Ferreira[1] **and Leandro Roza Livramento**[1]

[1]Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, Brazil

E-mail: livramento@usp.br

Abstract. We propose a generalization of the BPS Skyrme model [1], in which the Skyrme field takes values in any simple compact Lie group G that gives leads to a Hermitian symmetric space $G/H \otimes U(1)$, where H is a subgroup of G. In addition, the model includes extra scalar fields corresponding to the entries of a symmetric, positive, and invertible $\dim G \times \dim G$ matrix h. We investigate the self-dual sector of this theory within the generalized rational map ansatz proposed in [2]. Apart from the special case $G = SU(2)$, the self-dual equations do not fully determine the matrix h in terms of the Skyrme field, which is totally arbitrary. In general, the number of free components of h tends to grow with the dimension of G. Furthermore, we show how to construct particular self-dual solutions whenever the generalized rational map ansatz can be explicitly implemented, which includes the case $G = SU(p+q)$ with the Hermitian symmetric space $SU(p+q)/SU(p) \otimes SU(q) \otimes U(1)$, for all p, $q \geq 1$.

1 Introduction

The notion of self-duality has provided profound insights into the structure of topological solitons and it is a powerful tool to construct such a solutions in a broad class of nonlinear classical field theories. The self-duality appears in models that have two main features. First, the integer-valued homotopy invariant quantity that classifies the topological soliton, the so-called topological charge Q, must admits integral representation with a density with the form of the contraction $A_\alpha \tilde{A}_\alpha$, where A_α and $\tilde{A}_\alpha$ are real-valued functions of the real-valued fields and their first order derivatives. The meaning of the contraction in α index depends of the theory. Second, the static energy density must be written as a sum of squares involving the two objects A_α and $\tilde{A}_\alpha$ obtained from the splitting of the topological charge density. It follows that the so-called self-duality equations $A_\alpha = \pm \tilde{A}_\alpha$, also called BPS equations, imply second-order differential Euler-Lagrange equations and also correspond to the global minimizer of the static energy to each value of Q.

The self-duality significantly simplifies the construction of topological solutions, as the BPS equations are first-order differential equations. The reduction by one integration arises from

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a mathematical identity stemming from the homotopy invariance of the topological charge. A broad range of self-dual topological solutions is known. Notable examples include instantons and kinks in $(1+1)$ dimensions [3–6], as well as Abelian Chern–Simons vortices in $(2+1)$ dimensions [7].

By exploring the invariance of the topological charge density under the transformation $(A_\alpha,\, \tilde{A}_\alpha) \to (A'_\alpha,\, \tilde{A}'_\alpha) = (\kappa_{\alpha\beta}\, A_\beta,\, \kappa^{-1}_{\alpha\beta}\, \tilde{A}_\beta)$, where k is a real invertible matrix, one can construct more general models that contain extra fields using ideas of self-duality. The static energy density of these models have the form of the sum of $A'^{\,2}_\alpha = A_\alpha\, h_{\alpha\beta}\, A_\beta$ and $\tilde{A}'^{\,2}_\alpha = \tilde{A}_\alpha\, h^{-1}_{\alpha\beta}\, \tilde{A}_\beta$, where the extra fields corresponds to the entries of the positive symmetric matrix $h \equiv k\, k^T$. Examples of BPS models in $(3+1)$ dimensions constructed using this ideas are the self-dual Skyrme model proposed in [1], also known as the BPS Skyrme model, and the self-dual generalization of the Yang–Mills–Higgs system proposed in [8].

The standard Skyrme model is an effective classical field theory describing the triplet of pions in $(3+1)$ dimensions at low energies [4, 9–12]. The model is formulated in terms of three pseudoscalar pion fields assembled into the so-called Skyrme field, which takes values in the $SU(2)$ Lie group. The construction of finite energy solutions requires that the Skyrme field approach the same constant at spatial infinity, allowing its domain to be compactified to S^3. The Skyrme field $U : S^3 \to S^3$ is classified by a topological charge, which admits an integral representation. The standard version of the model contains only one quadratic and one quartic term in the space-time derivatives and supports stable topological solitons known as Skyrmions. Unfortunately, the standard Skyrme model does not admit self-dual solutions, as established in [11, 13].

