Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 413 (2024) 190–239

Journal of Differential Equations

www.elsevier.com/locate/jde

Asymptotics for singular solutions to conformally invariant fourth order systems in the punctured ball [☆]

João Henrique Andrade [a,b,c,*], João Marcos do Ó [c]

[a] Institute of Mathematics and Statistics, University of São Paulo, 05508-090, São Paulo-SP, Brazil
[b] Department of Mathematics, University of British Columbia, V6T 1Z2, Vancouver-BC, Canada
[c] Department of Mathematics, Federal University of Paraíba, 58051-900, João Pessoa-PB, Brazil

Received 8 December 2023; revised 26 July 2024; accepted 8 August 2024

Abstract

We study asymptotic profiles for singular solutions to a class of critical strongly coupled fourth order systems on the punctured ball. Assuming a superharmonicity condition, we prove that sufficiently close to the isolated singularity, singular solutions behave like the so-called Emden–Fowler solution to the blow-up limit problem. On the technical level, we use an involved spectral analysis to study the Jacobi fields' growth properties in the kernel of the linearization of our system around a blow-up limit solution, which may be of independent interest. Our main theorem positively answers a question posed by Frank and König (2019) [12] concerning the local behavior of singular solutions close to the isolated singularity for scalar solutions in the punctured ball. It also extends to the case of strongly coupled systems, the celebrated asymptotic classification due to Korevaar et al. (1999) [21].

☆ This work was partially supported by Fulbright Commission in Brazil #G-1-00001, Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) #312340/2021-4 and #429285/2016-7, Fundação de Apoio à Pesquisa do Estado da Paraíba (FAPESQ) #3034/2021, Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) #88882.440505/2019-01, and Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) # 2020/07566-3 and #2021/15139-0.

* Corresponding author at: Institute of Mathematics and Statistics, University of São Paulo, 05508-090, São Paulo-SP, Brazil.

E-mail addresses: andradejh@ime.usp.br, andradejh@math.ubc.ca, andradejh@mat.ufpb.br (J.H. Andrade), jmbo@pq.cnpq.br (J.M. do Ó).

https://doi.org/10.1016/j.jde.2024.08.029

MSC: 35J60; 35B09; 35J30; 35B40

Keywords: Bi-Laplacian; Coupled systems; Critical exponent; Asymptotic analysis; Local behavior

1. Description of the results

We study the local behavior for strongly positive singular solutions to the critical fourth order system,

$$\Delta^2 u_i = c(n)|\mathcal{U}|^{2^{**}-2}u_i \quad \text{in} \quad B_R^*, \tag{$\mathcal{S}_{p,R}$}$$

where $B_R^* := B_R^n(0) \setminus \{0\} \subset \mathbb{R}^n$ with $n \geqslant 5$ and $R < \infty$ is the punctured ball, Δ^2 is the bi-Laplacian, $\mathcal{U} = (u_1, \dots, u_p) : B_1^* \to \mathbb{R}^p$ is a p-map solution and $|\mathcal{U}| = (\sum_{i=1}^p u_i^2)^{1/2}$ is its Euclidean norm. System ($\mathcal{S}_{p,R}$) is strongly coupled by the Gross–Pitaevskii nonlinearity $f_i(\mathcal{U}) = c(n)|\mathcal{U}|^{2^{**}-2}u_i$ with associated potential $F(\mathcal{U}) = (f_1(\mathcal{U}), \dots, f_p(\mathcal{U}))$, where $2^{**} = 2n/(n-4)$ is the critical Sobolev exponent, and

$$c(n) = \frac{n(n-4)(n^2-4)}{16} \tag{1}$$

is a dimensional normalizing constant.

Let us introduce some terminology. We say that $\mathcal{U}$ is a classical solution to ($\mathcal{S}_{p,R}$) if each component $u_i \in C^{4,\zeta}(B_1^*)$, for some $\zeta \in (0,1)$, and satisfies ($\mathcal{S}_{p,R}$) pointwise. In addition, $\mathcal{U}$ is called a singular solution to ($\mathcal{S}_{p,1}$), if the origin is a non-removable singularity for $|\mathcal{U}|$, that is, $\lim_{|x|\to 0} |\mathcal{U}(x)| = \infty$. Otherwise, the origin is called a removable singularity, and $\mathcal{U}$ is a non-singular solution of ($\mathcal{S}_{p,R}$). By a strongly positive (nonnegative) solution $\mathcal{U}$ to ($\mathcal{S}_{p,R}$), we understand a classical solution such that $u_i > 0$ ($u_i \geqslant 0$) for all $i \in I := \{1, \dots, p\}$. We call $\mathcal{U}$ superharmonic in case $-\Delta u_i > 0$ for all $i \in I := \{1, \dots, p\}$. By the maximum principle, superharmonic nonnegative solutions are weakly positive, that is, for any $i \in I$ either $u_i > 0$ or $u_i \equiv 0$.

The first step to studying this local behavior is to classify the solutions to the blow-up limit system

$$\Delta^2 u_i = c(n)|\mathcal{U}|^{2^{**}-2}u_i \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \tag{$\mathcal{S}_{p,\infty}$}$$

These limiting profiles are often called Emden–Fowler solutions.

Let us compare our system with its scalar counterpart. Indeed, when $p = 1$, we get that ($\mathcal{S}_{p,R}$) reduces to the following fourth order critical equation,

$$\Delta^2 u = c(n)u^{2^{**}-1} \quad \text{in} \quad B_R^*. \tag{$\mathcal{S}_{1,R}$}$$

On this subject, C. S. Lin [23, Theorem 1.3] proved that all positive non-singular solutions to ($\mathcal{S}_{1,R}$) with $R = \infty$ are radially symmetric. He also obtained a closed expression for these solutions. Additionally, if the origin is a non-removable singularity, R. L. Frank and T. König [12, Theorem 2] (see also [14, Theorem 1.3]) proved that these solutions are also classified. Recently, T. Jin and J. Xiong [20, Theorems 1.1 and 1.2] used a Green identity for the poly-Laplacian and some localization methods to study an equivalent integral equation, proving asymptotic radial symmetry and sharp global estimates for singular solutions to ($\mathcal{S}_{1,R}$) with $R < \infty$. These results can be compiled in the following statement.

Theorem A. *Let u be a positive solution to ($\mathcal{S}_{1,R}$). Assume that*
Case (I): (*punctured space*) $R = \infty$.

(i) *If the origin is a removable singularity, then there exists $x_0 \in \mathbb{R}^n$ and $\mu > 0$ such that u is radially symmetric about x_0 and, up to a constant, is given by*

$$u_{x_0,\mu}(x) = \left(\frac{2\mu}{1+\mu^2|x-x_0|^2}\right)^{\frac{n-4}{2}}. \tag{2}$$

These are called the (fourth order) spherical solutions (or bubbles).

(ii) *If the origin is a non-removable singularity, then u is radially symmetric with respect to the origin. Moreover, there exist $a \in (0, a_0]$ and $T \in (0, T_a]$ such that*

$$u_{a,T}(x) = |x|^{\frac{4-n}{2}} v_a(\ln|x| + T). \tag{3}$$

Here $a_0 = [n(n-4)/(n^2-4)]^{n-4/8}$ and $T_a \in \mathbb{R}$ is the fundamental period of the unique periodic bounded solution v_a to the following fourth order Cauchy problem

$$\begin{cases} v^{(4)} - K_2 v^{(2)} + K_0 v = c(n) v^{2^{**}-1} \\ v(0) = a, \ v^{(1)}(0) = 0, \ v^{(2)}(0) = b, \ v^{(3)}(0) = 0, \end{cases} \tag{4}$$

where

$$K_0 = \frac{n^2(n-4)^2}{16} \quad \text{and} \quad K_2 = \frac{n^2-4n+8}{2}.$$

We call both $u_{a,T}$ and $v_{a,T}$ (fourth order) Emden–Fowler (or Delaunay-type) solutions.

Case (II): (*punctured ball*) $R < \infty$, *and the origin is a non-removable singularity. Suppose that u is superharmonic. Then, $u(x) = (1+\mathcal{O}(|x|))\overline{u}(|x|)$ as $x \to 0$, where $\overline{u}$ is the spherical average of u. Moreover, there exists $u_{a,T}$ as in (3) such that*

$$u(x) = (1 + \mathrm{o}(1)) u_{a,T}(|x|) \quad \text{as} \quad x \to 0. \tag{5}$$

Here $v^{(j)} = \frac{\mathrm{d}^{(j)}}{\mathrm{d}t^{(j)}}$ for $j \in \mathbb{N}$ denotes the j-th order ordinary derivative v with respect to t.

We now move to the vectorial case. In this situation, using sliding techniques and ODE analysis, in [2] the present authors obtained the classification for solutions to the limit blow-up system ($\mathcal{S}_{p,\infty}$). Before, we define $\mathbb{S}^{p-1}_{+,*} := \{x \in \mathbb{S}^{p-1} : x_i > 0\}$.

Theorem B. *Let $\mathcal{U}$ be a strongly positive solution to* ($\mathcal{S}_{p,\infty}$).
(i) *If the origin is a removable singularity. Then, there exists $\Lambda \in \mathbb{S}^{p-1}_{+,*}$ such that $\mathcal{U} = \Lambda u_{x_0,\mu}$, where $u_{x_0,\mu}$ (see (2)) is a positive solution ($\mathcal{S}_{1,R}$) with $R = \infty$;*
(ii) *If the origin is a non-removable singularity. Then, there exists $\Lambda \in \mathbb{S}^{p-1}_{+,*}$ such that $\mathcal{U} = \Lambda u_{a,T}$, where $u_{a,T}$ (see (3)) is a positive solution to ($\mathcal{S}_{1,R}$) with $R = \infty$.*

Our main result in this manuscript proves that strongly positive solutions to ($\mathcal{S}_{p,R}$) have a local asymptotic profile near the isolated singularity given by the radial solutions to ($\mathcal{S}_{p,\infty}$).

Theorem 1. *Let $\mathcal{U}$ be a strongly positive superharmonic singular solution to ($\mathcal{S}_{p,R}$). Then, there exist a solution $\mathcal{U}_{a,T}$ to ($\mathcal{S}_{p,\infty}$) and $0 < \beta_0^* < 1$ such that*

$$\mathcal{U}(x) = (1 + \mathcal{O}(|x|^{\beta_0^*}))\mathcal{U}_{a,T}(|x|) \quad \text{as} \quad x \to 0. \tag{6}$$

Let us mention that Theorem B and Theorem 1 extends Theorem A for the vectorial case $p > 1$. In addition, $p = 1$ improves the remainder error term in the estimate (5).

Remark 2. As a by-product of our arguments, one can improve the decay of the remainder term in (6), using deformed Emden–Fowler solutions in the sense of Appendix A (see [21, Section 7]). Precisely, under the assumptions of Theorem 1, we have the following refined asymptotics

$$\mathcal{U}(x) = (1 + \mathcal{O}(|x|^{\beta_1^*}))\mathcal{U}_{a,T,0}(|x|) \quad \text{as} \quad x \to 0, \tag{7}$$

for some $\beta_1^* > 1$ and $\mathcal{U}_{a,T,0}$ deformed Emden–Fowler solution to ($\mathcal{S}_{p,\infty}$).

From the geometric point of view, R. Schoen and S.-T. Yau [32] highlighted the importance of studying geometric singular equations and describing their asymptotic behavior near their singular sets. Indeed, a positive smooth solution $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ to ($\mathcal{S}_{1,R}$) with $R = \infty$ produces a conformally flat metric $\bar{g} = u^{4/(n-4)}\delta_0$ such that $\bar{g}$ has constant Q-curvature equals $Q_g = n(n^2-4)/8$, where δ_0 is the standard flat metric, and

$$Q_g = -\frac{1}{2(n-1)}\Delta_g R_g - \frac{2}{(n-2)^2}|\operatorname{Ric}_g|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2}R_g^2,$$

is a fourth order analog of the conformal Laplacian, where Δ_g, R_g, and Ric_g are the Laplace–Beltrami, scalar curvature and Ricci operator with respect to g. By the stereographic projection, ($\mathcal{S}_{1,R}$) is the particular case of the singular Q-curvature equation on the punctured round sphere

$$\begin{cases} P_g u = c(n)u^{2^{**}-1} & \text{on} \quad (\mathbb{S}^n \setminus \{p, -p\}, g_0) \\ \liminf\limits_{x \to \pm p} u(x) = \infty, \end{cases} \tag{8}$$

where g_0 is the standard round metric and

$$P_g u = \Delta_g^2 u - \operatorname{div}_g\left(\frac{(n-2)^2+4}{2(n-1)(n-2)}R_g g - \frac{4}{n-2}\operatorname{Ric}_g\right)\mathrm{d}u + \frac{n-4}{2}Q_g u$$

is the Paneitz–Branson operator (for more details, see [17] and the references therein). In this language, (6) with $p = 1$ states that any complete metric with nonnegative scalar curvature and constant Q-curvature is asymptotic to a Delaunay metric near an isolated singularity.

Now, we discuss some existing literature for second order equations. The singular Yamabe problem is a second order geometric PDE similar in spirit to (8). In the conformally flat case, [4] develops a measure-theoretic version of the Alexandrov technique to prove that solutions to this second order equation defined in the punctured ball are radially symmetry. Moreover, they classified these global singular solutions in the punctured space. They obtained the local behavior in the neighborhood of the isolated singularity, proving that any singular solution converges to an Emden–Fowler one.

Later, in [21], a more geometric approach for proving (6) is provided, based on the Jacobi field growth for the linearized operator around a blow-up limit solution. This was later extended to the case of general background metrics [25], at least for low dimensions. For second order strongly coupled systems, in [8,9] the blow-up solutions to an analog of ($\mathcal{S}_{p,R}$) are classified. Also, in [5], an asymptotic classification is obtained, similar in spirit to the one Theorem 1.

Strongly coupled systems also appear in several mathematical physics branches. For instance, in hydrodynamics, for modeling the behavior of deep-water and Rogue waves in the ocean [10, 24]. As well as it can be used as a model in the Hartree–Fock theory for Bose–Einstein double condensates [1,11].

The strategy to prove Theorem 1 relies on asymptotic analysis. Roughly speaking, this is a combination of classification results, some a priori estimates, and linear analysis. Using a simple scaling argument, we may assume that $R = 1$ in ($\mathcal{S}_{p,R}$), which gives us

$$\Delta^2 u_i = c(n)|\mathcal{U}|^{2^{**}-2}u_i \quad \text{in} \quad B_1^*, \tag{$\mathcal{S}_{p,1}$}$$

where $c(n) > 0$ is given by (1).

The first step is to show that the Jacobi fields (elements in the kernel of the linearization of ($\mathcal{S}_{p,1}$) around a blow-up solution) satisfy suitable growth properties:

Proposition 3. *For any $a \in (0, a_0]$, the projected (on the j-th eigenspace of spherical harmonics with $j \in \mathbb{N}$) linearized operator (see Lemma 12) satisfies:*
(i) *For $j = 0$, the homogeneous equation $\mathcal{L}_0^a(\Phi) = 0$ has a solutions basis with $2p$ elements, which are either bounded or at most linearly growing as $t \to \infty$;*
(ii) *For each $j \geqslant 1$, the homogeneous equation $\mathcal{L}_j^a(\Phi) = 0$ has a solutions basis with $4p$ elements, which are exponentially growing/decaying as $t \to \infty$.*

Inspired by [5,16], we use the spectral analysis of the linearized operator to prove the last proposition. The issue is that not all the Jacobi fields are generated by variations of some parameters in the classification of the Emden–Fowler solutions. To overcome this problem, we show that the spectrum of the linearized operator is purely absolutely continuous. More precisely, it is the union of spectral bands separated by gaps away from the origin. Therefore, the geometric Jacobi fields generate the zero frequency deficiency space. We also need to show that solutions to ($\mathcal{S}_{p,1}$) satisfy upper and lower bounds estimate near the isolated singularity

Proposition 4. *Let $\mathcal{U}$ be a strongly positive superharmonic solution to ($\mathcal{S}_{p,1}$). Then, $\mathcal{U}$ is radially symmetric with respect to the origin. Moreover, either the origin is a removable singularity, or there exists $C_1, C_2 > 0$, satisfying*

$$C_1|x|^{\frac{4-n}{2}} \leqslant |\mathcal{U}(x)| \leqslant C_2|x|^{\frac{4-n}{2}} \quad \text{for} \quad 0 < |x| < 1/2. \tag{9}$$

The main ingredients in the proof of Proposition 4 are the blow-up method based on the classification result for non-singular solutions to ($\mathcal{S}_{p,\infty}$) given by (2) and a removable singularity result relying on the sign of the Pohozaev invariant associated to ($\mathcal{S}_{p,1}$). The difficulties in our argument are numerous. The lack of maximum principle causes one due to the fourth order operator on the left-hand side of ($\mathcal{S}_{p,1}$). To handle the problem with the lack of maximum principle, we apply a Green identity to convert ($\mathcal{S}_{p,1}$) into an integral system [20]. Then, we prove that singular solutions satisfy an upper and lower bound near the isolated singularity; these arguments are based on an integral form of the moving spheres technique. We also need to deal with the nonlinear effects imposed by the coupling term on the right-hand side of ($\mathcal{S}_{p,1}$). The idea is to use Theorem B combined with some decoupling techniques from [8,9,13,18], which yields a comparison involving the norm of a p-map solution and each component.

Finally, the proof of Theorem 1 is a combination of Theorem B, Proposition 3, and Proposition 4, which is called Simon's (or slide-back technique) and arises in the theory of regularity for isolated singular points of minimal hypersurfaces.

Here is our plan for the rest of the paper. In Section 2, we introduce some tools to be used throughout the text. In section 3, we use an involved spectral analysis to prove Proposition 3. In Section 4, we use the integral moving spheres technique to prove Proposition 4. In Section 5, we apply Simon's technique to prove Theorem 1. In Appendix A, we prove a refined asymptotics for singular solutions.

2. Preliminaries

This section aims to introduce some necessary background for developing our methods.

2.1. Kelvin transform

The moving spheres technique we will use later is based on the fourth order Kelvin transform for a p-map. For $\Omega \subseteq \mathbb{R}^n$ a domain, before we define the Kelvin transform, we need to establish the concept of inversion through a sphere $\partial B_\mu(x_0)$, which is a map $\mathcal{I}_{x_0,\mu} : \Omega \to \Omega_{x_0,\mu}$ given by $\mathcal{I}_{x_0,\mu}(x) = x_0 + K_{x_0,\mu}(x)^2(x - x_0)$, where $K_{x_0,\mu}(x) = \mu/|x - x_0|$ and $\Omega_{x_0,\mu} := \mathcal{I}_{x_0,\mu}(\Omega)$ is the domain of the Kelvin transform. In particular, when $x_0 = 0$ and $\mu = 1$, we denote it simply by $\mathcal{I}_{0,1}(x) = x^*$ and $K_{0,1}(x) = x|x|^{-2}$.

The following definition is a generalization of the Kelvin transform.

Definition 5. For any $\mathcal{U} \in C^4(\Omega, \mathbb{R}^p)$, let us consider the fourth order Kelvin transform through the sphere with center at $x_0 \in \mathbb{R}^n$ and radius $\mu > 0$ defined on $\mathcal{U}_{x_0,\mu} : \Omega_{x_0,\mu} \to \mathbb{R}^p$ by

$$\mathcal{U}_{x_0,\mu}(x) = K_{x_0,\mu}(x)^{n-4}\mathcal{U}\left(\mathcal{I}_{x_0,\mu}(x)\right).$$

Now, we emphasize the invariance of System ($\mathcal{S}_{p,1}$) under the action of Kelvin transform.

Proposition 6. *Let $\mathcal{U}$ be a non-singular solution to ($\mathcal{S}_{p,1}$), then $\mathcal{U}_{x_0,\mu}$ satisfies*

$$\Delta^2(u_i)_{x_0,\mu} = c(n)|\mathcal{U}_{x_0,\mu}|^{2^{**}-2}(u_i)_{x_0,\mu} \quad \text{in} \quad (B_1^*)_{x_0,\mu} \quad \text{for} \quad i \in I,$$

where $\mathcal{U}_{x_0,\mu} = ((u_1)_{x_0,\mu}, \ldots, (u_p)_{x_0,\mu})$.

Proof. It is a direct consequence of the formula

$$\Delta^2 u_{x_0,\mu}(x) = K_{x_0,\mu}(x)^{n+4} \Delta^2 u\left(\mathcal{I}_{x_0,\mu}(x)\right) = K_{x_0,\mu}(x)^8 (\Delta^2 u)_{x_0,\mu}(x),$$

which is obtained by a simple computation. □

2.2. Cylindrical transformation

This subsection introduces a transformation that converts singular solutions to ($\mathcal{S}_{p,1}$) into non-singular solutions in a cylinder. Then, the local behavior of singular solutions near the origin reduces to understand the asymptotic global behavior for tempered solutions to a fourth order ODE defined on a cylinder.

Let us introduce the so-called (logarithmic) cylindrical transformation. First, we consider $\mathcal{C}_{0,1} = (0,1) \times \mathbb{S}^{n-1}$ and $\Delta^2_{\rm sph}$ the bi-Laplacian written in spherical (polar) coordinates,

$$\begin{aligned}\Delta^2_{\rm sph} = \partial_r^{(4)} + \frac{2(n-1)}{r}\partial_r^{(3)} + \frac{(n-1)(n-3)}{r^2}\partial_r^{(2)} - \frac{(n-1)(n-3)}{r^3}\partial_r \\ + \frac{1}{r^4}\Delta_\sigma^2 + \frac{2}{r^2}\partial_r^{(2)}\Delta_\sigma + \frac{2(n-3)}{r^3}\partial_r\Delta_\sigma - \frac{2(n-4)}{r^4}\Delta_\sigma,\end{aligned}$$

where Δ_σ denotes the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$. Then, we can rewrite ($\mathcal{S}_{p,1}$) as

$$\Delta^2_{\rm sph} u_i = c(n)|\mathcal{U}|^{2^{**}-2} u_i \quad \text{in} \quad \mathcal{C}_{0,1}.$$

In addition, we apply the Emden–Fowler change of variables (or logarithm cylindrical coordinates) given by $\mathcal{V}(t,\theta) = r^\gamma \mathcal{U}(r,\sigma)$, where $r = |x|$, $t = -\ln r$, $\theta = x/|x|$ and $\gamma = (n-4)/2$, which sends the problem to $\mathcal{C}_0 = (0,\infty) \times \mathbb{S}^{n-1}$.

Using this coordinate system and performing a lengthy computation, we arrive at the following fourth order nonlinear PDE system on the cylinder,

$$\Delta^2_{\rm cyl} v_i = c(n)|\mathcal{V}|^{2^{**}-2} v_i \quad \text{on} \quad \mathcal{C}_0. \tag{$\mathcal{C}_{p,0}$}$$

Here $\Delta^2_{\rm cyl}$ is the bi-Laplacian written in cylindrical coordinates given by

$$\Delta^2_{\rm cyl} = \partial_t^{(4)} - K_2\partial_t^{(2)} + K_0 + \Delta_\theta^2 + 2\partial_t^{(2)}\Delta_\theta - J_0\Delta_\theta, \tag{10}$$

where $K_0, K_2, J_0 \in \mathbb{R}$ are constants depending only on the dimension, which is defined by

$$K_0 = \frac{n^2(n-4)^2}{16}, \quad K_2 = \frac{n^2-4n+8}{2}, \quad \text{and} \quad J_0 = \frac{n(n-4)}{2}.$$

Furthermore, the superharmonicity condition $-\Delta u_i > 0$ is equivalent to

$$-\partial_t^{(2)} v_i + 2\partial_t v_i + \sqrt{K_0} - \Delta_\theta v_i > 0.$$

Notice that, in the blow-up limit case, solutions are rotationally symmetric, which transform ($\mathcal{C}_{p,0}$) into the following ODE system

$$v_i^{(4)} - K_2 v_i^{(2)} + K_0 v_i = c(n)|\mathcal{V}|^{2^{**}-2} v_i \quad \text{in} \quad \mathbb{R}, \qquad (\mathcal{C}_{p,\infty})$$

which can be taken with suitable initial conditions to become a well-posed Cauchy problem.

Along these lines, let us introduce the cylindrical transformation defined as follows

$$\mathfrak{F} : C_c^\infty(B_1^*, \mathbb{R}^p) \to C_c^\infty(\mathcal{C}_0, \mathbb{R}^p) \quad \text{given by} \quad \mathfrak{F}(\mathcal{U}) = r^\gamma \mathcal{U}(r, \sigma),$$

which sends singular solutions to ($\mathcal{S}_{p,1}$) into solutions to ($\mathcal{C}_{p,0}$).

Remark 7. In the geometric language, this change of variables corresponds to a restriction of the conformal diffeomorphism between the entire cylinder $\mathcal{C}_\infty := \mathbb{R} \times \mathbb{S}^{n-1}$ and the punctured space, namely, $\varphi : (\mathcal{C}_\infty, g_{\mathrm{cyl}}) \to (\mathbb{R}^n \setminus \{0\}, \delta_0)$ defined by $\varphi(t, \sigma) = e^{-t}\sigma$. Here $g_{\mathrm{cyl}} = \mathrm{d}t^2 + \mathrm{d}\sigma^2$ stands for the cylindrical metric and $\mathrm{d}\theta = e^{-2t}(\mathrm{d}t^2 + \mathrm{d}\sigma^2)$ for its volume element obtained via the pullback $\varphi^*\delta_0$, where δ_0 is the standard flat metric. In this fashion, our choice for the symbol $\Delta^2_{\mathrm{cyl}} = \Delta^2_{\mathrm{sph}} \circ \mathfrak{F}^{-1}$ is an abuse of notation since the cylindrical background metric is not flat, we should have $P_{\mathrm{cyl}} = \Delta^2_{\mathrm{sph}} \circ \mathfrak{F}^{-1}$, where P_{cyl} stands for the Paneitz–Branson operator of this metric in the new logarithmic cylindrical coordinate system.

2.3. Pohozaev invariant

In the next step, we define a type homological invariant associated with ($\mathcal{S}_{p,1}$). This invariant is the main ingredient in providing a removable singularity theorem and is one of the features for developing the convergence method. The existence of a Pohozaev-type invariant is closely related to a conservation law for the Hamiltonian energy of the ODE system ($\mathcal{C}_{p,\infty}$).

Initially, let us introduce a vectorial energy that is conserved in time for all p-map solutions $\mathcal{V}$ to system ($\mathcal{C}_{p,0}$), which depends on the angular variable.

