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Generalized φ-Pullback Attractors in Time-Dependent Spaces: Application to a Nonautonomous Wave Equation With Time-Dependent Propagation Velocity

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ABSTRACT

We present sufficient conditions to obtain a *generalized* $(\varphi, \mathfrak{D})$*-pullback attractor* for evolution processes on time-dependent phase spaces, where φ is a given *decay function* and $\mathfrak{D}$ is a given *universe*. We deal specifically with the case in which φ has either exponential or polynomial decay, the universe is the one of *uniformly bounded families*, and apply the abstract results to a non-autonomous wave equation with time-dependent propagation velocity.

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1 | Introduction

For nonautonomous systems, described in terms of *evolution processes*, in which the time variable appears as an independent parameter in the equations, the main feature of the solutions is that they depend separately on the initial and final times, and not simply on the elapsed time, as it occurs in the autonomous case, described by *semigroups*. Hence, there are a few different ways to define and study the *asymptotic behavior* of the system. One can work with the *forward attraction*, in which a compact set $\mathcal{A}$ called the *uniform attractor*, that is minimal in some sense, attracts all the solutions as the final time goes to infinity, uniformly for initial times and initial data in a bounded set of the phase space. We could also work with the *pullback attraction*, in which for each fixed final time t, we have a compact and invariant (in some sense) set $A(t)$ that attracts solutions when we let the initial time s tend to $-\infty$, uniformly for initial data in bounded subsets of the phase space. In this case, the family $\{A(t)\}_{t\in\mathbb{R}}$ is called the *pullback attractor* of the system. There exist several scenarios in which the pullback attraction can be studied and the forward attraction cannot, that is, we have the pullback attractor and not the uniform attractor. These are two possible frameworks that naturally appear when dealing with systems in a *fixed phase space*.

In order to apply the obtained abstract results to a nonautonomous wave equation with time-dependent propagation velocity, we make use of the theory of *time-dependent evolution*

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process and their attractors, in the pullback sense. It should be noted that this theory is a natural extension of the classical theory for nonautonomous dynamical systems in fixed phase spaces, and enables us to understand, for instance, the pullback dynamics of noncylindrical problems in PDEs and equations in which the coefficients depend on time, and influence in the choice of the phase space, as is the case of the problem we discuss in this work. Furthermore, the theory of pullback attractors, for instance as developed in [1–3], is the natural candidate to extend the theory of global attractors for semigroups (also called autonomous dynamical systems), especially when we consider it in the framework of time-dependent spaces because, to the extent of our current knowledge, there is no theory of uniform attractors for nonautonomous dynamical systems in time-dependent spaces.

Problems in a fixed space describe a vast number of phenomena, but they do not exhaust them all. For instance, in [4], the authors study the solutions of a Klein–Gordon equation with time-dependent Hamiltonian density, with applications to cosmological theory. Many other researchers have used this framework to study a number of problems. See, for instance, [5–10] and references therein. That is the reason why we have chosen this framework to develop our study.

Our goal is to generalize the theory developed in [11], for the autonomous case, and in [12, 13] for the nonautonomous case with a fixed phase space, to the nonautonomous framework with time-dependent phase spaces. As an application, following these three works mentioned and also [5, 6], we obtain the existence of the so-called *generalized φ-pullback attractor* for a nonautonomous wave equation with nonlocal weak damping and time-dependent propagation velocity.

To better outline our results, we introduce some terminology, definitions and notations. Let us consider the set $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ and denote $\mathbb{T}^2_d := \{(t,\tau) \in \mathbb{T}^2 : t \geqslant \tau\}$. We also consider a family $\mathscr{X} = \{(X_t, d_t)\}_{t\in\mathbb{T}}$ of complete metric spaces. A family $\mathcal{S} = \{S(t,\tau) : (t,\tau) \in \mathbb{T}^2_d\}$, consisting of maps $S(t,\tau) : X_\tau \to X_t$, is called an evolution process on $\mathscr{X}$ if it satisfies:

- $S(\tau,\tau)x = x$ for any $x \in X_\tau$ and $\tau \in \mathbb{T}$;
- $S(t,\tau) = S(t,s)S(s,\tau)$ for any $t \geqslant s \geqslant \tau$;
- $S(t,\tau) : X_\tau \to X_t$ is continuous for each $(t,\tau) \in \mathbb{T}^2_d$.

We say that the evolution process is time-discrete if $\mathbb{T} = \mathbb{Z}$ and time-continuous if $\mathbb{T} = \mathbb{R}$.

Let us denote by $\mathfrak{F}$ the class of all families $\hat{D} = \{D_t\}_{t\in\mathbb{T}}$ in which $D_t \subset X_t$ for each $t \in \mathbb{R}$. We say that a family $\hat{D} \in \mathfrak{F}$ is nonempty, open, closed, bounded, compact if each set D_t is nonempty, open, closed, bounded, compact, respectively, for each $t \in \mathbb{T}$. Furthermore, given $\hat{D}^1, \hat{D}^2 \in \mathfrak{F}$, we say that $\hat{D}^2 \subset \hat{D}^1$ if $D^2_t \subset D^1_t$ for each $t \in \mathbb{T}$. A universe $\mathfrak{D}$ is a subclass of $\mathfrak{F}$ which consists of families of nonempty sets. We say that a universe $\mathfrak{D}$ is of *inclusion-closed* if, for any $\hat{D}^1 \in \mathfrak{D}$ and $\hat{D}^2 \in \mathfrak{F}$ such that $\hat{D}^2 \subset \hat{D}^1$, we have $\hat{D}^2 \in \mathfrak{D}$. In this paper, we will consider only inclusion-closed universes.

Given a family $\mathscr{X}$ of complete metric spaces and $\mathcal{S}$ an evolution process on $\mathscr{X}$, we say that a family $\hat{D} \in \mathfrak{F}$ is positively invariant for $\mathcal{S}$ if

$$S(t,\tau)D_\tau \subset D_t \quad \text{for each} \quad (t,\tau) \in \mathbb{T}^2_d.$$

Also, for $t \in \mathbb{T}$ and $A, B \subset X_t$ two nonempty subsets of X_t, we denote the *Hausdorff semidistance between A and B* by

$$\mathrm{dist}^{X_t}_H(A,B) = \sup_{a\in A}\inf_{b\in B} d_t(a,b).$$

Lastly, we say that a function $\varphi : [a,\infty) \to [0,\infty)$, where $a \geqslant 0$, is a decay function if φ is nonincreasing, $\varphi(t) \to 0$ as $t \to \infty$, and for each $\omega > 0$ and $\eta \in \mathbb{R}$, we have

$$\limsup_{t\to\infty} \frac{\varphi(\omega t + \eta)}{\varphi(\omega t)} < \infty.$$

Examples of decay functions are $\varphi(t) = e^{-t}$ (exponential decay), $\varphi(t) = t^{-r}$ (polynomial decay), and $\varphi(t) = \frac{1}{\ln t}$ (logarithm decay) (see, for instance, [13, section 2]).

With that, we can introduce the object in which we focus our efforts in this paper.

Definition 1. Given a family $\mathscr{X}$ of complete metric spaces, $\mathcal{S}$ an evolution process on $\mathscr{X}$, and $\mathfrak{D}$ a universe, we say that a family $\hat{M} \in \mathfrak{F}$ is a generalized $(\varphi, \mathfrak{D})$-pullback attractor for $\mathcal{S}$ if $\hat{M}$ is compact, positively invariant for $\mathcal{S}$, and $\hat{M}$ is $(\varphi, \mathfrak{D})$-pullback attracting, that is, there exists a constant $\omega > 0$ such that for each $\hat{D} \in \mathfrak{D}$ and $t \in \mathbb{T}$, there exist $C = C(\hat{D}, t) \geqslant 0$ and $\tau_0 = \tau_0(\hat{D}, t) \geqslant 0$ such that

$$\mathrm{dist}^{X_t}_H\left(S(t,t-\tau)D_{t-\tau}, M_t\right) \leqslant C\varphi(\omega\tau) \quad \text{for all} \quad \tau \geqslant \tau_0.$$

When φ is an exponential decay function, we also call $\hat{M}$ a *generalized exponential $\mathfrak{D}$-pullback attractor*. The analogous definition holds when φ is a polynomial decay function.

Our goal is to find conditions under which a given evolution process $\mathcal{S}$ on $\mathscr{X}$ possesses a generalized $(\varphi, \mathfrak{D})$-pullback attractor, in the specific cases of polynomial and exponential decay functions, and a given universe $\mathfrak{D}$ that will later be defined. In Section 2, we state and prove abstract results that ensure the existence of such objects (namely, Theorem 2.5 for the exponential decay, and Theorem 2.6 for the polynomial decay), so that, in Section 3, we can apply them to the evolution process $\mathcal{S}_p$ generated by the family of nonautonomous wave equations with nonlocal damping and time-dependent speed of propagation, indexed by the parameter $p \geqslant 0$, given below:

$$\begin{cases} \varepsilon(t)u_{tt}(t,x) - \Delta u(t,x) + k(t)\|u_t(t,\cdot)\|^p_{L^2(\Omega)}u_t(t,x) \\ \quad + f(u(t,x)) = h(x), \quad (t,x) \in [s,\infty)\times\Omega, \\ u(t,x) = 0, \quad (t,x) \in [s,\infty)\times\partial\Omega, \\ u(s,x) = u_0(x), \ u_t(s,x) = u_1(x), \quad x \in \Omega. \end{cases} \tag{W–Eq}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$.

We consider $X_t = H^1_0(\Omega) \times L^2(\Omega)$ the Hilbert space with inner product

$$\langle (u_1,v_1),(u_2,v_2)\rangle_t = \int_\Omega \nabla u_1 \cdot \nabla u_2 dx + \varepsilon(t)\int_\Omega v_1 v_2 dx \quad for (u_1,v_1),(u_2,v_2) \in X_t,$$

and associated norm

$$\|(u,v)\|_t^2 = \|u\|^2_{H_0^1(\Omega)} + \varepsilon(t)\|v\|^2_{L^2(\Omega)} \quad \text{for } (u,v)\in X_t,$$

which is equivalent with the usual norm in $H_0^1(\Omega)\times L^2(\Omega)$, provided $\varepsilon(t)>0$ for each $t\in\mathbb{R}$. We consider the family of Hilbert spaces

$$\mathcal{X} = \{(X_t, \|\cdot\|_t)\}_{t\in\mathbb{R}} = \{H_0^1(\Omega)\times L^2(\Omega), \|\cdot\|_t\}_{t\in\mathbb{R}} \tag{1.1}$$

Taking into account (W−Eq), we assume the following:

H$_1$ $p \geqslant 0$,

H$_2$ $h\in L^2(\Omega)$ and we set $h_0 := \|h\|_{L^2(\Omega)}$,

H$_3$ $k\in C(\mathbb{R})$, and there exists $k_0, k_1>0$ such that $k_0\leqslant k(t)\leqslant k_1$ for all $t\in\mathbb{R}$;

H$_4$ $\varepsilon\in C^1(\mathbb{R})$ is such that $\varepsilon(t)>0$ and $\varepsilon'(t)\leqslant 0$ for all $t\in\mathbb{R}$, and

$$L := \sup_{t\in\mathbb{R}}\big(\varepsilon(t)+|\varepsilon'(t)|\big)<\infty;$$

H$_5$ $f\in C^1(\mathbb{R})$ is such that $f(0)=0$ and

$$\liminf_{|u|\to\infty} f'(u) > -\lambda_1,$$

where $\lambda_1>0$ is the first eigenvalue of the negative Laplacian operator $-\Delta$ with Dirichlet boundary conditions in Ω. Additionally, there exist $c_0>0$ and $1\leqslant q\leqslant 2$ such that

$$|f'(u)|\leqslant c_0(1+|u|^q) \; for\ all\ u\in\mathbb{R}.$$

H$_6$ there exist $\eta, c_1>0$ such that

$$F(t)=\int_0^t f(s)ds \geqslant \eta|t|^{q+2}-c_1 \quad for\ all\ t\in\mathbb{R}.$$

Now we define the universe we work with. Consider a family $\mathcal{X}=\{(X_t,d_t)\}_{t\in\mathbb{T}}$ of Banach spaces and denote by $\overline{B}_r^t$ the closed ball in X_t of radius $r>0$ centered at 0.