The BPS Skyrme model proposed in [1], mentioned above, generalizes the standard Skyrme model through the introduction of six additional scalar fields corresponding to the entries of the h matrix. This extension allows the model not only to possess an infinite number of self-dual solutions for any value of the topological charge, but also to be exactly solvable. In fact, let us introduce the matrix $\tau_{ab} \equiv R^a_i\, R^b_i$, which contracts the spatial components of the Maurer-Cartan form $R_\mu \equiv i\, \partial_\mu U\, U^{-1} \equiv R^a_\mu\, T_a$, with $a = 1,\, 2,\, 3$, where T_a are the generators of the Lie algebra of $SU(2)$. In any domain where τ is non-singular, the nine self-duality equations are equivalent to six algebraic equations that fully determine the matrix h in terms of the Skyrme field, which remains entirely arbitrary. This $SU(2)$ field remains entirely arbitrary even at points where τ is singular. However, in this case some of the components of the matrix h are also arbitrary. The arbitrariness of the Skyrme field makes the self-dual sector of the model remarkably rich, allowing for an infinite variety of exact self-dual solutions for any value of Q.

In this work, we propose a generalization of the BPS Skyrme model introduced in [1], in which the Skyrme fields map physical space into any simple compact Lie group G that leads to a Hermitian symmetric space $G/H \otimes U(1)$, where H is a subgroup of G. This construction includes the case $G = SU(N)$ with $N \geq 2$, thereby extending the model [1] to larger groups beyond $SU(2)$. Each component of h is a scalar field, and h is a $\dim G \times \dim G$ symmetric matrix that is invertible and positive definite. The dimension of h arises from the coupling of its row and column indices to the algebraic indices of the components of the Maurer–Cartan form. Examples of Lie groups that yield a Hermitian symmetric space are $G = A_r,\, B_r,\, C_r,\, D_r,\, E_6,\, E_7$, while counterexamples include $G = E_8,\, F_4,\, G_2$. Our second objective is to study the self-dual sector of the generalized BPS Skyrme model using the generalized rational map ansatz proposed in [2].

This work is organized as follows. In Section 2, we construct the generalized BPS Skyrme model using ideas of self-duality. In section 3 we study its self-dual sector within the generalized rational map ansatz proposed in [2]. In the section 4 we present our final considerations.

2 The model and its construction

The third homotopy group of any simple compact Lie group G is $\pi_3(G) = \mathbb{Z}$. Therefore, the maps $U : S^3 \to G$ are classified by the integers and the associated topological charge can be written in the integral representation as

$$Q = \frac{i}{48\,\pi^2} \int d^3x\, \varepsilon_{ijk}\, \widehat{\text{Tr}}\,(R_i\, R_j\, R_k) \tag{2.1}$$

where R_μ is a Maurer-Cartan form defined by

$$R_\mu \equiv i\,\partial_\mu U\, U^{-1} \equiv R^a_\mu\, T_a \tag{2.2}$$

with U taking its values on the group G, T_a being the generators of the corresponding compact simple Lie algebra $\mathcal{G}$ of the group G, with $a = 1, \ldots \dim,\, G$, satisfying $[T_a,\, T_b] = i\, f_{abc}\, T_c$, where f_{abc} is the structure constant. In addition, the generators are written in the orthogonal basis $\text{Tr}\,(T_a\, T_b) = \kappa\,\delta_{ab}$, and $\widehat{\text{Tr}}$ represents the normalized trace defined by $\widehat{\text{Tr}}\,(T_a\, T_b) \equiv \kappa^{-1}\, \text{Tr}\,(T_a\, T_b) = \delta_{ab}$.

The Maurer-Cartan form (2.2) satisfies by construction the Maurer-Cartan equation

$$\partial_\mu R_\nu - \partial_\nu R_\mu + i\,[\,R_\mu\,,\, R_\nu\,] = 0 \tag{2.3}$$

which allow us to write the topological charge (2.1) as

$$Q = \frac{1}{48\,\pi^2} \int d^3x\, \mathcal{A}^a_i\, \widetilde{\mathcal{A}}^a_i \tag{2.4}$$

with $\mathcal{A}^a_i \equiv R^b_i\, k_{ba}$ and $\widetilde{\mathcal{A}}^a_i \equiv \frac{i}{2}\, k^{-1}_{ab}\, \varepsilon_{ijk}\, \widehat{\text{Tr}}\,(T_b\,[\,R_j\,,\, R_k\,])$, where k_{ab} is some invertible real $\dim G \times \dim G$ dimensional matrix. Following the usual procedure of construction of self-dual theories from the integral representation of the topological charge [1,6], one can introduce the self-duality (BPS) equation

$$\lambda\, \mathcal{A}^a_i = \widetilde{\mathcal{A}}^a_i \qquad\Leftrightarrow\qquad \lambda\, R^b_i\, h_{ba} = \frac{i}{2}\, \varepsilon_{ijk}\, \widehat{\text{Tr}}\,(T_a\,[\,R_j\,,\, R_k\,]) \tag{2.5}$$