Definition 8. For any $\mathcal{V}$ strongly positive solution to ($\mathcal{C}_{p,\infty}$), let us consider its Hamiltonian Energy given by

$$\mathcal{H}(t, \theta, \mathcal{V}) := \mathcal{H}_{\mathrm{rad}}(t, \theta, \mathcal{V}) + \mathcal{H}_{\mathrm{ang}}(t, \theta, \mathcal{V}), \qquad (11)$$

where

$$\begin{aligned}\mathcal{H}_{\mathrm{rad}}(t, \theta, \mathcal{V}) &= -\langle \mathcal{V}^{(3)}(t, \theta), \mathcal{V}^{(1)}(t, \theta)\rangle + \frac{1}{2}|\mathcal{V}^{(2)}(t, \theta)|^2 + \frac{K_2}{2}|\mathcal{V}^{(1)}(t, \theta)|^2 \\ &\quad - \frac{K_0}{2}|\mathcal{V}(t, \theta)|^2 + \widehat{c}(n)|\mathcal{V}(t, \theta)|^{2^{**}},\end{aligned}$$

$$\mathcal{H}_{\mathrm{ang}}(t, \theta, \mathcal{V}) = |\Delta_\theta \mathcal{V}(t, \theta)|^2 + 2|\partial_t^{(2)} \nabla_\theta \mathcal{V}(t, \theta)|^2 - J_0 |\nabla_\theta \mathcal{V}(t, \theta)|^2, \quad \text{and} \quad \widehat{c}(n) = 2^{**-1} c(n).$$

A standard computation shows that the Hamiltonian energy is invariant on the variable t, that is, $\partial_t \mathcal{H}(t, \theta, \mathcal{V}) = 0$ for all solutions $\mathcal{V}$ to ($\mathcal{C}_{p,0}$). Hence, we can integrate (11) over the cylindrical slice to define another conserved quantity as follows

Definition 9. For any $\mathcal{V}$ strongly positive solution to ($\mathcal{C}_{p,0}$), let us define its cylindrical Pohozaev integral by

$$\mathcal{P}_{\mathrm{cyl}}(t,\mathcal{V}) = \int_{\mathbb{S}_t^{n-1}} \mathcal{H}(t,\theta,\mathcal{V})\mathrm{d}\theta.$$

Here $\mathbb{S}_t^{n-1} = \{t\} \times \mathbb{S}^{n-1}$ is the cylindrical ball with volume element given by $\mathrm{d}\theta = e^{-2t}\mathrm{d}\sigma_r$, where $\mathrm{d}\sigma_r$ is the volume element of the Euclidean ball of radius $r > 0$.

Since that by definition $\mathcal{P}$ also does not depend on t, let us consider the cylindrical Pohozaev invariant $\mathcal{P}_{\mathrm{cyl}}(\mathcal{V}) := \mathcal{P}_{\mathrm{cyl}}(t,\mathcal{V})$. Hence, applying the inverse of cylindrical transformation, we recover the classical spherical Pohozaev integral defined by $\mathcal{P}_{\mathrm{sph}}(r,\mathcal{U}) := \left(\mathcal{P}_{\mathrm{cyl}} \circ \mathfrak{F}^{-1}\right)(t,\mathcal{V})$, which satisfies the following Pohozaev-type identity:

Lemma 10. *Let $\mathcal{U},\widetilde{\mathcal{U}} \in C^4(B_1^*,\mathbb{R}^p)$ and $0 < r_1 \leqslant r_2 < 1$. Then, it follows*

$$\begin{aligned}
&\sum_{i=1}^{p} \int_{B_{r_2}\setminus B_{r_1}} \left[\Delta^2 u_i \langle x, \nabla\widetilde{u}_i\rangle + \Delta^2\widetilde{u}_i \langle x, \nabla u_i\rangle - \frac{n-4}{2}\left(\widetilde{u}_i\Delta^2 u_i + u_i\Delta^2\widetilde{u}_i\right)\right]\mathrm{d}x \\
&= \sum_{i=1}^{p}\left[\int_{\partial B_{r_2}} q(u_i,\widetilde{u}_i)\mathrm{d}\sigma_{r_2} - \int_{\partial B_{r_1}} q(u_i,\widetilde{u}_i)\mathrm{d}\sigma_{r_1}\right].
\end{aligned}$$

Here

$$\begin{aligned}
q(u_i,\widetilde{u}_i) = {} & \frac{2-n}{2}\langle\Delta u_i, \partial_\nu\widetilde{u}_i\rangle - \frac{r}{2}\langle\Delta u_i, \Delta\widetilde{u}_i\rangle + \frac{n-4}{2}\langle u_i, \partial_\nu\Delta\widetilde{u}_i\rangle + \langle x, \nabla u_i\rangle\partial_\nu\Delta\widetilde{u}_i \\
& - \Delta u_i \sum_{j=1}^{n} x_j\partial_\nu\widetilde{u}_{ij},
\end{aligned}$$

where $\widetilde{u}_{ij}$ is j-th coordinate function of u_i and ν is the outer normal vector to ∂B_{r_2}.

The last lemma is a vectorial version of the fourth order Pohozaev identity in [7, Proposition 4.1] (see also [20, Propositon A.1]).

Proof. See the proof in [7, Proposition 4.1]. □

Remark 11. Using the last lemma, we present an explicit formula for the spherical Pohozaev invariant

$$\mathcal{P}_{\mathrm{sph}}(\mathcal{U}) = \int_{\partial B_r}\left[\sum_{i=1}^{p} q(u_i,u_i) - r\widehat{c}(n)|\mathcal{U}|^{2^{**}}\right]\mathrm{d}\sigma_r, \tag{12}$$

where

$$q(u_i,u_i)=\frac{2-n}{2}\Delta u_i,\partial_\nu u_i-\frac{r}{2}|\Delta u_i|^2+\frac{n-4}{2}u_i,\partial_\nu\Delta u_i+\langle x,\nabla u_i\rangle\partial_\nu\Delta u_i-\Delta u_i\sum_{j=1}^n x_j\partial_\nu u_{ij}. \tag{13}$$

For easy reference, we summarize the following properties:
(i) There is a natural relation between these two invariants, $\mathcal{P}_{\rm sph}(\mathcal{U})=\omega_{n-1}\mathcal{P}_{\rm cyl}(\mathcal{V})$, where ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere;
(ii) In the blow-up limit case, one can check that if the non-singular solution is $\mathcal{U}_{x_0,\mu}=\Lambda u_{x_0,\mu}$ for some $\Lambda\in\mathbb{S}^{p-1}_+$ and $u_{x_0,\mu}$ a spherical solution, we obtain $\mathcal{P}_{\rm sph}(\mathcal{U}_{x_0,\mu})=0$. Also, if the singular solution has the form $\mathcal{U}_{a,T}=\Lambda u_{a,T}$ for some $\Lambda\in\mathbb{S}^{p-1}_+$ and $u_{a,T}$ an Emden–Fowler solution. Then, a direct computation shows $\mathcal{P}_{\rm sph}(\mathcal{U}_{a,T})=\mathcal{P}_{\rm sph}(u_{a,T})=\mathcal{P}_{\rm cyl}(v_{a,T})=c_n\mathcal{H}(v_{a,T})<0$ for some $c_n>0$ dimensional constant (for more details see [31, Corollary 4]).

3. Linear analysis

The objective of this section is to prove Proposition 3. More precisely, we show the linear stability of the linearized operator by studying its spectrum. Consequently, we can control the asymptotics for solutions using the growth of the Jacobi fields, computed using Floquet theory (or Bloch wave theory). Namely, we prove that spec$(\mathcal{L}^a)$ is a disjoint union of nondegenerate intervals with $0\in\Im_a$ an isolated point. The strategy is to use a decomposition to study the spectral bands of the Jacobi operator. We proceed by applying the Fourier–Laplace transform combined with some results from holomorphic functional analysis [27,28]. For complex numbers, we denote $\rho=\alpha+i\beta$, where $\Re(\rho)$, $\Im(\rho)$ stands for its real and imaginary parts, respectively.

3.1. Linearized operator

Now, we study the linearized operator around blow-up limit solutions. The heuristics are that when this operator is Fredholm, its indicial roots determine the rate at which singular solutions to the nonlinear problem ($\mathcal{S}_{p,1}$) converge to this limit solution near the isolated singularity. Here, we borrow some ideas from [3,28].

First, let us consider the following nonlinear operator acting on p-maps $\mathcal{N}(\mathcal{U}):=\Delta^2u_i-f_i(\mathcal{U})$, where we recall $f_i(\mathcal{V})=c(n)|\mathcal{V}|^{2^{**}-2}v_i$ for $i\in I$. Then, using the cylindrical transformation $\mathfrak{F}:C^\infty_c(B_1^*,\mathbb{R}^p)\to C^\infty_c(\mathcal{C}_0,\mathbb{R}^p)$ and the homogeneity of the Gross–Pitaevskii nonlinearity, we obtain

$$\mathcal{N}_{\rm cyl}(\mathcal{V}):=\Delta^2_{\rm cyl}v_i-f_i(\mathcal{V}). \tag{14}$$

In what follows, we drop the subscript since we often will be using the operator written in cylindrical coordinates.

Lemma 12. *The linearization of $\mathcal{N}:H^4(\mathcal{C}_0,\mathbb{R}^p)\to L^2(\mathcal{C}_0,\mathbb{R}^p)$ around an Emden–Fowler solution $\mathcal{V}_{a,T}$ to ($\mathcal{S}_{p,\infty}$) is given by*

$$\mathcal{L}^a_i(\Phi)=\phi_i^{(4)}-K_2\phi_i^{(2)}+K_0\phi_i-\left[c(n)v_{a,T}^{2^{**}-2}\phi_i+\frac{n(n^2-4)}{2}v_{a,T}^{2^{**}-2}\Lambda_i\langle\Lambda,\Phi\rangle\right] \tag{15}$$
$$+\Delta^2_\theta\phi_i+2\partial_t^{(2)}\Delta_\theta\phi_i-J_0\Delta_\theta\phi_i,$$

where $\Lambda = (\Lambda_1, \ldots, \Lambda_p) \in \mathbb{S}^{p-1}_{+,*}$*, and* $\mathrm{d}\mathcal{N}_{\rm cyl}\left[\mathcal{V}_{a,T}\right](\Phi) = \mathcal{L}^a(\Phi)$ *is the Fréchet derivative of* $\mathcal{N}$ *with* $\Phi = (\phi_1, \ldots, \phi_p)$*.*

Proof. By definition, we have that $\mathcal{L}^a_i(\Phi) := \mathcal{L}_i[\mathcal{V}_{a,T}](\Phi)$, where

$$\begin{aligned}\mathcal{L}_i[\mathcal{V}_{a,T}](\Phi) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{N}(\mathcal{V}_{a,T} + t\Phi) \\ &= \Delta^2_{\rm cyl}\phi_i - c(n)\left[\left(2^{**}-2\right)|\mathcal{V}_{a,T}|^{2^{**}-4}\langle\mathcal{V}_{a,T}, \Phi\rangle\Pi_i(\mathcal{V}_{a,T}) + |\mathcal{V}_{a,T}|^{2^{**}-2}\phi_i\right],\end{aligned} \tag{16}$$

where $\Pi_i(\mathcal{V}_{a,T})$ denotes the i-th component of the Emden–Fowler solution $\mathcal{V}_{a,T} \in C^4(\mathbb{R}, \mathbb{R}^p)$.

To prove this fact, we observe that since f_i is $(2^{**}-1)$-homogeneous, we find

$$\begin{aligned}&\mathcal{N}(\mathcal{V}_{a,T} + t\Phi) - \mathcal{N}(\mathcal{V}_{a,T}) \\ &\quad = \Delta^2_{\rm cyl}\mathcal{V}_{a,T} + t\Delta^2_{\rm cyl}\Phi - f_i\left(\mathcal{V}_{a,T} + t\Phi\right) - \Delta^2_{\rm cyl}\mathcal{V}_{a,T} + f_i\left(\mathcal{V}_{a,T}\right) \\ &\quad = t\Delta^2_{\rm cyl}\Phi + f_i\left(\mathcal{V}_{a,T}\right) - f_i\left(\mathcal{V}_{a,T} + t\Phi\right) \\ &\quad = t\Delta^2_{\rm cyl}\Phi - tc(n)\left[\left(2^{**}-2\right)|\mathcal{V}_{a,T}|^{2^{**}-4}\langle\mathcal{V}_{a,T}, \Phi\rangle\Pi_i(\mathcal{V}_{a,T}) + |\mathcal{V}_{a,T}|^{2^{**}-2}\phi_i\right] + \mathcal{O}\left(t^2\right),\end{aligned}$$

which implies (16).

Moreover, using the classification formula in Theorem B (ii), one can find a unit positive vector $\Lambda = (\Lambda_1, \ldots, \Lambda_p) \in \mathbb{S}^{p-1}_{+,*}$ and $v_{a,T} \in C^4(\mathbb{R})$ the unique T-periodic solution to (4) with $v_{a,T}(0) = a$ such that $\mathcal{V}_{a,T} = \Lambda v_{a,T}$, and so $\Pi_i(\mathcal{V}_{a,T}) = \Lambda_i v_{a,T}$. From this, we find

$$\begin{aligned}&\Delta^2_{\rm cyl}\phi_i - c(n)\left[\left(2^{**}-2\right)|\mathcal{V}_{a,T}|^{2^{**}-4}\langle\mathcal{V}_{a,T}, \Phi\rangle\Pi_i(\mathcal{V}_{a,T}) + |\mathcal{V}_{a,T}|^{2^{**}-2}\phi_i\right] \\ &= \phi_i^{(4)} - K_2\phi_i^{(2)} + K_0\phi_i \\ &- \frac{n(n-4)(n^2-4)}{16}\left[\frac{8}{n-4}|\Lambda v_{a,T}|^{2^{**}-4}\langle\Lambda v_{a,T}, \Phi\rangle\Lambda_i v_{a,T} + |\Lambda v_{a,T}|^{2^{**}-2}\phi_i\right] \\ &+ \Delta^2_\theta\phi_i + 2\partial_t^{(2)}\Delta_\theta\phi_i - J_0\Delta_\theta\phi_i \\ &= \phi_i^{(4)} - K_2\phi_i^{(2)} + K_0\phi_i - c(n)v_{a,T}^{2^{**}-2}\phi_i - \frac{n(n^2-4)}{2}\Lambda_i\langle\Lambda, \Phi\rangle v_{a,T}^{2^{**}-2} \\ &+ \Delta^2_\theta\phi_i + 2\partial_t^{(2)}\Delta_\theta\phi_i - J_0\Delta_\theta\phi_i.\end{aligned}$$

Hence, we can simplify (16) to obtain (15), which proves the lemma. □

3.2. Jacobi fields

Unfortunately, the linearized operator is not generally Fredholm since it does not have a closed range [29, Theorem 5.40]. Nontrivial elements on its kernel cause this issue; these are called the Jacobi fields [3]. Therefore, we need to introduce suitable weighted Sobolev and Hölder spaces on which the linearized operator has a well-defined right-inverse, up to a discrete set on the complex plane. For more details, see [30, Section 2].

Definition 13. Given $k, p, q \geqslant 1$ and $\beta \in \mathbb{R}$, for any $\mathcal{V} \in L^q_{\mathrm{loc}}(\mathcal{C}_0, \mathbb{R}^p)$ define the following weighted Lebesgue norm

$$\|\mathcal{V}\|^q_{L^q_\beta(\mathcal{C}_0,\mathbb{R}^p)} = \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{-2\beta t}|\mathcal{V}(t,\theta)|^q \mathrm{d}\theta \mathrm{d}t.$$

Also, let us define the weighted Lebesgue space by $L^q_\beta(\mathcal{C}_0, \mathbb{R}^p) = \{\mathcal{V} \in L^q_{\mathrm{loc}}(\mathcal{C}_0) : \|\mathcal{V}\|_{L^q_\beta(\mathcal{C}_0,\mathbb{R}^p)} < \infty\}$. Similarly consider the Sobolev spaces $W^{k,q}_\beta(\mathcal{C}_0, \mathbb{R}^p)$ of p-maps with k weak derivatives in L^q having finite weighted norms. Here we also denote the Hilbert space $W^{k,2}_\beta(\mathcal{C}_0, \mathbb{R}^p) = H^k_\beta(\mathcal{C}_0, \mathbb{R}^p)$ and $W^{k,q}(\mathcal{C}_0) = W^{k,q}(\mathcal{C}_0, \mathbb{R})$. Notice that when $\beta = 0$, we recover the classical Sobolev spaces of p-maps.

Definition 14. Given $m, p \geqslant 1$, $\beta \in \mathbb{R}$ and $\zeta \in (0, 1)$, for any $u \in C^{0,\beta}_{\mathrm{loc}}(\mathcal{C}_0, \mathbb{R}^p)$ define the following norm

$$\|\mathcal{V}\|_{C^{0,\zeta}_\beta(\mathcal{C}_0,\mathbb{R}^p)} = \sup_{T>1} \sup \left\{ \frac{e^{-\beta t_1}|\mathcal{V}(t_1,\theta_1)| - e^{-\beta t_2}|\mathcal{V}(t_2,\theta_2)|}{\mathrm{d_{cyl}}((t_1,\theta_1),(t_2,\theta_2))^\zeta} : (t_1,\theta_1),(t_2,\theta_2) \in \mathcal{C}_{T-1,T+1} \right\},$$

where $\mathcal{C}_{T-1,T+1} = (T-1, T+1) \times \mathbb{S}^{n-1}$. Also, let us define the (zeroth order) weighted Holdër space by

$$C^{0,\zeta}_\beta(\mathcal{C}_0, \mathbb{R}^p) = \left\{ \mathcal{V} \in C^{0,\beta}_{\mathrm{loc}}(\mathcal{C}_0, \mathbb{R}^p) : \|\mathcal{V}\|_{C^{0,\beta}_{\mathrm{loc}}(\mathcal{C}_0,\mathbb{R}^p)} < \infty \right\}.$$

One can similarly define higher order weighted Holdër spaces $C^{m,\zeta}_\beta(\mathcal{C}_0, \mathbb{R}^p)$.

Remark 15. The functional spaces defined above are suitable to obtain the asymptotic results in Theorem 1, since $v \in W^{k,q}_\beta(\mathcal{C}_0)$ is equivalent to $v \in W^{k,q}(\mathcal{C}_0)$ together with the decay $v = \mathcal{O}(e^{-\beta t})$ as $t \to \infty$. Additionally, by regularity theory, we can indistinguishably work with both the Sobolev or the Hölder spaces.

Definition 16. The Jacobi fields in the kernel of $\mathcal{L}^a : H^4_\beta(\mathcal{C}_0, \mathbb{R}^p) \to L^2_\beta(\mathcal{C}_0, \mathbb{R}^p)$, are the solutions $\Phi \in H^4_\beta(\mathcal{C}_0, \mathbb{R}^p)$ to the following fourth order linear system

$$\mathcal{L}^a(\Phi) = 0 \quad \text{on} \quad \mathcal{C}_0. \tag{17}$$

3.3. Fourier eigenmodes

We study the kernel of a linearized operator around an Emden–Fowler solution by decomposing into its Fourier eigenmodes, a separation of variables technique. First, let us consider $\{\lambda_j, \chi_j(\theta)\}_{j\in\mathbb{N}}$ the eigendecomposition of the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$ with the normalized eigenfunctions,

$$\Delta_\theta \chi_j(\theta) + \lambda_j \chi_j(\theta) = 0. \tag{18}$$

Here the eigenfunctions $\{\chi_j(\theta)\}_{j\in\mathbb{N}}$ are called spherical harmonics with associated sequence of eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ given by $\lambda_j = j(j+n-2)$ counted with multiplicity $\mathfrak{m}_j$, which are defined by

$$\mathfrak{m}_0 = 1 \quad \text{and} \quad \mathfrak{m}_j = \frac{(2j+n-2)(j+n-3)!}{(n-2)!j!}.$$

In particular, we have $\lambda_0 = 0$, $\lambda_1 = \cdots = \lambda_n = n-1$, $\lambda_j \geqslant 2n$, if $j > n$ and $\lambda_j \leqslant \lambda_{j+1}$. Moreover, these eigenfunctions are the restrictions to $\mathbb{S}^{n-1}$ of homogeneous harmonic polynomials in $\mathbb{R}^n$. Here we denote by V_j the eigenspace spanned by $\chi_j(\theta)$. Using (18), it is easy to observe that the eigendata of the bi-Laplace–Beltrami operator Δ^2_θ is given by $\{\lambda_j^2, \chi_j(\theta)\}_{j\in\mathbb{N}}$. Namely, for all $j \in \mathbb{N}$, it follows

$$\Delta^2_\theta \chi_j(\theta) - \lambda_j^2 \chi_j(\theta) = 0. \tag{19}$$

3.3.1. Scalar case

When $p = 1$, the nonlinear operator (14) becomes

$$\mathcal{N}(v) := \Delta^2_{\mathrm{cyl}} v - c(n) v^{2^{**}-1} \quad \text{and} \quad \mathcal{L}^a(\phi) = \Delta^2_{\mathrm{cyl}}\phi - \widetilde{c}(n) v_{a,T}^{2^{**}-2}\phi,$$

where $\tilde{c}(n) = (2^{**}-1)c(n) > 0$ is a positive constant. Furthermore, using the decomposition (10) combined with (18) and (19), we get

$$\mathcal{L}^a(\phi) = \partial_t^{(4)}\phi - K_2\partial_t^{(2)}\phi + K_0\phi + \Delta^2_\theta\phi + 2\partial_t^{(2)}\Delta_\theta\phi - J_0\Delta_\theta\phi - \widetilde{c}(n) v_{a,T}^{2^{**}-2}\phi,$$

which by projecting on the eigenspaces V_j gives us

$$\mathcal{L}^a_j(\phi) = \phi^{(4)} - (K_2 + 2\lambda_j)\phi^{(2)} + \left[K_0 + \lambda_j(\lambda_j + J_0) - \widetilde{c}(n) v_{a,T}^{2^{**}-2}\right]\phi. \tag{20}$$

Moreover, for any $\phi \in L^2(\mathbb{S}^{n-1})$, we write

$$\phi(t,\theta) = \sum_{j=0}^{\infty} \phi_j(t)\chi_j(\theta), \quad \text{where} \quad \phi_j(t) = \int_{\mathbb{S}^{n-1}} \phi(t,\theta)\chi_j(\theta)\mathrm{d}\theta.$$

In other terms, ϕ_j is the projection of ϕ on the eigenspace V_j. Thus, to understand the kernel of $\mathcal{L}^a$, we consider the induced family of ODEs $\mathcal{L}^a_j(\phi_j) = 0$ for $j \in \mathbb{N}$.

Remark 17. For $p = 1$, some (low-frequency) Jacobi fields are generated by the variation of a two-parameter family of Emden–Fowler solutions. When $j = 0$, they are given by $\phi^+_{a,0}(t) = \partial_T\big|_{T=0} v_{a,T}(t)$ and $\phi^-_{a,0}(t) = \partial_a\big|_{a=0} v_{a,T}(t)$, where $v_{a,T} \in \mathcal{C}(\mathbb{R})$ is the Emden–Fowler solution defined as the unique T-periodic solution to (4) with $v_{a,T}(0) = a$ for any $a \in (0, a_0]$. However, the other two Jacobi fields cannot be directly constructed as variations of some family of solutions to the limit equation. One can show that they are not based on this zero-frequency case.

3.3.2. System case

For $p>1$ and $\Phi \in L^2(\mathcal{C}_0, \mathbb{R}^p)$, we write

$$\Phi(t,\theta) = \sum_{j=0}^{\infty} \Phi_j(t)\chi_j(\theta), \quad \text{where} \quad \Phi_j(t) = \int_{\mathbb{S}^{n-1}} \Phi(t,\theta)\chi_j(\theta)\mathrm{d}\theta. \tag{21}$$

Hence, for all $i \in I$ and $j \in \mathbb{N}$, we decompose (17) as

$$\begin{aligned}\mathcal{L}^a_{ij}(\Phi) = \phi_i^{(4)} - (K_2 + 2\lambda_j)\phi_i^{(2)} + \left[K_0 + \lambda_j(\lambda_j + J_0) - c(n)v_{a,T}^{2^{**}-2}\right]\phi_i \\ - \frac{n(n^2-4)}{2}\Lambda_i\langle\Lambda, \Phi\rangle v_{a,T}^{2^{**}-2}.\end{aligned} \tag{22}$$

Whence, to understand the kernel of (22), we again consider the induced equations $\mathcal{L}^a_{ij}(\phi_j) = 0$. Therefore, studying the kernel of $\mathcal{L}^a$ reduces to solving infinitely many ODEs. In Fourier analysis, it is convenient to divide any $\Phi \in L^2(\mathcal{C}_0, \mathbb{R}^p)$ into its frequency modes by

$$\pi_0[\Phi](t,\theta) = \Phi_0(t)\chi_0(\theta), \ \pi_1[\Phi](t,\theta) = \sum_{j=1}^{\mathfrak{m}_1} \Phi_j(t)\chi_j(\theta), \ \text{and}$$

$$\pi_l[\Phi](t,\theta) = \sum_{j=\mathfrak{m}_l+1}^{\mathfrak{m}_{l+1}} \Phi_j(t)\chi_j(\theta).$$

In particularly, the projections π_0, π_1 and $\sum_{l=2}^{\infty}\pi_l$ are called respectively the zero-frequency, low-frequency, and high-frequency modes.

3.4. Fourier–Laplace transform

Following [28, Section 4] (see also [19, Section 3]), we consider the Fourier–Laplace transform, which is the suitable transformation to invert the linearized operator in the frequency space. We can use the real parameter $\alpha = \Re z$ for $\rho \in \mathfrak{R}_a$ to move the weight of the Sobolev space and invert this transform up to some region in the complex plane. Before, we need to introduce some background notation and tools. Here, we recall that $T_a \in \mathbb{R}$ is the fundamental period of the Emden–Fowler solution v_a given by (4).

Definition 18. Let $\Phi \in H^k_{\widetilde{\beta}}(\mathcal{C}_\infty, \mathbb{R}^p)$ extended to be zero on the region $\mathcal{C}_\infty \setminus \mathcal{C}_0$. We define its Fourier–Laplace transform as

$$\mathcal{F}_a(\Phi)(t,\theta,\rho) = \sum_{l\in\mathbb{Z}} e^{-il\rho}\Phi\left(t + lT_a, \theta\right), \tag{23}$$

where $\rho \in \mathfrak{R}_a := \{\alpha + i\beta \in \mathbb{C} : \beta < -\widetilde{\beta}T_a\} \subset \mathbb{C}$ for some $\widetilde{\beta} \in \mathbb{R}$. For the sake of simplicity, we fix the notation $\widehat{\Phi}(t,\theta,\rho) := \mathcal{F}_a(\Phi)(t,\theta,\rho)$.

Due to the periodicity properties of the linearized operator, it makes sense to define the following spaces, whose elements are sometimes referred to as Bohr α-quasi-periodic p-maps.

Definition 19. Fixing the notation $\mathcal{C}_{0,T_a} := [0, T_a] \times \mathbb{S}^{n-1}$ and $\alpha \in \mathbb{R}$, let us introduce the functional (Hilbert) space $L^2_\alpha\left(\mathcal{C}_{0,T_a}, \mathbb{C}^p\right)$ defined as the L^2-completion of $\mathcal{C}^0_\alpha\left(\mathcal{C}_{0,T_a}, \mathbb{C}^p\right)$, where $\mathcal{C}^0_\alpha\left(\mathcal{C}_{0,T_a}, \mathbb{C}^p\right) := \left\{\Phi \in C^0\left(\mathcal{C}_{0,T_a}, \mathbb{C}^p\right) : \Phi(T_a, \theta) = e^{i\alpha t}\Phi(0,\theta)\right\}$.

The main proposition of this subsection is a direct integral (in the sense of Hilbert spaces) decomposition of $L^2(\mathcal{C}_0, \mathbb{R}^p)$ in terms of the parameter $\alpha \in \mathbb{R}$ in the Fourier–Laplace transform (*cf.* [15, Definition 7.18]).