Definition 2. A family $\hat{D}\in\mathfrak{F}$ is called uniformly bounded if there exists $r>0$ such that $D_t\subset\overline{B}_r^t$ for all $t\in\mathbb{T}$. The universe of uniformly bounded families is denoted by $\mathfrak{D}_{ub}$.

The main application of our results, proved in Section 3, is the following theorem:

Theorem A. *Assume that 1–1 hold. Then (W−Eq) generates an evolution process $\mathcal{S}_p$ on the family $\mathcal{X}$ defined in (1.1), for each $p\geqslant 0$. Moreover:*

a. *$\mathcal{S}_0$ has a generalized exponential $\mathfrak{D}_{ub}$-pullback attractor $\hat{M}_0$;*
b. *for $p>0$, $\mathcal{S}_p$ has a generalized polynomial $\mathfrak{D}_{ub}$-pullback attractor $\hat{M}_p$.*

In any case, $\mathcal{S}_p$ has also a $\mathfrak{D}_{ub}$-pullback attractor $\hat{A}_p$, with $\hat{A}_p\subset\hat{M}_p$.

2 | A Review on the Abstract Results

This section is devoted to stating the results concerning the existence of *generalized $(\varphi,\mathfrak{D})$-pullback attractors* associated with evolution processes in time-dependent phase spaces. We point out that not all the results stated in this section are proven, as many of them follow similarly to those presented in [13]. In the proofs we do present here, we carefully indicate where the differences lie when working in time-dependent phase spaces.

To begin, we provide some definitions. Given a family $\mathcal{X}$ of complete metric spaces and an evolution process $\mathcal{S}$ on $\mathcal{X}$, we say that a family $\hat{B}\in\mathfrak{F}$ is

- positively invariant for $\mathcal{S}$ if $S(t,\tau)B_\tau\subset B_t$ for each $(t,\tau)\in\mathbb{T}^2_d$;
- invariant for $\mathcal{S}$ if $S(t,\tau)B_\tau = B_t$ for each $(t,\tau)\in\mathbb{T}^2_d$;
- $\mathfrak{D}$-pullback absorbing for a universe $\mathfrak{D}$, if given $\hat{D}\in\mathfrak{D}$ and $t\in\mathbb{T}$, there exists $\tau_0=\tau_0(\hat{D},t)\geqslant 0$ such that

$$S(t,t-\tau)D_{t-\tau}\subset B_t \quad forall \tau\geqslant\tau_0.$$

When there exists $\tau_0=\tau_0(\hat{D})\geqslant 0$ *independent of* $t\in\mathbb{T}$ for which the inclusion above holds for all $\tau\geqslant\tau_0$ and all $t\in\mathbb{T}$, we say that $\hat{B}$ is uniformly $\mathfrak{D}$-pullback absorbing.

Remark 1. We point out that the definition of a *uniformly $\mathfrak{D}$-pullback absorbing family* presented here slightly differs from [13, definition 2.3]. The property we ask here is stronger. However, in the application presented in [13], they obtain a uniformly $\mathfrak{D}$-pullback absorbing family according to our definition.

We will also need the following definition: if $\mathcal{X}=\{(X_t,\|\cdot\|_t)\}_{t\in\mathbb{T}}$ is a family of Banach spaces, we say that a family $\hat{D}\in\mathfrak{F}$ is backwards bounded if given $t\in\mathbb{T}$ there exists $r_t>0$ such that $D_s\subset\overline{B}^s_{r_t}$ for all $s\leqslant t$.

As for the existence of generalized $(\varphi,\mathfrak{D})$-pullback attractors, we can see, analogously to what already has been presented in the literature so far, that in the scenario in which the evolution properties enjoy some compactness property, the following result holds.

Proposition 2.1. (*cf.*, [13, proposition 2.1]). *Let $\mathcal{X}$ be a family of Banach spaces, $\mathcal{S}$ an evolution process on $\mathcal{X}$, and $\mathfrak{D}$ a given universe. Assume also that*

- *$\mathcal{S}$ is an eventually compact evolution process, that is, there exists $\tau>0$ such that $S(t,t-\tau): X_{t-\tau}\to X_t$ is a compact map for each $t\in\mathbb{T}$;*
- *there exists a backwards bounded $\mathfrak{D}$-pullback absorbing family $\hat{B}\in\mathfrak{D}$.*

Then, given any decay function φ, $\mathcal{S}$ has a generalized $(\varphi,\mathfrak{D})$-pullback attractor $\hat{M}$. Additionally if $\hat{B}$ is closed, then $\hat{M}\in\mathfrak{D}$.

We recall that, given a metric space (X,d) and $B\subset X$ bounded, we define its *Kuratowski measure of noncompactness* by

$$\kappa(B) = \inf\{\delta > 0 : B \text{admits a finite cover by sets of diameter less than or equal to } \delta\},$$

and its *ball measure of noncompactness* by

$$\beta(B) = \inf\{r > 0 : B \text{ admits a finite cover by open balls of radius } r\}.$$

We could replace *less than or equal to* by *less than* and *open balls* by *closed balls* in the definitions of κ and β, respectively, and obtain the same quantities.

We work in the framework of a family of spaces $\mathcal{X} = \{(X_t, d_t)\}_{t\in\mathbb{T}}$ and hence, each X_t has its own Kuratowski measure of noncompactness and ball measure of noncompactness, which we could denote by κ_t and β_t, for each $t \in \mathbb{T}$. However, to simplify notation, and in the understanding that this will not cause any confusion, we denote the Kuratowski measure of noncompactness and ball measure of noncompactness in every X_t simply by κ and β, respectively.

As in [12, 13], inspired by [11] and [14], we make the following definition:

Definition 2. Given a family $\mathcal{X}$ of complete metric spaces, an evolution process $\mathcal{S}$ on $\mathcal{X}$, and a universe $\mathfrak{D}$, we say that $\mathcal{S}$ is $(\varphi, \mathfrak{D})$-pullback κ-dissipative if there exists $\omega > 0$ such that for any $\hat{D} \in \mathfrak{D}$ and $t \in \mathbb{T}$, there exist $C = C(\hat{D}, t) \geqslant 0$ and $\tau_0 = \tau_0(\hat{D}, t) \geqslant 0$ such that

$$\kappa\left(\bigcup_{\sigma\geqslant\tau} S(t, t-\sigma)D_{t-\sigma}\right) \leqslant C\varphi(\omega\tau) \quad \text{for all } \tau \geqslant \tau_0.$$

As in [13], the following result holds.

Proposition 2.3. (cf., [13, proposition 2.8]). *Consider a family $\mathcal{X}$ of complete metric spaces, $\mathcal{S}$ an evolution process on $\mathcal{X}$, and $\mathfrak{D}$ a universe. Assume that $\hat{B}$ is a uniformly $\mathfrak{D}$-pullback absorbing family for $\mathcal{S}$, and that there exist a decay function φ and $\omega > 0$ such that for each $t \in \mathbb{R}$, there exist $C = C(t) \geqslant 0$ and $\tau_0 = \tau_0(t) > 0$ such that*

$$\kappa(S(t, t-\tau)B_{t-\tau}) \leqslant C\varphi(\omega\tau) \quad for\ all \tau \geqslant \tau_0.$$

Then $\mathcal{S}$ is $(\varphi, \mathfrak{D})$-pullback κ-dissipative.

Also, with very similar proof as in [13] (which is inspired by [15, 16]), we have the following result presenting conditions to ensure the existence of a generalized $(\varphi, \mathfrak{D})$-pullback attractor.

Proposition 2.4. (*cf.,* [13, theorem 2.6]). *Consider a family of complete metric spaces $\mathcal{X}$, a time-continuous evolution process $\mathcal{S}$ on $\mathcal{X}$, and a given universe $\mathfrak{D}$. Assume also the following conditions:*

- $\mathbf{C_1}$ *there exists a closed and bounded family $\hat{B} \in \mathfrak{D}$ such that $\hat{B}$ is uniformly $\mathfrak{D}$-pullback absorbing;*
- $\mathbf{C_2}$ *there exist $\gamma, L > 0$ for which*

  $$d_{s+\tau}(S(s+\tau, s)x, S(s+\tau, s)y) \leqslant Ld_s(x, y) \quad for all x, y \in B_s, \ s \in \mathbb{R} and \tau \in [0, \gamma];$$
- $\mathbf{C_3}$ *for a given decay function φ, $\mathcal{S}$ is $(\varphi, \mathfrak{D})$-pullback κ-dissipative.*

Then, there exists a generalized $(\varphi, \mathfrak{D})$-pullback attractor $\hat{M}$ for $\mathcal{S}$, with $\hat{M} \subset \hat{B}$.

We now state and outline the proof of the two main results of this section, in which we present sufficient conditions to obtain a generalized exponential and polynomial $\mathfrak{D}_{ub}$-pullback attractor, which are based on [12, theorem 2.5] and [13, theorem 2.7], respectively.

Before that, we just introduce two concepts that are needed. For a complete metric space X and $B \subset X$, a function $\psi : X \times X \to \mathbb{R}^+$ is called *contractive* on B if for each sequence $\{x_n\}_{n\in\mathbb{N}} \subset B$ we have

$$\liminf_{m,n\to\infty} \psi(x_n, x_m) = 0.$$

We denote the set of such functions by $\mathrm{contr}(B)$.

A *pseudometric* in a set X is a function $\rho : X \times X \to [0, \infty)$ that satisfies:

- $\rho(x, x) = 0$ for all $x \in X$;
- $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
- $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$.

Also, for $B \subset X$, we say that ρ is *precompact on B* if any sequence in B has a Cauchy subsequence with respect to ρ.

Theorem 2.5. (The exponential case). *Consider a family of Banach spaces $\mathcal{X}$, a time-continuous evolution process $\mathcal{S}$ on $\mathcal{X}$, and the universe $\mathfrak{D}_{ub}$. Assume also that conditions 2.4 and 2.4 of Proposition 2.4 hold, and that $\hat{B}$ in 2.4 is also positively invariant and $\hat{B} \in \mathfrak{D}_{ub}$.*

Furthermore, assume that there exist constants $\mu \in (0, 1)$, $T > 0$ and $r > 0$ satisfying: given $t \in \mathbb{R}$ and $n \in \mathbb{N}$, there exist $m(n) \in \mathbb{N}$, functions $g_n : (\mathbb{R}^+)^{m(n)} \to \mathbb{R}^+$, $\psi_n : X_{t-nT} \times X_{t-nT} \to \mathbb{R}^+$ and pseudometrics $\rho_1^{(n)}, \ldots, \rho_{m(n)}^{(n)}$ on X_{t-nT} such that

1. *g_n is nondecreasing with respect to each variable, $g_n(0, \ldots, 0) = 0$ and g_n is continuous at $(0, \ldots, 0)$;*
2. *the pseudometrics $\rho_1^{(n)}, \ldots, \rho_{m(n)}^{(n)}$ are precompact on B_{t-nT};*
3. *$\psi_n \in \mathrm{contr}(B_{t-nT})$;*
4. *for $x, y \in B_{t-nT}$ we have*

   $$d_{t-(n-1)T}(S_n x, S_n y)^r \leqslant \mu d_{t-nT}(x, y)^r + g_n(\rho_1^{(n)}(x, y), \ldots, \rho_{m(n)}^{(n)}(x, y)) + \psi_n(x, y),$$

   where $S_n := S(t-(n-1)T, t-nT)$.