where $\lambda \equiv \pm m\, e$ and we have introduced a $\dim G \times \dim G$ dimensional matrix $h_{ab} = \left(k\, k^T\right)_{ab} = k_{ac}\, k_{bc}$, which is by consctruction invertible, symmetric and positive. The BPS equations (2.5) implies the Euler-Lagrange equations associated to the following static energy functional

$$E \;=\; \frac{1}{2} \int d^3x \left[m^2\, (\mathcal{A}^a_i)^2 + \frac{1}{e^2}\, \left(\widetilde{\mathcal{A}}^a_i\right)^2 \right] \tag{2.6}$$

which is the static energy of a generalized version of the Skyrme model [9, 10]. Therefore, the self-dual sector composed by the solutions of the self-dual equations (2.5) of the model (2.6) is a subset of the static sector composed by the static solutions of the Euler-Lagrange equations associated to (2.6). The action associated to energy (2.6) that defines our generalized BPS Skyrme model corresponds to

$$S \;=\; \int d^4x \left[\frac{m^2}{2}\, h_{ab}\, R^a_\mu\, R^{b\,,\,\mu} - \frac{1}{4\, e^2}\, h^{-1}_{ab}\, H^a_{\mu\nu}\, H^{b\,,\,\mu\nu} \right] \tag{2.7}$$

where we have introduced $H^a_{\mu\nu} \equiv -i\, \widehat{\text{Tr}}\,(T_a\,[\,R_\mu\,,\, R_\nu\,])$.

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Using the definition (2.6) one can introduce the BPS bound for the static energy by

$$E \;=\; \frac{1}{2\,e^2}\int d^3x\left[\lambda\,\mathcal{A}_i^a-\widetilde{\mathcal{A}}_i^a\right]^2+\text{sign}\,(\lambda)\;48\,\pi^2\,\frac{m}{e}\,Q\;\geq\;\text{sign}\,(\lambda)\;48\,\pi^2\,\frac{m}{e}\,Q \qquad (2.8)$$

which is saturated for the the self-dual solutions (2.5), reducing the static energy to

$$E=48\,\pi^2\,\frac{m}{e}\;\mid Q\mid \qquad (2.9)$$

where we have used the fact that E is non-negative, which due to the BPS bound (2.8) implies $\text{sign}\,(Q\,\lambda)=1$. Since the self-dual energy (2.9) is proportional to the modulus of the topological charge, is so follows that the binding energy of the BPS Skyrmions per topological charge unit $E_B=E_{Q=1}-E_Q/\mid Q\mid$ vanishes for each value of Q.

We can write the self-duality equations (2.5) in an alternative form by contracting (2.5) with R_i^c, which leads to

$$\lambda\,\tau_{cb}\,h_{ba}=\sigma_{ca} \qquad (2.10)$$

with

$$\tau_{ab}\equiv R_i^a\,R_i^b\,;\qquad\qquad \sigma_{ab}\equiv-\frac{1}{2}\,\varepsilon_{ijk}\,R_i^a\,H_{jk}^b \qquad (2.11)$$

The hermitian symmetric spaces have the form of a coset G/K, where K is the little group $K=H\otimes U(1)$. Using the usual algebraic structure of a symmetric space we have that

$$[\,\mathcal{G}\,,\,\mathcal{G}\,]\subset\mathcal{G}\qquad\qquad[\,\mathcal{G}\,,\,\mathcal{P}\,]\subset\mathcal{P}\qquad\qquad[\,\mathcal{P}\,,\,\mathcal{P}\,]\subset\mathcal{G} \qquad (2.12)$$

The hermitian character of such symmetric spaces is that $\mathcal{P}$ is even dimensional and it is split by Λ into two parts according the its eigenvalues

$$\mathcal{P}=\mathcal{P}_{+}+\mathcal{P}_{-}\qquad\qquad[\,\Lambda\,,\,P_{\pm}\,]=\pm P_{\pm}\qquad\qquad P_{\pm}\in\mathcal{P}_{\pm} \qquad (2.13)$$

It turns out that $\mathcal{P}_{-}$ is like the hermitian conjugate of $\mathcal{P}_{+}$, and so both spaces have the same dimension, i.e. $\dim\mathcal{P}_{+}=\dim\mathcal{P}_{-}=\frac{\dim\mathcal{P}}{2}$. Consequently, Λ provides a gradation of the Lie algebra $\mathcal{G}$ into subspaces of grades 0 and ±1, and so $\mathcal{P}_{\pm}$ are abelian (see [2] for more details). The compact irreducible hermitian symmetric spaces $G/H\otimes U(1)$ are

$$\begin{array}{ll} SU(p+q)/SU(p)\otimes SU(q)\otimes U(1)\,; & Sp(N)/SU(N)\otimes U(1)\,;\\ SO(N+2)/SO(N)\otimes U(1)\,; & E_6/SO(10)\otimes U(1)\,;\\ SO(2N)/SU(N)\otimes U(1)\,; & E_7/E_6\otimes U(1)\end{array} \qquad (2.14)$$