Proposition 20. *For any $a \in [0, a_0]$, it follows $L^2(\mathcal{C}_0, \mathbb{R}^p) = \int^{\oplus}_{\alpha\in[0,2\pi]} L^2_\alpha\left(\mathcal{C}_{0,T_a}, \mathbb{C}^p\right) \mathrm{d}\alpha$, where $\int^{\oplus}$ denotes the direct integral of Hilbert spaces.*

Proof. We divide the proof into a sequence of claims:
Claim 1: If $\rho \in \mathfrak{R}_a$, then the Fourier–Laplace $\mathcal{F}_a(\rho) : L^2(\mathcal{C}_\infty, \mathbb{R}^p) \to L^2(\mathcal{C}_\infty, \mathbb{C}^p)$ transform is well-defined.
In fact, since $\Phi \in H^k_{\widetilde{\beta}}(\mathcal{C}_0, \mathbb{R}^p)$ we know that $|\Phi(t,\theta)| = \mathcal{O}(e^{\widetilde{\beta}t})$, which yields

$$|\mathcal{F}_a(\Phi)(t,\theta,\rho)| = \sum_{l\in\mathbb{Z}} \left|e^{-i(\alpha+i\beta)l}\Phi(t+lT_a,\theta)\right| = \sum_{l\in\mathbb{Z}} e^{\beta l}|\Phi(t+lT_a,\theta)| \leqslant Ce^{\widetilde{\beta}t}\sum_{l\in\mathbb{Z}} e^{(\alpha+\widetilde{\beta}T_a)l},$$

where we used that each choice of $\rho \in \mathbb{C}$ only gives finitely many zeros, and since $\rho \in \mathfrak{R}_a$, we use the growth property to conclude that all the exponents in the series are negative. Therefore, the last sum must converge uniformly in $\mathfrak{R}_a$. We can rephrase this conclusion like $\mathcal{F}_a$ is analytic whenever $\Phi \in H^k_\beta(\mathcal{C}_0, \mathbb{R}^p)$.

We invert the Fourier–Laplace transform using a contour integral in the following claim.
Claim 2: Let $\Phi \in H^k_\beta(\mathcal{C}_0, \mathbb{R}^p)$ and $\rho \in \mathfrak{R}_a$. For each t choose $\bar{t} \in [0, T_a)$ such that $t = \bar{t}$ mod T_a, that is, there exists $l_0 \in \mathbb{Z}$ satisfying $t = \bar{t} + l_0T_a$. Then, we get

$$\Phi(t,\theta) = \frac{1}{2\pi}\int_{\alpha=0}^{2\pi} e^{il_0T_a(\alpha+i\beta)}\widehat{\Phi}(\bar{t}, \alpha+i\beta, \theta)\mathrm{d}\alpha.$$

Indeed, since $\rho \in \mathfrak{R}_a$ for all $i \in I$, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{\alpha=0}^{2\pi} e^{\widetilde{\beta}}\widehat{\phi_i}(\bar{t},\theta,\rho)\mathrm{d}\alpha &= \frac{1}{2\pi}\int_{\alpha=0}^{2\pi} e^{il_0\rho}\sum_{l\in\mathbb{Z}} e^{-il\rho}\phi_i\left(\bar{t}+lT_a,\theta\right)\mathrm{d}\alpha \\ &= \sum_{l\in\mathbb{Z}}\frac{1}{2\pi}\int_{\alpha=0}^{2\pi} e^{i(\alpha+i\beta)(l_0-l)}\phi_i\left(\bar{t}+lT_a,\theta\right)\mathrm{d}\alpha = \phi_i(t,\theta).\end{aligned}$$

We prove a type of Parseval–Plancherel identity for p-maps.
Claim 3: For each $\theta \in \mathbb{S}^{n-1}$, it follows

$$\left\|\widehat{\Phi}(t,\theta,\rho)\right\|^2_{L^2(\mathcal{S}_a,\mathbb{C}^p)} \simeq 2\pi\left\|\widehat{\Phi}(t,\theta)\right\|^2_{L^2_{\beta/T_a}(\mathbb{R},\mathbb{R}^p)}, \tag{24}$$

where $\mathcal{S}_a = [0, T_a] \times [0, 2\pi]$.
As a matter of fact, for all $i \in I$, it holds

$$\begin{aligned}
&\int_0^{T_a}\int_0^{2\pi} \left|\widehat{\phi}_i(t,\theta,\rho)\right|^2 \mathrm{d}\alpha \mathrm{d}t \\
&= \int_0^{T_a}\int_0^{2\pi} \left(\sum_{l\in\mathbb{Z}} e^{-il\alpha} e^{l\beta}\phi_i\left(t+lT_a,\theta\right)\right)\left(\sum_{l\in\mathbb{Z}} e^{-il\alpha} e^{l\beta}\phi_i\left(t+lT_a,\theta\right)\right) \mathrm{d}\alpha \mathrm{d}t \\
&= \int_0^{T_a}\int_0^{2\pi} \sum_{l\in\mathbb{Z}}\sum_{\ell=-l}^{l} \binom{l}{\ell} e^{i(\ell-l)\alpha} e^{(\ell+l)\beta}\phi_i\left(t+lT_a,\theta\right)\phi_i\left(t+lT_a,\theta\right) \mathrm{d}\alpha \mathrm{d}t \\
&= \int_0^{T_a}\int_0^{2\pi} \sum_{l\in\mathbb{Z}} e^{2\beta l} \left|\phi_i\left(t+lT_a,\theta\right)\right|^2 \mathrm{d}\alpha \mathrm{d}t \\
&\simeq 2\pi \int_{\mathbb{R}} \left|e^{\beta t/T_a}\phi_i(t,\theta)\right|^2 \mathrm{d}t.
\end{aligned}$$

Next, we prove that $\widehat{\Phi}$ is a section of the flat bundle $\mathbb{T}_a^n = \mathbb{S}_a^1 \times \mathbb{S}^{n-1}$ with holonomy $\rho \in \mathbb{C}$ around the $\mathbb{S}^1$ loop, where we identify $\mathbb{S}_a^1 = \mathbb{R}/T_a\mathbb{Z}$.
Claim 4: For each $\theta \in \mathbb{S}^{n-1}$, we have

$$\left\|\widehat{\Phi}(t,\theta,\rho)\right\|^2_{L^2(\mathcal{S}_a,\mathbb{C}^p)} = 2\pi \left\|\widehat{\Phi}(t,\theta)\right\|^2_{L^2(\mathbb{R},\mathbb{R}^p)}. \tag{25}$$

Indeed, by taking $\beta = 0$ in (24) and using (23), we get $\Phi\left(t+T_a,\theta\right) = \mathcal{F}_a^{-1}\left(e^{i\rho}\mathcal{F}_a\left(\Phi\right)\right)(t,\theta)$, which concludes the proof of the claim.

Finally, the proof of the proposition is a consequence of Claims 2,3, and 4. □

Remark 21. We stress that the dependence of the inversion on the parameter $\alpha > 0$ will allow us to change the growth rate of the solution produced later using the Green's function of the twisted operator.

3.5. Spectral analysis

Now inspired by [27, Section 4.2], we study the geometric structure of the spectrum of the linearized operator around an Emden–Fowler solution. The idea is to construct a twisted operator, which captures the periodicity property of this linear operator, it is unitarily equivalent to the linearized operator, and for which a Fredholm theory is available. In this direction, let us first introduce the suitable domain of definition for this operator.

Definition 22. For each $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, let us define the set of quasi-periodic p-maps $H^{k,4}_{\alpha}\left([0,T_a],\mathbb{C}^p\right)$ to be the completion of the space of $C^{\infty}\left([0,T_a],\mathbb{C}^p\right)$ under the H^k-norm with boundary conditions given by $\Phi^{(j)}(T_a) = e^{iT_a\alpha}\Phi^{(j)}(0)$ for $j = 0, 1, \ldots, k-1$. We also denote by $\mathcal{L}^a_{ij,\alpha}$ the restriction of $\mathcal{L}^a_{ij}$ to $H^m_\alpha([0, T_a])$.

Initially, for any $\rho \in \mathfrak{R}_a$ we use the inversion of the Fourier–Laplace transform to define $\widehat{\mathcal{L}}^a(\rho) = \mathcal{F}_a \circ \mathcal{L}^a \circ \mathcal{F}_a^{-1}$, or equivalently $\widehat{\mathcal{L}}^a(\rho)(\widehat{\Phi}) = \widehat{\mathcal{L}^a(\Phi)}$, which by (25) yields

$$\widehat{\mathcal{L}}^a(\rho)(e^{i\rho}\widehat{\Phi})(t,\theta,\rho) = e^{i\rho}\widehat{\mathcal{L}}^a(\rho)(e^{i\rho}\widehat{\Phi})(t,\theta,\rho) \quad \text{and} \quad e^{-i\rho}\widehat{\mathcal{L}}^a(\rho)(e^{i\rho}\widehat{\Phi}) = \widehat{\mathcal{L}}^a(\rho)(\widehat{\Phi}).$$

Using the last relation, we set $\widetilde{\mathcal{L}}^a(\rho) : H^{k+4}\left(\mathbb{T}^n_a, \mathbb{C}^p\right) \to H^k\left(\mathbb{T}^n_a, \mathbb{C}^p\right)$, given by

$$\widetilde{\mathcal{L}}^a(\rho)(\widehat{\Phi}) = e^{i\rho}\mathcal{F}_a \circ \mathcal{L}^a \circ \mathcal{F}_a^{-1}\left(e^{-i\rho t}\widehat{\Phi}\right). \tag{26}$$

Remark 23. Notice that $\widehat{\mathcal{L}}^a$ has the same coordinate expression as $\mathcal{L}^a$. Thus, their Fourier eigenmodes decomposition $\widehat{\mathcal{L}}^a_j$ and $\widetilde{\mathcal{L}}^a_j$ are also unitarily equivalent. Moreover, by Claim 3 of Proposition 20 one has that $\mathcal{L}^a_j(\alpha)$ coincides with the restriction of $\widehat{\mathcal{L}}^a_j(\alpha)$ to $[0, T_a]$. Furthermore, $\widetilde{\mathcal{L}}^a(\rho)$ acts on the same functional space for all $\rho \in \mathbb{C}$.

This motivates the following definition:

Definition 24. For each $a \in (0, a_0)$, $j \in \mathbb{Z}$, and $\alpha \in \mathbb{R}$, let us denote by $\sigma_k(a, j, \alpha)$ the eigenvalues of $\mathcal{L}^a_{j,\alpha}$. In addition, since for each $a \in (0, a_0)$, $j \in \mathbb{Z}$, one has $\mathcal{L}^a_{j,0} = \mathcal{L}^a_{j,2\pi}$, it follows that $\sigma_k(a, j, \cdot) : \mathbb{S}^1 \to \mathbb{R}$. Therefore, let us define the k-th spectral band of $\mathcal{L}^a_j$ by

$$\mathfrak{B}_k(a, j) = \{\sigma_k \in \mathbb{R} : \sigma_k = \sigma_k(a, j, \alpha) \text{ for some } \alpha \in [0, 2\pi/T_a]\}.$$

Remark 25. Notice that $\operatorname{spec}(\mathcal{L}^a) = \operatorname{spec}(\widetilde{\mathcal{L}}^a) = \cup_{j,k\in\mathbb{N}}\mathfrak{B}_k(a, j)$.

Remark 26. The eigenfunction Φ_k corresponding to the eigenvalue $\sigma_k(a, j, \alpha)$ satisfies

$$\Phi\left(t + 2\pi/T_a\right) = e^{i\alpha}\Phi(t) = e^{(2\pi-\alpha)i}\Phi(t) \quad \text{and} \quad \widehat{\Phi}(t + 2\pi) = e^{-i\alpha}\widehat{\Phi}(t).$$

Furthermore, $\sigma_k(a, j, 2\pi - \alpha) = \sigma_k(a, j, \alpha)$, since $\mathcal{L}^a_j$ has real coefficients; thus, we can restrict $\sigma_k : \mathbb{S}^1 \to \mathbb{R}$ to the half-circle corresponding to $\alpha \in [0, \pi]$.

Now, we have conditions to enunciate and prove one of the most important results in this section.

Proposition 27. *For any $a \in [0, a_0]$, $0 \in \mathfrak{I}_a$ is an isolated indicial root of $\mathcal{L}^a$.*

Proof. The proof follows by estimating the endpoints of the spectral bands of $\mathcal{L}^a$, and it will be divided into a sequence of claims:
Claim 1: For any $a \in (0, a_0)$ and $j, k \in \mathbb{N}$, the band $\mathfrak{B}_k(a, j)$ is a nondegenerate interval.
In fact, each $\mathcal{L}^a_j$ is a fourth order ordinary differential operator such that the ODE system $\mathcal{L}^a_j(\Phi) = \sigma_k(a, j, \alpha)\Phi$ has a $4p$-dimensional solution space. Suppose that $\mathfrak{B}_k(a, j)$ reduces to

a single point, then σ_k would be constant on $[0, 2\pi]$ and $\mathcal{L}_j^a(\Phi) = \sigma_k(a, j, \alpha)\Phi$ would have an infinite dimensional solution space, which is contradiction.
Claim 2: For any $a \in (0, a_0)$ and $j, k \in \mathbb{N}$, it follows that

$$\mathfrak{B}_{2k}(a, j) = [\sigma_{2k}(a, j, 0), \sigma_{2k}(a, j, \pi)] \quad \text{and} \quad \mathfrak{B}_{2k+1}(a, j) = \left[\sigma_{2k+1}(a, j, \pi), \sigma_{2k+1}(a, j, 0)\right].$$

This is a consequence of Floquet theory, since $\mathfrak{B}_{2k}$ are nondecreasing for any $k \in \mathbb{Z}$, whereas $\mathfrak{B}_{2k+1}$ are all nonincreasing. Thus, we conclude $\sigma_0(a, j, 0) \leqslant \sigma_0(a, j, \pi) \leqslant \sigma_1(a, j, \pi) \leqslant \sigma_1(a, j, 0) \leqslant \dots$
Claim 3: For any $a \in (0, a_0)$ and $j, k \in \mathbb{N}$, we find the lower bound

$$\sigma_k(a, j, 0) > \sigma_0(a, 0, \alpha) + J_0\lambda_j + \lambda_j^2. \tag{27}$$

As a matter of fact, we can relate $\mathfrak{B}_k(a, 0)$ to $\mathfrak{B}_k(a, j)$ since $\mathcal{L}_j^a - \mathcal{L}_0^a = -2\lambda_j\partial_t^{(2)} + J_0\lambda_j + \lambda_j^2$, which for an eigenvalue Φ of $\mathcal{L}_{j,\alpha}^a$ implies

$$\sigma_k(a, j, \alpha)\Phi = \mathcal{L}_0^a(\Phi) - 2\lambda_j\Phi^{(2)} + \left(J_0\lambda_j + \lambda_j^2\right)\Phi. \tag{28}$$

Using the decomposition $\Phi = \sum_{l=0}^{\infty} c_l\Phi_l$, where $\mathcal{L}_0^a(\Phi_l) = \sigma_l(a, 0, \alpha)\Phi_l$ we can reformulate (28) as

$$\sum_{l\in\mathbb{N}} c_l\sigma_k(a, j, \alpha)\Phi_l = \sum_{l\in\mathbb{N}} c_l\left[\sigma_l(a, 0, \alpha)\Phi_l - 2\lambda_j\Phi_l^{(2)} + \left(J_0\lambda_j + \lambda_j^2\right)\Phi_l\right],$$

which provides $2\lambda_j\Phi_l^{(2)} = -\left[\sigma_k(a, j, \alpha) - \sigma_l(a, 0, \alpha) - J_0\lambda_j - \lambda_j^2\right]\Phi_l$. Finally, noticing that the last equation admits quasi-periodic solutions, if, and only if, $\sigma_k(a, j, 0) > \sigma_0(a, 0, \alpha) + J_0\lambda_j + \lambda_j^2$, we conclude the proof of the claim.
Claim 4: For any $a \in (0, a_0)$ and $j, k \in \mathbb{N}$, it follows that $\mathfrak{B}_k(a, j) \subset (0, \infty)$.
This is the most delicate part; thus, we separate the proof into some steps. By the classification in Theorem B (ii), we can reduce our analysis to the case $p = 1$. From now on, we denote $v_a = v_{a,T_a}$.
Step 1: For each $a \in (0, a_0]$, it follows

$$\check{c}(n)\left(\frac{1}{T_a}\int_0^{T_a} v_a^{2^{**}}\mathrm{d}t\right)^{1-2/2^{**}} \leqslant \sigma_0(a, 0, 0) < 0, \tag{29}$$

where $\check{c}(n) = c(n) - \widetilde{c}(n) = -n\left(n^2 - 4\right)/2 < 0$. Moreover, either $\sigma_1(a, 0, 0) = 0$ or $\sigma_2(a, 0, 0) = 0$.
In fact, we start by the upper bound. Using the Rayleigh quotient of $\widetilde{\mathcal{L}}_0^a$, we get

$$\sigma_0(a, 0, 0) = \inf_{\phi\in H^4(\mathbb{T}_a^n)} \frac{\int_0^{T_a}\phi\widetilde{\mathcal{L}}_0^a(\phi)\mathrm{d}t}{\int_0^{T_a}\phi^2\mathrm{d}t}. \tag{30}$$

Since v_a is a periodic, it can be taken as a test function on the right-hand side of (30); this provides

$$\begin{aligned}\mathcal{L}_0^a(v_a) &= v_a^{(4)} - K_2 v_a^{(2)} + K_0 v_a - \widetilde{c}(n) v_{a,T}^{2^{**}-1}\\ &= v_a^{(4)} - K_2 v_a^{(2)} + K_0 v_a - c(n) v_{a,T}^{2^{**}-1} + \check{c}(n) v_{a,T}^{2^{**}-1}\\ &= \check{c}(n) v_{a,T}^{2^{**}-1},\end{aligned}$$

where we used that $\mathcal{L}_0^a$ and $\widetilde{\mathcal{L}}_0^a$ have the same coordinate expression. Hence, since $\check{c}(n) < 0$, the estimate (29) is a consequence of (30).

To prove the lower bound estimate, we observe that a combination of the results in [14] with the classification given by (3) implies the variational characterization below

$$v_a = \inf_{\phi \in H_0^4([0,T_a])} \frac{\int_0^{T_a} \left(\left|\phi^{(2)}\right|^2 - K_2 \left|\phi^{(1)}\right|^2 + K_0 |\phi|^2 \right) \mathrm{d}t}{\left(\int_0^{T_a} \phi^{2^{**}} \mathrm{d}t \right)^{2/2^{**}}}.$$

Moreover, since v_a satisfies (4), for all $\phi \in H_0^4([0, T_a])$, we find

$$\frac{\int_0^{T_a} \left(\left|\phi^{(2)}\right|^2 - K_2 \left|\phi^{(1)}\right|^2 + K_0 |\phi|^2 \right) \mathrm{d}t}{\left(\int_0^{T_a} \phi^{2^{**}} \mathrm{d}t \right)^{2/2^{**}}} \geqslant c(n) \left(\int_0^{T_a} v_a^{2^{**}} \mathrm{d}t \right)^{1-2/2^{**}}. \tag{31}$$

On the other hand, using the Hölder inequality, we get

$$\int_0^{T_a} \phi^2 \mathrm{d}t \leqslant T_a^{1-2/2^{**}} \left(\int_0^{T_a} \phi^{2^{**}} \mathrm{d}t \right)^{2/2^{**}}. \tag{32}$$

Then, for all $\phi \in H_0^4([0, T_a])$ a combination of (31) and (32) yields

$$\begin{aligned}&\int_0^{T_a} \phi \mathcal{L}_0^a \phi \mathrm{d}t\\ &= \int_0^{T_a} \left(\phi^{(4)} - K_2 \phi^{(2)} + K_0 \phi - \widetilde{c}(n) v_{a,T}^{2^{**}-1} \phi \right) \mathrm{d}t\\ &\geqslant c(n) \left(\int_0^{T_a} v_a^{2^{**}} \mathrm{d}t \right)^{1-2/2^{**}} \left(\int_0^{T_a} \phi^{2^{**}} \mathrm{d}t \right)^{2/2^{**}} - \widetilde{c}(n) \int_0^{T_a} \phi^{2^{**}} \mathrm{d}t\end{aligned}$$

$$= \left(\int_0^{T_a} \phi^{2^{**}} \mathrm{d}t\right)^{2/2^{**}} \left[c(n) \left(\int_0^{T_a} v_a^{2^{**}} \mathrm{d}t\right)^{1-2/2^{**}} - \widetilde{c}(n) \left(\int_0^{T_a} \phi^{2^{**}}\right)^{1-2/2^{**}} \mathrm{d}t \right]$$

$$\geqslant T_a^{2/2^{**}-1} \left[c(n) \int_0^{T_a} \phi^2 \mathrm{d}t \left(\int_0^{T_a} v_a^{2^{**}} \mathrm{d}t\right)^{1-2/2^{**}} - \widetilde{c}(n) \int_0^{T_a} \phi^2 \mathrm{d}t \left(\int_0^{T_a} \phi^{2^{**}}\right)^{1-2/2^{**}} \mathrm{d}t \right].$$

In particular, taking $\phi \in H_0^4([0, T_a])$ such that $\|\phi\|_{H_0^4([0,T_a])} = \|v_a\|_{H_0^4([0,T_a])}$, we obtain

$$\int_0^{T_a} \phi \mathcal{L}_0^a(\phi) \mathrm{d}t \geqslant \check{c}(n) T_a^{2/2^{**}-1} \|\phi\|^2_{L^2([0,T_a])} \|v_a\|^{2^{**}-2}_{L^{2^{**}}([0,T_a])},$$

which directly implies the lower bound estimate.

Finally, since $\phi_{a,0}^+ = \partial_a v_a$ is a periodic solution to $\mathcal{L}_0^a(\phi_{a,0}^+) = 0$, we have that there exists an eigenfunction with associated eigenvalue $\lambda = 0$ subjected to periodic boundary conditions provided by $\alpha = 0$. Besides, this eigenfunction has two nodal domains within the interval $[0, T_a]$, which is associated either to $\sigma_1(a, 0)$ or to $\sigma_2(a, 0)$.

In the remaining steps, we provide more precise localization of the spectral bands of $\mathcal{L}^a$:

Step 2: For any $a \in (0, a_0)$, it follows that $\mathfrak{B}_k(a, 0) \subset (0, \infty)$ for each $k \geqslant 3$ and $\mathfrak{B}_k(a, 0) \subset [0, \infty)$ for each $k \geqslant 2$.

This is a direct consequence of Claim 2 and Step 1.

Step 3: For any $a \in (0, a_0)$ and $j, k \in \mathbb{N}$, it holds that $\mathfrak{B}_k(a, j) \subset (0, \infty)$.

In fact, when $j > n$ we have $\lambda_j > 2n$, which by Claim 3 implies $\sigma_k(a, j, 0) > \sigma_0(a, 0, 0) + n^3$ for all $k \in \mathbb{N}$. On the other hand, since $0 < v_a(t) < 1$, for all $t \in \mathbb{R}$, using the lower bound estimate, we find that $\sigma_0(a, 0, 0) \geqslant \check{c}(n)$ and $\sigma_k(a, j, 0) > \sigma_0(a, 0, 0) + n^3 \geqslant n^3 + \check{c}(n) > 0$. When $1 \leqslant j \leqslant n$, it follows from the construction for the geometric Jacobi fields Remark 17, since $\mathcal{L}_j^a(\phi_{a,j}^\pm) = 0$ and $\phi_{a,j}^\pm = e^{\pm t}\left(\pm v_a^{(1)} + \gamma v_a\right) + \mathcal{E}_\pm$, where $\mathcal{E}_+(t) = \mathcal{O}(1)$ and $\mathcal{E}_-(t) = \mathcal{O}\left(e^{-2t}\right)$ as $t \to \infty$ are positive periodic solutions to $\mathcal{L}_j^a$.

The last claim relates the spectral bands $\mathfrak{B}_k(a, j)$ and the indicial roots $\mathfrak{I}_j^a$.

Claim 5: The ODE $\mathcal{L}_j^a(\Phi) = 0$ admits a quasi-periodic solution, if and only if, for some $k \in \mathbb{N}$, $0 \in \mathfrak{B}_k(a, j)$

Indeed, we have that $\Phi = \mathcal{F}_a^{-1}\left(e^{-i\alpha t}\widehat{\Phi}\right)$ solves $\mathcal{L}_j^a(\Phi) = 0$ since

$$0 = \mathcal{L}_{j,\alpha}^a(\widehat{\Phi}) = e^{i\alpha t}\mathcal{F}_a\left(\mathcal{L}_j^a\left(\mathcal{F}_a^{-1}\left(e^{-i\alpha t}\widehat{\Phi}\right)\right)\right) \quad \text{and} \quad \mathcal{L}_j^a\left(\mathcal{F}_a^{-1}\left(e^{-i\alpha t}\widehat{\Phi}\right)\right) = 0.$$

Therefore, by Remark 26, the proof of the claim follows. □

3.6. Fredholm theory

We investigate the spectrum of the linearized operator. Our goal is to conclude that $\mathcal{L}^a$ is Fredholm, which follows by showing that $\mathfrak{I}_a \subset \mathbb{R}$ is a discrete set. The last assessment is not trivial to prove; in fact, we need to use the Fourier-Laplace transform results to find a right-inverse for the linearized operator. The strategy is based on some results from holomorphic functional analysis.

To invert the twisted operator, we use the analytic Fredholm theorem. Since the symbol of $\mathcal{L}^a$ in cylindrical coordinates is given by $\partial_t^{(4)} + \Delta_\theta^2$, we conclude that if $k \in \mathbb{N}$, $a \in (0, a_0)$ and $\beta \in \mathbb{R}$, then $\mathcal{L}^a : H_\beta^{k+4}(\mathcal{C}_0, \mathbb{R}^p) \to H_\beta^k(\mathcal{C}_0, \mathbb{R}^p)$ is a bounded linear elliptic self-adjoint operator.

The main result of this subsection states the invertibility of the linearized operator.

Proposition 28. *If $k \in \mathbb{N}$ and $\beta \notin \mathfrak{I}^a$, then $\mathcal{L}^a : H_\beta^{k+4}(\mathcal{C}_0, \mathbb{R}^p) \to H_\beta^k(\mathcal{C}_0, \mathbb{R}^p)$ is Fredholm.*

Proof. To apply the analytic Fredholm theorem, we use the twisted operator from (26),

$$\widetilde{\mathcal{L}}^a(\rho) : H^{k+4}(\mathbb{T}_a^n, \mathbb{C}^p) \to H^k(\mathbb{T}_a^n, \mathbb{C}^p) \quad \text{given by} \quad \widetilde{\mathcal{L}}^a(\rho)(\widehat{\Phi}) = e^{i\rho}\mathcal{F}_a \circ \mathcal{L}^a \circ \mathcal{F}_a^{-1}\left(e^{-i\rho t}\widehat{\Phi}\right).$$

In what follows, we divide the proof into some claims:

Claim 1: If $\alpha \in (0, 2\pi)$, then $\widetilde{\mathcal{L}}_\alpha^a$ is Fredholm.