Then, for the decay function $\varphi(s) = \mu^s$, $\mathcal{S}$ is $(\varphi, \mathfrak{D}_{ub})$-pullback κ-dissipative, and it possesses a uniformly bounded generalized $(\varphi, \mathfrak{D}_{ub})$-pullback attractor $\hat{M} \in \mathfrak{D}_{ub}$.

Proof. Because $\hat{B} \in \mathfrak{D}_{ub}$, there exists a positive constant R such that $B_s \subset \overline{B}^s_R$ for all $s \in \mathbb{R}$. As in [12, theorem 2.5], for $A \subset B_{t-nT}$ we obtain

$$\kappa(S(t, t-nT)A)^r \leqslant 2^r \mu^n \kappa(A)^r \leqslant 2^r \mu^n (2R)^r,$$

which, in particular, implies that

$$\kappa(S(t, t-nT)B_{t-nT}) \leqslant 4R\mu^{\frac{n}{r}}.$$

Finally, for $s \geqslant T$ let $n \in \mathbb{N}$ be such that $s - T < nT \leqslant s$.

From the positive invariance of $\hat{B}$, we obtain

$$\begin{aligned}\kappa(S(t,t-s)B_{t-s}) &= \kappa\big(S(t,t-nT)S(t-nT,t-s)B_{t-s}\big)\\ &\leqslant \kappa\big(S(t,t-nT)B_{t-nT}\big) \leqslant 4R\mu^{\frac{n}{r}} \leqslant 4R\mu^{-\frac{1}{r}}\mu^{\frac{s}{rT}} \leqslant C\mu^{\omega s},\end{aligned}$$

where $C = 4R\mu^{-\frac{1}{r}}$ and $\omega = \frac{1}{rT}$. It follows from Proposition 2.3 that S is $(\varphi, \mathfrak{D}_{ub})$-pullback κ-dissipative, with decay function φ given by $\varphi(s) = \mu^s$, and it follows from Proposition 2.4 that S possesses a generalized $(\varphi, \mathfrak{D}_{ub})$-pullback attractor $\hat{M}$ with $\hat{M} \subset \hat{B}$. Because $\hat{B} \in \mathfrak{D}_{ub}$, so does $\hat{M}$. □

Theorem 2.6. (The polynomial case). *Consider a family of Banach spaces $\mathcal{X}$, a time-continuous evolution process $\mathcal{S}$ on $\mathcal{X}$, and the universe $\mathfrak{D}_{ub}$. Assume also that conditions 2.4 and 2.4 of Proposition 2.4 hold, and that $\hat{B}$ in 2.4 is also positively invariant and $\hat{B} \in \mathfrak{D}_{ub}$.*

Furthermore, suppose also that there exist $\beta \in (0,1)$, $r > 0$, $T > 0$, $C > 0$ satisfying: given $t \in \mathbb{R}$ and $n \in \mathbb{N}$, there exist $m(n) \in \mathbb{N}$, functions $g_n, h_n : (\mathbb{R}^+)^{m(n)} \to \mathbb{R}^+$, $\psi_n, \theta_n : X_{t-nT} \times X_{t-nT} \to \mathbb{R}^+$ and pseudometrics $\rho_1^{(n)}, \dots, \rho_{m(n)}^{(n)}$ on X_{t-nT} such that

i. *g_n, h_n are nondecreasing with respect to each variable, $g_n(0,\dots,0) = h_n(0,\dots,0) = 0$ and g_n, h_n are continuous at $(0,\dots,0)$;*
ii. *$\rho_1^{(n)}, \dots, \rho_{m(n)}^{(n)}$ are precompact on B_{t-nT};*
iii. *$\psi_n, \theta_n \in \mathrm{contr}(B_{t-nT})$;*
iv. *for all $x, y \in B_{t-nT}$ we have*

$$d_{t-(n-1)T}(S_n x, S_n y)^r \leqslant d_{t-nT}(x,y)^r + g_n(\rho_1^{(n)}(x,y), \dots, \rho_{m(n)}^{(n)}(x,y)) + \psi_n(x,y);$$

and

$$\begin{aligned}d_{t-(n-1)T}(S_n x, S_n y)^r \leqslant C\Big[&d_{t-nT}(x,y)^r - d_{t-(n-1)T}(S_n x, S_n y)^r\\ &+ g_n(\rho_1^{(n)}(x,y), \dots, \rho_{m(n)}^{(n)}(x,y)) + \psi_n(x,y)\Big]^\beta\\ &+ h_n(\rho_1^{(n)}(x,y), \dots, \rho_{m(n)}^{(n)}(x,y)) + \theta_n(x,y),\end{aligned}$$

where $S_n := S(t-(n-1)T, t-nT)$.

Then, for the decay function $\varphi(s) = s^{\frac{\beta}{r(\beta-1)}}$, $\mathcal{S}$ is $(\varphi, \mathfrak{D}_{ub})$-pullback κ-dissipative. Also, $\mathcal{S}$ has a uniformly bounded generalized $(\varphi, \mathfrak{D}_{ub})$-pullback attractor $\hat{M} \in \mathfrak{D}_{ub}$.

Proof. As in [13, theorem 2.7], consider the function $u : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $u(s) = (3C)^{-1/\beta} s^{1/\beta} + s$. Because u is an increasing bijective function, it has an inverse function that we will denote $v : \mathbb{R}^+ \to \mathbb{R}^+$, which also is increasing. The composite functions u^n and v^n are also increasing for $n \geqslant 2$ and satisfy $u \leqslant u^2 \leqslant u^3 \leqslant \cdots$ and $v \geqslant v^2 \geqslant v^3 \geqslant \cdots$ for any $s \geqslant 0$.

Because $\hat{B} \in \mathfrak{D}_{ub}$, there exists a positive constant R such that $B_s \subset \overline{B}^s_R$ for all $s \in \mathbb{R}$. For $n \in \mathbb{N}$ and $A \subset B_{t-nT}$ we have

$$\kappa(S(t,t-nT)A)^r \leqslant 2^r v^n(\kappa(A)^r) \leqslant 2^r v^n(2^r R^r). \tag{2.1}$$

From [13, proposition A.2], there exists $n_0 \in \mathbb{N}$, such that for $n \geqslant n_0$, we have

$$v^n(2^r R^r) \leqslant \left[(n-n_0)\left(\frac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}} + (2R)^{\frac{r(\beta-1)}{\beta}}\right]^{\frac{\beta}{\beta-1}}.$$

Thus for all $n \geqslant n_0$, it holds that

$$\kappa(S(t,t-nT)B_{t-nT}) \leqslant 2\left[(n-n_0)\left(\frac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}} + (2R)^{\frac{r(\beta-1)}{\beta}}\right]^{\frac{\beta}{r(\beta-1)}}.$$

If $s \geqslant (n_0+1)T$, take $n \in \mathbb{N}$ such that $\frac{s}{T} - 1 < n \leqslant \frac{s}{T}$. We have $n > n_0$, and from the positive invariance of $\hat{B}$, we obtain

$$\begin{aligned}\kappa(S(t,t-s)B_{t-s}) &= \kappa\big(S(t,t-nT)S(t-nT,t-s)B_{t-s}\big)\\ &\leqslant \kappa\big(S(t,t-nT)B_{t-nT}\big)\\ &\leqslant 2\left[(n-n_0)\left(\frac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}} + (2R)^{\frac{r(\beta-1)}{\beta}}\right]^{\frac{\beta}{r(\beta-1)}}\\ &\leqslant 2\left[\left(\frac{s}{T}-1-n_0\right)\left(\frac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}}\right]^{\frac{\beta}{r(\beta-1)}}\\ &= 2(\omega s - \eta)^{\frac{\beta}{r(\beta-1)}},\end{aligned}$$

where

$$\omega = \frac{1}{T}\left(\frac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}} \text{ and } \eta = (1+n_0)\left(\frac{1}{\beta}-1\right)(1+3C)^{-\frac{1}{\beta}}.$$

Now, there exist $C_1 > 0$ and $s_0 > 0$ such that $2(\omega s - \eta)^{\frac{\beta}{r(\beta-1)}} \leqslant C_1(\omega s)^{\frac{\beta}{r(\beta-1)}}$ for $s \geqslant s_0$. Then, taking $s \geqslant s_1 := \max\{s_0, (n_0+1)T\}$, we have

$$\kappa(S(t,t-s)B_{t-s}) \leqslant C_1(\omega s)^{\frac{\beta}{r(\beta-1)}}$$

for $s \geqslant s_1$.

It follows from Proposition 2.3 that $\mathcal{S}$ is $(\varphi, \mathfrak{D}_{ub})$-pullback κ-dissipative, with the decay function φ given by $\varphi(s) = s^{\frac{\beta}{r(\beta-1)}}$, and therefore, from Theorem 2.4, there exists a generalized $(\varphi, \mathfrak{D}_{ub})$-pullback attractor $\hat{M}$ for $\mathcal{S}$ with $\hat{M} \subset \hat{B}$. Because $\hat{B}$ belongs to $\mathfrak{D}_{ub}$, so does $\hat{M}$. □

3 | Application to a Nonautonomous Wave Equation With Nonlocal Damping and Time-Dependent Speed of Propagation: The Proof of Theorem A

3.1 | Well-Posedness of the Problem

In order to obtain existence, uniqueness and continuity with respect to initial data for (W–Eq), we first present some estimates related to the functions f and F defined in 1 and 1. The proof of the next result is standard, and can be found in detail, for instance, in [17].

Proposition 3.1. *For all $u, v \in \mathbb{R}$, we have*

i. $|f(u)| \leqslant 2c_0(1 + |u|^{q+1})$;

ii. $|f(u) - f(v)| \leqslant 2c_0(1 + |u|^q + |v|^q)|u - v|$;

iii. $|F(u)| \leqslant 4c_0(1 + |u|^{q+2})$;

iv. $|F(u) - F(v)| \leqslant 8c_0(1 + |u|^{q+1} + |v|^{q+1})|u - v|$;

v. *For each $K > 0$,*

$$\int_{-K}^{K} |f(u)|du \leqslant 8c_0(1 + K^{q+2});$$

vi. *There exist $\mu_0 \in (0, \lambda_1)$ and $M > 0$ such that*

$$f'(u) > -\mu_0 \text{ and } \frac{f(u)}{u} > -\mu_0 \text{ for all } |u| > M; \tag{3.1}$$

vii. *For μ_0 and M given above, there exists a constant $e_0 > 0$ such that*

$$F(u) \geqslant -\left(\frac{\mu_0 + \lambda_1}{4}\right)u^2 - 8c_0(1 + M^{q+2}) \text{ for all } |u| > M \tag{3.2}$$

$$F(u) \leqslant uf(u) + \frac{\mu_0}{2}u^2 + e_0 \text{ for all } |u| > M \tag{3.3}$$

and

$$|F(u)| \leqslant 8c_0(1 + M^{q+2}) \text{ for all } |u| \leqslant M \tag{3.4}$$

In what follows $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L^2(\Omega)$ and $\langle \cdot, \cdot \rangle_1$ is the inner product in $H_0^1(\Omega)$, where $\langle u, v \rangle_1 := \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v \rangle = \langle \nabla u, \nabla v \rangle$. Furthermore, to simplify notation, we set $\| \cdot \|_{L^p} := \| \cdot \|_{L^p(\Omega)}$, for $p \geqslant 1$, and $\| \cdot \|_{H_0^1} := \| \cdot \|_{H_0^1(\Omega)}$.