3 The self-dual sector whithin the generalized rational map ansatz

In this paper we use a generalization of the rational map ansätze proposed in [2] on the Euclidean space $\mathbb{R}^3$, for any compact simple Lie group G such that $G/H\otimes U(1)$ is a compact Hermitian symmetric space, for some subgroup H of G. For a given element $U\in G$, the ansatz has the form

$$U=e^{i\,f(r)\,g\,\Lambda\,g^{-1}} \qquad (3.1)$$

where $f\,(r)$ is a radial profile function, Λ is the generator of the $U(1)$ subgroup of G appearing in Hermitian symmetric space $G/H\otimes U(1)$, and g is a matrix that parameterizes the coset $G/H\otimes U(1)$. Using the coordinates $(r,\,z,\,\bar{z})$ defined by

$$x_1=r\,\frac{i\,(\bar{z}-z)}{1+\mid z\mid^2}\;;\qquad x_2=r\,\frac{z+\bar{z}}{1+\mid z\mid^2}\;;\qquad x_3=r\,\frac{\left(-1+\mid z\mid^2\right)}{1+\mid z\mid^2} \qquad (3.2)$$

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the matrix g is defined by

$$g(z,\,\bar{z}) \equiv \mathbb{1} + \frac{1}{\vartheta}\left[i\left(S + S^{\dagger}\right) - \frac{1}{\vartheta+1}\left(S\,S^{\dagger} + S^{\dagger}\,S\right)\right]; \qquad \vartheta \equiv \sqrt{1+\omega} \tag{3.3}$$

with S being a matrix in some special representation of G such that it satisfies

$$S^2 = 0\,; \qquad \left(S\,S^{\dagger}\right)\,S = \omega\,S \tag{3.4}$$

with ω being a real and non-negative eigenvalue. The matrix S is either holomorphic, $\partial_{\bar{z}}S = 0$, or anti-holomorphic, $\partial_z S = 0$. For simplicity, we can write $S = S(\chi)$, where $\chi = z$ ($\chi = \bar{z}$) in the case that S is (anti-)holomorphic. In addition, S and $S^{\dagger}$ belongs to the $\mathcal{P}_+$ and $\mathcal{P}_-$ abelian subalgebras, respectively. For later convenience we can also introduce the sign function $\eta = 1$ for $\chi = z$, and $\eta = -1$ for $\chi = \bar{z}$. Therefore, the matrix g defines holomorphic, or anti-holomorphic, maps from the two-spheres S^2 in $\mathbb{R}^3$, parametrized by z and $\bar{z}$, to the Hermitian symmetric space $G/H \otimes U(1)$.

The components of the Maurer-Cartan form (2.2) associated to the generalized rational map ansatz (3.1) becomes

$$R^b_\alpha = \widehat{\mathrm{Tr}}\,(T_b\,R_i) = -\widehat{\mathrm{Tr}}\left(V\,T_b\,V^{-1}\,\Sigma_\alpha\right) = -\widehat{\mathrm{Tr}}\,(T_c\,\Sigma_\alpha)\;d_{cb}\,(V) \tag{3.5}$$

where we have introduced the matrix for the group elements in the adjoint representation of G, i.e. $g\,T_a\,g^{-1} = T_b\,d_{ba}\,(g)$. In addition, $V \equiv e^{-i\,f\,\Lambda/2}\,g^{-1}$, and the components Σ_α are given by

$$\Sigma_r = f'\;\Lambda\,; \qquad \Sigma_\chi = -2\,\sin\frac{f}{2}\,P^{(+)}_\chi\,; \qquad \Sigma_{\bar{\chi}} = 2\,\sin\frac{f}{2}\,P^{(-)}_{\bar{\chi}} \tag{3.6}$$

where $P^{(+)}_\chi = i\,\vartheta^{-1}\,(1+\vartheta)^2\,\partial_\chi\left((1+\vartheta)^{-2}\,S\right) \in \mathcal{P}_+$ and $P^{(-)}_{\bar{\chi}} = -\left(P^{(-)}_{\bar{\chi}}\right)^{\dagger} \in \mathcal{P}_-$. We define a sign function η' and fix the boundary conditions of the profile function as $f(0) = 2\,\pi\,m$ and $f(\infty) = 0$ for $\eta' = -1$, where $m \in \mathbb{N}$, and $f(0) = 0$ and $f(\infty) = 2\,\pi\,m$ for $\eta' = 1$. The topological charge (2.1) becomes

$$Q \;=\; \eta'\,m\,Q_{\rm top} \qquad \text{with} \qquad Q_{\rm top} \equiv \eta\,\frac{i}{4\,\pi}\int dz\,d\bar{z}\;\widehat{\mathrm{Tr}}\left(P^{(+)}_\chi\,P^{(-)}_{\bar{\chi}}\right) \tag{3.7}$$