For each $\alpha \in (0, 2\pi)$, the operator $\widetilde{\mathcal{L}}^a(\rho)$ is linear, bounded, elliptic and depends holomorphically on ρ. Thus, this operator is either never Fredholm or it is Fredholm for ρ outside a discrete set. We take $\rho = \alpha \in (0, 2\pi)$ and suppose there exists $\widehat{\Phi} \in H^{k+4}(\mathbb{T}_a^n, \mathbb{R}^p)$ such that $\widetilde{\mathcal{L}}^a(\rho)(\widehat{\Phi}) = 0$; thus, $\mathcal{L}^a(\rho)(\Phi)$, where $\Phi = \mathcal{F}_a^{-1}(e^{-i\rho t}\widehat{\Phi})$. Then, Φ is quasi-periodic; in particular, Φ is bounded. However, by Proposition 27, any bounded Jacobi field is a multiple of Φ_0^+, which is not quasi-periodic. Hence, $\mathcal{L}^a(\alpha)$ is injective. Finally, since this operator is formally self-adjoint, it follows that $\widetilde{\mathcal{L}}^a(\rho)$ is Fredholm.

Claim 2: If $a \in (0, a_0)$ and $\beta \in \mathfrak{I}^a$, then there exists $\widetilde{\mathcal{G}}^a(\rho) : H^k(\mathbb{T}_a^n, \mathbb{C}^p) \to H^{k+4}(\mathbb{T}_a^n, \mathbb{C}^p)$ such that $\widetilde{\mathcal{G}}^a(\rho)$ is a right-inverse for $\widetilde{\mathcal{L}}^a(\rho)$.

Using Claim 1, we can find a discrete set $\mathfrak{D}_a \subset \mathfrak{R}_a$ and a meromorphic operator $\widetilde{\mathcal{G}}^a(\rho) : H^k(\mathbb{T}_a^n, \mathbb{C}^p) \to H^{k+4}(\mathbb{T}_a^n, \mathbb{C}^p)$ such that $\widehat{\Phi} = \left(\widetilde{\mathcal{G}}^a(\rho) \circ \widetilde{\mathcal{L}}^a(\rho)\right)(\widehat{\Phi})$ for $\rho \notin \mathfrak{D}_a$.

Claim 3: If $a \in (0, a_0)$ and $\beta \in \mathfrak{I}^a$, then there exists $\mathcal{G}^a : H_\beta^k(\mathcal{C}_0, \mathbb{R}^p) \to H_\beta^{k+4}(\mathcal{C}_0, \mathbb{R}^p)$ right-inverse for $\mathcal{L}^a$.

Indeed, notice that $\mathfrak{I}^a = \{\beta \in \mathbb{R} : \beta = \Im(\rho) \text{ for some } \rho \in \mathfrak{D}_a\}$, which provides

$$\mathcal{G}^a(\Phi) = \mathcal{F}_a^{-1}\left(e^{-i\rho T_a t}\left(\widetilde{\mathcal{G}}^a\left(e^{-i\rho T_a t}\left(\mathcal{F}_a(\Phi)\right)\right)\right)\right).$$

Furthermore, by construction, we obtain that $\widehat{\Phi} = \mathcal{G}^a(\Phi) \in H_{-\Im(\rho)}^{k+4}(\mathcal{C}_0, \mathbb{R}^p)$, which by the Fredholm alternative concludes the proof of the claim. The last claim proves the proposition. □

Proposition 29. *The set $\mathfrak{I}^a$ is discrete.*

Proof. Note that each element in $\mathfrak{I}^a$ is the imaginary part of a pole to $\widetilde{\mathcal{G}}_a$, which by the analytic Fredholm theory is a discrete subset of $\mathbb{C}$. On the other hand, the operator $\mathcal{L}^a(\rho)$ is unitarily equivalent to $\mathcal{L}^a(\rho + 2\pi l)$ for each $l \in \mathbb{Z}$; thus, ρ is a pole of $\mathcal{G}_a$, if and only if $\rho + 2\pi l$ also is for any $l \in \mathbb{Z}$. Therefore, $\mathcal{G}_a$ can only have finitely many poles in each horizontal strip. □

Corollary 30. *If $k \in \mathbb{N}$ and $\beta \notin (0, 1)$, then*

(i) *the operator $\mathcal{L}^a : H_\beta^{k+4}(\mathcal{C}_0, \mathbb{R}^p) \to H_\beta^k(\mathcal{C}_0, \mathbb{R}^p)$ is surjective;*

(ii) *the operator $\mathcal{L}^a : H_{-\beta}^{k+4}(\mathcal{C}_0, \mathbb{R}^p) \to H_{-\beta}^k(\mathcal{C}_0, \mathbb{R}^p)$ is injective.*

Proof. It follows from the proof of Proposition 28 that $\mathcal{L}^a(\rho)$ is injective for each $\rho \in \mathbb{C}$ with $-1 < \Im(\rho) < 0$, which implies $\mathcal{L}^a : H^{k+4}_{-\beta}(\mathcal{C}_0, \mathbb{R}^p) \to H^k_{-\beta}(\mathcal{C}_0, \mathbb{R}^p)$ is injective, and thus (ii) is proved. Besides, since by duality the operator $\mathcal{L}^a : H^{k+4}_{\beta}(\mathcal{C}_0, \mathbb{R}^p) \to H^k_{\beta}(\mathcal{C}_0, \mathbb{R}^p)$ is formally self-adjoint; thus, the surjectiveness follows, and (i) is proved. □

3.7. *Existence of singular solutions*

We prove the existence of solutions to $(\mathcal{S}_{p,1})$. We proceed by studying the spectral properties of the linearized operator around an Emden–Fowler solution. Let us remark that by the implicit function theorem, the existence of solutions to $(\mathcal{S}_{p,1})$ can be obtained by showing that the linearized operator $\mathcal{L}^a$ is Fredholm. We already know that $\mathcal{L}^a$ sometimes does not satisfy this property since its kernel is not closed. To overcome this issue, we introduce the following definition:

Definition 31. For each $v_{a,T}$, let us consider the deficiency space generated by the Jacobi fields basis of the linearized operator. In other words,
(i) for $j = 0$, we have $D_{a,0}(\mathcal{C}_0, \mathbb{R}^p) = \mathrm{span}\{\Phi^+_{a,0}, \Phi^-_{a,0}\}$;
(ii) for $j \geqslant 1$, we have $D_{a,j}(\mathcal{C}_0, \mathbb{R}^p) = \mathrm{span}\{\Phi^+_{a,j}, \Phi^-_{a,j}, \widetilde{\Phi}^+_{a,j}, \widetilde{\Phi}^-_{a,j}\}$.

The fact that there are only two Jacobi fields in (i) of the last definition is a consequence of Proposition 27. Namely, any zero-frequency Jacobi field with growth less than exponential (tempered) is generated by the ones obtained by varying geometric parameters in the Emden–Fowler solution.

Now, we can present the main result of the subsection:

Proposition 32. *Let $\mathcal{V}_{a,T}$ be an Emden–Fowler solution.*
(i) *If $\beta \in (\beta_{a,0}, \beta_{a,1})$, then $\mathcal{L}^a : C^{4,\zeta}_{\beta}(\mathcal{C}_0, \mathbb{R}^p) \oplus D_{a,0}(\mathcal{C}_0, \mathbb{R}^p) \to C^{0,\zeta}_{\beta}(\mathcal{C}_0, \mathbb{R}^p)$ is a surjective Fredholm mapping with bounded right-inverse, given by*

$$\mathcal{G}^a_0 : C^{0,\zeta}_{\beta}(\mathcal{C}_0, \mathbb{R}^p) \to C^{4,\zeta}_{\beta}(\mathcal{C}_0, \mathbb{R}^p) \oplus D_{a,0}(\mathcal{C}_0, \mathbb{R}^p).$$

(ii) *If $\beta \in (\beta_{a,1}, \beta_{a,2})$, then $\mathcal{L}^a : C^{4,\zeta}_{\beta}(\mathcal{C}_0, \mathbb{R}^p) \oplus D_{a,0}(\mathcal{C}_0, \mathbb{R}^p) \oplus D_{a,1}(\mathcal{C}_0) \to C^{0,\zeta}_{\beta}(\mathcal{C}_0, \mathbb{R}^p)$ is a surjective Fredholm mapping with bounded right-inverse, given by*

$$\mathcal{G}^a_1 := C^{0,\zeta}_{\beta}(\mathcal{C}_0, \mathbb{R}^p) \to C^{4,\zeta}_{\beta}(\mathcal{C}_0, \mathbb{R}^p) \oplus D_{a,0}(\mathcal{C}_0, \mathbb{R}^p) \oplus D_{a,1}(\mathcal{C}_0, \mathbb{R}^p).$$

Proof. We proceed as in Proposition 28. First, we decompose the linearized operator into Fourier modes and apply the Laplace–Fourier transform. Then, by conjugation, let us define a family of transformations satisfying the assumptions of classical analytic Fredholm theory. We can therefore invert the conjugated operator $\widetilde{\mathcal{L}}^a_j(\rho)$. Afterward, we reconstruct the function by undoing the Fourier–Laplace transform inverse. In other terms, for all $j \in \mathbb{N}$, we take the right-inverse,

$$\mathcal{G}^a_j(\Phi) = \mathcal{F}^{-1}_a \left(e^{-i\rho t} \left(\widetilde{\mathcal{G}}_j{}^a \left(e^{-i\rho t} \left(\mathcal{F}_a(\Phi) \right) \right) \right) \right).$$

This provides the proof of the proposition. □

Remark 33. The necessity of adding the deficiency spaces $D_{a,j}(\mathcal{C}_0, \mathbb{R}^p)$ comes from a simple form of the linear regularity theorem from [28, Lemma 4.18] and some ODE theory. In addition, note that if $\beta = \pm\beta_{a,j}$, then $\mathcal{L}^a$ does not have closed range. Moreover, we have Schauder estimates in the sense of weighted spaces. More precisely, if $\mathcal{V}$ is solution to the inhomogeneous problem $\mathcal{L}^a(\mathcal{V}) = \Psi$, then $\mathcal{V} \in C^{4,\zeta}_\beta(\mathcal{C}_0, \mathbb{R}^p)$ whenever $\Psi \in C^{0,\zeta}_\beta(\mathcal{C}_0, \mathbb{R}^p)$. More generally, it should be possible to find an inverse like $\mathcal{G}^a_j := C^{0,\zeta}_\beta(\mathcal{C}_0, \mathbb{R}^p) \to C^{4,\zeta}_\beta(\mathcal{C}_0, \mathbb{R}^p) \bigoplus_{l=0}^{j} D_{a,l}(\mathcal{C}_0, \mathbb{R}^p)$, which would give us refined information.

As a consequence of our results, we present the main result of this subsection.

Corollary 34. *There exists at least one strongly positive solution $\mathcal{V}$ to ($\mathcal{C}_{p,0}$).*

Another application is the following improved regularity theorem for solutions to ($\mathcal{S}_{p,1}$) in cylindrical coordinates:

Corollary 35. *Let $\mathcal{V}$ be a strongly positive solution to $\mathcal{L}^a(\mathcal{V}) = \Psi$. Assume that $\mathcal{V} \in C^{4,\zeta}_{\widetilde{\beta}}(\mathcal{C}_0, \mathbb{R}^p)$ and $\Psi \in C^{0,\zeta}_{\widehat{\beta}}(\mathcal{C}_0, \mathbb{R}^p)$.*
(i) *If $0 < \widetilde{\beta} < \widehat{\beta} < 1$, then $\mathcal{V} \in C^{4,\zeta}_{\beta_2}(\mathcal{C}_0, \mathbb{R}^p)$;*
(ii) *If $0 < \widetilde{\beta} < 1 < \widehat{\beta} < \beta_{a,2}$, then $\mathcal{V} \in C^{4,\zeta}_{\widehat{\beta}}(\mathcal{C}_0, \mathbb{R}^p) \oplus D_{a,1}(\mathcal{C}_0, \mathbb{R}^p)$.*

Proof. First, we use the right-inverse operator $\mathcal{G}^a_0$ in Proposition 32 to obtain that $\widetilde{\mathcal{V}} + c\Phi^+_{a,0} = \mathcal{G}^a_0(\Psi) \in C^{4,\zeta}_\beta(\mathcal{C}_0, \mathbb{R}^p) \oplus D_{a,1}(\mathcal{C}_0, \mathbb{R}^p)$ is also a solution to $\mathcal{G}^a_0(\mathcal{V}) = \Psi$, which implies that $\widehat{\mathcal{V}} = \mathcal{V} - \widetilde{\mathcal{V}}$ satisfies $\mathcal{G}^a_0(\widehat{\mathcal{V}}) = 0$. Then, $\widehat{\mathcal{V}}$ is exponentially decaying, that is, $\widehat{\mathcal{V}} \in C^{4,\zeta}_1(\mathcal{C}_0, \mathbb{R}^p)$. Finally, $\mathcal{V} \in C^{4,\zeta}_{\beta_2}(\mathcal{C}_0, \mathbb{R}^p)$ since $\mathcal{V} = \widehat{\mathcal{V}} + \widetilde{\mathcal{V}}$, which finishes the proof of (i). The proof of (ii) follows the same argument, so we omit it. □

3.8. Growth properties for the Jacobi fields

In this part, we apply the spectral analysis developed before to investigate the growth/decay rate in which the Jacobi fields on the kernel of the linearized operator grow/decay.

Let us begin with some considerations concerning the scalar case $p = 1$. First, by [12, Theorem 2], the operator (20) has periodic coefficients. Second, by Proposition 27, we can use classical Floquet theory (or Boch wave theory) to study the asymptotic behavior of the Jacobi fields on the projection over V_j. For this, we transform the fourth order operator (20) into a first order operator on $\mathbb{R}^4$. More precisely, defining $X = (\phi, \phi^{(1)}, \phi^{(2)}, \phi^{(3)})$, we conclude that the fourth order equation $\mathcal{L}^a_j(\phi) = 0$ is equivalent to the first order system $X'(t) = N_{a,j}(t)X(t)$. Here

$$N_{a,j}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -B_j & +C_{a,j}(t) \end{bmatrix},$$

where $B_j := K_2 + 2\lambda$ and $C_{a,j}(t) = K_0 + \lambda_j(\lambda_j + J_0) - \widetilde{c}(n) v_{a,T}(t)^{2^{**}-2}$.

Notice that $N_{a,j}(t)$ is a T_a-periodic matrix. Hence, the monodromy matrix associate to this ODE system with periodic coefficients is given by $M_{a,j}(t) = \exp \int_0^t N_{a,j}(\tau)\mathrm{d}\tau$. Finally, we define the Floquet exponents, denoted by $\widetilde{\mathfrak{I}}_j^a$, as the complex frequencies associated with the eigenvectors of $M_{a,j}(t)$, which forms a four-dimensional basis for the kernel of $\mathcal{L}_j^a$. Using Abel's identity, we get that $N_{a,j}(t)$ is constant, which yields

$$\det(M_{a,j}(t)) = \exp \int_0^T \operatorname{tr} N_{a,j}(\tau)\mathrm{d}\tau = \exp\left(-\int_0^T C_{a,j}(\tau)\mathrm{d}\tau\right) = 1.$$

Since $N_{a,j}(t)$ has real coefficients, all its eigenvalues are pairs of complex conjugates. Equivalently, $\widetilde{\mathfrak{I}}_j^a = \{\pm\rho_{a,j}, \pm\bar{\rho}_{a,j}\}$, where $\rho_{a,j} = \alpha_{a,j} + i\beta_{a,j}$ and $\widetilde{\rho}_{a,j} = \widetilde{\alpha}_{a,j} - i\beta_{a,j}$. Then, the set of indicial roots of $\mathcal{L}_j^a$ are given by $\mathfrak{I}_j^a = \{-\beta_{a,j}, \beta_{a,j}\}$. Moreover, for any $\phi \in \ker(\mathcal{L}_j^a)$, we have

$$\phi(t) = b_1\phi_{a,j}^+(t) + b_2\phi_{a,j}^-(t) + b_3\widetilde{\phi}_{a,j}^+(t) + b_4\widetilde{\phi}_{a,j}^-(t),$$

where $\phi_{a,j}^{\pm}(t) = e^{\pm\rho_{a,j}t}$ and $\widetilde{\phi}_{a,j}^{\pm}(t) = e^{\pm\widetilde{\rho}_{a,j}t}$. Hence, the Jacobi fields' exponential decay/growth rate is controlled by $|\beta_{a,j}|$. Therefore, the asymptotic properties of $\mathcal{L}_j^a$ are obtained by the study of $\mathfrak{I}^a$. Remember that in the definition of indicial roots, we are not considering the multiplicity; that is, $\beta_{a,2}$ stands for all the low-frequency ($j = 1$) Jacobi fields.

In the following lemma, we give some structure to the set of indicial roots of $\mathcal{L}^a$.

Lemma 36. *Let $a \in [0, a_0]$.*
(i) *If $j = 0$, then $0 \in \mathfrak{I}_0^a$.*
(ii) *If $j = 1$, then $\{-1, 1\} \subset \mathfrak{I}_1^a$.*
(iii) *If $j > 1$, then $\min_{j>1} \mathfrak{I}_j^a > 1$.*
Moreover, $\mathfrak{I}^a$ is a discrete set, namely, $\mathfrak{I}^a = \{\dots, -\beta_{a,2}, -1, 0, 1, -\beta_{a,2}, \dots\}$. In particular, the indicial root 0 is isolated.

Proof. Let us divide the proof into three case steps, namely, $a = 0$, $a = a_0$ and $a \in (0, a_0)$. In the first two ones by (3) we know that $v_{a,T}$ is constant; thus, the indicial exponents are the solutions to a fourth order characteristic equation. In this fashion, let us also introduce the following notation for the discriminant of this indicial equation, $D_{a,j} := B_j^2 - 4C_{a,j}$.

We also divide each step into three cases with respect the Fourier eigenmodes, namely $j = 0$ (zero-frequency), $j = 1$ (low-frequency) and $j > 1$ (high-frequency).

Step 1: (spherical solution) $a = 0$.
When $a = 0$, we have that $v_{a,T} \equiv 0$, then the linearized operator becomes

$$\mathcal{L}_j^0(\phi) = \phi^{(4)} - (K_2 + 2\lambda_j)\phi^{(2)} + \left(K_0 + \lambda_j^2 + \lambda_j J_0\right)\phi.$$

Therefore, we shall compute the roots of the characteristic equation $\rho^4 - B_j\rho^2 - C_{0,j}\rho = 0$.
Case 1: $j = 0$, $\mathfrak{m}_0 = 1$ and $\lambda_0 = 0$.
The operator has the following expression

$$\mathcal{L}_0^0(\phi) = \phi^{(4)} - K_2\phi^{(2)} + K_0\phi.$$

Notice that when $D_{a,j} > 0$, then $\beta_{a,j} = 0$. More generally, the sign of $D_{a,j}$ controls the nature of the complex roots. It is straightforward to check that the indicial roots of this operator are $\rho_{0,0} = \frac{n}{2}$ and $\widetilde{\rho}_{0,0} = \frac{n-4}{2}$.
Case 2: $j = 1$, $\mathfrak{m}_1 = n$ and $\lambda_1 = \cdots \lambda_n = n-1$.
Here we obtain,

$$\mathcal{L}_1^0(\phi) = \phi^{(4)} - B_1\phi^{(2)} + C_{0,1}\phi,$$

and the indicial roots are given by $\rho_{0,1} = \frac{1}{2}(n+2)$ and $\widetilde{\rho}_{0,1} = \frac{1}{2}(n-2)$.
Case 3: $j > 1$, $\mathfrak{m}_j > n$ and $\lambda_j = \ell(n-2+\ell)$, for some $\ell \in \mathbb{N}$.
Here we obtain,

$$\mathcal{L}_j^0(\phi) = \phi^{(4)} - B_j\phi^{(2)} + C_{0,j}\phi,$$

and the indicial roots are given by $\rho_{0,\ell} = \frac{1}{2}(2+\sqrt{D_{0,\ell}})$ and $\widetilde{\rho}_{0,\ell} = \frac{1}{2}\left(2-\sqrt{D_{0,\ell}}\right)$, where $D_{0,\ell} = n^2 - 4n + 4 + 4\ell(n+\ell-2)$. Notice that $D_{0,\ell} > 0$ for $\ell > 1$. Using a direct argument, we can check that $\Im\rho_{0,j} > \Im\rho_{0,1}$ and $\Im\rho_{0,j} > \Im\widetilde{\rho}_{0,1}$ for all $j > 1$, which by the last case concludes the proof of Step 1.
Step 2: (cylindrical solution) $a = a_0$.
Since $v_{a_0,T} \equiv a_0 = [n(n-4)/(n^2-4)]^{n-4/8}$, we proceed identically as in the last step. First, we have

$$\mathcal{L}_j^{a_0}(\phi) = \phi^{(4)} - (K_2 + 2\lambda_j)\phi^{(2)} + \left(K_0 + \lambda_j^2 + \lambda_j J_0 - \widetilde{c}(n)a_0^{2^{**}-2}\right)\phi.$$

As before, we shall divide our approach as follows
Case 1: $j = 0$, $\mathfrak{m}_0 = 1$ and $\lambda_0 = 0$.

$$\rho_{a_0,0} = \frac{1}{2}\sqrt{n^2 - 4n + 8 + \sqrt{D_{a_0,0}}} \quad \text{and} \quad \widetilde{\rho}_{a_0,0} = \frac{1}{2}\sqrt{n^2 - 4n + 8 - \sqrt{D_{a_0,0}}},$$

where $D_{a_0,0} = n^4 - 64n + 64$.
Case 2: $j = 1$, $\mathfrak{m}_1 = n$ and $\lambda_1 = \cdots \lambda_n = n-1$.

$$\rho_{a_0,1} = \sqrt{\frac{n^2+2}{2}} \quad \text{and} \quad \widetilde{\rho}_{a_0,1} = 1.$$

Case 3: $j > 1$, $\mathfrak{m}_j > n$ and $\lambda_j = \ell(n-2+\ell)$, for some $\ell \in \mathbb{N}$.

$$\rho_{a_0,j} = \frac{1}{4}\sqrt{[(n+2(\ell-1))]^2 + \sqrt{D_{a_0,\ell}}} \text{ and } \widetilde{\rho}_{a_0,j} = \frac{1}{4}\sqrt{[(n+2(\ell-1))]^2 - \sqrt{D_{a_0,\ell}}},$$

where $D_{a_0,\ell} = n^4 + 64(\ell-1)(n+\ell-1)$.
Step 3: (Emden–Fowler solution) $a \in (0, a_0)$.
This is the most delicate case since $v_{a,T}$ is periodic, so the zeroth order term in the operator $\mathcal{L}_j^a$ is also T_a-periodic. In this case, it is impossible to compute the Floquet exponents explicitly. Nonetheless, we can show that they are strictly bigger than one when $j > 1$.

Case 1: $j = 0$, $\mathfrak{m}_0 = 1$ and $\lambda_0 = 0$.
By Remark 17, it follows that $\phi^+_{a,0}$ is bounded and $\phi^-_{a,0}$ is linearly growing, then $0 \in \mathfrak{I}^a$ with multiplicity 2.
Case 2: $j = 1$, $\mathfrak{m}_1 = n$ and $\lambda_1 = \cdots \lambda_n = n - 1$.
Again using Remark 17, it follows $\phi^\pm_{a,1} = \cdots = \phi^\pm_{a,n}$ is exponentially growing/decaying, then $\{-1, 1\} \subset \mathfrak{I}^a$.
Case 3: More generally, note that the indicial roots form an increasing sequence,

$$\beta_{a,0} \leqslant \beta_{a,1} \leqslant \cdots \leqslant \beta_{a,j} \leqslant \beta_{a,j+1} \to \infty \quad \text{as} \quad j \to \infty,$$

which is a consequence of a comparison principle on $a \in (0, a_0)$ for the linearized operator in cylindrical coordinates. □

Lemma 37. *The indicial set $\mathfrak{I}^a_j$ is discrete. Moreover, $\mathfrak{I}^a_j = \{\ldots, -\beta_{a,2}, -1, 0, 1, \beta_{a,2}, \ldots\}$. In particular, the indicial root 0 is isolated.*

Proof. It follows directly by Proposition 27. □

Notice that Lemma 36 only provides exponential growth/decay for the Jacobi fields. Nevertheless, we need something slightly more robust to apply Simon's technique. Namely, for $j = 0$, we must show that the Jacobi fields are either periodic (bounded) or linearly growing. For the first two Jacobi fields $\phi^+_{a,0}$ and $\phi^-_{a,0}$, this follows because they arise respectively as the variation of the necksize and translation parameters that appear in the classification for the Emden–Fowler solutions. The difficulty here is to show that they generate the zero-frequency space. We overcome this issue observing that by the direct computation in Lemma 36, we know that $0 \in \mathfrak{I}^a_0$ with multiplicity two.

Next, we proceed as in [26, Proposition 4.14] to prove the following asymptotic expansion

Proposition 38. *Let $\psi \in C^\infty_0(\mathcal{C}_0)$, $\beta \in (0, 1)$ and $\phi \in H^4_{-\beta}(\mathcal{C}_0)$ satisfying $\mathcal{L}^a(\phi) = \psi$. Then, ϕ has an asymptotic expansion $\phi = \sum_{j\in\mathbb{N}} \phi_j$ with $\mathcal{L}^a\left(\phi_j\right) = 0$ and $\phi_j \in H^4_{-\beta}(\mathcal{C}_0)$ for any $\beta \in (0, \beta_{a,j})$.*

Proof. We divide the proof into some steps.
Step 1: For $\beta \in (0, 1)$, it follows $\phi \in H^{k+4}_{-\beta}(\mathcal{C}_0)$.
Indeed, take $\rho \in \mathbb{C}$ with $0 < \beta < \Im(\rho)$ and consider the transformed equation $\widetilde{\mathcal{L}}^a(\rho)(\widehat{\phi}) = \widetilde{\psi}$, where $\widetilde{\phi} = e^{i\rho t}\widehat{\phi}$ and $\widetilde{\psi} = e^{i\rho t}\widehat{\psi}$. By applying the inverse operator $\widetilde{\mathcal{G}}^a(\rho)$ in both sides of the last equation, we get that $\widehat{\phi} = \widetilde{\mathcal{L}}^a(\rho)(\widetilde{\psi})$. Then, since $\psi \in C^\infty_0(\mathcal{C}_0)$, it follows that $\widetilde{\psi}(\rho)$ is an entire function on ρ and smooth on (t, θ). Notice that $\widetilde{\phi}$ is analytic on the half-plane $\Im(\rho) > \beta$, since the poles of $\mathcal{G}^a(\rho)$ coincide with the zeros of $\widetilde{\psi}$. Finally, take $\beta' \in (\beta, 1)$ and since $\mathcal{G}^a(\rho)$ has no poles in $\Im(\rho) \in (\beta', \beta)$, by the Cauchy formula, we can define the contour integral $\mathcal{F}^{-1}_a$ up to $\Im(\rho)$.
Step 2: For each $\beta \in (0, 1)$, there exist $\beta'' \in (1, \beta_{a,2})$, $\phi'' \in H^{k+4}_{-\beta''}(\mathcal{C}_0)$, and $\phi' \in H^{k+4}_{-\beta}(\mathcal{C}_0)$ with $\mathcal{L}^a(\phi') = 0$ satisfying $\phi = \phi' + \phi''$.
Choose $\beta'' \in (1, \beta_{a,2})$ and ρ'' such that $\Im(\rho'') = \beta''$. Now let us define $\widetilde{\phi}'' = \widetilde{\mathcal{G}}^a(\rho'')$. Finally, we apply the inverse $\mathcal{F}^{-1}_a$ on the two contour lines $\Im(\rho) = \beta$ and $\Im(\rho) = \beta''$, which by periodicity does not take into account the vertical sides of the rectangle $[\beta, \beta''] \times [0, 2\pi] \subset \mathbb{C}$. In fact, $\widetilde{\phi} - \widetilde{\phi}'' = \widetilde{\mathcal{G}}^a(\rho) - \widetilde{\mathcal{G}}^a(\rho'')$ is the residue of a meromorphic function with pole at $-i$.