Proposition 3.2. *There exist $C_0 \geqslant 0$ and $q_0 \geqslant 0$ such that*

$$\langle F(u), 1 \rangle = \int_\Omega F(u)dx \geqslant -\left(\frac{\mu_0 + \lambda_1}{4}\right)\|u\|_{L^2}^2 - C_0 \geqslant -\left(\frac{\mu_0 + \lambda_1}{4\lambda_1}\right)\|u\|_{H_0^1}^2 - C_0,$$

and

$$\langle F(u), 1 \rangle \leqslant \langle f(u), u \rangle + \frac{\mu_0}{2}\|u\|_{L^2}^2 + q_0 \leqslant \langle f(u), u \rangle + \frac{\mu_0}{2\lambda_1}\|u\|_{H_0^1}^2 + q_0.$$

Proof. The first inequality follows directly from (3.2), by taking $C_0 = 8c_0(1 + M^{q+2})|\Omega| > 0$. The second one is a consequence of inequalities (3.3) and (3.4) taking $q_0 = e_0|\Omega_1| + 8c_0(1 + M^{q+2})|\Omega_2| + 8Mc_0(1 + M^{q+2})$. □

Lemma 3.3. *There exists a constant $C_R \geqslant 0$ such that*

$$\|f(u) - f(v)\|_{L^2} \leqslant C_R\|u - v\|_{H_0^1}$$

for all $u, v \in H_0^1(\Omega)$ satisfying $\|u\|_{H_0^1} \leqslant R$ and $\|v\|_{H_0^1} \leqslant R$.

Proof. This result is a consequence of item 3.1 of Proposition 3.1. □

Definition 3.4. A weak solution of (W–Eq), associated to the initial condition $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, is a function u such that for all $T > s$,

$$u \in L^\infty(s, T; H_0^1(\Omega)),$$
$$u_t \in L^\infty(s, T; L^2(\Omega)), \quad u_{tt} \in L^2(s, T; H^{-1}(\Omega)),$$

which, for a.e. $t \in (s, T)$ and all $v \in H_0^1(\Omega)$, satisfies

$$\varepsilon(t)\langle u_{tt}, v \rangle_{-1} + \langle \nabla u, \nabla v \rangle + k(t)\langle \|u_t(t, \cdot)\|_{L^2}^p u_t, v \rangle + \langle f(u), v \rangle = \langle h, v \rangle, \tag{3.5}$$

where $\langle \cdot, \cdot \rangle_{-1}$ denotes the duality from $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and it satisfies the initial conditions $u(s) = u_0$ and $u_t(s) = u_1$.

The proof of the following theorem is based on [18, theorem 6.1, p. 222] and [19, section 7.2.2].

Proposition 3.5. (Existence of weak solutions). *Suppose that conditions 1–1 hold. Then, for each $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists at least one weak solution u of (W–Eq).*

Sketch of the proof using the Faedo–Galerkin method. Consider an orthonormal basis $\{w_k\}$ of $L^2(\Omega)$, consisting of eigenvectors of $-\Delta$, each associated to the eigenvalue λ_k, with $w_k \in H_0^1(\Omega)$ for all $k \in \mathbb{N}$. Let $u^m = \sum_{k=1}^m \gamma_k^m(t)w_k \in W^m := \text{span}\{w_1, \ldots, w_m\}$ and consider the following ordinary differential equations

$$\begin{cases} \langle \varepsilon(t)u_{tt}^m, w_k \rangle + \langle \nabla u^m, \nabla w_k \rangle \\ \quad + \langle k(t)\|u_t^m(t, \cdot)\|_{L^2}^p u_t^m, w_k \rangle + \langle f(u^m), w_k \rangle = \langle h, w_k \rangle, \\ u^m(s) = P^m u_0, \ u_t^m(s) = P^m u_1, \end{cases} \tag{3.6}$$

for $k = 1, \ldots, m$, where $P^m : L^2(\Omega) \to W^m$ is the orthogonal projection of $L^2(\Omega)$ into W^m. It follows from [20, theorem 3.4 pg. 287] that, for each $T > 0$, (3.6) has a unique maximal solution $\gamma^m = (\gamma_1^m, \ldots, \gamma_m^m) \in C([s, t_m], \mathbb{R}^m)$ with $s < t_m \leqslant T$. The *a priori* estimates below will show that $t_m = T$.

We multiply the kth equation of (3.6) by $\frac{d}{dt}\gamma_k^m(t)$ and add the equations for $k = 1, \dots, m$ to obtain

$$\frac{\varepsilon(t)}{2}\frac{d}{dt}\|u_t^m\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u^m\|_{L^2}^2 + k(t)\|u_t^m(t,\cdot)\|_{L^2}^{p+2} + \langle f(u^m), u_t^m\rangle = \langle h, u_t^m\rangle.$$

Note that

$$\langle f(u^m), u_t^m\rangle = \frac{d}{dt}\int_\Omega F(u^m)dx \quad \text{and} \quad \eta|u^m|^{q+2} - c_1 \leqslant F(u^m) \leqslant 32c_0(1+|u^m|^{q+2}),$$

and also that $\|P^m u_0\|_{H_0^1} \leqslant \|u_0\|_{H_0^1}$ and $\|P^m u_1\|_{L^2} \leqslant \|u_1\|_{L^2}$ for all $m \in \mathbb{N}$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Therefore, it follows from 1 and the Gronwall inequality that

$$\begin{cases} \{u_t^m\} \text{ is bounded in } L^\infty(s,T;L^2(\Omega)), \\ \{u^m\} \text{ is bounded in } L^\infty(s,T;H_0^1(\Omega)), \\ \{u^m\} \text{ is bounded in } L^{q+2}(s,T;L^{q+2}(\Omega)). \end{cases} \tag{3.7}$$

Using that $|f(u^m)| \leqslant 8c_0(1+|u^m|^{q+1})$, (3.7), Proposition 3.1, and Banach–Alaoglu's Theorem, it follows that, up to a subsequence,

$$\begin{cases} u^m \overset{*}{\rightharpoonup} u \text{ in } L^\infty(s,T;H_0^1(\Omega)), \\ u_t^m \overset{*}{\rightharpoonup} u_t \text{ in } L^\infty(s,T;L^2(\Omega)), \\ f(u^m) \rightharpoonup \mathcal{X}_1 \text{ in } L^{\frac{q+2}{q+1}}(s,T;L^{\frac{q+2}{q+1}}(\Omega)), \\ \|u_t^m\|_{L^2(\Omega)}^p u_t^m \rightharpoonup \mathcal{X}_2 \text{ in } L^{\frac{p+2}{p+1}}(s,T;L^2(\Omega)). \end{cases} \tag{3.8}$$

Note that, for $v \in H_0^1(\Omega)$, we have

$$\begin{aligned} |\langle \varepsilon(t)u_{tt}^m, P^m v\rangle| &= |\langle h, P^m v\rangle - (\nabla u^m, \nabla P^m v) \\ &\quad - \langle k(t)\|u_t^m\|_{L^2(\Omega)}^p u_t^m, P^m v\rangle - \langle f(u^m), P^m v\rangle| \\ &\leqslant \|h\|_{L^2}\|v\|_{L^2} + \|u^m\|_{H_0^1}\|v\|_{H_0^1} \\ &\quad + k(t)\|u_t^m\|_{L^2}^{p+1}\|v\|_{L^2} + \|f(u^m)\|_{L^{\frac{q+2}{q+1}}}\|v\|_{L^{q+2}} \\ &\leqslant \|h\|_{L^2}\|v\|_{L^2} + \|u^m\|_{H_0^1}\|v\|_{H_0^1} \\ &\quad + k(t)\|u_t^m\|_{L^2}^{p+1}\|v\|_{L^2} + \|f(u^m)\|_{L^{\frac{q+2}{q+1}}}\|v\|_{H_0^1}, \end{aligned}$$

because $H_0^1(\Omega) \subset L^{q+2}(\Omega)$ continuously. Then, the sequence $\{u_{tt}^m\}$ is bounded in $L^2(s,T;H^{-1}(\Omega))$. From this last fact, we deduce

$$u_{tt}^m \rightharpoonup u_{tt} \text{ in } L^2(s,T;H^{-1}(\Omega)) \tag{3.9}$$

Now, let us fix $k \in \mathbb{N}$. Using (3.6), (3.9), and making $m \to \infty$, we obtain

$$\langle \varepsilon(t)u_{tt}, w_k\rangle_{-1} + \langle \nabla u, \nabla w_k\rangle + \langle k(t)\mathcal{X}_2, w_k\rangle + \langle \mathcal{X}_1, w_k\rangle = \langle h, w_k\rangle.$$

Hence, for all $v \in H_0^1(\Omega)$, we obtain

$$\langle \varepsilon(t)u_{tt}, v\rangle_{-1} + \langle \nabla u, \nabla v\rangle + \langle k(t)\mathcal{X}_2, v\rangle + \langle \mathcal{X}_1, v\rangle = \langle h, v\rangle \tag{3.10}$$

Finally, because $f \in C^1(\mathbb{R})$ and $u^m \to u$ a.e. in $\Omega \times (s,T)$, we obtain $f(u^m) \to f(u)$ a.e. in $\Omega \times (s,T)$. Thus, by (3.8) we have $\mathcal{X}_1 = f(u)$. With an analogous reasoning we deduce that $\mathcal{X}_2 = \|u_t\|_{L^2(\Omega)}^p u_t$. Therefore, from (3.10) and (3.9), u is a weak solution of (W–Eq).

Remark 3.6. Any weak solution u to (W–Eq) has a continuous representative, in the sense that $u \in C([\tau,T], H_0^1(\Omega))$ and $u_t \in C([\tau,T], L^2(\Omega))$. Moreover, by a density argument, we can multiply (W–Eq) by u_t to we obtain the following *energy equality*:

$$\begin{aligned} &\frac{\varepsilon(t)}{2}\frac{d}{dt}\|u_t\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|u\|_{H_0^1}^2 \\ &\quad + k(t)\|u_t(t,\cdot)\|_{L^2}^{p+2} \\ &\quad + \langle f(u), u_t\rangle = \langle h, u_t\rangle \text{for a.e. } t \in (s,T). \end{aligned} \tag{3.11}$$

Also, the uniqueness of weak solutions, their continuous dependence on initial data, and the fact that weak solutions are *globally defined* (that is, defined in $[s,\infty)$), will be proven later (see Proposition 3.7).

Translated problem. We use the translations of the original nonautonomous problem in order to obtain a problem defined in $[0,\infty)$ rather than on $[s,\infty)$. To be more specific, fixing $s \in \mathbb{R}$ and setting $v(t,x) := u(t+s,x)$ for $t \geqslant 0$ and $x \in \Omega$, we formally have

$$\begin{aligned} &\varepsilon_s(t)v_{tt}(t,x) - \Delta v(t,x) + k_s(t)\|v_t(t,\cdot)\|_{L^2}^p v_t(t,x) + f(v(t,x)) - h(x) \\ &= \varepsilon(t+s)u_{tt}(t+s,x) - \Delta u(t+s,x) \\ &\quad + k(t+s)\|u_t(t+s,\cdot)\|_{L^2}^p u_t(t+s,x) + f(u(t+s,x)) - h(x) = 0, \end{aligned}$$

where the boundary and initial conditions become

$$v(t,x) = u(t+s,x) = 0 \quad for (t,x) \in [0,\infty) \times \partial\Omega,$$

$$v(0,x) = u(s,x) = u_0(x), \ v_t(0,x) = u_t(s,x) = u_1(x) \quad for \ x \in \Omega.$$

Thus, we will study the boundary and initial conditions problem

$$\begin{cases} \varepsilon_s(t)v_{tt}(t,x) - \Delta v(t,x) + k_s(t)\|v_t(t,\cdot)\|_{L^2}^p v_t(t,x) \\ \quad + f(v(t,x)) = h(x), \ (t,x) \in [0,\infty) \times \Omega, \\ v(t,x) = 0, \ (t,x) \in [0,\infty) \times \partial\Omega, \\ v(0,x) = u_0(x), \ v_t(0,x) = u_1(x), \ x \in \Omega, \end{cases} \tag{tNWE}$$

instead of (W–Eq). This problem is equivalent to the initial one, but with the nonautonomous terms being $\varepsilon_s(\cdot) = \varepsilon(\cdot + s)$ and $k_s(\cdot) = k(\cdot + s)$ instead of ε and k.