Using (3.5) and (3.6) the BPS equations (2.5) can be written as

$$\lambda\,\widetilde{\tau}_{cb}\,\widetilde{h}_{ba} = \widetilde{\sigma}_{ca} \tag{3.8}$$

where we have introduced

$$\widetilde{h}_{ab} \equiv d_{ac}\,(V)\;h_{cd}\,d^{-1}_{db}\,(V)\;; \quad \widetilde{\tau}_{ab} \equiv d_{ac}\,(V)\;\tau_{cd}\,d^{-1}_{db}\,(V)\;; \quad \widetilde{\sigma}_{ab} \equiv d_{ac}\,(V)\;\sigma_{cd}\,d^{-1}_{db}\,(V) \tag{3.9}$$

The matrices $\widetilde{h}_{ab}$ and $\widetilde{\tau}_{ab}$ have the same eigenvalues of h and τ, respectively, and are symmetric, since the adjoint representation of a compact simple Lie group is unitary and real, and so d is an orthogonal matrix, i.e. $d^T = d^{-1}$. In so follows that $\widetilde{\tau}_{ab} = \Sigma^a_i\,\Sigma^b_i$ and $\widetilde{\sigma}_{ab} = -\frac{i}{2}\,\widehat{\mathrm{Tr}}\,(T_a\,\Sigma_i)\;\varepsilon_{ijk}\,\widehat{\mathrm{Tr}}\,(T_b\,[\,\Sigma_j\,,\,\Sigma_k\,])$.

For convenience, we adopt the convention that the index $\mathcal{H}$, when appearing in the row or column indices of the matrices $\widetilde{\tau}$ and $\widetilde{\sigma}$, denotes any index a labelling the generators T_a of the subalgebra $\mathcal{H}$, and so on. A crucial consequence of the generalized rational map ansatz is that the relations (3.6) and (3.9) imply $\widetilde{\tau}_{a\,\mathcal{H}} = \widetilde{\sigma}_{a\,\mathcal{H}} = 0$ for all $a = 1, ..., \dim G$. Consequently,

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none of the self-dual equations (3.8) depends on the $\tilde{h}_{\mathcal{H}\mathcal{H}}$ fields, which remain entirely arbitrary. Therefore, in case $\mathcal{H} \neq \emptyset$, it follows that $\widetilde{\tau}$ is not invertible. Otherwise, we could contract the self-dual equations (3.8) with $\widetilde{\tau}^{-1}_{dc}$, which would fix $\tilde{h}$ entirely, contradicting the fact that $\widetilde{h}_{\mathcal{H}\mathcal{H}}$ is free.

Taking the row index c as $\mathcal{H}$ in the BPS equations (3.8), one obtains that $\lambda\,\widetilde{\tau}_{\mathcal{H}\,b}\,\tilde{h}_{b\,a} = \widetilde{\sigma}_{\mathcal{H}\,a}$, which are automatically satisfied using (3.6). On the other hand, using (3.6), the remaining BPS equations in (3.8) are reduced to

$$\widetilde{\sigma}_{\Lambda\,b} = \lambda\,\widetilde{\tau}_{\Lambda\Lambda}\,\tilde{h}_{\Lambda\,b} \qquad \Rightarrow \qquad \tilde{h}_{\Lambda\,b} = \frac{\widetilde{\sigma}_{\Lambda\,b}}{\lambda\,\widetilde{\tau}_{\Lambda\Lambda}} \tag{3.10}$$

$$0 = \widetilde{\tau}_{\mathcal{P}_+\,\mathcal{P}_-}\,\tilde{h}_{\mathcal{P}_-\,\mathcal{P}_+} = \widetilde{\tau}_{\mathcal{P}_+\,\mathcal{P}_-}\,\tilde{h}_{\mathcal{P}_-\,\mathcal{H}} = \widetilde{\tau}_{\mathcal{P}_-\,\mathcal{P}_+}\,\tilde{h}_{\mathcal{P}_+\,\mathcal{P}_-} = \widetilde{\tau}_{\mathcal{P}_-\,\mathcal{P}_+}\,\tilde{h}_{\mathcal{P}_+\,\mathcal{H}} \tag{3.11}$$

$$0 = \widetilde{\tau}_{\mathcal{P}_-\,\mathcal{P}_+}\left(\tilde{h}_{\mathcal{P}_+\,\mathcal{P}_+} + \eta\,\lambda^{-1}\,f'\,\mathbb{1}\right) = \widetilde{\tau}_{\mathcal{P}_+\,\mathcal{P}_-}\left(\tilde{h}_{\mathcal{P}_-\,\mathcal{P}_-} + \eta\,\lambda^{-1}\,f'\,\mathbb{1}\right) \tag{3.12}$$