We can continue this process by shifting the contour integral to the other poles in the strip. □

Corollary 39. *Let $\beta \in (0,1)$, $\psi \in C_c^\infty(\mathcal{C}_0) \cap L^2_{-\beta}(\mathcal{C}_0)$ and $\phi \in H^4_{-\beta}(\mathcal{C}_0)$ satisfying $\mathcal{L}^a(\phi) = \psi$. Then, there exist $\phi' \in H^4_{-\beta}(\mathcal{C}_0)$ and $\phi'' \in D_{a,0}(\mathcal{C}_0)$ such that $\phi = \phi' + \phi''$.*

Corollary 40. *The following properties hold for the projected scalar linearized operator:*
(i) *Assume $j = 0$, then the homogeneous equation $\mathcal{L}_0^a(\phi) = 0$ has a solutions basis with two elements, which are either bounded or at most linearly growing as $t \to \infty$;*
(ii) *Assume $j \geqslant 1$, then the homogeneous equation $\mathcal{L}_j^a(\phi) = 0$ has a solutions basis with four elements, which are exponentially growing/decaying as $t \to \infty$.*

Proof. For (i), we use Corollary 39. Notice that (ii) follows directly from Lemma 36. □

When $p > 1$, we can use a similar strategy to study (17). For $p = 1$, we have constructed a Jacobi field basis with four elements (two in the zero-frequency case). Now, we must find a base with $4p$ elements ($2p$ in the zero-frequency case), sharing the same growth properties in Corollary 40.

Proof of Proposition 3. First, notice that by Theorem B (ii), there exist $a \in [0, a_0]$, $T \in [0, T_a]$ and $\Lambda \in \mathbb{S}^{p-1}_+$ that provides $p+1$ families of solutions given by $T \mapsto \Lambda v_a(t+T)$, $a \mapsto \Lambda v_{a,T}(t)$, and $\theta \mapsto \Lambda(\theta) v_{a,T}(t)$, which by differentiation gives rise to some elements of the basis. Namely $\Lambda \partial_T \big|_{T=0} v_{a,T}(t)$, $\Lambda 1/a \partial_a \big|_{a=0} v_{a,T}(t)$, and $\partial_{\theta_i} \Lambda(\theta) v_{a,T}(t)$ for $i = 1, \ldots, p-1$.

Second, to construct all the Jacobi fields basis, let us consider $\{\mathbf{e}_i\}_{i \in I} \subset \mathbb{S}^{p-1}$ a linearly independent set in $\mathbb{R}^p$ with $\mathbf{e}_1 = \Lambda$, which provides four families with p Jacobi fields each given by $\Phi^+_{a,j,i} = \mathbf{e}_i \phi^+_{a,j}$, $\Phi^-_{a,j,i} = \mathbf{e}_i \phi^-_{a,j}$, $\widetilde{\Phi}^+_{a,j,i} = \mathbf{e}_i \widetilde{\phi}^+_{a,j}$, and $\widetilde{\Phi}^-_{a,j,i} = \mathbf{e}_i \widetilde{\phi}^-_{a,j}$.

Then, using Theorem B (ii), it is easy to check that $\mathcal{B}^a_j = \cup_{i \in I} \{\Phi^\pm_{a,j,i}, \widetilde{\Phi}^\pm_{a,j,i}\}$ is a basis to the kernel of $\mathcal{L}^a_j$ with $4p$ elements for each $j \geqslant 1$, and $\mathcal{B}^a_0 = \cup_{i \in I} \{\Phi^\pm_{a,0,i}\}$ a basis with $2p$ elements when $j = 0$. □

4. Qualitative properties and a priori estimates

This section is devoted to proving Proposition 4. We show that solutions to ($\mathcal{S}_{p,1}$) are asymptotic radially symmetric and satisfy an upper and lower bound estimate near the isolated singularity. Our strategy is to convert ($\mathcal{S}_{p,1}$) into a system of integral equations. Then, we use the Kelvin transform to perform a moving sphere technique and the Pohozaev invariant with a barrier construction to obtain the a priori estimates. Here, we are inspired by some techniques from [20]. The main difference in the proofs is that we need to deal with many components of System ($\mathcal{S}_{p,1}$).

4.1. Integral representation formulas

We now use a Green identity to transform the fourth order differential system ($\mathcal{S}_{p,1}$) into an integral one. In this way, we can avoid using the classical form of the maximum principle. Besides, it is also possible to prove regularity using a barrier construction in this setting.

For $n \geqslant 3$, the following expression for the Green function of the Laplacian in the unit ball is well-known

$$G_1(x,y)=\frac{1}{(n-2)\omega_{n-1}}\left(|x-y|^{2-n}-\left|\frac{x}{|x|}-x|y|\right|^{2-n}\right),$$

where ω_{n-1} is the surface area of the unit sphere. In addition, for any $u\in C^2(B_1)\cap C\left(\bar{B}_1\right)$, the next decomposition holds,

$$u(x)=\int_{B_1} G_1(x,y)\Delta u(y)\mathrm{d}y+\int_{\partial B_1} H_1(x,y)u(y)\mathrm{d}\sigma_y,$$

where

$$H_1(x,y)=-\partial_{\nu_y}G_1(x,y)=\frac{1-|x|^2}{\omega_{n-1}|x-y|^n}\quad \text{for}\quad x\in B_1\quad \text{and}\quad y\in\partial B_1,$$

with ν_y the outward normal vector at y.

Similarly, in the fourth order case with $n\geqslant 5$, for any $u\in C^4(B_1)\cap C^2\left(\bar{B}_1\right)$, it follows

$$u(x)=\int_{B_1} G_2(x,y)\Delta^2 u(y)\mathrm{d}y+\int_{\partial B_1} H_1(x,y)u(y)\mathrm{d}\sigma_y-\int_{\partial B_1} H_2(x,y)\Delta u(y)\mathrm{d}\sigma_y,$$

where

$$G_2(x,y)=\int_{B_1\times B_1} G_1(x,y_1)G_1(y_1,y)\,\mathrm{d}y_1\quad \text{and}\quad H_2(x,y)=\int_{B_1\times B_1} G_1(x,y_1)H_1(y_1,y)\,\mathrm{d}y_1.$$

By a direct computation, we have

$$G_2(x,y)=C(n,2)|x-y|^{4-n}-A(x,y), \tag{33}$$

where $C(n,2)=\frac{\Gamma(n-4)}{2^4\pi^{n/2}\Gamma(2)}$, $A: B_1\times B_1\to\mathbb{R}$ is a smooth map and $H_i(x,y)\geqslant 0$ for $i=1,2$.

In the following lemma, we find an integral representation for solutions to ($\mathcal{S}_{p,1}$).

Lemma 41. *Let $\mathcal{U}\in C^4(\bar{B}_1^*,\mathbb{R}^p)\cap L^1(B_1,\mathbb{R}^p)$ be a strongly positive singular solution to ($\mathcal{S}_{p,1}$). Then, $|x|^{-q}u_i^{2^{**}-1}\in L^1(B_1)$ for any $q<n-4\frac{2^{**}-1}{2^{**}-2}$ and $i\in I$. Moreover,*

$$u_i(x)=\int_{B_1} G_2(x,y)\Delta^2 u_i(y)\mathrm{d}y+\int_{\partial B_1} H_1(x,y)u_i(y)\mathrm{d}\sigma_y-\int_{\partial B_1} H_2(x,y)\Delta u_i(y)\mathrm{d}\sigma_y.$$

Proof. It is a straightforward adaptation of [33, Lemma 2.1] to the context of systems. □

In the following proposition, we use the Green identity to convert ($\mathcal{S}_{p,1}$) into an integral system, which is the main result of this section. Here the superharmonicity condition is assumed.

Proposition 42. *Let $\mathcal{U}$ be a strongly positive superharmonic solution to ($\mathcal{S}_{p,1}$). Then, there exists $r_0 > 0$ such that*

$$u_i(x) = \int_{B_{r_0}} |x-y|^{4-n} f_i(\mathcal{U})\mathrm{d}y + \psi_i(x) \quad \text{in} \quad B_1^*, \tag{$\mathcal{I}_{p,1}$}$$

where $\psi_i > 0$ satisfies $\Delta^2\psi_i = 0$ in B_{r_0}. Moreover, one can find a constant $c > 0$, depending on $\widetilde{r}$ such that

$$\|\nabla \ln \psi_i\|_{C^0(B_{\widetilde{r}})} \leqslant c \quad \text{for all} \quad i \in I \quad \text{and} \quad 0 < \widetilde{r} < r_0. \tag{34}$$

Proof. Using that $-\Delta u_i > 0$ in B_1^* and $u_i > 0$ in $\bar{B}_1$, it follows from the maximum principle that $c_1 := \inf_{B_1} u_i = \min_{\partial B_1} u_i > 0$. In addition, by Lemma 41, we get that $f_i(\mathcal{U}) \in L^1(B_1)$, which implies that there exists $r_0 < 1/4$ satisfying the following inequality $\int_{B_{r_0}} |A(x,y)| f_i(\mathcal{U})\mathrm{d}y \leqslant c_1/2$ for $x \in B_{r_0}$, where $A(x,y)$ is given by (33). Hence, for $x \in B_{r_0}$, we get

$$\begin{aligned}
\psi_i(x) &= -\int_{B_{r_0}} A(x,y) f_i(\mathcal{U})\mathrm{d}y + \int_{B_1\setminus B_{r_0}} G_2(x,y) f_i(\mathcal{U})\mathrm{d}y \\
&\quad + \int_{\partial B_1} H_1(x,y)u_i(y)\mathrm{d}\sigma_y - \int_{\partial B_1} H_2(x,y)\Delta u_i(y)\mathrm{d}\sigma_y \\
&\geqslant -\frac{c_1}{2} + \int_{\partial B_1} H_1(x,y)u_i(y)\mathrm{d}\sigma_y \\
&\geqslant -\frac{c_1}{2} + \inf_{B_1} u_i = \frac{c_1}{2}.
\end{aligned}$$

By hypothesis ψ_i is biharmonic, then a removable singularity theorem, and elliptic regularity shows that $\psi_i \in C^\infty(B_{r_0})$ for all $i \in I$, which provides that $|\nabla\psi_i| \leqslant c_2$ in $B_{\widetilde{r}}$ for all $\widetilde{r} < r_0$ and $i \in I$, where $c_2 > 0$ depends only on $n, r_0 - \widetilde{r}$, and in the L^1 norm of $f_i(\mathcal{U})$. Consequently, $\|\nabla \ln \psi_i\|_{C^0(B_{\widetilde{r}})} \leqslant 2\frac{c_2}{c_1}$ for all $i \in I$, which finishes the proof. □

4.2. Upper bound estimate

The objective is to prove the upper bound estimate in Proposition 4.

First, we use the integral form of the moving spheres technique.

Lemma 43. *Let $\mathcal{U}$ be a strongly positive solution to ($\mathcal{I}_{p,1}$). For any $x \in B_1$, $z \in B_2 \setminus (\{0\} \cup B_\mu(x))$ and $\mu < 1$, it follows that $u_i(z) - (u_i)_{x,\mu}(z) > 0$ for all $i \in I$.*

Proof. If $\mathcal{U}$ is a strongly positive solution to ($\mathcal{I}_{p,1}$), then, replacing $u_i(x)$ by $r^\gamma u_i(rx)$ for $r = 1/2$ and $i \in I$, we may consider the equation defined in B_2^* for convenience. Namely, we have

$$u_i(x) = \int_{B_2} |x-y|^{4-n} f_i(\mathcal{U}(y))\mathrm{d}y + \psi_i(x) \quad \text{in} \quad B_2^* \tag{$\mathcal{I}_{p,2}$}$$

such that $u_i \in C\left(B_2^*\right) \cap L^{2^{**}-1}(B_2)$ and $|\nabla \ln \psi_i| \leqslant c$ in $B_{3/2}$. Extending u_i to be identically zero outside B_2, we find

$$u_i(x) = \int_{\mathbb{R}^n} |x-y|^{4-n} f_i(\mathcal{U}(y))\mathrm{d}y + \psi_i(x) \quad \text{in} \quad B_2^*.$$

Using the identities in [22, page 162], one has

$$\left(\frac{\mu}{|z-x|}\right)^{n-4} \int_{|y-x|\geqslant\mu} \left|\mathcal{I}_{x,\mu}(z)-y\right|^{n-4} f_i(\mathcal{U}(y))\mathrm{d}y = \int_{|y-x|\leqslant\mu} |z-y|^{n-4} f_i(\mathcal{U}(z))\mathrm{d}y \tag{35}$$

and

$$\left(\frac{\mu}{|z-x|}\right)^{n-4} \int_{|y-x|\leqslant\mu} \left|\mathcal{I}_{x,\mu}(z)-y\right|^{n-4} f_i(\mathcal{U}(y))\mathrm{d}y = \int_{|y-x|\geqslant\mu} |z-y|^{n-4} f_i(\mathcal{U}(y))\mathrm{d}y, \tag{36}$$

which implies

$$(u_i)_{x,\mu}(z) = \int_{\mathbb{R}^n} |z-y|^{n-4} f_i(\mathcal{U}(y))\mathrm{d}y + (\psi_i)_{x,\mu}(z) \quad \text{for} \quad z \in \mathcal{I}_{x,\mu}(B_2).$$

Consequently, for any $x \in B_1$ and $0 < \mu < 1$, it follows

$$u_i(z) - (u_i)_{x,\mu}(z) = \int_{|y-x|\geqslant\mu} E(x,y,\mu,z)\left[f_i(\mathcal{U}) - f_i(\mathcal{U}_{x,\mu})\right]\mathrm{d}y + \left[(\psi_i)_{x,\mu}(z) - \psi_i(z)\right],$$

for $z \in B_2^* \cup B_\mu(x)$, where

$$E(x,y,z,\mu) = |z-y|^{4-n} - \left(\frac{|z-x|}{\mu}\right)^{4-n} \left|\mathcal{I}_{x,\mu}(z)-y\right|^{4-n}$$

is used to estimate the difference between a p-map $\mathcal{U}$ and its Kelvin transform $\mathcal{U}_{x,\mu}$. Finally, it is straightforward to check that $E(x,y,z,\mu) > 0$ for all $|z-x| > \mu > 0$, which concludes the proof of the lemma. □

Second, we use a contradiction argument based on the blow-up classification.

Proposition 44. *Let $\mathcal{U} \in C\left(B_2^*, \mathbb{R}^p\right) \cap L^{2^{**}-1}(B_2, \mathbb{R}^p)$ be a strongly positive solution to ($\mathcal{I}_{p,2}$). Suppose that $\psi_i \in C^1(B_2)$ is a positive function satisfying (34) for any $i \in I$. Then,* $\limsup_{|x|\to 0} |x|^\gamma |\mathcal{U}(x)| < \infty$.

Proof. We assume by contradiction that there exist $i \in I$ and $\{x_k\}_{k\in\mathbb{N}} \subset B_2$ such that $\lim_{k\to\infty} |x_k| = 0$ and $|x_k|^\gamma u_i(x_k) \to \infty$ as $k \to \infty$. In this fashion, for $|x - x_k| \leqslant 1/2|x_k|$, we define $\widetilde{u}_{ki}(x) := (|x_k|/2 - |x - x_k|)^\gamma u_i(x)$. Hence, using that u_i is positive and continuous in $\bar{B}_{|x_k|/2}(x_k)$, there exists a maximum point $\bar{x}_k \in B_{|x_k|/2}(x_k)$ of $\widetilde{u}_{ki}$, that is, $\widetilde{u}_{ki}(\bar{x}_k) = \max_{|x-x_k|\leqslant|x_k|/2} \widetilde{u}_{ki}(x) > 0$. Taking $2\mu_k := |x_k|/2 - |\bar{x}_k - x_k| > 0$, we get

$$0 < 2\mu_k \leqslant \frac{|x_k|}{2} \quad \text{and} \quad \frac{|x_k|}{2} - |x - x_k| \geqslant \mu_k \quad \text{for } |x - \bar{x}_k| \leqslant \mu_k. \tag{37}$$

Moreover, it follows that $2^\gamma u_i(\bar{x}_k) \geqslant u_i(x)$ for $|x - \bar{x}_k| \leqslant \mu_k$ and

$$(2\mu_k)^\gamma u_i(\bar{x}_k) = \widetilde{u}_{ki}(\bar{x}_k) \geqslant \widetilde{u}_{ki}(x_k) = 2^{-\gamma}|x_k|^\gamma u_i(x_k) \to \infty \quad \text{as} \quad k \to \infty. \tag{38}$$

We consider $w_{ki}(y) = u_i(\bar{x}_k)^{-1} u_i(\bar{x}_k + y u_i(\bar{x}_k)^{-1/\gamma})$ and $h_{ki}(y) = u_i(\bar{x}_k)^{-1} \psi_i(\bar{x}_k + y u_i(\bar{x}_k)^{-1/\gamma})$ in Ω_k, where $\Omega_k = \{y \in \mathbb{R}^n : \bar{x}_k + y u_i(\bar{x}_k)^{-1/\gamma} \in B_2^*\}$. Now, extending w_{ki} to be zero outside of Ω_k and using Proposition 42, we get

$$w_{ki}(y) = \int_{\mathbb{R}^n} f_i(\mathcal{W}_k)|y - x|^{4-n} \mathrm{d}x + h_{ki}(y) \quad \text{for} \quad y \in \Omega_k \tag{39}$$

and $w_{ki}(0) = 1$ for $k \in \mathbb{N}$, where $\mathcal{W}_k = \mathbf{e_1} w_{ki}$. Moreover, from (37) and (38), it holds $\|h_{ki}\|_{C^1(\Omega_k)} \to 0$ and $w_{ki}(y) \leqslant 2^\gamma$ in $B_{R_{ki}}$, where $R_{ki} := \mu_{ki} u_i(\bar{x}_k)^{1/\gamma} \to \infty$ as $k \to \infty$. Using the regularity results in [22], one can find $w_{0i} > 0$ such that $w_{ki} \to w_{0i}$ as $k \to \infty$ in $C^{4,\zeta}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\zeta \in (0, 1)$, where $w_{0i} > 0$ satisfies

$$w_{0i}(y) = c(n) \int_{\mathbb{R}^n} |y - x|^{4-n} w_{0i}^{2^{**}-1} \mathrm{d}x \quad \text{in} \quad \mathbb{R}^n,$$

or, equivalently $\Delta^2 w_{0i} = f_i(w_{0i})$ in $\mathbb{R}^n$. Furthermore, by construction, we conclude $w_{0i}(0) = 1$, which by Theorem B (i), implies that there exist $\mu > 0$ and $y_0 \in \mathbb{R}^n$ such that

$$w_{0i}(y) = \left(\frac{2\mu}{1 + \mu^2|y - y_0|^2}\right)^\gamma. \tag{40}$$

In the next claim, we use the last classification formula to apply the moving spheres technique.

Claim 1: For any $\mu > 0$, it holds that $(w_{0i})_{x,\mu}(y) \leqslant w_{0i}(y)$ for $|y - x| \geqslant \mu$.

Indeed, for a fixed $\mu_0 > 0$, we have $0 < \mu_0 < R_k/10$ when $k \gg 1$. We also consider $\widetilde{\Omega}_k := \{y \in \mathbb{R}^n : \bar{x}_k + y u_i(\bar{x}_k)^{-1/\gamma} \in B_1^*\} \subset\subset \Omega_k$.

Now let us divide the proof of the claim into three steps as follows:

Step 1: For $k \gg 1$, it holds that $(w_{ki})_{x,\mu_0}(y) \leqslant w_{ki}(y)$ for $y \in \widetilde{\Omega}_k$ such that $|y| \geqslant \mu_0$.

In fact, by Lemma 43, there exists $\bar{r} > 0$ such that for all $0 < \mu \leqslant \bar{r}$ and $\bar{x} \in B_{1/100}$,

$$\left(\frac{\mu}{|y|}\right)^{n-4} \psi_{ki}\left(\mathcal{I}_{0,\mu}(y) + \bar{x}\right) \leqslant \psi_{ki}(y + \bar{x}) \quad \text{for} \quad |y| \geqslant \mu \quad \text{and} \quad y \in B_{149/100}. \tag{41}$$

Let $k \gg 1$ be sufficiently large such that $\mu_0 u_i(\bar{x}_k)^{(4-n)^{-1}} < \bar{r}$. Hence, for any $0 < \mu \leqslant \mu_0$, it holds

$$(\psi_{ki})_{x,\mu}(y) \leqslant \psi_{ki}(y) \quad \text{in} \quad \widetilde{\Omega}_k \setminus B_\mu, \tag{42}$$

which by passing to the limit as $k \to \infty$ concludes the proof of Step 1.

Step 2: For $k \gg 1$, there exists $\mu_1 > 0$, independent of k, such that $(w_{ki})_{x,\mu}(y) \leqslant w_{ki}(y)$ in $\widetilde{\Omega}_k \setminus B_\mu$ for $0 < \mu < \mu_1$.

As matter of fact, since $w_{ki} \to w_{0i}$ as $k \to \infty$ in $C^{4,\zeta}$-topology and w_{0i} is given by (40) we get that there exists $c_1 > 0$ satisfying $w_{ki} \geqslant c_1 > 0$ on B_1 for $k \gg 1$. On the other hand, by (39) and standard regularity results, it follows that $\left|D^{(j)} w_{ki}\right| \leqslant c_1 < \infty$ on B_1 for $j = 1, 2, 3, 4$. Using Lemma 43, there exists $r_0 > 0$, not depending on $k \gg 1$, such that for all $0 < \mu \leqslant r_0$, it holds

$$(w_{ki})_{x,\mu}(y) < w_{ki}(y) \quad \text{for} \quad 0 < \mu < |y| \leqslant r_0. \tag{43}$$

Again, since $w_{ki} \geqslant c_1 > 0$ on B_1 for $k \gg 1$, there exists $c_2 > 0$ satisfying

$$w_{ki}(x) \geqslant c_1^{2^{**}-1} \int_{B_1} |x - y|^{4-n} \mathrm{d}y \geqslant \frac{1}{c_2}(1 + |x|)^{4-n} \quad \text{in} \quad \Omega_k.$$

Therefore, one can find $0 < \mu_1 \leqslant r_0 \ll 1$ sufficiently small such that for all $0 < \mu < \mu_1$, we have

$$(w_{ki})_\mu(y) \leqslant \left(\frac{\mu_1}{|y|}\right)^{n-4} \max_{B_{r_0(x)}} w_{ki} \leqslant c_1 \left(\frac{\mu_1}{|y|}\right)^{n-4} \leqslant w_{ki}(y) \quad \text{for} \quad y \in \Omega_k \quad \text{and} \quad |y| \geqslant r_0,$$

which combined with (43) proves Step 2.

Step 3: For $k \gg 1$, it holds that $\mu^* = \mu_0$, where

$$\mu^* := \sup\left\{0 < \mu \leqslant \mu_0 : (w_{ki})_{x,\mu}(y) \leqslant w_{ki}(y),\ y \in \widetilde{\Omega}_k \text{ with } |y - x_0| \geqslant \mu \text{ and } 0 < \mu < \mu_0\right\}.$$

Indeed, using (35), (36) and (42), it follows

$$\begin{aligned} & w_{ki}(y) - (w_{ki})_{0,\mu}(y) \\ & = \int_{\mathbb{R}^n \setminus B_\mu} E(0, y, z, \mu)\left[w_{ki}(z)^{2^{**}-1} - (w_{ki})_{0,\mu}(z)^{2^{**}-1}\right] \mathrm{d}z + \left[\psi_{ki}(y) - (\psi_{ki})_{0,\mu}(y)\right] \\ & \geqslant \int_{\widetilde{\Omega}_k \setminus B_\mu} E(0, y, z, \mu)\left[w_{ki}(z)^{2^{**}-1} - (w_{ki})_{0,\mu}(z)^{2^{**}-1}\right] \mathrm{d}z + J(\mu, w_{ki}, y), \end{aligned} \tag{44}$$

for any $\mu^* \leqslant \mu \leqslant \mu^* + 1/2$ and $y \in \widetilde{\Omega}_k$ with $|y| > \mu$, where

$$
\begin{aligned}
J(\mu, w_{ki}, y) &= \int_{\mathbb{R}^n \setminus \widetilde{\Omega}_k} E(0, y, z, \mu) \left[w_{ki}(z)^{2^{**}-1} - (w_{ki})_{0,\mu}(z)^{2^{**}-1} \right] \mathrm{d}z \\
&= \int_{\Omega_k \setminus \widetilde{\Omega}_k} E(0, y, z, \mu) \left[w_{ki}(z)^{2^{**}-1} - (w_{ki})_{0,\mu}(z)^{2^{**}-1} \right] \mathrm{d}z \\
&- \int_{\mathbb{R}^n \setminus \Omega_k} E(0, y, z, \mu) (w_{ki})_{0,\mu}(z)^{2^{**}-1} \mathrm{d}z.
\end{aligned}
$$

For $z \in \mathbb{R}^n \setminus \widetilde{\Omega}_k$ and $\mu^* \leqslant \mu \leqslant \mu^* + 1$, we obtain that $|z| \geqslant 1/2u(\bar{x}_k)^{-1/\gamma}$ and thus

$$
(w_{ki})_{0,\mu}(z) \leqslant \left(\frac{\mu}{|z|} \right)^{n-4} \max_{B_{\mu^*+1}} w_{ki} \leqslant c_2 u_i(\bar{x}_k)^{-2}.
$$

In addition, since $u_i \geqslant c_1 > 0$ in $B_2 \setminus B_{1/2}$, by the definition of w_{ki}, we have $w_{ki}(y) \geqslant \frac{c_1}{u_i(\bar{x}_k)}$ in $\Omega_k \setminus \widetilde{\Omega}_k$, which in turns yields

$$
\begin{aligned}
J(\mu, w_{ki}, y) &\geqslant \frac{1}{2} \left(\frac{c_1}{u_i(\bar{x}_k)} \right)^{2^{**}-1} \int_{\Omega_k \setminus \widetilde{\Omega}_k} E(0, y, z, \mu) \mathrm{d}z - c_2 \int_{\mathbb{R}^n \setminus \Omega_k} E(0, y, z, \mu) \left(\frac{\mu}{|z|} \right)^{n+4} \mathrm{d}z \\
&\geqslant \begin{cases} \frac{1}{c_2}(|y| - \mu) u_i(\bar{x}_k)^{-1}, & \text{if} \quad \mu \leqslant |y| \leqslant \mu^* + 1 \\ \frac{1}{c_2} u_i(\bar{x}_k)^{-1}, & \text{if} \quad |y| > \mu^* + 1 \quad \text{and} \quad y \in \widetilde{\Omega}_k. \end{cases}
\end{aligned}
\tag{45}
$$

Indeed, $E(0, y, z, \mu) = 0$ for $|y| = \mu$, and $y \nabla_y E(0, y, z, \mu)\big|_{|y|=\mu} = (n-4)|y-z|^{4-n-2}(|z|^2 - |y|^2) > 0$, for $|z| \geqslant \mu^* + 2$, which together with the positivity and smoothness of E implies the existence of $0 < \delta_1 \leqslant \delta_2 < \infty$ satisfying

$$
\delta_1 |y-z|^{4-n}(|y| - \mu) \leqslant E(0, y, z, \mu) \leqslant \delta_2 |y-z|^{4-n}(|y| - \mu), \tag{46}
$$

for $\mu^* \leqslant \mu \leqslant |y| \leqslant \mu^* + 1$, $\mu^* + 2 \leqslant |z| \leqslant R < \infty$. Furthermore, if $R \gg 1$ is large, it follows that $0 < c_1 \leqslant y \cdot \nabla_y \left(|y-z|^{n-4} E(0, y, z, \mu) \right) \leqslant c_1 < \infty$ for all $|z| \geqslant \mu$, $\mu^* \leqslant \mu \leqslant |y| \leqslant \mu^* + 1$. Thus, (46) holds for $\mu^* \leqslant \mu \leqslant |y| \leqslant \mu^* + 1$ and $|z| \geqslant R$. Besides, by the definition of $E(0, y, z, \mu)$, there exists $0 < \delta_3 \leqslant 1$ such that

$$
\delta_3 |y-z|^{4-n} \leqslant E(0, y, z, \mu) \leqslant |y-z|^{4-n}, \tag{47}
$$

for $|y| \geqslant \mu^* + 1$ and $|z| \geqslant \mu^* + 2$. Therefore, for $k \gg 1$ and $\mu \leqslant |y| \leqslant \mu^* + 1$, we find $c_3, c_4 > 0$ satisfying

$$
J(\mu, w_{ki}, y) \geqslant \frac{1}{2} \left(\frac{c_1}{u_i(\bar{x}_k)} \right)^{2^{**}-1} \int_{\Omega_k \setminus \widetilde{\Omega}_k} \delta_1 |y-z|^{4-n}(|y| - \mu) \mathrm{d}z
$$

$$
\begin{aligned}
&- c_2 \int\limits_{\mathbb{R}^n \setminus \Omega_k} \delta_2 |y-z|^{4-n} (|y|-\mu) \left(\frac{\mu}{|z|}\right)^{n+4} \mathrm{d}z \\
&\geqslant \frac{1}{c_3} (|y|-\mu) u_i\left(\bar{x}_k\right)^{-1} - \frac{1}{c_4} (|y|-\mu) u_i\left(\bar{x}_k\right)^{-2^{**}} \\
&\geqslant \frac{1}{2c_3} (|y|-\mu) u_i\left(\bar{x}_k\right)^{-1}.
\end{aligned}
$$

Similarly, for $|y| \geqslant \mu^* + 1$ and $y \in \widetilde{\Omega}_k$, since $u_i\left(\bar{x}_k\right) \to \infty$ as $k \to \infty$, there exist $c_5, c_6 > 0$ such that $J\left(\mu, w_{ki}, y\right) \geqslant \frac{1}{c_5} u_i\left(\bar{x}_k\right)^{-1} - \frac{1}{c_6} u_i\left(\bar{x}_k\right)^{-2^{**}} \geqslant \frac{1}{2c_5} u_i\left(\bar{x}_k\right)^{-1}$, which verifies (45).