Fix $s \in \mathbb{R}$ and consider the initial data $V_0 := (u_0, u_1) \in X_s$. For V_0, we denote by $V_p(\cdot, V_0) : [0,\infty) \to X_{\cdot+s}$ a weak solution of (tNWE), which for now we assume that is defined in $[0,\infty)$.

Let $V_1, V_2 \in X_s$ be the initial data for (W–Eq) such that $\|V_1\|_{X_s} \leqslant R$ and $\|V_2\|_{X_s} \leqslant R$. We have that there exists a constant $c_R \geqslant 0$ such that for all $\tau \geqslant 0$, $\left\|V_p(\tau,V_1)\right\|_{X_{\tau+s}} \leqslant c_R$ and $\left\|V_p(\tau,V_2)\right\|_{X_{\tau+s}} \leqslant c_R$ (see Proposition 3.14, which, in particular, proves that weak solutions are globally defined). The next result refers to the continuity of the problem with respect to initial data and, in particular, gives us the uniqueness of solutions.

Proposition 3.7. *Given $R > 0$, there exists a constant $K = K(R) \geqslant 0$ such that*

$$\left\|V_p(\tau,V_1) - V_p(\tau,V_2)\right\|_{X_{\tau+s}} \leqslant e^{K\tau}\|V_1 - V_2\|_{X_s}$$

for every $V_1, V_2 \in X_s$ where $\|V_1\|_{X_s} \leqslant R$ and $\|V_2\|_{X_s} \leqslant R$.

Proof. If $V_p(\tau, V_1) = (v(\tau), v_t(\tau))$ and $V_p(\tau, V_2) = (w(\tau), w_t(\tau))$ for $\tau \geqslant 0$, denote $Z(\tau) = (z(\tau), z_t(\tau))$, where $z = v - w$. Then $V_p(\tau, V_1) - V_p(\tau, V_2) = (v(\tau) - w(\tau), v_t(\tau) - w_t(\tau)) = (z(\tau), z_t(\tau)) = Z(\tau)$. Because v and w are weak solutions of (W−Eq), we formally obtain that

$$\varepsilon_s(\tau) z_{tt} - \Delta z + k_s(\tau)(\|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t) + f(v) - f(w) = 0 \tag{3.12}$$

Multiplying formally (3.12) by z_t and integrating over Ω, we have

$$\begin{aligned} \varepsilon_s(\tau)\langle z_t, z_{tt}\rangle - \langle \Delta z, z_t\rangle + k_s(\tau)\langle \|v_t\|_{L^2}^p v_t \\ - \|w_t\|_{L^2}^p w_t, z_t\rangle + \langle f(v) - f(w), z_t\rangle = 0. \end{aligned} \tag{3.13}$$

Now observe that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|Z(\tau)\|_{X_{\tau+s}}^2 &= \frac{1}{2}\frac{d}{dt}\left(\|z\|_{H_0^1}^2 + \varepsilon_s(\tau)\|z_t\|_{L^2}^2\right) \\ &= -\langle \Delta z, z_t\rangle + \frac{1}{2}\varepsilon_s'(\tau)\|z_t\|_{L^2}^2 + \varepsilon_s(\tau)\langle z_t, z_{tt}\rangle \end{aligned}$$

and (3.13) becomes

$$\begin{aligned} \frac{d}{dt}\|Z(\tau)\|_{X_{\tau+s}}^2 &= \varepsilon_s'(\tau)\|z_t\|_{L^2}^2 - 2k_s(\tau)\langle \|v_t\|_{L^2}^p v_t \\ &\quad - \|w_t\|_{L^2}^p w_t, z_t\rangle - 2\langle f(v) - f(w), z_t\rangle. \end{aligned}$$

We know from [21, lemma 2.2] that

$$\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle \geqslant 2^{-p}\|z_t\|_{L^2}^{p+2},$$

which implies

$$-2k_s(\tau)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle \leqslant -2^{1-p}k_s(\tau)\|z_t\|_{L^2}^{p+2},$$

and by Lemma 3.3 we can show that there exists a constant $C_R \geqslant 0$ such that

$$\begin{aligned} -2\langle f(v) - f(w), z_t\rangle &\leqslant 2\|f(v) - f(w)\|_{L^2}\|z_t\|_{L^2} \\ &\leqslant 2C_R\|z\|_{H_0^1}\|z_t\|_{L^2}. \end{aligned}$$

Then,

$$\begin{aligned} \frac{d}{d\tau}\|Z(\tau)\|_{X_{\tau+w}}^2 &\leqslant \varepsilon_s'(\tau)\|z_t\|_{L^2}^2 - 2^{1-p}k_s(\tau)\|z_t\|_{L^2}^{p+2} \\ &\quad + 2C_R\|z\|_{H_0^1}\|z_t\|_{L^2} \\ &\overset{(*)}{\leqslant} \varepsilon_s'(\tau)\|z_t\|_{L^2}^2 - k(\tau)2^{1-p}\|z_t\|_{L^2}^{p+2} \\ &\quad + k_s(\tau)2^{1-p}\|z_t\|_{L^2}^{p+2} + \hat{C}_R\|z\|_{H_0^1}^{\frac{p+2}{p+1}} \\ &\leqslant \varepsilon_s'(\tau)\|z_t\|_{L^2}^2 + \hat{C}_R\|z\|_{H_0^1}^2 \overset{(**)}{\leqslant} C\varepsilon_s(\tau)\|z_t\|_{L^2}^2 \\ &\quad + \hat{C}_R\|z\|_{H_0^1}^2 = \tilde{C}_R\|Z(\tau)\|_{X_{\tau+s}}^2, \end{aligned} \tag{3.14}$$

with $\tilde{C}_R := \max\{C, \hat{C}_R\}$.

To obtain $(*)$ we note that, applying Young's inequality,

$$\begin{aligned} 2C_R\|z\|_{H_0^1}\|z_t\|_{L^2} &= [k_s(\tau)(p+2)]^{\frac{1}{p+2}} 2^{\frac{1-p}{p+2}}\|z_t\|_{L^2} 2 \\ &\quad C_R[k_s(\tau)(p+2)]^{-\frac{1}{p+2}} 2^{\frac{p-1}{p+2}}\|z\|_{H_0^1} \\ &\leqslant \frac{\left([k_s(\tau)(p+2)]^{\frac{1}{p+2}} 2^{\frac{1-p}{p+2}}\|z_t\|_{L^2}\right)^{p+2}}{p+2} \\ &\quad + \frac{\left(2C_R[k_s(\tau)(p+2)]^{-\frac{1}{p+2}} 2^{\frac{p-1}{p+2}}\|z\|_{H_0^1}\right)^{\frac{p+2}{p+1}}}{\frac{p+2}{p+1}} \\ &= k_s(\tau)2^{1-p}\|z_t\|_{L^2}^{p+2} + \hat{C}_R\|z\|_{H_0^1}^{\frac{p+2}{p+1}}, \end{aligned}$$

where $\hat{C}_R = \left(\frac{p+1}{p+2}\right)(2C_R)^{\frac{p+2}{p+1}}[k(p+2)]^{-\frac{1}{p+1}} 2^{\frac{p-1}{p+1}}$. In $(**)$ we observe that for each $t \in \mathbb{R}$ we have $\varepsilon_s'(t) \leqslant 0 \leqslant C\varepsilon_s(t)$, by hypothesis.

Integrating (3.14) from 0 to τ, we obtain

$$\|Z(\tau)\|_{X_{\tau+s}} \leqslant e^{\frac{\tilde{C}_R}{2}\tau}\|Z(0)\|_{X_s},$$

and the result is proved taking $K = \frac{\tilde{C}_R}{2}$. □

Hence, under conditions 1–1, for $p \geqslant 0$, $s \in \mathbb{R}$, $V_0 \in X_s$ and $t \geqslant 0$, we can define the map $S_p(t,s) : X_s \to X_t$ by

$$S_p(t,s)V_0 = V_p(t-s, V_0).$$

The family $\mathcal{S}_p = \{S_p(t,s) : t \geqslant s\}$ defines a time-continuous evolution process on the family $\mathcal{X} = \{(X_t, \|\cdot\|_t)\}$ defined in (1.1).

The proof of the existence of a generalized $(\varphi, \mathfrak{D}_{ub})$-pullback attractor for the evolution process $\mathcal{S}_p$ involves, as a key step, showing the existence of a uniformly $\mathfrak{D}_{ub}$-pullback absorbing family for $\mathcal{S}_p$, as required by condition 2.4 of Proposition 2.4, and this is done in Section 3.2. The obtained family, while closed and uniformly bounded, does not have the property of positive invariance. However, this is not an issue, as Proposition 3.23 enables the construction of a new family that preserves these properties and it is also positively invariant (hence, satisfying 2.4).

We provide, separately, the proofs for the cases $p > 0$ and $p = 0$. The case $p > 0$ follows the ideas of [22] and [13] while the case $p = 0$ is inspired by the work developed in [6] and [12].

3.2 | Existence of a Uniformly $\mathfrak{D}_{ub}$-Pullback Absorbing Family

Case $p > 0$. Consider a constant $\delta > 0$. Multiplying formally (tNWE) by $v_t + \delta v$ and integrating over Ω, we obtain

$$\begin{aligned} &\varepsilon_s(t)\langle v_t, v_{tt}\rangle_{-1} + \langle v_t, Av\rangle + k_s(t)\|v_t\|_{L^2}^{p+2} + \langle v_t, f(v)\rangle \\ &+ \delta\varepsilon_s(t)\langle v, v_{tt}\rangle_{-1} + \delta\langle v, Av\rangle + \delta k_s(t)\|v_t\|_{L^2}^p\langle v, v_t\rangle \\ &+ \delta\langle v, f(v)\rangle = \langle h, v_t\rangle + \delta\langle h, v\rangle. \end{aligned} \tag{3.15}$$

Noting that

$$\frac{d}{dt}\left(\frac{1}{2}\|v\|_{H_0^1}^2+\frac{1}{2}\varepsilon_s(t)\|v_t\|_{L^2}^2\right)$$
$$-\frac{1}{2}\varepsilon_s'(t)\|v_t\|_{L^2}^2=\varepsilon_s(t)\langle v_t,v_{tt}\rangle_{-1}+\langle v_t,Av\rangle,$$

$$\frac{d}{dt}\langle F(v),1\rangle=\langle\frac{d}{dt}F(v),1\rangle+\langle F(v),0\rangle$$
$$=\langle f(v)v_t,1\rangle=\langle f(v),v_t\rangle,$$
$$\delta\langle v,Av\rangle=\delta\langle A^{\frac12}v,A^{\frac12}v\rangle=\delta\|v\|_{H_0^1}^2,$$
$$\frac{d}{dt}\big(\delta\varepsilon_s(t)\langle v_t,v\rangle\big)=\delta\varepsilon_s'(t)\langle v_t,v\rangle+\delta\varepsilon_s(t)\|v_t\|_{L^2}^2+\delta\varepsilon_s(t)\langle v,v_{tt}\rangle_{-1},$$

and

$$\frac{d}{dt}\langle h,v\rangle=\langle h,v_t\rangle,$$

Equation (3.15) becomes

$$\begin{aligned}\frac{d}{dt}&\left(\frac{1}{2}\|v\|_{H_0^1}^2+\frac{1}{2}\varepsilon_s(t)\|v_t\|_{L^2}^2+\langle F(v),1\rangle+\delta\varepsilon_s(t)\langle v_t,v\rangle-\langle h,v\rangle\right)\\&=-k_s(t)\|v_t\|_{L^2}^{p+2}-\delta k_s(t)\|v_t\|_{L^2}^{p}\langle v,v_t\rangle-\delta\|v\|_{H_0^1}^2\\&\quad+\delta\varepsilon_s'(t)\langle v_t,v\rangle+\delta\varepsilon_s(t)\|v_t\|_{L^2}^2+\frac{1}{2}\varepsilon_s'(t)\|v_t\|_{L^2}^2\\&\quad-\delta\langle v,f(v)\rangle+\delta\langle h,v\rangle.\end{aligned}\tag{3.16}$$

For each initial data $V_0=(u_0,u_1)\in X_s$, we set the function $E_s(\cdot,V_0):[0,\infty)\to\mathbb{R}_+$ by

$$E_s(t)=\frac{1}{2}\|v\|_{H_0^1}^2+\frac{1}{2}\varepsilon_s(t)\|v_t\|_{L^2}^2+\langle F(v),1\rangle+\frac{\lambda_1+\mu_0}{4}\|v\|_{L^2}^2+C_0,$$

where the constant C_0 was obtained in Proposition 3.2. and $(v(t),v_t(t))$ is the solution related to this initial data. Understanding this function is paramount to prove the existence of a uniformly $\mathfrak{D}_{ub}$-pullback absorbing family for the evolution process S_p related to (W−Eq).