Although there is an implicit sum over the line index of the $\tilde{h}$ matrix, leading to a linear system to the $\tilde{h}_{ab}$ fields, in (3.10) this sum is performed over a single generator, which corresponds to the $U(1)$ generator Λ. Therefore, $\widetilde{h}_{\Lambda\,a}$ is fully determined by

$$\tilde{h}_{\Lambda\,\Lambda} = \alpha\,\eta\,\mathrm{Tr}\left(P^{(+)}_{\chi}\,P^{(-)}_{\bar{\chi}}\right); \quad \tilde{h}_{\Lambda\,\mathcal{H}} = \alpha\,\eta\,\mathrm{Tr}\left(\mathcal{H}\,\left[P^{(+)}_{\chi}\,,\,P^{(-)}_{\bar{\chi}}\right]\right); \quad \tilde{h}_{\Lambda\,\mathcal{P}_\pm} = 0 \tag{3.13}$$

where $\alpha \equiv \frac{2\,\sin^2\frac{f}{2}}{\lambda\,f'\,\mathrm{Tr}(\Lambda^2)}\,\frac{(1+|z|^2)^2}{r^2}$ and the normalized trace $\widehat{\mathrm{Tr}}$ has been replaced by the usual trace Tr, since the κ factor cancels in the self-duality equations (3.8).

Obviously, when the matrix $\widetilde{\tau}_{\mathcal{P}_-\,\mathcal{P}_+}$ is invertible, equations (3.11) and (3.12) imply that

$$\tilde{h}_{\mathcal{P}_\pm\,\mathcal{H}} = 0 \qquad\qquad \tilde{h}_{\mathcal{P}\,\mathcal{P}} = -\eta\,\lambda^{-1}\,f'\,\mathbb{1} \tag{3.14}$$

Nonetheless, in the case that $\widetilde{\tau}_{\mathcal{P}_-\,\mathcal{P}_+}$ is not invertible, the fields (3.13) and (3.14) continue to provide a particular self-dual solution of (3.8). Regardless, the self-duality equation for the fields $\tilde{h}_{\mathcal{P}_\pm\,\mathcal{H}}$ in (3.11) can be expressed as

$$\mathrm{Tr}\left(\mathcal{P}_-\,P^{(+)}_{\chi}\right)\,\tilde{h}_{\mathcal{P}_-\,\mathcal{H}} = \mathrm{Tr}\left(\mathcal{P}_+\,P^{(-)}_{\bar{\chi}}\right)\,\tilde{h}_{\mathcal{P}_+\,\mathcal{H}} = 0 \tag{3.15}$$

Since there is no other BPS equation involving the fields $\tilde{h}_{\mathcal{P}_\pm\,\mathcal{H}}$, there are only $\dim\mathcal{H}$ equations to determine the components $\tilde{h}_{\mathcal{P}_+\,\mathcal{H}}$, and there is an independent set of $\dim\mathcal{H}$ equations to determine the components $\tilde{h}_{\mathcal{P}_-\,\mathcal{H}}$. As a result, there are at least $2\dim\mathcal{H}\,(\dim\mathcal{P}_+ - 1)$ components of $\tilde{h}_{\mathcal{H}\mathcal{P}}$ free. The remaining self-duality equations in (3.11) correspond to $\dim\mathcal{P}_+$ equations for the fields $\tilde{h}_{\mathcal{P}_-\,\mathcal{P}_+}$ which can be written as

$$\mathrm{Tr}\left(\mathcal{P}_-\,P^{(+)}_{\chi}\right)\,\tilde{h}_{\mathcal{P}_-\,\mathcal{P}_+} = 0 \tag{3.16}$$