Next, by (44) and (45), there exists $\varepsilon_1 \in (0, 1/2)$, depending on k, such that for $|y| \geqslant \mu^* + 1$, it holds $w_{ki}(y) - \left(w_{ki}\right)_{0,\mu^*}(y) \geqslant \varepsilon_1 |y|^{4-n}$ in $\widetilde{\Omega}_k$. Using the last inequality and the formula for $\left(w_{ki}\right)_{0,\mu}$, there exists $0 < \varepsilon_2 < \varepsilon_1 \ll 1$ such that for $|y| \geqslant \mu^* + 1$, $\mu^* \leqslant \mu \leqslant \mu^* + \varepsilon_2$, we get

$$
w_{ki}(y) - \left(w_{ki}\right)_{0,\mu}(y) \geqslant \varepsilon_1 |y|^{4-n} + \left[\left(w_{ki}\right)_{0,\mu^*}(y) - \left(w_{ki}\right)_{\mu}(y)\right] \geqslant \frac{\varepsilon_1}{2} |y|^{4-n}. \tag{48}
$$

For $\varepsilon \in (0, \varepsilon_3]$ that will be chosen later, by (44) and (45) combined with the inequality, we obtain $c_7 > 0$ such that $\left|\left(w_{ki}\right)(z)^{2^{**}-1} - \left(w_{ki}\right)_{0,\mu}(z)^{2^{**}-1}\right| \leqslant c_7(|z|-\mu)$, which implies

$$
\begin{aligned}
w_{ki}(y) - \left(w_{ki}\right)_{0,\mu}(y) \geqslant & \int\limits_{\mu \leqslant |z| \leqslant \mu^*+1} E(0, y, z, \mu) \left(w_{ki}(z)^{2^{**}-1} - \left(w_{ki}\right)_{0,\mu}(z)^{2^{**}-1}\right) \mathrm{d}z \\
& + \int\limits_{\mu^*+2 \leqslant |z| \leqslant \mu^*+3} E(0, y, z, \mu) \left(w_{ki}(z)^{2^{**}-1} - \left(w_{ki}\right)_{0,\mu}(z)^{2^{**}-1}\right) \mathrm{d}z \\
& \geqslant -c_7 \int\limits_{\mu \leqslant |z| \leqslant \mu+\varepsilon} E(0, y, z, \mu)(|z|-\mu) \mathrm{d}z \\
& + \int\limits_{\mu+\varepsilon \leqslant |z| \leqslant \mu^*+1} E(0, y, z, \mu) \left(w_{ki}(z)^{2^{**}-1} - \left(w_{ki}\right)_{0,\mu}(z)^{2^{**}-1}\right) \mathrm{d}z \\
& + \int\limits_{\mu^*+2 \leqslant |z| \leqslant \mu^*+3} E(0, y, z, \mu) \left(w_{ki}(z)^{2^{**}-1} - \left(w_{ki}\right)_{0,\mu}(z)^{2^{**}-1}\right) \mathrm{d}z,
\end{aligned}
$$

for $\mu^* \leqslant \mu \leqslant \mu^* + \varepsilon$ and $\mu \leqslant |y| \leqslant \mu^* + 1$. From (48), there exists $\delta_5 > 0$ such that for $\mu^* + 2 \leqslant |z| \leqslant \mu^* + 3$, it follows $w_{ki}(z)^{2^{**}-1} - \left(w_{ki}\right)_{0,\mu}(z)^{2^{**}-1} \geqslant \delta_5$. Moreover, since there exists some constant $c_8 > 0$, not depending on ε, such that $\|w_{ki}\|_{C^1(B_2)} \leqslant c_8$ (independent of k) for $\mu^* \leqslant \mu \leqslant \mu^* + \varepsilon$, we get $|\left(w_{ki}\right)_{0,\mu^*}(z)^{2^{**}-1} - \left(w_{ki}\right)_{0,\mu}(z)^{2^{**}-1}| \leqslant c_8(\mu - \mu^*) \leqslant c_8 \varepsilon$, for $\mu \leqslant |z| \leqslant \mu^* + 1$. Also, for $\mu \leqslant |y| \leqslant \mu^* + 1$, we find

$$
\int\limits_{\mu+\varepsilon \leqslant |z| \leqslant \mu^*+1} E(0, y, z, \mu) \mathrm{d}z \leqslant \left| \int\limits_{\mu+\varepsilon \leqslant |z| \leqslant \mu^*+1} \left(|y-z|^{4-n} - |\mathcal{I}_{0,\mu}(y) - z|^{4-n}\right) \mathrm{d}z \right|
$$

$$+\int_{\mu+\varepsilon\leqslant|z|\leqslant\mu^*+1}\left|\left(\frac{\mu}{|y|}\right)^{n-4}-1\right|\left|\mathcal{I}_{0,\mu}(y)-z\right|^{n-4}\mathrm{d}z$$
$$\leqslant c_8\left(\varepsilon^3+|\ln\varepsilon|+1\right)(|y|-\mu)$$

and

$$\int_{\mu\leqslant|z|\leqslant\mu+\varepsilon}E(0,y,z,\mu)(|z|-\mu)\mathrm{d}z\leqslant\left|\int_{\mu\leqslant|z|\leqslant\mu+\varepsilon}\left(\frac{|z|-\mu}{|y-z|^{n-4}}-\frac{|z|-\mu}{\left|\mathcal{I}_{0,\mu}(y)-z\right|^{n-4}}\right)\mathrm{d}z\right|$$
$$+\varepsilon\int_{\mu\leqslant|z|\leqslant\mu+\varepsilon}\left|\left(\frac{\mu}{|y|}\right)^{n-4}-1\right|\left|\mathcal{I}_{0,\mu}(y)-z\right|^{4-n}\mathrm{d}z$$
$$\leqslant I+c_8\varepsilon(|y|-\mu),$$

where, for $|y|\geqslant\mu+10\varepsilon$, we arrive at

$$I=\left|\int_{\mu\leqslant|z|\leqslant\mu+\varepsilon}\left(\frac{|z|-\mu}{|y-z|^{n-4}}-\frac{|z|-\mu}{\left|\mathcal{I}_{0,\mu}(y)-z\right|^{n-4}}\right)\mathrm{d}z\right|\leqslant c_8\varepsilon\left(\varepsilon^3+|\ln\varepsilon|+1\right)(|y|-\mu).$$

On the other hand, for $\mu\leqslant|y|\leqslant\mu+10\varepsilon$, it follows

$$I\leqslant\left|\int_{\mu\leqslant|z|\leqslant\mu+10(|y|-\mu)}\left(\frac{|z|-\mu}{|y-z|^{n-4}}-\frac{|z|-\mu}{\left|\mathcal{I}_{0,\mu}(y)-z\right|^{n-4}}\right)\mathrm{d}z\right|$$
$$+\left|\int_{\mu+10(|y|-\mu)\leqslant|z|\leqslant\mu+\varepsilon}\left(\frac{|z|-\mu}{|y-z|^{n-4}}-\frac{|z|-\mu}{\left|\mathcal{I}_{0,\mu}(y)-z\right|^{n-4}}\right)\mathrm{d}z\right|$$
$$\leqslant c_8(|y|-\mu)\int_{\mu\leqslant|z|\leqslant\mu+10(|y|-\mu)}\left(\frac{1}{|y-z|^{n-4}}+\frac{1}{\left|\mathcal{I}_{0,\mu}(y)-z\right|^{n-4}}\right)\mathrm{d}z$$
$$+c_8\left|y-\mathcal{I}_{0,\mu}(y)\right|\int_{\mu+10(|y|-\mu)\leqslant|z|\leqslant\mu+\varepsilon}\frac{|z|-\mu}{|y-z|^{n-3}}\mathrm{d}z$$
$$\leqslant c_8(|y|-\mu)\sup_{\widetilde{z}\in\mathbb{R}^n}\int_{\mu\leqslant|z|\leqslant\mu+100\varepsilon}|\widetilde{z}-z|^{n-4}\mathrm{d}z$$
$$\leqslant c_8(|y|-\mu)\varepsilon^{4/n}.$$

Finally, using (47), for $\mu<|y|\leqslant\mu^*+1$ and $0<\varepsilon\ll1$ sufficiently small, it holds

$$w_{ki}(y)-(w_{ki})_{0,\mu}(y)\geqslant -c_8\varepsilon^{\frac{4}{n}}(|y|-\mu)+\delta_1\delta_5(|y|-\mu)\int\limits_{\mu+2\leqslant|z|\leqslant\mu^*+3}|y-z|^{4-n}\mathrm{d}z$$
$$\geqslant\left(\delta_1\delta_5-c_8\varepsilon^{\frac{4}{n}}\right)(|y|-\mu)\geqslant 0,$$

which together with (48) contradicts the definition of $\mu^*>0$, if $\mu^*<\mu_0$. Hence, the proof of Step 3 is finished, so Claim 1. Besides, this is also a contradiction with (40), which concludes the proof of the proposition. □

Consequently, we obtain the upper bound estimate of Proposition 4.

Corollary 45. *Let $\mathcal{U}$ be a strongly positive singular solution to ($\mathcal{S}_{p,1}$). Then, there exists $C_2>0$ satisfying $|\mathcal{U}(x)|\leqslant C_2|x|^{-\gamma}$ for $0<|x|<1/2$.*

4.3. Asymptotic radial symmetry

Here, we prove the convergence of singular solutions to ($\mathcal{S}_{p,1}$) to their spherical average $\overline{\mathcal{U}}(x)=\fint_{\partial B_1}\mathcal{U}(r\theta)\mathrm{d}\theta$. In particular, this approximation implies that they are asymptotic radially symmetric with respect to the origin.

Proposition 46. *Let $\mathcal{U}$ be a strongly positive singular solution to ($\mathcal{S}_{p,1}$). Then, $|\mathcal{U}|$ is radially symmetric with respect to the origin and $|\mathcal{U}(x)|=(1+\mathcal{O}(|x|))|\overline{\mathcal{U}}(x)|$ as $x\to 0$, where $|\overline{\mathcal{U}}|$ is its spherical average.*

Proof. First, we prove the following claim:
Claim 1: There exists $0<\varepsilon<\min\{1/10,\bar{r}\}$ such that $|\mathcal{U}_{x,\mu}(y)|\leqslant|\mathcal{U}(y)|$ in $B_1(x)\setminus B_\mu(x)$ for $0<\mu<|x|<\varepsilon$, where $\bar{r}$ is such that (41) holds for all $0<\mu\leqslant\bar{r}$.

We divide the proof of the claim into two steps as follows:
Step 1: The critical parameter

$$\mu^*(x):=\sup\left\{0<\mu\leqslant|x|:|\mathcal{U}_{x,\mu}(y)|\leqslant|\mathcal{U}(y)|\text{ for }y\in B_2\setminus\left(B_\mu(x)\cup\{0\}\right)\text{ and }0<\mu<\mu^*\right\}$$

is well-defined and positive.
Indeed, using Lemma 43, for every $x\in B^*_{1/10}$ one can find $0<r_x<|x|$ such that for all $0<\mu\leqslant r_x$, it follows $|\mathcal{U}_{x,\mu}(y)|\leqslant|\mathcal{U}(y)|$ for $0<\mu<|y-x|\leqslant r_x$. Moreover, as a consequence of ($\mathcal{I}_{p,2}$), we get

$$|\mathcal{U}(x)|\geqslant 4^{4-n}\int\limits_{B_2}|f_i(\mathcal{U})|(y)\mathrm{d}y:=c_1>0, \tag{49}$$

which implies that there exists $0<\mu_1\ll r_x$ such that, for every $0<\mu\leqslant\mu_1$, it holds

$$|\mathcal{U}_{x,\mu}(y)|\leqslant|\mathcal{U}(y)|\quad\text{for}\quad y\in B_2\setminus\left(B_{r_x}(x)\cup\{0\}\right). \tag{50}$$

Hence, as a combination of (49) and (50), it follows the proof of Step 1.

Step 2: There exists $\varepsilon > 0$ such that $\mu^* = |x|$ for all $|x| \leqslant \varepsilon$, $\mu^* \leqslant \mu < |x| \leqslant \bar{r}$, and $y \in B_1$, we get

$$u_i(y) - (u_i)_{x,\mu}(y) \geqslant \int_{B_1 \setminus B_\mu(x)} E(0, y, z, \mu)\left[f_i(\mathcal{U}(z)) - f_i(\mathcal{U}_{x,\mu}(z))\right]\mathrm{d}z + J(\mu, u, y),$$

for any $i \in I$, where

$$J(\mu, u_i, y) = \int_{B_2 \setminus B_1} E(x, y, z, \mu)\left[f_i(\mathcal{U}(z)) - f_i(\mathcal{U}_{x,\mu}(z))\right]\mathrm{d}z - \int_{\mathbb{R}^n \setminus B_2} E(x, y, z, \mu) f_i(\mathcal{U}_{x,\mu}(z))\mathrm{d}z.$$

In fact, for $y \in \mathbb{R}^n \setminus B_1$ and $\mu < |x| < \varepsilon < 1/10$, we have

$$\mathcal{I}_{x,\mu}(y) = \left| x + \frac{\mu^2(y-x)}{|y-x|^2} \right| \geqslant |x| - \frac{10}{9}\mu^2 \geqslant |x| - \frac{10}{9}|x|^2 \geqslant \frac{8}{9}|x|.$$

Using Proposition 43, there exists $c_2 > 0$ such that $|\mathcal{U}(\mathcal{I}_{x,\mu}(y))| \leqslant c_2|x|^{-\gamma}$, which, for all $y \in \mathbb{R}^n \setminus B_1$, yields

$$|\mathcal{U}_{x,\mu}(y)| = \left(\frac{\mu}{|y-x|}\right)^{n-4} |\mathcal{U}\left(\mathcal{I}_{x,\mu}(y)\right)| \leqslant c_2\mu^{n-4}|x|^{-\gamma} \leqslant c_2|x|^{\gamma} \leqslant c_2\varepsilon^{\gamma}.$$

By (49), we find $|\mathcal{U}_{x,\mu}(y)| < |\mathcal{U}(y)|$ for $y \in B_2 \setminus B_1$. In addition, combining (49) and (50) with the proof of (45), there exist $c_3 > 0$, independent of x, such that

$$\begin{aligned}
&J(\mu, |\mathcal{U}|, y) \\
&\geqslant \int_{B_2 \setminus B_1} E(x, y, z, \mu)\left(c_1^{2^{**}-1} - c_2^{2^{**}-1}\varepsilon^{n+4}\right)\mathrm{d}z \\
&- c_2 \int_{\mathbb{R}^n \setminus B_2} E(x, y, z, \mu)\left(|x-z|^{4-n}|x|^{\gamma}\right)^{2^{**}-1}\mathrm{d}z \\
&\geqslant \frac{1}{2}c_1^{2^{**}-1} \int_{B_2 \setminus B_1} E(x, y, z, \mu)\mathrm{d}z - \varepsilon^{\frac{n+4}{2}} c_2 \int_{\mathbb{R}^n \setminus B_2} E(x, y, z, \mu)|x-z|^{n+4}\mathrm{d}z \\
&\geqslant \frac{1}{2}c_1^{2^{**}-1} \int_{B_{19/10} \setminus B_{11/10}} E(0, y-x, z, \mu)\mathrm{d}z - \varepsilon^{\frac{n+4}{2}} c_2 \int_{\mathbb{R}^n \setminus B_{19/10}} E(0, y-x, z, \mu)|x-z|^{n+4}\mathrm{d}z \\
&\geqslant \frac{1}{c_3}(|y-x| - \mu),
\end{aligned}$$

for $y \in B_1 \setminus \left(B_\mu(x) \cup \{0\}\right)$ and $0 < \varepsilon \ll 1$. Eventually, if $\mu^* < |x|$, by Lemma 43, we get a contradiction with the definition of μ^*. Hence, Step 2 is proved, and so Claim 1.

Finally, for any $i \in I$, choose $0 < r_i < \varepsilon^2$ and $x_{1i}, x_{2i} \in \partial B_{r_i}$ satisfying $u_i(x_{1i}) = \max_{\partial B_{r_i}} u_i$ and $u_i(x_{2i}) = \min_{\partial B_{r_i}} u_i$. Now choosing

$$x_{3i} = x_{1i} + \frac{\varepsilon_i\,(x_{1i} - x_{2i})}{4\,|x_{1i} - x_{2i}|} \quad \text{and} \quad \mu_i = \sqrt{\frac{\varepsilon_i}{4}\left(|x_{1i} - x_{2i}| + \frac{\varepsilon_i}{4}\right)},$$

it follows from Claim 1 that $(u_i)_{x_{3i},\mu_i}(x_{2i}) \leqslant u_i(x_{2i})$. Furthermore, we have

$$\begin{aligned}(u_i)_{x_{3i},\mu_i}(x_{2i}) &= \left(\frac{\mu_i}{|x_{1i} - x_{2i}| + \varepsilon_i/4}\right)^{n-4} u_i(x_{1i}) = \left(\frac{1}{4\,|x_{1i} - x_{2i}|\,\varepsilon_i^{-1} + 1}\right)^{\gamma} u_i(x_{1i}) \\ &\geqslant \left(\frac{1}{8r\varepsilon_i^{-1} + 1}\right)^{\gamma} u_i(x_{1i}),\end{aligned}$$

which implies $\max_{\partial B_{r_i}} u_i \leqslant \left(8r_i\varepsilon^{-1} + 1\right)^{\gamma} \min_{\partial B_{r_i}} u_i$ and this proves the proposition. □

As a consequence of the upper estimate, we prove the following Harnack inequality for solutions to $(\mathcal{S}_{p,1})$, whose scalar version can be found in [6, Theorem 3.6].

Corollary 47. *Let $\mathcal{U}$ be a strongly positive solution to $(\mathcal{S}_{p,1})$. Then, there exists $c > 0$ such that*

$$\max_{|x|=r} |\mathcal{U}(x)| \leqslant c \min_{|x|=r} |\mathcal{U}(x)| \quad \text{for} \quad 0 < r < 1/4.$$

Moreover, $|D^{(j)}\mathcal{U}(x)| \leqslant c|x|^{-j}|\mathcal{U}(x)|$ for $j = 1, 2, 3, 4$ and $0 < r < 1/4$.

Proof. For any $i \in I$, let us define $\widetilde{u}_i(y) = r^{\gamma} u_i(ry)$. Thus,

$$\widetilde{u}_i(y) = \int_{B_2} |y - z|^{4-n} f_i(\widetilde{\mathcal{U}}(z))\mathrm{d}z + \widetilde{h}_i(y), \tag{51}$$

where $\widetilde{\psi}_i(y) = r^{\gamma}\psi_i(ry)$. By Corollary 45, there exists $C_2 > 0$, such that $\widetilde{u}_i \leqslant C_2$ in $B_2 \setminus B_{1/10}$. Taking $|x| = 1$, let us consider $(\varrho_i)_x(y) = \int_{B_{2/r}(x)\setminus B_{9/10}(x)} |y - z|^{4-n} f_i(\widetilde{\mathcal{U}}(z))\mathrm{d}z$. Hence, for any $y_1, y_2 \in B_{1/2}(x)$, there exists $c_1 > 0$ such that

$$(\varrho_i)_x(y_1) \leqslant c_1 \int_{B_{2/r}(x)\setminus B_{9/10}(x)} |y - z|^{4-n} f_i(\widetilde{\mathcal{U}}(z))\mathrm{d}z \leqslant c_1(\varrho_i)_x(y_2),$$

which implies that ϱ_i satisfies the Harnack inequality in $B_{1/2}(x)$. On the other hand, ψ_i also satisfies the Harnack inequality in $B_{1/2}(x)$ and

$$\widetilde{u}_i(y) = \int_{B_{9/10}(x)} |y - z|^{4-n} f_i(\widetilde{\mathcal{U}})\mathrm{d}z + (\varrho_i)_x(y) + \widetilde{h}_i(y) \quad \text{in} \quad B_{1/2}(x).$$

Now by [22, Theorem 2.3], there exists $c_2 > 0$ such that $\sup_{B_{1/2}(x)} \widetilde{u}_i \leqslant c_2 \inf_{B_{1/2}(x)} \widetilde{u}_i$, which by a covering argument provides $\sup_{1/2 \leqslant |y| \leqslant 3/2} \widetilde{u}_i \leqslant c_2 \inf_{1/2 \leqslant |y| \leqslant 3/2} \widetilde{u}_i$, and, by rescaling back to u, the proof of the first part follows.

Next, for any fixed x and $i \in I$, let $r = |x|$ and $\widetilde{u}_i(y) = r^{\gamma} u_i(ry)$. Thus, $\widetilde{\mathcal{U}}$ satisfies (51) and, by Corollary 45, it holds that $\widetilde{u}_i \leqslant C_2$ in $B_{3/2} \setminus B_{1/2}$ for all $i \in I$. Finally, using the local estimates from [22, Section 2.1] and the smoothness of ψ_i, one can find $c_3 > 0$ satisfying $|D^{(j)} \widetilde{u}_i(x)| \leqslant c_3$ for $|x| = 1$ and $j = 1, 2, 3, 4$, which, by scaling back to u_i, concludes the proof of this corollary. □

4.4. Lower bound estimate

Next, we use the Pohozaev invariant, the Harnack inequality, and a barrier argument to prove a removable classification result, which implies the lower bound estimate in Proposition 4.

Lemma 48. *Let $\mathcal{U} \in C\left(B_2^*, \mathbb{R}^p\right) \cap L^{2^{**}-1}(B_2, \mathbb{R}^p)$ be a strongly positive solution to $(\mathcal{I}_{p,2})$. Assume $\psi_i \in C^{\infty}(B_1)$ for all $i \in I$. If $\limsup_{|x| \to 0} |\mathcal{U}(x)| = \infty$, then $\liminf_{|x| \to 0} |\mathcal{U}(x)| = \infty$.*

Proof. Let us consider $\{x_k\}_{k \in \mathbb{N}} \subset B_1$ satisfying $r_k = |x_k| \to 0$ and $u_i(x_k) \to \infty$ as $k \to \infty$. By the Harnack inequality, we have $\inf_{\partial B_{r_k}} u_i \to \infty$ as $k \to \infty$. Thus, we obtain $-\Delta(u_i - \psi_i) \geqslant 0$ in B_2^* for all $i \in I$. Hence, since $\psi_i \in C^{\infty}(B_1)$ for all $i \in I$, it follows $\min_{B_{r_k} \setminus B_{r_{k+1}}} u_i(x) \to \infty$ as $k \to \infty$. Therefore, we conclude $\min_{B_{r_k} \setminus B_{r_{k+1}}} (u_i - \psi_i) = \min_{\partial B_{r_k} \cup \partial B_{r_{k+1}}} (u_i - \psi_i)$, which proves the lemma. □

Lemma 49. *Let $\mathcal{U} \in C\left(B_2^*, \mathbb{R}^p\right) \cap L^{2^{**}-1}(B_2, \mathbb{R}^p)$ be a strongly positive solution to $(\mathcal{I}_{p,2})$. If $\lim_{|x| \to 0} |x|^{\gamma} |\mathcal{U}(x)| = 0$, then $|\mathcal{U}|$ can be extended as a continuous function to the whole B_1.*

Proof. Let us consider the barrier functions from [22]. For any $i \in I$ and $\delta > 0$, we choose $0 < \rho \ll 1$ such that $u_i(x) \leqslant \delta |x|^{-\gamma}$ in B_ρ^*. Fixing $\varepsilon > 0$, $\kappa \in (0, \gamma)$ and $c_1 \gg 1$ to be chosen later, we define

$$\varsigma_i(x) = \begin{cases} c_1 |x|^{-\kappa} + \varepsilon |x|^{4-n-\kappa}, & \text{if } 0 < |x| < \rho \\ u_i(x), & \text{if } \rho < |x| < 2. \end{cases}$$

Notice that for every $0 < \kappa < n - 4$ and $0 < |x| < 2$, one can find $c_2 > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^n} |x - y|^{4-n} |y|^{-4-\kappa} \mathrm{d}y &= |x|^{4-n} \int_{\mathbb{R}^n} \left| |x|^{-1} x - |x|^{-1} y \right|^{4-n} |y|^{\kappa-4} \mathrm{d}y \\ &= |x|^{-\kappa+4} \int_{\mathbb{R}^n} \left| |x|^{-1} x - z \right|^{4-n} |z|^{\kappa-4} \mathrm{d}z \\ &\leqslant c_2 \left(\frac{1}{n-4-\kappa} + \frac{1}{\kappa} + 1 \right) |x|^{-\kappa}, \end{aligned}$$

which, for $0 < |x| < 2$ and $0 < \delta \ll 1$, yields

$$\int_{B_\rho} u_i^{2^{**}-2}(y)\varsigma_i(y)|x-y|^{4-n}\mathrm{d}y \leqslant \delta^{2^{**}-2}\int_{\mathbb{R}^n}\varsigma_i(y)|x-y|^{n-4}|y|^{-4}\mathrm{d}y \leqslant c_2\delta^{2^{**}-2}\varsigma_i(x) < \frac{1}{2}\varsigma_i(x).$$

Moreover, for $0<|x|<\rho$ and $\bar{x}=\rho x|x|^{-1}$, we get

$$\begin{aligned}\int_{B_2\setminus B_\rho} u_i^{2^{**}-2}(y)\varsigma_i(y)|x-y|^{4-n}\mathrm{d}y &= \int_{B_2\setminus B_\rho}\frac{|\bar{x}-y|^{n-4}}{|x-y|^{n-4}}\frac{u_i^{2^{**}-1}(y)}{|\bar{x}-y|^{n-4}}\mathrm{d}y \leqslant 2^{n-4}\int_{B_2\setminus B_\rho}\frac{u_i^{2^{**}-1}(y)}{|\bar{x}-y|^{n-4}}\mathrm{d}y\\ &\leqslant 2^{n-4}u_i(\bar{x})\\ &\leqslant 2^{n-4}\max_{\partial B_\rho} u_i.\end{aligned}$$

The last inequality implies that for $0<|x|<\tau$ and $c_1 \geqslant \max_{\partial B_\rho} u_i$, we have

$$\varsigma_i(x)+\int_{B_2}\frac{u_i^{2^{**}-2}(y)\varsigma_i(y)}{|x-y|^{4-n}}\mathrm{d}y \leqslant \varsigma_i(x)+2^{n-4}\max_{\partial B_\rho} u+\frac{1}{2}\varsigma_i(x) < \varsigma_i(x).$$

In the following claim, we show that ς_i can be taken indeed as a barrier for any u_i.