For this purpose, we first define, for a positive δ (which will be conveniently chosen later), the auxiliary function $\Psi_\delta^s(\cdot,V_0):[0,\infty)\to\mathbb{R}$ by

$$\Psi_\delta^s(t)=\frac{1}{2}\|v\|_{H_0^1}^2+\frac{1}{2}\varepsilon_s(t)\|v_t\|_{L^2}^2+\langle F(v),1\rangle+\delta\varepsilon_s(t)\langle v_t,v\rangle-\langle h,v\rangle.$$

To simplify notation we write $E_s(t)$ and $\Psi_\delta^s(t)$ instead of $E_s(t,V_0)$ and $\Psi_\delta^s(t,V_0)$, but we keep in mind the dependence of both functions on the initial data V_0. The next result establishes a relation between these two functions.

Proposition 3.8. *For a constant* $0<\delta\leqslant\delta_0:=\min\left\{\frac{\lambda_1-\mu_0}{8L},\frac{\lambda_1-\mu_0}{8\lambda_1},\frac{\lambda_1-\mu_0}{6L^2},\frac{k_0}{4L}\right\}$, *there exists* $d_0>0$ *such that, for* $t\geqslant0$,

$$\left(\frac{\lambda_1-\mu_0}{4\lambda_1}\right)E_s(t)-d_0\leqslant\Psi_\delta^s(t)\leqslant\frac{11}{8}E_s(t)+d_0.$$

Proof. Note that

$$\frac{1}{2}\|v\|_{H_0^1}^2+\frac{1}{2}\varepsilon_s(t)\|v_t\|_{L^2}^2$$
$$+\langle F(v),1\rangle=E_s(t)-\left(\frac{\lambda_1+\mu_0}{4}\right)\|v\|_{L^2}^2-C_0\leqslant E_s(t)$$

and using the Poincaré's inequality, we obtain

$$\begin{aligned}\delta\varepsilon_s(t)|\langle v_t,v\rangle|&\leqslant\delta\varepsilon_s(t)\|v_t\|_{L^2}\|v\|_{L^2}\\&\leqslant\frac{\delta\varepsilon_s(t)}{\sqrt{\lambda_1}}\|v_t\|_{L^2}\|v\|_{H_0^1}=\delta\varepsilon_s(t)\|v_t\|_{L^2}\frac{1}{\sqrt{\lambda_1}}\|v\|_{H_0^1}\\&\leqslant\frac{\delta\varepsilon_s(t)}{2}\left(\|v_t\|_{L^2}^2+\frac{1}{\lambda_1}\|v\|_{H_0^1}^2\right)\\&\leqslant\frac{\delta L}{2\lambda_1}\|v\|_{H_0^1}^2+\frac{\delta\varepsilon_s(t)}{2}\|v_t\|_{L^2}^2\\&\leqslant\max\left\{\frac{\delta L}{2\lambda_1},\frac{\delta}{2}\right\}\left(\|v\|_{H_0^1}^2+\varepsilon_s(t)\|v_t\|_{L^2}^2\right)\\&\leqslant\max\left\{\frac{\delta L}{\lambda_1},\delta\right\}E_s(t)\leqslant\frac{\lambda_1-\mu_0}{8\lambda_1}E_s(t)\leqslant\frac{1}{4}E_s(t).\end{aligned}$$

Additionally, we have that

$$\begin{aligned}|\langle h,v\rangle|&\leqslant\frac{1}{16}\left(1-\frac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2+\frac{4}{\lambda_1-\mu_0}h_0^2\leqslant\frac{1}{8}\left(1-\frac{\mu_0}{\lambda_1}\right)E_s(t)\\&\quad+\frac{4}{\lambda_1-\mu_0}h_0^2\leqslant\frac{1}{8}E_s(t)+\frac{4}{\lambda_1-\mu_0}h_0^2.\end{aligned}$$

Then, for $t\geqslant0$ and $0<\delta\leqslant\delta_0$,

$$\Psi_\delta^s(t)\leqslant E_s(t)+\delta\varepsilon_s(t)\langle v_t,v\rangle-\langle h,v\rangle\leqslant\frac{11}{8}E_s(t)+\frac{4}{\lambda_1-\mu_0}h_0^2.$$

On the other hand,

$$\begin{aligned}\Psi_\delta^s(t)&=E_s(t)-\left(\frac{\lambda_1+\mu_0}{4}\right)\|v\|_{L^2}^2-C_0\\&\quad+\delta\varepsilon_s(t)\langle v_t,v\rangle-\langle h,v\rangle\\&\geqslant E_s(t)-\left(\frac{\lambda_1+\mu_0}{2\lambda_1}\right)E_s(t)-C_0\\&\quad-\left(\frac{\lambda_1-\mu_0}{8\lambda_1}\right)E_s(t)-\left(\frac{\lambda_1-\mu_0}{8\lambda_1}\right)E_s(t)-\frac{4}{\lambda_1-\mu_0}h_0^2\\&=\left(\frac{\lambda_1-\mu_0}{4\lambda_1}\right)E_s(t)-C_0-\frac{4}{\lambda_1-\mu_0}h_0^2.\end{aligned}$$

The proof is concluded if we set $d_0:=C_0+\frac{4}{\lambda_1-\mu_0}h_0^2$. □

The following Lemmas 3.9, 3.10, and 3.11 will be necessary for the proof of Proposition 3.12 .

Lemma 3.9. *There exists a constant* $g_0>0$ *such that for all* $t\geqslant0$, *we have*

$$\begin{aligned}-\langle f(v),v\rangle&\leqslant-\langle F(v),1\rangle-\left(\frac{\lambda_1+\mu_0}{4}\right)\|v\|_{L^2}^2-C_0\\&\quad+\frac{1}{4}\left(\frac{3\mu_0}{\lambda_1}+1\right)\|v\|_{H_0^1}^2+g_0\\&\leqslant\frac{1}{4}\left(\frac{3\mu_0}{4\lambda_1}+1\right)\|v\|_{H_0^1}^2+g_0,\end{aligned}$$

where C_0 is given in Proposition 3.2.

Proof. It is a consequence of Proposition 3.1. Here, $g_0 := e_0|\Omega_1| + 8c_0(1+M^{q+2})|\Omega_2| + 8Mc_0(1+M^{q+2}) + C_0 > 0$. □

Lemma 3.10. *There exists a constant $K^* > 0$ such that for $t \geqslant 0$ and $\delta > 0$, we have*

$$-k_s(t)\|v_t\|_{L^2}^{p+2} - \delta k_s(t)\|v_t\|_{L^2}^{p}\langle v, v_t\rangle \leqslant -k_0\|v_t\|_{L^2}^{p+2}\left(1 - \frac{\delta K^* k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\right) + \frac{\delta}{12}\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2.$$

Proof. The proof is analogous to that one of [13, lemma 3.16]□

Lemma 3.11. *For all $0 < \delta \leqslant \delta_0$,*

$$|\delta\varepsilon_s'(t)\langle v_t, v\rangle| \leqslant \frac{1}{2}\|v_t\|_{L^2}^2 + \frac{\delta}{12}\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2. \tag{3.17}$$

Proof.

$$\begin{aligned}
|\delta\varepsilon_s'(t)\langle v_t, v\rangle| &= \delta|\varepsilon_s'(t)||\langle v_t, v\rangle| \leqslant \delta L\|v_t\|_{L^2}\|v\|_{L^2}\\
&\leqslant \frac{1}{2}(\|v_t\|_{L^2}^2 + \delta^2L^2\|v\|_{L^2}^2)\\
&\leqslant \frac{1}{2}\|v_t\|_{L^2}^2 + \frac{\delta^2L^2}{2\lambda_1}\|v\|_{H_0^1}^2\\
&= \frac{1}{2}\|v_t\|_{L^2}^2 + \frac{\delta^2L^2}{2\lambda_1}.\frac{6}{\lambda_1 - \mu_0}.\frac{\lambda_1 - \mu_0}{6}\|v\|_{H_0^1}^2\\
&\leqslant \frac{1}{2}\|v_t\|_{L^2}^2 + \frac{\lambda_1 - \mu_0}{12\lambda_1}.\frac{6\delta^2L^2}{\lambda_1 - \mu_0}\|v\|_{H_0^1}^2\\
&\leqslant \frac{1}{2}\|v_t\|_{L^2}^2 + \frac{\delta}{12}\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2,
\end{aligned} \tag{3.18}$$

where in the last inequality we used that $\frac{6\delta^2L^2}{\lambda_1-\mu_0} \leqslant \delta$ when $\delta \leqslant \delta_0 \leqslant \frac{\lambda_1-\mu_0}{6L^2}$. □

Proposition 3.12. *There exist $\eta_1, \eta_2 > 0$ such that for $0 < \delta \leqslant \delta_0$ and $t \geqslant 0$, we have*

$$\frac{d}{dt}\Psi_\delta^s(t) \leqslant -k_0\|v_t\|_{L^2}^{p+2}\left(\frac{1}{2} - \delta\eta_1\left(\Psi_\delta^s(t) + d_0\right)^{\frac{p}{2(p+1)}}\right) - \frac{8}{11}\left(1 - \frac{\mu_0}{\lambda_1}\right)\delta\Psi_\delta^s(t) + \eta_2.$$

Proof. From the hypotheses imposed on the function ε, we have

$$\frac{1}{2}\varepsilon_s'(t)\|v_t\|_{L^2}^2 \leqslant \frac{1}{2}L\|v_t\|_{L^2}^2 \tag{3.19}$$

$$\delta\varepsilon_s(t)\|v_t\|_{L^2}^2 \leqslant \delta_0L\|v_t\|_{L^2}^2 \tag{3.20}$$

and

$$\frac{\delta}{2}\varepsilon_s(t)\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v_t\|_{L^2}^2 \leqslant \frac{\delta_0}{2}L\|v_t\|_{L^2}^2 \tag{3.21}$$

Adding and subtracting $\frac{\delta}{2}\varepsilon_s(t)\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v_t\|_{L^2}^2$ to the Equation (3.16), we obtain

$$\begin{aligned}
\frac{d}{dt}\Psi_\delta^s(t) &= -k_s(t)\|v_t\|_{L^2}^{p+2} - \delta k_s(t)\|v_t\|_{L^2}^{p}\langle v, v_t\rangle\\
&\quad - \delta\|v\|_{H_0^1}^2 + \frac{1}{2}\varepsilon_s'(t)\|v_t\|_{L^2}^2\\
&\quad + \delta\varepsilon_s'(t)\langle v_t, v\rangle + \delta\varepsilon_s(t)\|v_t\|_{L^2}^2 - \delta\langle v, f(v)\rangle + \delta\langle h, v\rangle\\
&\quad + \frac{\delta}{2}\varepsilon_s(t)\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v_t\|_{L^2}^2 - \frac{\delta}{2}\varepsilon_s(t)\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v_t\|_{L^2}^2.
\end{aligned} \tag{3.22}$$