Although there is also a set of $\dim\mathcal{P}_+$ linear equations for the fields $\mathrm{Tr}\left(\mathcal{P}_+\,P^{(-)}_{\bar{\chi}}\right)\,\tilde{h}_{\mathcal{P}_+\,\mathcal{P}_-} = 0$ in (3.11), this set of equations corresponds to the complex conjugate of (3.16). From (3.6) and the definition of $\widetilde{\tau}$, we can write the remaining BPS equations, given in (3.12), as

$$\mathrm{Tr}\left(\mathcal{P}_+\,P^{(-)}_{\bar{\chi}}\right)\,\tilde{h}_{\mathcal{P}_+\,\mathcal{P}_+} = -\eta\,\lambda^{-1}\,f'\,\mathrm{Tr}\left(\mathcal{P}_+\,P^{(-)}_{\bar{\chi}}\right) \tag{3.17}$$

$$\mathrm{Tr}\left(\mathcal{P}_-\,P^{(+)}_{\chi}\right)\,\tilde{h}_{\mathcal{P}_-\,\mathcal{P}_-} = -\eta\,\lambda^{-1}\,f'\,\mathrm{Tr}\left(\mathcal{P}_-\,P^{(+)}_{\chi}\right) \tag{3.18}$$

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Therefore, there are $\dim\mathcal{P}_+$ linear equations to determine $\dim\mathcal{P}_+\,(\dim\mathcal{P}_+ + 1)/2$ fields $\tilde{h}_{\mathcal{P}_+\,\mathcal{P}_+}$, and the same follows for the fields $\tilde{h}_{\mathcal{P}_-\,\mathcal{P}_-}$. Once that the block $\tilde{h}_{\mathcal{P}\mathcal{P}}$ forms a $\dim\mathcal{P}\times\dim\mathcal{P}$ dimensional symmetric matrix, the relations (3.16), (3.17) and (3.18) together compose a set of $3\dim\mathcal{P}_+$ linear equations to determined $\dim\mathcal{P}_+\,(2\dim\mathcal{P}_+ + 1)$ components of the $\tilde{h}$-fields, leaving at least $2\dim\mathcal{P}_+\,(\dim\mathcal{P}_+ - 1)$ components undetermined.

The analysis above demonstrates that having $\dim\mathcal{P}_+ = 1$ is required for the fields $\tilde{h}_{\mathcal{P}\mathcal{P}}$ to be entirely fixed by the self-duality equations (3.8). Moreover, as shown earlier, in the cases where $\widetilde{\tau}_{\mathcal{P}_-\,\mathcal{P}_+}$ is invertible, it follows that $\tilde{h}_{\mathcal{P}\mathcal{P}}$ must be entirely determined by (3.14). Consequently, $\dim\mathcal{P}_+ = 1$ is also a necessary condition for $\widetilde{\tau}_{\mathcal{P}_-\,\mathcal{P}_+}$ to be invertible, which is the case for $G = SU(2)$.

Note that for any representation of the group G where its is possible to construct the $S\in\mathcal{P}_+$ matrix satisfying (3.4), the field configuration given in (3.13) and (3.14) that fixes the components $\tilde{h}_{\Lambda\,\Lambda}$, $\tilde{h}_{\Lambda\,\mathcal{H}}$, $\tilde{h}_{\Lambda\,\mathcal{P}_\pm}$, $\tilde{h}_{\mathcal{P}_\pm\,\mathcal{H}}$, $\tilde{h}_{\mathcal{P}\,\mathcal{P}}$, but do not fixes any component of $\tilde{h}_{\mathcal{H}\,\mathcal{H}}$, corresponds to a particular solution of the BPS equations (3.8). Since the profile function remains arbitrary, this leads to an infinite number of exact topological solutions for any simple compact Lie G that leads to a Hermitian symmetric space, given in (2.14). In addition, the fields that parametrizes the S matrix may be also arbitrary, enlarging the set of distinct exact BPS solutions of the model.

By example, in the case of the Hermitian symmetric space $SU\,(p+q)\,/SU\,(p)\otimes SU\,(q)\otimes U\,(1)$, the S matrix was constructed in [2] in the fundamental $(p+q)\times(p+q)$ representation of the group $G = SU(p+q)$. In such a case, the S matrix is parametrized by p complex scalar fields $u_a = u_a(\chi)$, with $a = 1,\,...,\,p$, and q complex scalar fields $v_b = v_b(\chi)$, with $b = 1,\,...,\,q$, corresponding with the entries of the objects $u^T = (u_1,\,...,\,u_p)$ and $v^T = (v_1,\,...,\,v_q)$. The Λ and S matrices, which defines the generalized rational map ansatz (3.1) and (3.3), are given by

$$\Lambda = \frac{1}{p+q}\begin{pmatrix} q\,\mathbb{1}_{p\times p} & O_{p\times q} \\ O_{q\times p} & -p\,\mathbb{1}_{q\times q}\end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} O_{p\times p} & u\otimes v \\ O_{q\times p} & O_{q\times q}\end{pmatrix} \tag{3.19}$$