Claim 1: For any $i\in I$, it holds that $u_i(x)\leqslant \varsigma_i(x)$ in B_ρ^*.
Indeed, assume it does not hold. Since $u_i(x)\leqslant \delta|x|^{-\gamma}$ in B_ρ^*, by the definition of ς_i, there exists $\bar{\tau}\in(0,\rho)$, depending on ε, such that $\varsigma_i \geqslant u_i$ in $B_{\bar{\rho}}^*$ and $\varsigma_i > u_i$ near ∂B_ρ. Let us consider $\bar{\tau} := \inf\left\{\tau>1 : \tau\psi_i > u_i \text{ in } B_\rho^*\right\}$. Then, we have that $\bar{\tau}\in(1,\infty)$ and there exists $\bar{x}\in B_\rho\setminus \bar{B}_{\tilde{\tau}}$ such that $\bar{\tau}\varsigma_i(\bar{x})=u_i(\bar{x})$. Furthermore, for $0<|x|<\tau$, it follows

$$\bar{\tau}\varsigma_i(x) \geqslant \int_{B_2} u_i^{2^{**}-2}(y)\bar{\tau}\varsigma_i(y)|x-y|^{4-n}\mathrm{d}y+\bar{\tau}\varsigma_i(x) \geqslant \int_{B_2} u_i^{2^{**}-2}(y)\bar{\tau}\varsigma_i(y)|x-y|^{4-n}\mathrm{d}y+\varsigma_i(x),$$

which gives us $\bar{\tau}\varsigma_i(x)-u_i(x)\geqslant \int_{B_2} u_i^{2^{**}-2}(y)(\bar{\tau}\varsigma_i(y)-u_i(y))|x-y|^{4-n}\mathrm{d}y$. Finally, by evaluating the last inequality at $\bar{x}\in B_\rho\setminus\bar{B}_{\bar{\tau}}$, we get a contradiction, which proves the claim.

As a consequence of the claim, we get that $u_i(x)\leqslant\varsigma_i(x)\leqslant c_1|x|^{-\kappa}+\varepsilon|x|^{4-n-\kappa}$ in B_ρ^*, which, by passing to the limit as $\varepsilon\to 0$, implies that $u_i^{2^{**}-2}\in L^s(B_\rho^*)$ for some $s>n/4$ and any $i\in I$. Hence, using standard elliptic regularity, the proof of the lemma follows. □

Proposition 50. *Let $\mathcal{U}\in C\left(B_2^*\right)\cap L^{2^{**}-1}(B_2)$ be a strongly positive solution to ($\mathcal{S}_{p,1}$). Assume $\psi_i\in C^\infty(B_1)$ is a positive function in $\mathbb{R}^n$ satisfying $\Delta^2\psi_i=0$ in B_2 for all $i\in I$. If* $\liminf_{|x|\to 0}|x|^\gamma|\mathcal{U}(x)|=0$*, then* $\lim_{|x|\to 0}|x|^\gamma|\mathcal{U}(x)|=0$.

Proof. Assume by contradiction that there exists $c_1>0$ such that $\limsup_{x\to 0}|x|^\gamma|\mathcal{U}(x)|=c_1>0$; thus, from Lemma 48, we get $\liminf_{|x|\to 0}|\mathcal{U}(x)|=\infty$. Using the assumption and the Harnack inequality in Lemma 47, there exists $\{r_k\}_{k\in\mathbb{N}}$ such that $r_k\to 0$ and $r_k^\gamma\bar{u}_i(r_k)\to 0$ as $k\to\infty$. As well as, r_k is a local minimum point of $r^\gamma\bar{u}_i(r)$. Furthermore, let $\mathbf{e}_n=(0,\dots,1)\in\mathbb{R}^n$ and define $\varphi_{ki}(y)=\frac{u_i(r_k y)}{u_i(r_k\mathbf{e}_n)}$, which combined with ($\mathcal{I}_{p,2}$), gives us

$$\varphi_{ki}(y)=\int\limits_{B_{2/r_k}}\left(r_k^{\gamma}u_i\left(r_k\mathbf{e}_n\right)\right)^{2^{**}-2}\varphi_{ki}(\eta)^{2^{**}-1}|y-z|^{4-n}\mathrm{d}z+\psi_{ki}(y)\quad\text{in}\quad B^*_{2/r_k},$$

where $\psi_{ki}(y)=u_i(r_k\mathbf{e}_n)^{-1}\psi_i\left(r_k y\right)$.

Claim 1: For any $i\in I$, it follows that $\lim_{k\to\infty}\varphi_{ki}(y)=1/2(|y|^{4-n}+1)$ in $C^2_{\mathrm{loc}}\left(\mathbb{R}^n\setminus\{0\}\right)$.

In fact, since $u_i\left(r_k\mathbf{e}_n\right)\to\infty$, we have that $\psi_{ki}(y)\to 0$ as $k\to\infty$ in $C^n_{\mathrm{loc}}\left(\mathbb{R}^n\right)$. Next, using the Harnack inequality, we obtain that $r_k^{\gamma}u_i\left(r_k\mathbf{e}_n\right)\to 0$, and φ_{ki} is locally uniformly bounded in B^*_{2/r_k}. Hence, $\lim_{k\to\infty}\left(r_k^{\gamma}u_i\left(r_k\mathbf{e}_n\right)\right)^{2^{**}-2}\varphi_{ki}(y)^{2^{**}-1}=0$ in $C^n_{\mathrm{loc}}\left(\mathbb{R}^n\setminus\{0\}\right)$. Thus, for any $\tau>1$, $0<|y|<\tau$ and $0<\varepsilon<|y|/100$, up to subsequences, it follows

$$\begin{aligned}\lim_{k\to\infty}\int\limits_{B_\tau}\frac{\left(r_i^{\gamma}u\left(r_k\mathbf{e}_n\right)\right)^{2^{**}-2}\varphi_{ki}(z)^{2^{**}-1}}{|y-z|^{n-4}}\mathrm{d}z&=\lim_{k\to\infty}\int\limits_{B_\varepsilon}\frac{\left(r_k^{\gamma}u\left(r_k\mathbf{e}_n\right)\right)^{2^{**}-2}\varphi_{ki}(z)^{2^{**}-1}}{|y-z|^{n-4}}\mathrm{d}z\\&=\frac{(1+\mathcal{O}(\varepsilon))}{|y|^{n-4}}\lim_{k\to\infty}\int\limits_{B_\varepsilon}\left(r_k^{\gamma}u\left(r_k\mathbf{e}_n\right)\right)^{2^{**}-2}\varphi_{ki}(z)^{2^{**}-1}\mathrm{d}z.\end{aligned}$$

By sending $\varepsilon\to 0$, we have that $\lim_{k\to\infty}\int_{B_\tau}\left(r_k^{\gamma}u\left(r_k\mathbf{e}_n\right)\right)^{2^{**}-2}\varphi_{ki}(z)^{2^{**}-1}|y-z|^{4-n}\mathrm{d}z=c_2|y|^{4-n}$, for some $c_2\geqslant 0$. Moreover, since the left-hand side of the last equation is locally uniformly bounded in $C^{n+1}_{\mathrm{loc}}\left(B_\tau\right)$, for any $i\in I$, there exists $\varrho_i\in C^2\left(B_\tau\right)$ satisfying

$$\lim_{k\to\infty}\int\limits_{B_2\setminus B_\tau}\left(r_k^{\gamma}u_i\left(r_k\mathbf{e}_n\right)\right)^{2^{**}-2}\varphi_{ki}(z)^{2^{**}-1}|y-z|^{4-n}\mathrm{d}z=\varrho_i(y)\geqslant 0\quad\text{in}\quad C^n_{\mathrm{loc}}\left(B_\tau\right).$$

In addition, for any fixed $R\gg 1$ and $y\in B_\tau$, we have

$$\lim_{k\to\infty}\int\limits_{t\leqslant|y|\leqslant R}\left(r_k^{\gamma}u_i\left(r_k\mathbf{e}_n\right)\right)^{2^{**}-2}\varphi_{ki}(z)^{2^{**}-1}|y-z|^{4-n}\mathrm{d}z=0,$$

and for any $y_1,y_2\in B_\tau$, we obtain

$$\begin{aligned}&\int\limits_{B_{2/r_k}\setminus B_R}\left(r_k^{\gamma}u_i\left(r_k\mathbf{e}_n\right)\right)^{2^{**}-2}\varphi_{ki}(z)^{2^{**}-1}|y_1-z|^{4-n}\mathrm{d}z\\&\leqslant\left(\frac{R+\tau}{R-\tau}\right)^{n-4}\int\limits_{B_{2/r_k}\setminus B_R}\left(r_k^{\gamma}u_i\left(r_k\mathbf{e}_n\right)\right)^{2^{**}-2}\varphi_{ki}(z)^{2^{**}-1}|y_2-z|^{4-n}\mathrm{d}z.\end{aligned}$$

Therefore, it follows $\varrho_i\left(y_1\right)\leqslant\left(\frac{R+\tau}{R-\tau}\right)^{n-4}\varrho_i\left(y_2\right)$, which, by passing to the limit as $R\to\infty$ and exchanging the roles of y_1 and y_2, implies $\varrho_i\left(y_2\right)=\varrho_i\left(y_1\right)$. Whence, $\varrho_i(y)\equiv\varrho_i(0)$ for all $y\in B_\tau$ and $i\in I$. Since φ_{ki} is locally uniformly bounded in B^*_{2/r_k}, it is also locally uniformly bounded in $C^{n+1}(B^*_{2/r_k})$. Hence, up to subsequence, $\varphi_{ki}\to\varrho_i$ as $k\to\infty$ in $C^n_{\mathrm{loc}}\left(\mathbb{R}^n\setminus\{0\}\right)$, for some ϱ_i, which yields that $\lim_{k\to\infty}\varphi_{ki}(y)=c_2|y|^{4-n}+\psi_i(0)$ in $C^n_{\rm loc}(\mathbb{R}^n\setminus\{0\})$. Using that $\varphi_{ki}(\mathbf{e}_n)=1$ and

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{r^{\gamma}\bar{\varphi}_{ki}(r)\right\}\Big|_{r=1}=r_k^{-\gamma+1}u_i\left(r_k\mathbf{e}_n\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r^{\gamma}\bar{u}_i(r)\right\}\Big|_{r=r_k}=0,$$

by taking the limit $k\to\infty$, it follows that $c_2=\psi_i(0)=1/2$, which proves the claim.
Claim 2: $\lim_{k\to\infty}\mathcal{P}_{\rm sph}(r_k,\mathcal{U})=0$.
In fact, for any $i\in I$, let us consider $\widetilde{u}_i(x)=\int_{B_2}f_i(\mathcal{U})|x-y|^{4-n}\mathrm{d}y+\psi_i(x)$ in $\mathbb{R}^n\setminus\{0\}$, which provides that $\widetilde{u}_i=u_i$ in B_2^*, and $\widetilde{u}_i(x)=\int_{B_2}f_i(\widetilde{\mathcal{U}})|x-y|^{4-n}\mathrm{d}y+\psi_i(x)$ in $\mathbb{R}^n\setminus\{0\}$. Consequently, using that $\Delta^2\psi_i=0$ in B_2 for any $i\in I$, it follows that $\Delta^2\widetilde{u}_i=f_i(\mathcal{U})$ in B_2^*. On the other hand, we know that $\mathcal{P}_{\rm sph}(r_k,\mathcal{U})$ is a constant on r. Moreover, since there exists $c_3>0$ such that $|D^{(j)}\varphi_{ki}|\leqslant C$ near ∂B_1 and $r_k^{\gamma}u(r_k\mathbf{e}_n)=\mathrm{o}(1)$ as $k\to\infty$, we have $|D^{(j)}u_i(x)|\leqslant c_3r_k^{-j}u(r_k\mathbf{e}_n)=\mathrm{o}(1)r_k^{-\gamma-k}$ for all $|x|=r_k$ and $j=1,2,3,4$, which proves the second claim.

Hence, using Claim 2, it holds that $\mathcal{P}_{\rm sph}(r_k,\mathcal{U})=0$ for $k\in\mathbb{N}$. Thus, by (12), we get

$$\sum_{i=1}^{p}\int_{\partial B_1}q\left(\varphi_{ki}(x),\varphi_{ki}(x)\right)\mathrm{d}x+\widehat{c}(n)\left(r_k^{\gamma}u_i\left(r_ie_n\right)\right)^{2^{**}-2}\sum_{i=1}^{p}\int_{\partial B_1}|\varphi_{ki}(x)|^{2^{**}}\mathrm{d}x=0,$$

where we recall that q is defined by (13). Next, sending $k\to\infty$, and doing some manipulation, we obtain $\int_{\partial B_1}q(|x|^{4-n}+1,|x|^{4-n}+1)\mathrm{d}x=0$.

On the other hand, by Theorem B (i), we know that $\mathcal{U}_{0,\mu}(x)=\Lambda\left(\frac{2\mu}{1+|x|^2\mu^2}\right)^{\gamma}$ satisfies $(\mathcal{S}_{p,\infty})$ for some $\Lambda\in\mathbb{S}^{p-1}_{+,*}$, which, for any $\mu>0$, implies that $\mathcal{P}_{\rm sph}(1,\mathcal{U}_{0,\mu})=\lim_{r\to\infty}\mathcal{P}_{\rm sph}(r,\mathcal{U}_{0,\mu})=0$. Hence, we find

$$\sum_{i=1}^{p}\int_{\partial B_1}q\left(\mu^{-\gamma}(u_i)_{0,\mu},\mu^{-\gamma}(u_i)_{0,\mu}\right)\mathrm{d}x+\widehat{c}(n)\mu^{4-n}\sum_{i=1}^{p}\int_{\partial B_1}(u_i)_{0,\mu}^{2^{**}}\mathrm{d}x=0,$$

which, by taking the limit as $\mu\to 0$, provides

$$\begin{aligned}0&=\int_{\partial B_1}\left[q\left(|x|^{4-n}+1,|x|^{4-n}+1\right)-q\left(|x|^{4-n},|x|^{4-n}\right)\right]\mathrm{d}\sigma\\&=(n-4)\int_{\partial B_1}\partial_{\nu}\Delta\left(|x|^{4-n}\right)\mathrm{d}\sigma\neq 0,\end{aligned}$$

which is a contradiction. This concludes the proof of the proposition. □

Using the last lemma, we can present a removable singularity theorem.

Corollary 51. *Let $\mathcal{U}$ be a strongly positive solution to $(\mathcal{S}_{p,1})$. Then, $\mathcal{P}_{\rm sph}(\mathcal{U})\leqslant 0$ and $\mathcal{P}_{\rm sph}(\mathcal{U})=0$, if, and only if, $\mathcal{U}$ has a removable singularity at the origin.*

The lower bound estimate is a direct consequence of the last results.

Corollary 52. *Let $\mathcal{U}$ be a strongly positive singular solution to ($\mathcal{S}_{p,1}$). Then, there exists $C_1 > 0$ such that $C_1|x|^{-\gamma} \leqslant |\mathcal{U}(x)|$ for $0 < |x| < 1/2$.*

Ultimately, we have the proof of the main theorem in this section.

Proof of Proposition 4. It is a combination of Corollaries 45 and 52. □

5. Local asymptotic behavior

In this section, we present the proof of Theorem 1. To this end, we use the growth properties of the Jacobi fields in Proposition 3 and the a priori estimates in Proposition 4. We know that in the high-frequency case, there is a basis of four linearly independent Jacobi fields, two of which grow unbounded and the others exponentially decay. Surprisingly, this Jacobi field basis has only two elements in the zero-frequency case.

We employ the so-called Simon's convergence technique for strongly positive solutions to ($\mathcal{C}_{p,\infty}$), which can be summarized in following steps:
(a) There exist $C_1, C_2 > 0$ such that any strongly positive solution to ($\mathcal{C}_{p,0}$) satisfies the uniform estimate $C_1 \leqslant |\mathcal{V}(t,\theta)| \leqslant C_2$;
(b) If $\tau_k \to \infty$ and $\mathcal{V}_k(t,\theta) := \mathcal{V}(t+\tau_k,\theta)$. Then, the slide back sequence $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ converges uniformly on compact sets to a bounded solution $\mathcal{V}_\infty$ to ($\mathcal{C}_{p,\infty}$);
(c) Any angular derivative $|\partial_\theta \mathcal{V}(t,\theta)|$ converges to 0 as $t \to \infty$;
(d) There exists $S > 0$ such that for any infinitesimal rotation ∂_θ of $\mathbb{S}^{n-1}$, and for any $\tau_k \to \infty$, if we set $A_k = \sup_{t\geqslant 0} |\partial_\theta \mathcal{V}(t,\theta)|$, and if $|\partial_\theta \mathcal{V}_k(\tau_k,\theta)| = A_k$ for some $(\tau_k,\theta_k) \in \mathcal{C}_0$, then $s_k \leqslant S$;
(e) $|\partial_\theta \mathcal{V}(t,\theta)|$ converges to 0 exponentially as $t \to \infty$, as well as $|\mathcal{V}(t,\theta) - \overline{\mathcal{V}}(t)|$, where $\overline{\mathcal{V}}$ is a spherical average of $\mathcal{V}$;
(f) There exists a bounded solution $\mathcal{V}_{a,T}$ of ($\mathcal{C}_{p,\infty}$) and $\sigma \geqslant 0$ such that $\mathcal{V}(t,\theta)$ converges to $\mathcal{V}(t+\sigma)$ exponentially as $t \to \infty$.

The steps above are verified by combining Theorem B, Proposition 3 and Proposition 4. Namely, we prove a result equivalent to Theorem 1 written in cylindrical coordinates.

Proposition 53. *Let $\mathcal{V}$ be a strongly positive singular solution to ($\mathcal{C}_{p,0}$). Then, there exists $\beta_0^* > 0$ and an Emden–Fowler solution $\mathcal{V}_{a,T}$ such that*

$$\mathcal{V}(t) = (1 + \mathcal{O}(e^{\beta_0^* t}))\mathcal{V}_{a,T}(t) \quad \text{as} \quad t \to \infty. \tag{52}$$

Proof. Initially, by Remark 11, the origin is a non-removable singularity. Thus, using Corollary 51, we have that $\mathcal{P}_{\mathrm{sph}}(\mathcal{U}) < 0$. Let $\mathcal{V} = \mathfrak{F}(\mathcal{U})$ and $\{\tau_k\}_{k\in\mathbb{N}}$ be such that $\tau_k \to \infty$ as $k \to \infty$. We define the sequence of translations $\mathcal{V}_k(t,\theta) = \mathcal{V}(t+\tau_k,\theta)$ defined in $\mathcal{C}_{\tau_k} := (-\tau_k,\infty) \times \mathbb{S}^{n-1}$. Hence, applying Proposition 4, we get $C_1 \leqslant |\mathcal{V}_k(t,\theta)| \leqslant C_2$, which yields that $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $C^{4,\zeta}_{\mathrm{loc}}(\mathcal{C}_0,\mathbb{R}^p)$ for some $\zeta \in (0,1)$. Hence, by standard elliptic regularity, there exists a limit solution $\mathcal{V}_\infty \in C^{4,\zeta}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^p)$ such that, up to subsequence, $\mathcal{V}_k \to \mathcal{V}_\infty$ as $k \to \infty$, where $\mathcal{V}_\infty$ satisfies ($\mathcal{C}_{p,0}$). Thus, by Theorem B (ii), $\mathcal{V}_\infty$ is an Emden–Fowler solution, that is, there exist $a \in (0,a_0)$ and $T \in (0,T_a)$ such that $\mathcal{V}_\infty = \mathcal{V}_{a,T}$ and does not depend on the variable θ, where $\mathcal{V}_{a,T} = \Lambda v_{a,T}$ with $v_{a,T}$ a solution to (4) and $\Lambda = (\Lambda_1, \dots, \Lambda_p) \in \mathbb{S}^{p-1}_{+,*}$, that is, $\Lambda_i > 0$ for all $i \in I$.

We divide the rest of the proof into some claims.

Claim 1: The following elliptic estimates hold:
(i) $\mathcal{V}_k(t,\theta) = \overline{\mathcal{V}}_k(t)(1 + \mathrm{o}(1))$;
(ii) $\nabla\mathcal{V}_k(t,\theta) = -\overline{\mathcal{V}}_k^{(1)}(t)(1 + \mathrm{o}(1))$;
(iii) $\Delta\mathcal{V}_k(t,\theta) = \overline{\mathcal{V}}_k^{(2)}(t)(1 + \mathrm{o}(1))$;
(iv) $\nabla\Delta\mathcal{V}_k(t,\theta) = -\overline{\mathcal{V}}_k^{(3)}(t)(1 + \mathrm{o}(1))$.
Indeed, if (i) is not valid, there would exist $\bar{\varepsilon} > 0$ and $\tau_k \to \infty, \theta_k \to \infty$ such that $|\frac{\mathcal{V}_k(\tau_k,\theta_k)}{\overline{\mathcal{V}}_k(\tau_k)} - 1| \geqslant \bar{\varepsilon}$, which is a contradiction since $\mathcal{V}_k \to \mathcal{V}_\infty$ and $\mathcal{V}_\infty$ is radially symmetric. The same argument holds for (ii), (iii), and (iv). This estimate implies (b), that is, any angular derivative $|\partial_\theta\mathcal{V}_k|$ converges uniformly to zero.

Claim 2: The necksize of $\mathcal{V}_\infty$ does not depend on $k \in \mathbb{N}$.
In fact, this is a consequence of the following identity

$$\mathcal{P}_{\mathrm{cyl}}(\mathcal{V}_\infty) := \mathcal{P}_{\mathrm{cyl}}(0,\mathcal{V}_\infty) = \lim_{k\to\infty}\mathcal{P}_{\mathrm{cyl}}(0,\mathcal{V}_k) = \lim_{k\to\infty}\mathcal{P}_{\mathrm{cyl}}(\tau_k,\mathcal{V}) = \mathcal{P}_{\mathrm{cyl}}(\mathcal{V}),$$

which says that for each $\{\tau_k\}_{k\in\mathbb{N}}$, the correspondent sequence $\{\mathcal{V}_k\}_{k\in\mathbb{N}}$ converges to $\mathcal{V}_{a,T} := \Lambda v_{a,T}$ as $k \to \infty$, where T does not depend on k.

In the following claim, we prove (c), (d), (e), and (f).

Claim 3: There exist $\sigma \in \mathbb{R}$ and $\beta_0^*, c_2 > 0$ such that $|\mathcal{V}_\sigma(t,\theta) - \mathcal{V}_{a,T}(t)| \leqslant c_2 e^{\beta_0^* t}$ on $\mathcal{C}_0$.
As a matter of fact, we divide this argument into three steps. First, let $T_a \in \mathbb{R}$ be the fundamental period of the Emden–Fowler solution $v_{a,T}$ and define $A_\tau = \sup_{t\geqslant 0}|\partial_\theta\mathcal{V}_\tau|$. Since $|\partial_\theta\mathcal{V}_\tau|$ converges uniformly to zero as $t \to \infty$, we have $A_\tau < \infty$.

Step 1: For every $c > 0$, there exists an integer $N > 0$ such that, for any $\tau > 0$ either:
(i) $A_\tau \leqslant ce^{-2\tau}$, or
(ii) A_τ is attained at some point in $\overline{\mathcal{C}}_{0,I_N} := I_N \times \mathbb{S}^{n-1}$, where $I_N = [0, NT_a]$.
Suppose that the claim is not true. Then, there exists $c_1 > 0$ and $\tau_k, \theta_k \to \infty$ such that $|\partial_\theta\mathcal{V}_\tau(s_k,\theta_k)| = A_{\tau_k}$ and $A_{\tau_k} > c_1 e^{-2\tau_k}$ as $k \to \infty$. We define $\widetilde{\mathcal{V}}_k(t,\theta) = \mathcal{V}_k(t + s_k,\theta)$ and $\Phi_k = A_{\tau_k}^{-1}\partial_\theta\widetilde{\mathcal{V}}_k$. In addition, we have that $|\Phi_k| \leqslant 1$ and $\widetilde{\mathcal{V}}_k$ satisfy ($\mathcal{C}_{p,0}$), which by differentiation with respect to θ implies that $\mathcal{L}^a(\Phi_k) = 0$. Now, using standard elliptic regularity, we can extract a subsequence $\{\Phi_k\}_{k\in\mathbb{N}}$ which converges in compact subsets to a nontrivial bounded Jacobi field satisfying $\mathcal{L}^a(\Phi) = 0$. Finally, since Φ has no zero eigencomponent relative to Δ_θ and thus is unbounded, this contradiction proves Step 1.

Assuming that $\mathcal{V}_k(t,\theta)$ converges to $\mathcal{V}_a(t + T)$ as $k \to \infty$, we define $\mathcal{W}_k(t,\theta) = \mathcal{V}_k(t,\theta) - \mathcal{V}_a(t + T)$, $\eta_k = \varrho\max_{I_N}|\mathcal{W}_k|$, and $\Phi_k = \eta_k^{-1}\mathcal{V}_k$, where $\varrho > 0$ will be chosen later and satisfies $|\Phi_k| \leqslant \varrho^{-1}$ in I_N. Then, by Theorem B (ii), it follows

$$\Delta^2_{\mathrm{cyl}}\mathcal{W}_k - \left[f_i(\mathcal{V}_k) - \Lambda v_{a,T}^{2^{**}-1}\right] = 0, \tag{53}$$

where

$$f_i(\mathcal{V}_k) - \Lambda v_{a,T}^{2^{**}-1} = |\mathcal{V}_k|^{2^{**}-2}v_{ki} + \Lambda_i v_{a,T}\frac{|\mathcal{V}_k|^{2^{**}-2} - v_{a,T}^{2^{**}-2}}{|\mathcal{V}_k|^2 - v_{a,T}^2}\sum_{j=1}^{p}\left(v_{kj} + \Lambda_i v_{a,T}\right).$$

Multiplying (53) by η_k^{-1} and taking the limit as $k \to \infty$ we get $\mathcal{L}^a(\Phi^*) = 0$, where $\Phi^* = \lim_{k\to\infty} \Phi_k$.