Joining the inequalities (3.19)–(3.21), Lemmas 3.9, 3.10, and 3.11 and Equation (3.22), we obtain

$$\begin{aligned}
\frac{d}{dt}\Psi_\delta^s(t) &\leqslant -k_0\|v_t\|_{L^2}^{p+2}\left(1 - \frac{\delta K^* k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\right)\\
&\quad + \frac{\delta}{12}\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2 - \delta\|v\|_{H_0^1}^2\\
&\quad - \delta\left[\langle F(v), 1\rangle + \left(\frac{\lambda_1 + \mu_0}{4}\right)\|v\|_{L^2}^2 + C_0\right]\\
&\quad + \frac{\delta}{4}\left(\frac{3\mu_0}{\lambda_1} + 1\right)\|v\|_{H_0^1}^2 + \delta_0 g_0\\
&\quad + \frac{\delta}{16}\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2 + \frac{4\delta_0}{\lambda_1 - \mu_0}h_0^2\\
&\quad + \frac{1}{2}\|v_t\|_{L^2}^2 + \frac{\delta}{12}\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2\\
&\quad + \frac{L}{2}\|v_t\|_{L^2}^2 + \delta_0L\|v_t\|_{L^2}^2 + \frac{\delta_0L}{2}\|v_t\|_{L^2}^2\\
&\quad - \frac{\delta}{2}\varepsilon_s(t)\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v_t\|_{L^2}^2.
\end{aligned} \tag{3.23}$$

Now observe that setting $C_1 := \frac{1}{2} + \frac{L}{2} + \delta_0L + \frac{\delta_0L}{2} \geqslant 0$ and $C_2 := \delta_0g_0 + \frac{4}{\lambda_1-\mu_0}h_0^2\delta_0 \geqslant 0$, we have

$$\begin{aligned}
\frac{d}{dt}\Psi_\delta^s(t) &\leqslant -k_0\|v_t\|_{L^2}^{p+2}\left(1 - \frac{\delta K^* k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\right)\\
&\quad + C_1\|v_t\|_{L^2}^2 - \delta\left[\langle F(v), 1\rangle + \left(\frac{\lambda_1 + \mu_0}{4}\right)\|v\|_{L^2}^2 + C_0\right]\\
&\quad + C_2 - \frac{\delta}{2}\varepsilon_s(t)\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v_t\|_{L^2}^2 - \frac{25\delta}{48}\left(1 - \frac{\mu_0}{\lambda_1}\right)\|v\|_{H_0^1}^2\\
&\leqslant -k_0\|v_t\|_{L^2}^{p+2}\left(1 - \frac{\delta K^* k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\right) + \frac{k_0}{2}\|v_t\|_{L^2}^{p+2}\\
&\quad + C_3 - \delta\left[\langle F(v), 1\rangle + \left(\frac{\lambda_1 + \mu_0}{4}\right)\|v\|_{L^2}^2 + C_0\right]\\
&\quad + C_2 - \frac{\delta}{2}\left(1 - \frac{\mu_0}{\lambda_1}\right)\left(\|v\|_{H_0^1}^2 + \varepsilon_s(t)\|v_t\|_{L^2}^2\right)\\
&\leqslant -k_0\|v_t\|_{L^2}^{p+2}\left(\frac{1}{2} - \frac{\delta K^* k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\right)\\
&\quad + C_3 - \delta\left[\langle F(v), 1\rangle + \left(\frac{\lambda_1 + \mu_0}{4}\right)\|v\|_{L^2}^2 + C_0\right]\\
&\quad + C_2 - \frac{\delta}{2}\left(1 - \frac{\mu_0}{\lambda_1}\right)\left(\|v\|_{H_0^1}^2 + \varepsilon_s(t)\|v_t\|_{L^2}^2\right),
\end{aligned} \tag{3.24}$$

where

$$C_3 := \frac{p}{p+2}\left(\frac{4}{k_0(p+2)}\right)^{\frac{2}{p}} C_1^{\frac{p+2}{p}}$$

arises from an application of the Young's inequality to obtain $C_1\|v_t\|_{L^2}^2 \leqslant \frac{k_0}{2}\|v_t\|_{L^2}^{p+2} + C_3$.

From Proposition 3.8 and from the fact that $\|v\|_{H_0^1}^{\frac{p}{p+1}} \leqslant (2E_s(t))^{\frac{p}{2(p+1)}}$, we have

$$\begin{aligned}
&-k_0\|v_t\|_{L^2}^{p+2}\left(\frac{1}{2} - \frac{\delta K^* k_1}{k_0}\|v\|_{H_0^1}^{\frac{p}{p+1}}\right)\\
&\leqslant -k_0\|v_t\|_{L^2}^{p+2}\left[\frac{1}{2} - \frac{\delta K^* k_1}{k_0}\left(\frac{8\lambda_1}{\lambda_1-\mu_0}\right)^{\frac{p}{2(p+1)}}\left(\Psi_\delta^s(t)+d_0\right)^{\frac{p}{2(p+1)}}\right].
\end{aligned} \tag{3.25}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{2}\left(1-\frac{\mu_0}{\lambda_1}\right)(\|v\|_{H_0^1}^2 + \varepsilon_s(t)\|v_t\|_{L^2}^2)\\
&\quad + \langle F(v),1\rangle + \left(\frac{\lambda_1+\mu_0}{4}\right)\|v\|_{L^2}^2 + C_0\\
&= \frac{1}{2}\left(1-\frac{\mu_0}{\lambda_1}\right)(\|v\|_{H_0^1}^2 + \varepsilon_s(t)\|v_t\|_{L^2}^2)\\
&\quad + E_s(t) - \frac{1}{2}\|v\|_{H_0^1}^2 - \frac{1}{2}\varepsilon_s(t)\|v_t\|_{L^2}^2\\
&= -\frac{\mu_0}{2\lambda_1}\left(\|v\|_{H_0^1}^2 + \varepsilon_s(t)\|v_t\|_{L^2}^2\right) + E_s(t)\\
&\geqslant -\frac{\mu_0}{\lambda_1}E_s(t) + E_s(t) = E_s(t)\left(1-\frac{\mu_0}{\lambda_1}\right),
\end{aligned}$$

which implies, again using Proposition 3.8, that

$$\begin{aligned}
&-\delta\left[\frac{1}{2}\left(1-\frac{\mu_0}{\lambda_1}\right)(\|v\|_{H_0^1}^2 + \varepsilon_s(t)\|v_t\|_{L^2}^2)\right.\\
&\left.+\langle F(v),1\rangle + \left(\frac{\lambda_1+\mu_0}{4}\right)\|v\|_{L^2}^2 + C_0\right]\\
&\leqslant -\left(1-\frac{\mu_0}{\lambda_1}\right)\delta E_s(t)\\
&\leqslant -\frac{8}{11}\left(1-\frac{\mu_0}{\lambda_1}\right)\delta\Psi_\delta^s(t) + \frac{8}{11}d_0\delta_0\left(1-\frac{\mu_0}{\lambda_1}\right).
\end{aligned} \tag{3.26}$$

Joining (3.25) and (3.26), we conclude that

$$\begin{aligned}
\frac{d}{dt}\Psi_\delta^s(t) \leqslant{}& -k_0\|v_t\|_{L^2}^{p+2}\\
&\left[\frac{1}{2} - \frac{\delta K^* k_1}{k_0}\left(\frac{8\lambda_1}{\lambda_1-\mu_0}\right)^{\frac{p}{2(p+1)}}\left(\Psi_\delta^s(t)+d_0\right)^{\frac{p}{2(p+1)}}\right]\\
&-\frac{8}{11}\left(1-\frac{\mu_0}{\lambda_1}\right)\delta\Psi_\delta^s(t) + \frac{8}{11}d_0\delta_0\left(1-\frac{\mu_0}{\lambda_1}\right) + C_2 + C_3.
\end{aligned}$$

The proof is done taking $\eta_1 = \frac{K^* k_1}{k_0}\left(\frac{8\lambda_1}{\lambda_1-\mu_0}\right)^{\frac{p}{2(p+1)}}$ and $\eta_2 = \frac{8}{11}\delta_0 g_0\left(1-\frac{\mu_0}{\lambda_1}\right) + C_2 + C_3$. □

From Proposition 3.12 and with a similar reasoning of [13, lemma 3.18], we have the following lemma.

Lemma 3.13. *Assume $0 < \delta \leqslant \delta_0$. If for a fixed $\tau \geqslant 0$ we have*

$$\Psi_\delta^s(\tau) \leqslant (2\eta_1\delta)^{-\frac{2(p+1)}{p}} - \frac{11}{8}\frac{\lambda_1}{\lambda_1-\mu_0}\eta_2\delta^{-1} - d_0,$$

then for all $t \geqslant \tau$ we have

$$\Psi_\delta^s(t) \leqslant (2\eta_1\delta)^{-\frac{2(p+1)}{p}} - d_0.$$

Proposition 3.14. *Given $R > 0$, there exists a constant $c_R \geqslant 0$ such that $E_s(t) \leqslant c_R$ for all $V_0 \in \overline{B}_R^s$ and $t \geqslant 0$.*

Proof. Using 3.1 from Proposition 3.1 and considering the continuous inclusion $H_0^1(\Omega) \hookrightarrow L^{q+2}(\Omega)$, we can prove that there exists a constant $\alpha_0 \geqslant 0$ such that $|\langle F(v),1\rangle| \leqslant \alpha_0(1+\|v\|_{H_0^1}^{q+2})$. Fix $R > 0$ and let $V_0 = (u_0,u_1) \in X_s$ such that $V_0 \in \overline{B}_R^s$. We deduce

$$\begin{aligned}
E_s(0) &= \frac{1}{2}\left(\|u_0\|_{H_0^1}^2 + \varepsilon(s)\|u_1\|_{L^2}^2\right)\\
&\quad + \langle F(u_0),1\rangle + \frac{\lambda_1+\mu_0}{4}\|u_0\|_{L^2}^2 + C_0\\
&\leqslant \frac{1}{2}\|V_0\|_{X_s}^2 + \alpha_0(1+\|u_0\|_{H_0^1}^{q+2}) + \frac{\lambda_1+\mu_0}{4\lambda_1}\|u_0\|_{H_0^1}^2 + C_0\\
&\leqslant \frac{1}{2}R^2 + \alpha_0 R^{q+2} + \frac{\lambda_1+\mu_0}{4\lambda_1}R^2 + \alpha_0 + C_0.
\end{aligned}$$

By Proposition 3.8,

$$\begin{aligned}
\Psi_\delta^s(0) &\leqslant \frac{11}{8}E_s(0) + d_0\\
&\leqslant \frac{11}{8}\left(\frac{1}{2}R^2 + \alpha_0 R^{q+2} + \frac{\lambda_1+\mu_0}{4\lambda_1}R^2 + \alpha_0 + C_0\right) + d_0 := \gamma_R.
\end{aligned}$$

By an analogous process to the one carried out in the proof of [13, proposition 3.14], we obtain that there exists a constant $c_R \geqslant 0$ such that $E_s(t) \leqslant c_R$ for $t \geqslant 0$. □

Again, analogous proofs to [13, proposition 3.20] and [13, theorem 3.12] yield the next two results, where the second is almost an immediate consequence of the first.

Proposition 3.15. *There exists a constant $R_0 > 0$ such that for any $R > 0$ and $s \in \mathbb{R}$, we have*

$$\limsup_{\tau\to\infty}\left(\sup_{\|V_0\|_{X_s}\leqslant R} E_s(\tau)\right) \leqslant R_0.$$

Proposition 3.16. (Existence of a pullback absorbing family). *There exists $r_0 > 0$ such that for each $R > 0$, there exists $\tau_0 = \tau_0(R) \geqslant 0$ with*

$$\left\|S_p(t,t-\tau)(u_0,u_1)\right\|_{X_t} \leqslant r_0$$

for all $\tau \geqslant \tau_0$, $t \in \mathbb{R}$ and $(u_0,u_1) \in X_{t-\tau}$ with $\|(u_0,u_1)\|_{X_{t-\tau}} \leqslant R$.