where $O_{p\times q}$ is a $p\times q$ zero matrix, and so on. When S is (anti-)holomorphic, both the fields u and v are (anti-)holomorphic, and S satisfies (3.4) with $\omega =\mid u\mid^2\mid v\mid^2$. Therefore, (3.13) and (3.14) corresponds to a particular solutions of the BPS Skyrme model, where the profile function f and the matrix S given in (3.19) are arbitrary, i.e., u and v are arbitrary rational maps. The topological charge (3.7) becomes

$$Q = \eta'\,m\,Q_{\rm top}\,;\qquad Q_{\rm top} = -\frac{i\,\eta}{2\,\pi}\int\frac{dz\,d\bar{z}}{\vartheta^2}\,\left[\partial_\chi\partial_{\bar\chi}\omega - \frac{\partial_\chi\omega\,\partial_{\bar\chi}\omega}{1+\omega}\right] \tag{3.20}$$

A special class of very simple self-dual solutions can be constructed choosing all the components of the fields u are v equal to the same (anti-)holomorphic rational maps $u_1(\chi) = p_u(\chi)/q_u(\chi)$ and $v_1(\chi) = p_v(\chi)/q_v(\chi)$, respectively, between the Riemann spheres S^2. Therefore, the topological charge (3.20) is reduced to $Q = \eta'\,\eta\,m\,n$, where $n = \deg\,(u_1\,v_1)$ is the degree of the rational map $u_1\,v_1$ which can be any natural number depending of the choice of the rational maps u_1 and v_2. On top of this freedom, the profile function is also free, leadind to an infinit number of exact solutions to each integer value of the topological charge Q. Using this approach, and also choosing $u\,v = \chi/\sqrt{p\,q}$, the density of the topological charge and the static energy density become spherically symmetric. Therefore, by adjusting the boundary condition of the profile function, one can construct spherically symmetric BPS Skyrmions for any value of Q.

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4 Conclusion

We have shown that in our generalization of the BPS Skyrme model for any compact Lie group G leading to a Hermitian symmetric space, the full determination of the h fields in terms of the Skyrme fields occurs only in the case $G = SU(2)$. The generalized rational map ansatz employed in these arguments drastically simplifies the self-dual equations, allowing the exact determination of all components $\tilde{h}_{\Lambda\Lambda}$, $\tilde{h}_{\Lambda\mathcal{P}_\pm}, \tilde{h}_{\Lambda\mathcal{H}}$ in terms of the Skyrme field, while yielding linear algebraic equations for the components of $\tilde{h}_{\mathcal{H}\mathcal{P}_\pm}, \tilde{h}_{\mathcal{P}_\pm\mathcal{P}_\pm}, \tilde{h}_{\mathcal{P}_\pm\mathcal{P}_\mp}$. Outside the case $G = SU(2)$, these equations are underdetermined, leaving at least $2\dim\mathcal{P}_+ \,(\dim\mathcal{P}_+ - 1)$ components of $\tilde{h}_{\mathcal{P}\mathcal{P}}$ and $2\dim\mathcal{H}\,(\dim\mathcal{P}_+ - 1)$ components of $\tilde{h}_{\mathcal{H}\mathcal{P}}$ entirely arbitrary.

In any representation of G where a matrix S satisfying (3.4) can be constructed, we have shown how to generate an infinite number of exact topological solutions within the generalized rational map ansatz. We have explicitly constructed the S matrix for the Hermitian symmetric space $SU(p+q)/SU(p)\otimes SU(q)\otimes U(1)$, where S is expressed in terms of p rational maps $u_i(\chi)$ and q rational maps $v_i(\chi)$, which can be chosen freely. This allows us to construct a wide variety of topological solutions for any value of Q. In a special case, choosing all components of u_i and v_i are equal to $u_1(\chi)$ and $v_1(\chi)$, respectively, we have shown how to obtain any value of the topological charge by changing either the boundary conditions of the profile function or the degree $n = \deg(u_1\, v_1)$ of the rational map $u_1\, v_1$.

The study of the generalized Skyrme model is greatly facilitated by the generalized rational map ansatz, which also may play a key role in the construction of multi-Skyrmions in extensions of this model. For instance, the global energy minimizer in such extensions may preserve all the BPS equations, as occurs in the generalized False Vacuum Skyrme model [14]. Notably, our model allows the construction of spherically symmetric multi-Skyrmions for any value of Q in the case of $G = SU(N)$, which has been employed to construct $SU(N)$ False Vacuum Skyrmions [14].

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