Step 2: The Jacobi filed Φ^* is bounded for all $t \geqslant 0$.

Using Proposition 3 and the Fourier decomposition (21), we get $\Phi^* = b_1\Phi^+_{a,0} + b_2\Phi^-_{a,0} + \check{\Phi}$, where $\check{\Phi}$ is the projection on the subspace generated by the eigenfunctions associated to the nonzero eigenvalues of Δ_θ. We will use Proposition 3 to show that $\check{\Phi}$ is bounded. Indeed, we need to verify that $\partial_\theta\check{\Phi} = \partial_\theta\Phi$ is bounded for $t \geqslant 0$. In this fashion, we have that $\partial_\theta\Phi = \lim_{k\to\infty} \eta_k\partial_\theta\mathcal{V}_k$. Furthermore, the result follows if $\partial_\theta\Phi$ is zero. Then, we suppose that $\partial_\theta\Phi$ is nontrivial. In this case, if (i) of Step 1 happens, we get $\sup_{t\geqslant 0}\left(\eta_k^{-1}|\partial_\theta\mathcal{V}_k|\right) \leqslant c\eta_k^{-1}e^{-2\tau_k} \leqslant c$. On the other hand, if (ii) of Step 1 happens, since $\eta_k^{-1}|\partial_\theta\mathcal{V}_k|$ converges in the $C^{4,\zeta}$-topology, we have $\sup_{t\geqslant 0}\left(\eta_k^{-1}|\partial_\theta\mathcal{V}_k|\right) \leqslant \sup_{I_N}\left(\eta_k^{-1}|\partial_\theta\mathcal{V}_k|\right) \leqslant c$. The last two inequalities imply that $\check{\Phi}$ is bounded.

To finish the proof of Step 2, we must show that $b_2 = 0$. Indeed, since $\Phi_k = \eta_k^{-1}\mathcal{W}_k \to \Phi$ as $k \to \infty$, we obtain

$$\mathcal{V}_k = \mathcal{V}_{a,T} + \eta_k\Phi^* + \mathrm{o}(\eta_k) = \mathcal{V}_{a,T} + \eta_k(b_1\Phi^+_{a,0} + b_2\Phi^-_{a,0} + \check{\Phi}) + \mathrm{o}(\eta_k).$$

On the other hand,

$$\mathcal{P}_{\mathrm{cyl}}(0,\mathcal{V}_k) = \mathcal{P}_{\mathrm{cyl}}(\tau_k,\mathcal{V}) = \mathcal{P}_{\mathrm{cyl}}(\mathcal{V}) + \mathcal{O}(e^{-2\tau_k}) = \mathcal{P}_{\mathrm{cyl}}(T,\mathcal{V}_a) + \mathcal{O}(e^{-2\tau_k}).$$

Then, if $b_2 \neq 0$, we would have a contradiction, since $\eta_k^{-1}e^{-2\tau_k} = \mathrm{o}(1)$ as $k \to \infty$ and the two sides of the last equality would differ for sufficiently large k.

Let us define $\mathcal{W}_\tau(t,\theta) = \mathcal{V}(t+\tau,\theta) - \mathcal{V}_a(t+T)$ and $\eta(\tau) = \varrho\max_{I_N}|\mathcal{W}_\tau|$, where I_N is defined in Step 1 and $\varrho > 0$ will again be chosen later. For a fixed $c_2 > 0$, we have the following:

Step 3: Assume that $N, \varrho, \tau \gg 1$ and $0 < \eta \ll 1$. Then, there exists $\delta > 0$ such that for $|\delta| \leqslant c_2\eta(\tau)$, it holds

$$2\eta(\tau + NT_a + \delta) \leqslant \eta(\tau). \tag{54}$$

Suppose that (54) does not hold. Then, there would exist $\tau_k \to \infty$ such that $\eta(\tau_k) \to 0$ and for $s > 0$ satisfying $|s| \leqslant c_2\eta(\tau_k)$ we have $\eta(\tau_k + NT_a + s) > 1/2\eta(\tau_k)$. Similarly to the previous step, let us define $\Phi_k = \eta(\tau_k)^{-1}\mathcal{W}_{\tau_k}$; thus by Step 2, we can suppose that $\{\Phi_k\}_{k\in\mathbb{N}}$ converges to a bounded Jacobi Field Φ^*, which provides

$$\Phi^* = b_1\Phi^+_{a,0} + \check{\Phi}, \tag{55}$$

where $\check{\Phi}$ has exponential decay. Since $|\check{\Phi}| < \varrho^{-1}$ on I_N, we get that b_1 is uniformly bounded and independent of $\tau_k > 0$. Moreover, we know that $\Phi^+_{a,0} = \partial_a\mathcal{V}_{a,T}$ is bounded and $\Phi^-_{a,0} = \partial_\theta\mathcal{V}_{a,T}$ is linearly growing. Setting $s_k = -\eta(\tau_k)b_1$, we can choose $c_2 \gg 1$ sufficiently large such that $|s_k| < |c_2\eta(\tau_k)|$. Hence, for $t \in [0, 2NT_a]$, we get

$$\begin{aligned}\mathcal{W}_{\tau_k+s_k}(t,\theta) &= \mathcal{V}(t+\tau_k-\eta(\tau_k)b_1,\theta) - \mathcal{V}_{a,T}(t)\\ &= \mathcal{V}_{\tau_k}(t-\eta(\tau_k)b_1,\theta) - \mathcal{V}_{a,T}(t-\eta(\tau_k)b_1) - \eta(\tau_k)b_1\frac{\mathcal{V}_{a,T}(t-\eta(\tau_k)b_1)-\mathcal{V}_{a,T}(t)}{-\eta(\tau_k)b_1}\end{aligned}$$

$$= \eta(\tau_k)\Phi_k(t - \eta(\tau_k)b_1, \theta) - \eta(\tau_k)b_1\Phi^+_{a,0} + o(\eta(\tau_k))$$
$$= \mathcal{W}_{\tau_k}(t, \theta) - \eta(\tau_k)b_1\Phi^+_{a,0} + o(\eta(\tau_k)).$$

Therefore, by (55), we obtain $\mathcal{W}_{\tau_k+s_k} = \eta(\tau_k)\check{\Phi} + o(\eta(\tau_k))$ on $[0, 2NT_a]$, which implies $\max_{I_N} |\mathcal{W}_{\tau_k+s_k+NT_a}| = \max_{[NT_a,2NT_a]} |\mathcal{W}_{\tau_k+s_k}| \leqslant \eta(\tau_k) \max_{I_N}(|\check{\Phi}|) + o(\eta(\tau_k))$. Then, since $|\check{\Phi}|$ decreases exponentially in a fixed rate, one can choose $N, \varrho \gg 1$ sufficiently large satisfying $\max_{I_N} |\mathcal{W}_{\tau_k+s_k+NT_a}| \leqslant 2^{-1}\eta(\tau_k)$, which implies $\eta(\tau + NT_a + s) \leqslant \eta(\tau)$. This is a contradiction, and this step is proved.

Now, we use Step 3 to construct a sequence that converges to the correct translation parameter.
Step 4: There exists $\sigma > 0$ such that $|\mathcal{W}_\sigma(t, \theta)|$ converges exponentially to 0 as $t \to \infty$.
First, choose $\tau_0, N \gg 1$ such that Step 3 is satisfied and $c_2\eta(\tau_0) \leqslant 2^{-1}NT_a$. Set $s_0 = -\eta(\tau_0)b_1$ as above; thus $|s_0| \leqslant c_2\eta(\tau_0) \leqslant 2^{-1}NT_a$. Let us define inductively three sequences:

$$\sigma_k = \tau_0 + \sum_{i=0}^{k-1} s_i, \quad \tau_k = \tau_{k-1} + s_{k-1} + NT_a = \sigma_k + kNT_a, \quad s_k = -\eta(\tau_k)b_1.$$

Notice that by induction, it follows that $\eta(\tau_k) \leqslant 2^{-k}\eta(\tau_0)$ and $|s_k| < 2^{k-1}NT_a$. Then, there exists $\sigma = \lim_{k\to\infty} \sigma_k \leqslant \tau_0 + NT_a$ and so $\tau_k \to \infty$ as $k \to \infty$. Now choosing $k \in \mathbb{N}$ such that $t = kNT_a + [t]$ with $[t] \in I_N$, we can write

$$\mathcal{W}_\sigma(t, \theta) = \mathcal{V}(t + \sigma, \theta) - \Lambda v_{a,T} = \mathcal{V}(t + \sigma, \theta) - \mathcal{V}(t + \sigma_k, \theta) + \mathcal{V}(t + \sigma_k, \theta) - \Lambda v_{a,T}(t).$$

Since $\partial_t\mathcal{V}$ is uniformly bounded, we get $\mathcal{V}(t+\sigma, \theta) - \mathcal{V}(t+\sigma_k, \theta) = \partial_t\mathcal{V}(t_0) \sum_{i=k}^{\infty} s_i = \mathcal{O}(2^{-k})$, for some $t_0 > 0$ and $\mathcal{V}(t + \sigma_k, \theta) - \mathcal{V}_{a,T}(t) = \mathcal{V}(\tau_k + [t], \theta) - \mathcal{V}_{a,T}([t]) = \mathcal{W}([t], \theta)$, which provides $\mathcal{W}_\sigma(t, \theta) = \mathcal{W}_{\tau_k}([t], \theta) + \mathcal{O}(2^{-k})$. Therefore, using that $b \max_{I_N} |\mathcal{W}_{\tau_k}| = \eta(\tau_k) \leqslant 2^{-k}\eta(\tau_0)$, we obtain that $|\mathcal{W}_\sigma(t, \theta)| = \mathcal{O}(2^{-k})$ as $k \to \infty$, or in terms of $t = kNT_a + [t]$, it follows $|\mathcal{W}_\sigma(t, \theta)| \leqslant c_2 e^{-\frac{\ln 2}{NT_a}t}$, which, by taking $\beta_0^* = -\ln 2/NT_a$, concludes the proof of Step 4.

Finally, we observe that Claim 3 directly implies (52), and so the theorem is proved. □

At last, the proof of our main result is followed by a direct consequence using the inverse of the cylindrical transformation.

Proof of Theorem 1. It follows by undoing the cylindrical transformation in (52) and rescaling back to the original ball. □

Data availability

No data was used for the research described in the article.

Acknowledgments

The paper was completed while the first-named author was visiting the Department of Mathematics at Princeton University, whose hospitality he gratefully acknowledges. He would also like to express his gratitude to Professor F. C. Marques. After finishing this manuscript, we learned that J. Ratzkin [31] had independently obtained a scalar version of Theorem 1.

Appendix A. Deformed Emden–Fowler solutions

In this appendix, we follow [21] to introduce the family of deformed Emden–Fowler solutions.

Here, recall that $\gamma = (n-4)/2$ the Fowler rescaling exponent. We construct a $2n$-parameter family of solutions using the pullback of a composition of three conformal transformations described below. First, take $p = 1$ and consider an Emden–Fowler solution with $T = 0$, given by $u_{a,0}(x) = |x|^{-\gamma} v_a(-\ln|x|)$; thus, using an inversion about the unit sphere, we obtain $\tilde{u}_{a,0}(x) = |x|^{-\gamma} v_a(\ln|x|)$. Second, we employ an Euclidean translation about $x_0 \in \mathbb{R}^n \setminus \{0\}$ to get $\tilde{u}_{a,0,x_0}(x) = |x|x|^{-2} - x_0|^{-\gamma} v_a(\ln|x|x|^{-2} - x_0|)$. Finally, applying another inversion, we find $u_{a,0,x_0}(x) = |x|^{-\gamma}|\theta - x_0|x||^{-\gamma} v_a(-\ln|x| + \ln|\theta - x_0 x|)$, where $\theta = x|x|^{-1}$. Moreover, in cylindrical coordinates, we have

$$v_{a,0,x_0}(t,\theta) = |\theta - x_0 e^{-t}|^{-\gamma} v_a(t + \ln|\theta - x_0 e^{-t}|). \tag{56}$$

Finally, taking a time translation $T \in (0, T_a)$, we construct the families u_{a,T,x_0} and v_{a,T,x_0}. Second, in the case $p > 1$, we can proceed similarly to define the family of vectorial deformed Emden–Fowler solutions $\mathcal{U}_{a,T,x_0}$ and $\mathcal{V}_{a,T,x_0}$. The parameters $x_0 \in \mathbb{R}^n$ and $T \in (0, T_a)$ correspond to conformal motions. In contrast, the so-called *Fowler parameter* $a \in (0, a_0)$ does not have a geometrical interpretation.

Remark 54. In the light of Theorem B (ii), for $p > 1$, it follows that $\mathcal{U}_{a,T,x_0} = \Lambda u_{a,T,x_0}$ and $\mathcal{V}_{a,T,x_0} = \Lambda v_{a,T,x_0}$, where $\Lambda \in \mathbb{S}^{p-1}_{+,*}$ and u_{a,T,x_0} and v_{a,T,x_0} are scalar Emden–Fowler solutions.

Lemma 55. *For any $a \in (0, a_0)$ and $x_0 \in \mathbb{R}^n$, we have $\mathcal{U}_{a,0,x_0}(x) = (1 + \mathcal{O}(|x|))\mathcal{U}_{a,0}(|x|)$ as $|x| \to 0$.*

Proof. Initially, we take $p = 1$ and calculate the Taylor series of $u_{a,0,x_0}$ nearby $|x| = 0$,

$$|x|x|^{-1} - x_0|x||^{\gamma} = 1 + \gamma(x_0 \cdot x) + \mathcal{O}(|x|^2). \tag{57}$$

Similarly, $\ln|x|x|^{-1} - x_0|x|| = -(x_0 \cdot x) + \mathcal{O}(|x|^2)$, which implies

$$v_a(-\ln|x| - (x_0 \cdot x) + \mathcal{O}(|x|^2)) = v_a(-\ln|x|) - v_a^{(1)}(-\ln|x|)(x_0 \cdot x) + \mathcal{O}(|x|^2). \tag{58}$$

Combining, (57) and (58), we obtain

$$\begin{aligned} u_{a,0,x_0}(x) &= |x|^{-\gamma}[v_a(-\ln|x|) + (x_0 \cdot x)(v_a^{(1)}(-\ln|x|) + \gamma v_a(-\ln|x|)) + \mathcal{O}(|x|^2)] \\ &= u_{a,0}(x) + |x|^{-\gamma}(x_0 \cdot x)(-v_a^{(1)} + \gamma v_a) + \mathcal{O}(|x|^{-\gamma-2}), \end{aligned}$$

which together with Theorem B (ii) yields

$$\mathcal{U}_{a,0,x_0}(x) = \mathcal{U}_{a,0}(x) + |x|^{-\gamma}(x_0 \cdot x)(-\mathcal{V}_a^{(1)} + \gamma \mathcal{V}_a) + \mathcal{O}(|x|^{-\gamma-2}); \tag{59}$$

this concludes the proof of the Proposition. □

Remark 56. Since the Jacobi fields and the indicial roots of the linearized operator $\mathcal{L}^a$ are not counted with multiplicity, we have $\Phi^+_{a,1} = \cdots = \Phi^+_{a,n}$ and (59) can be reformulated as

$$\mathcal{U}_{a,0,x_0}(x) = |x|^\gamma \left[\mathcal{V}_a \left(-\ln|x| + |x| \left(\sum_{j=1}^n x_j \chi_j(\theta) \Phi_{a,j} \right) + \mathcal{O}(|x|^2) \right) \right] \tag{60}$$
$$= |x|^\gamma [\mathcal{V}_a(-\ln|x| + (x_0 \cdot x)\Phi^+_{a,1}(-\ln|x|) + \mathcal{O}(|x|^2))] \quad \text{as} \quad |x| \to 0.$$

In cylindrical coordinates, we can rewrite

$$\mathcal{V}_{a,0,x_0}(t,\theta) = \mathcal{V}_a(t) + e^{-t}\langle\theta, a\rangle(-\mathcal{V}_a^{(1)} + \gamma\mathcal{V}_a) + \mathcal{O}(e^{-2t}) \quad \text{as} \quad t \to \infty. \tag{61}$$

Nevertheless, for the translation $\bar{\mathcal{V}}_{a,x_0}(t,\theta) = \mathcal{V}_a(t - t_0, \theta)$ with $t_0 = -\ln|x_0|$, we have

$$\bar{\mathcal{V}}_{a,x_0}(t,\theta) = e^t(-\mathcal{V}_a^{(1)} + \gamma\mathcal{V}_a) + \mathcal{O}(1) \quad \text{as} \quad t \to \infty.$$

Also, notice that when $\langle a, \theta\rangle > 0$, then $|\mathcal{V}_{a,0,x_0}(t,\theta)| > |\mathcal{V}_a(t)|$ and $|\bar{\mathcal{V}}_{a,0,x_0}(t,\theta)| > |\bar{\mathcal{V}}_a(t)|$. Moreover, the opposite inequality holds when $\langle a, \theta\rangle < 0$.

We discuss the statement in Remark 2 to complete our analysis. Based on the surjectiveness of the linearized operator stated in Proposition 30, we provide a higher order expansion for solutions to ($\mathcal{C}_{p,\infty}$), which can be stated as

Proposition 57. *Let $\mathcal{U}$ be a strongly positive superharmonic singular solution to ($\mathcal{S}_{p,R}$). Then, for any $x_0 \in \mathbb{R}^n$ there exists an Emden–Fowler solution $\mathcal{V}_{a,T}$ such that*

$$\mathcal{U}(x) = |x|^{-\gamma} \left[\mathcal{V}_a(-\ln|x| + T) + (x_0 \cdot x)\Phi^+_{a,1}(-\ln|x| + T) + \mathcal{O}(|x|^{\beta_1^*}) \right] \quad \text{as} \quad |x| \to 0, \tag{62}$$

where $\beta_1^* := \min\{2, \beta_{a,2}\} > 1$.

Proof. We start with $p = 1$. Using the asymptotics proved in Theorem 1, we deduce

$$u(x) = |x|^{-\gamma} v_a(-\ln|x|) = |x|^{-\gamma} [v_a(-\ln|x| + T_a) + w(-\ln|x|)],$$

where $\phi \in C^{4,\delta}_{-\beta}(\mathcal{C})$ for some $\beta > 0$. Moreover, since v_a satisfies (4), we get $\mathcal{L}^a(\phi) = \psi(\phi)$, where

$$\psi(\phi) = (v_a + \phi)^{2^{**}-1} - v_a^{2^{**}-1} - \tilde{c}(n) v_a^{2^{**}-2}\phi.$$

It is straightforward to see that if $\phi \in C^{m,\delta}_{-\beta}(\mathcal{C})$, then $\psi(\phi) \in C^{m,\delta}_{-2\beta}(\mathcal{C})$ for any $m \in \mathbb{N}$. Now, we can run an iterative method. First, assume that $\beta \in (0, 1/2)$, then using Claim 1, we obtain $\psi(\phi) \in C^{0,\delta}_{-2\beta}(\mathcal{C})$. In addition, by (i) of Corollary 35, we have $\phi \in C^{4,\delta}_{-2\beta}(\mathcal{C})$ and $\psi(\phi) \in C^{4,\delta}_{-4\beta}(\mathcal{C})$, which implies $w \in C^{4,\delta}_{-4\beta}(\mathcal{C})$. After some steps, we conclude that $w \in C^{4,\delta}_{\beta'}(\mathcal{C})$ for some $\beta' \in (1/2, 1)$. Therefore, $\psi(\phi) \in C^{4,\delta}_{-2\beta'}(\mathcal{C})$ and by (ii) of Corollary 35, we find that $\phi \in C^{4,\delta}_{-2\beta'}(\mathcal{C}) \oplus$

$D_{a,1}(\mathcal{C})$, which provides $\phi \in C^{4,\delta}_{\beta}(\mathcal{C})$ for $\beta' = \min\{2\beta', \beta_{a,2}\}$. In addition, we observe that $\beta' > \beta$ is optimal.

Finally, (62) follows as consequence of Theorem B (ii). □

In conclusion, we have the following refined asymptotics.

Corollary 58. *Let $\mathcal{V}$ be a solution to ($\mathcal{C}_{p,0}$) and $\mathcal{V}_{a,T}$ an Emden–Fowler solution to ($\mathcal{C}_{p,\infty}$). Then, there exists $\beta_1^* > 1$ such that*

$$|\mathcal{V}(t,\theta) - \mathcal{V}_{a,T}(t) - \pi_0[\mathcal{V}](t,\theta) - \pi_1[\mathcal{V}](t,\theta)| \leqslant Ce^{-\beta_1^* t} \quad \text{for} \quad t > 0,$$

where $\beta^*_{a,1} = \min\{2, \beta_{a,2}\} > 1$.

References

[1] M.J. Ablowitz, B. Prinari, A.D. Trubatch, Discrete and Continuous Nonlinear Schrödinger Systems, London Mathematical Society Lecture Note Series, vol. 302, Cambridge University Press, Cambridge, 2004.

[2] J.H. Andrade, J.M. do Ó, Qualitative properties for solutions to conformally invariant fourth order critical systems, Discrete Contin. Dyn. Syst. 43 (2023) 3008–3042.

[3] S. Baraket, S. Rebhi, Construction of dipole type singular solutions for a biharmonic equation with critical Sobolev exponent, Adv. Nonlinear Stud. 2 (2002) 459–476.

[4] L.A. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Commun. Pure Appl. Math. 42 (1989) 271–297.

[5] R. Caju, J.M. do Ó, A. Santos, Qualitative properties of positive singular solutions to nonlinear elliptic systems with critical growth, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 36 (2019) 1575–1601.

[6] G. Caristi, E. Mitidieri, Harnack inequality and applications to solutions of biharmonic equations, in: Partial Differential Equations and Functional Analysis, in: Oper. Theory Adv. Appl., vol. 168, Birkhäuser, Basel, 2006, pp. 1–26.

[7] Z. Djadli, A. Malchiodi, M.O. Ahmedou, Prescribing a fourth order conformal invariant on the standard sphere. II. Blow up analysis and applications, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 1 (2002) 387–434.

[8] O. Druet, E. Hebey, Stability for strongly coupled critical elliptic systems in a fully inhomogeneous medium, Anal. PDE 2 (2009) 305–359.

[9] O. Druet, E. Hebey, J. Vétois, Bounded stability for strongly coupled critical elliptic systems below the geometric threshold of the conformal Laplacian, J. Funct. Anal. 258 (2010) 999–1059.

[10] K.B. Dysthe, Note on modification to the nonlinear Schrödinger equation for application for deep water waves, Proc. R. Soc. A 369 (1979) 105–114.

[11] B.D. Esry, C.H. Greene, J.P. Burke Jr., J.L. Bohn, Hartree-Fock theory for double condensates, Phys. Rev. Lett. 78 (1997) 3594–3597.

[12] R.L. Frank, T. König, Classification of positive singular solutions to a nonlinear biharmonic equation with critical exponent, Anal. PDE 12 (2019) 1101–1113.

[13] M. Ghergu, S. Kim, H. Shahgholian, Isolated singularities for semilinear elliptic systems with power-law nonlinearity, Anal. PDE 13 (2020) 701–739.

[14] Z. Guo, X. Huang, L. Wang, J. Wei, On Delaunay solutions of a biharmonic elliptic equation with critical exponent, J. Anal. Math. 140 (2020) 371–394.

[15] B.C. Hall, Quantum Theory for Mathematicians, Graduate Texts in Mathematics, vol. 267, Springer, New York, 2013.

[16] Z.-C. Han, Y. Li, E.V. Teixeira, Asymptotic behavior of solutions to the σ_k-Yamabe equation near isolated singularities, Invent. Math. 182 (2010) 635–684.

[17] F. Hang, P.C. Yang, Lectures on the fourth-order Q-curvature equation, in: Geometric Analysis Around Scalar Curvatures, in: Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., vol. 31, World Sci. Publ., Hackensack, NJ, 2016, pp. 1–33.

[18] E. Hebey, Critical elliptic systems in potential form, Adv. Differ. Equ. 11 (2006) 511–600.

[19] A. Hernández, M. Kowalczyk, Nondegeneracy and the Jacobi fields of rotationally symmetric solutions to the Cahn-Hillard equation, Indiana Univ. Math. J. 68 (2019) 1047–1087.

[20] T. Jin, J. Xiong, Asymptotic symmetry and local behavior of solutions of higher order conformally invariant equations with isolated singularities, Ann. Inst. Henri Poincaré, Anal. Non Linéaire (2020).

[21] N. Korevaar, R. Mazzeo, F. Pacard, R. Schoen, Refined asymptotics for constant scalar curvature metrics with isolated singularities, Invent. Math. 135 (1999) 233–272.

[22] Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, J. Eur. Math. Soc. 6 (2004) 153–180.

[23] C.-S. Lin, A classification of solutions of a conformally invariant fourth order equation in $\mathbf{R}^n$, Comment. Math. Helv. 73 (1998) 206–231.

[24] E. Lo, C. Mei, A numerical study of water-wave modulation based on a higher-order nonlinear Schrodinger equation, J. Fluid Mech. 150 (1985) 395–416.

[25] F.C. Marques, Isolated singularities of solutions to the Yamabe equation, Calc. Var. Partial Differ. Equ. 32 (2008) 349–371.

[26] R. Mazzeo, F. Pacard, Constant scalar curvature metrics with isolated singularities, Duke Math. J. 99 (1999) 353–418.

[27] R. Mazzeo, F. Pacard, Bifurcating Nodoids, Topology and Geometry: Commemorating SISTAG, Contemp. Math., vol. 314, Amer. Math. Soc., Providence, RI, 2002, pp. 169–186.

[28] R. Mazzeo, D. Pollack, K.K. Uhlenbeck, Moduli spaces of singular Yamabe metrics, J. Am. Math. Soc. 9 (1996) 303–344.

[29] R.B. Melrose, The Atiyah-Patodi-Singer Index Theorem, Research Notes in Mathematics, vol. 4, A K Peters, Ltd., Wellesley, MA, 1993.

[30] F. Pacard, T. Rivière, Linear and Nonlinear Aspects of Vortices: the Ginzburg-Landau Model, Progress in Nonlinear Differential Equations and Their Applications, vol. 39, Birkhäuser Boston, Inc., Boston, MA, 2000.

[31] J. Ratzkin, On constant Q-curvature metrics with isolated singularities, arXiv:2001.07984 [math.DG], 2020.

[32] R. Schoen, S.-T. Yau, Conformally flat manifolds, Kleinian groups and scalar curvature, Invent. Math. 92 (1988) 47–71.

[33] H. Yang, Asymptotic behavior of positive solutions to a nonlinear biharmonic equation near isolated singularities, Calc. Var. Partial Differ. Equ. 59 (2020) 130.