This implies that the family $\hat{B} = \{B_t : B_t \subset X_t, t \in \mathbb{R}\}$ with $B_t = \overline{B}_{r_0}^t$ for all $t \in \mathbb{R}$ is a uniformly $\mathfrak{D}_{ub}$-pullback absorbing family for $\mathcal{S}_p$ which is, additionally, uniformly bounded, that is, $\hat{B} \in \mathfrak{D}_{ub}$.

Case $p=0$. For each initial data $V_0=(u_0,u_1)\in X_s$, we set the function $E_s(\cdot,V_0):[0,\infty)\to\mathbb{R}_+$ by

$$E_s(t)=\|v\|^2_{H_0^1}+\varepsilon_s(t)\|v_t\|^2_{L^2}.$$

Additionally, for a positive δ (which we will choose later) we define the auxiliary function $\Psi^s_\delta(\cdot,V_0):[0,\infty)\to\mathbb{R}$ by

$$\Psi^s_\delta(t)=E_s(t)+\delta k_s(t)\|v\|^2_{L^2}+2\delta\varepsilon_s(t)\langle v_t,v\rangle+2\langle F(v),1\rangle-2\langle h,v\rangle.$$

Proposition 3.17. *For $0<\delta\leqslant\delta_0$ and $\|v\|_{H_0^1}\leqslant R$, there exists $\hat{C}_R\geqslant 0$ (depending on R) such that for all $t\geqslant 0$, it holds that*

$$\left(\frac{\lambda_1-\mu_0}{4\lambda_1}\right)E_s(t)-\left(\frac{4h_0^2}{\lambda_1-\mu_0}+C_0\right)\leqslant\Psi^s_\delta(t)\leqslant\hat{C}_RE_s(t)+2q_0,$$

where C_0 and q_0 are the constants that arise from Proposition 3.2 and δ_0 is again taken as in Proposition 3.8.

Proof. Firstly, note that

$$\begin{aligned}2|\langle h,v\rangle|&\leqslant 2\|h\|_{L^2}\|v\|_{L^2}\\&=2\left(\frac{2}{\sqrt{\lambda_1-\mu_0}}\|h\|_{L^2}\frac{\sqrt{\lambda_1-\mu_0}}{2}\|v\|_{L^2}\right)\leqslant\frac{4h_0^2}{\lambda_1-\mu_0}\\&\quad+\frac{\lambda_1-\mu_0}{4\lambda_1}\|v\|^2_{H_0^1}\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}2\delta\varepsilon_s(t)\left|\langle v_t,v\rangle\right|&\leqslant 2\delta\varepsilon_s(t)\|v_t\|_{L^2}\|v\|_{L^2}\\&=2\varepsilon_s(t)\left(\frac{1}{\sqrt{2}}\|v_t\|_{L^2}\delta\sqrt{2}\|v\|_{L^2}\right)\\&\leqslant\varepsilon_s(t)\left(\frac12\|v_t\|^2_{L^2}+2\delta^2\|v\|^2_{L^2}\right)\\&\leqslant\frac{\varepsilon_s(t)}{2}\|v_t\|^2_{L^2}+2\delta^2\varepsilon_s(t)\|v\|^2_{L^2}\\&\leqslant\frac{\varepsilon_s(t)}{2}\|v_t\|^2_{L^2}+2\delta^2L\|v\|^2_{L^2}\\&\leqslant\frac{\varepsilon_s(t)}{2}\|v_t\|^2_{L^2}+\frac{\delta k_0}{2}\|v\|^2_{L^2},\end{aligned}\tag{3.28}$$

where in the last inequality we consider that $0<\delta\leqslant\frac{k_0}{4L}$.

It follows from inequalities (3.27), (3.28), and Proposition 3.2 that

$$\begin{aligned}\Psi^s_\delta(t)&=\|v\|^2_{H_0^1}+\varepsilon_s(t)\|v_t\|^2_{L^2}+\delta k_s(t)\|v\|^2_{L^2}\\&\quad+2\delta\varepsilon_s(t)\langle v_t,v\rangle+2\langle F(v),1\rangle-2\langle h,v\rangle\\&\geqslant\|v\|^2_{H_0^1}+\varepsilon_s(t)\|v_t\|^2_{L^2}+\delta k_0\|v\|^2_{L^2}-\frac{\varepsilon_s(t)}{2}\|v_t\|^2_{L^2}-\frac{\delta k_0}{2}\|v\|^2_{L^2}\\&\quad-\left(\frac{\mu_0+\lambda_1}{2\lambda_1}\right)\|v\|^2_{H_0^1}-C_0-\frac{4h_0^2}{\lambda_1-\mu_0}-\left(\frac{\lambda_1-\mu_0}{4\lambda_1}\right)\|v\|^2_{H_0^1}\\&\geqslant\frac{\lambda_1-\mu_0}{4\lambda_1}\|v\|^2_{H_0^1}+\frac{\varepsilon_s(t)}{2}\|v_t\|^2_{L^2}+\frac{\delta k_0}{2}\|v\|^2_{L^2}-\frac{4h_0^2}{\lambda_1-\mu_0}-C_0\\&\geqslant\frac{\lambda_1-\mu_0}{4\lambda_1}\left(\|v\|^2_{H_0^1}+\varepsilon_s(t)\|v_t\|^2_{L^2}\right)\\&\quad-\frac{4h_0^2}{\lambda_1-\mu_0}-C_0=\frac{\lambda_1-\mu_0}{4\lambda_1}E_s(t)-\left(\frac{4h_0^2}{\lambda_1-\mu_0}+C_0\right),\end{aligned}$$

because $\frac12\geqslant\frac{\lambda_1-\mu_0}{4\lambda_1}$ and $\frac{\delta k_0}{2}\|v\|^2_{L^2}\geqslant 0$.

On the other hand, it follows from Proposition 3.2 and Lemma 3.3 that for $v\in H_0^1(\Omega)$ satisfying $\|v\|_{H_0^1}\leqslant R$, we have

$$\begin{aligned}2\langle F(v),1\rangle&\leqslant 2\langle f(v),v\rangle+\frac{\mu_0}{\lambda_1}\|v\|^2_{H_0^1}+2q_0\\&\leqslant 2\|f(v)\|_{L^2}\|v\|_{L^2}+\frac{\mu_0}{\lambda_1}\|v\|^2_{H_0^1}+2q_0\\&\leqslant\|f(v)\|^2_{L^2}+\|v\|^2_{L^2}+\frac{\mu_0}{\lambda_1}\|v\|^2_{H_0^1}+2q_0\\&\leqslant(C_R^2+\frac{1}{\lambda_1}+\frac{\mu_0}{\lambda_1})\|v\|^2_{H_0^1}+2q_0.\end{aligned}\tag{3.29}$$

Applying (3.27), (3.28), and (3.29) to the expression for $\Psi^s_\delta(t)$, we obtain

$$\begin{aligned}\Psi^s_\delta(t)&\leqslant\|v\|^2_{H_0^1}+\varepsilon_s(t)\|v_t\|^2_{L^2}+\delta k_1\|v\|^2_{L^2}\\&\quad+\frac{\varepsilon_s(t)}{2}\|v_t\|^2_{L^2}+\frac{\delta k_0}{2}\|v\|^2_{L^2}\\&\quad+(C_R^2+\frac{1}{\lambda_1}+\frac{\mu_0}{\lambda_1})\|v\|^2_{H_0^1}+2q_0\\&\quad+\frac{4h_0^2}{\lambda_1-\mu_0}+\frac{\lambda_1-\mu_0}{4\lambda_1}\|v\|^2_{H_0^1}\\&\leqslant\|v\|^2_{H_0^1}+\varepsilon_s(t)\|v_t\|^2_{L^2}+\frac{\delta k_1}{\lambda_1}\|v\|^2_{H_0^1}\\&\quad+\frac{\varepsilon_s(t)}{2}\|v_t\|^2_{L^2}+\frac{\delta k_0}{2\lambda_1}\|v\|^2_{H_0^1}\\&\quad+(C_R^2+\frac{1}{\lambda_1}+\frac{\mu_0}{\lambda_1})\|v\|^2_{H_0^1}+2q_0\\&\quad+\frac{4h_0^2}{\lambda_1-\mu_0}+\frac{\lambda_1-\mu_0}{4\lambda_1}\|v\|^2_{H_0^1}\\&\leqslant\left(1+\frac{\delta_0k_1}{\lambda_1}+\frac{\delta_0k_0}{2\lambda_1}+C_R^2+\frac{1}{\lambda_1}+\frac{\mu_0}{\lambda_1}+\frac{\lambda_1-\mu_0}{4\lambda_1}\right)\|v\|^2_{H_0^1}\\&\quad+\frac{3\varepsilon_s(t)}{2}\|v_t\|^2_{L^2}+2q_0\\&\leqslant\hat{C}_RE_s(t)+2q_0,\end{aligned}$$

where $\hat{C}_R=max\left\{1+\frac{\delta_0k_1}{\lambda_1}+\frac{\delta_0k_0}{2\lambda_1}+C_R^2+\frac{1}{\lambda_1}+\frac{\mu_0}{\lambda_1}+\frac{\lambda_1-\mu_0}{4\lambda_1},\frac32\right\}$. □

Proposition 3.18. *For any fixed $\delta>0$,*

$$\begin{aligned}&\frac{d}{dt}\Psi^s_\delta(t)+\left(2k_s(t)-\varepsilon_s'(t)-2\delta\varepsilon_s(t)\right)\|v_t\|^2_{L^2}\\&\quad+2\delta\|v\|^2_{H_0^1}+2\delta\langle f(v),v\rangle-2\delta\langle h,v\rangle=2\delta\varepsilon_s'(t)\langle v_t,v\rangle.\end{aligned}$$

Proof. This result is obtained by multiplying $\varepsilon_s(t)v_{tt}+k_s(t)v_t+Av+f(v)=h$ by $2v_t+2\delta v$ and integrating over Ω. □

Proposition 3.19. *The function*

$$\begin{aligned}\Gamma^s_\delta(t)&=\left(k_s(t)-\varepsilon_s'(t)-3\delta\varepsilon_s(t)-\frac{k_0}{k_0+1}\right)\|v_t\|^2_{L^2}\\&\quad+\left(\delta-\frac{(1+k_0)\delta^2L^2}{k_0\lambda_1}\right)\|v\|^2_{H_0^1}-\delta^2k_s(t)\|v\|^2_{L^2}-2\delta^2\varepsilon_s(t)\langle v_t,v\rangle\\&\quad-\frac{\delta\mu_0}{\lambda_1}\|v\|^2_{H_0^1}\end{aligned}$$

is positive for $0 < \delta \leqslant \delta_0$ sufficiently small fixed.

Proof. Note that, because $-2\delta^2\varepsilon_s(t)\langle v_t, v\rangle \geqslant -\delta^2\varepsilon_s(t)\|v_t\|_{L^2}^2 - \delta^2 L\|v\|_{L^2}^2 \geqslant -\delta^2\varepsilon_s(t)\|v_t\|_{L^2}^2 - \frac{\delta^2 L}{\lambda_1}\|v\|_{H_0^1}^2$ and $-\delta^2 k_s(t)\|v\|_{L^2}^2 \geqslant -\delta^2 k_1\|v\|_{L^2}^2 \geqslant -\frac{\delta^2 k_1}{\lambda_1}\|v\|_{H_0^1}^2$, we have

$$\Gamma_\delta^s(t) \geqslant \left(k_s(t) - \varepsilon_s'(t) - 3\delta\varepsilon_s(t) - \frac{1}{\eta} - \delta^2\varepsilon_s(t)\right)\|v_t\|_{L^2}^2 + \left(\delta - \frac{\delta^2 k_1}{\lambda_1} - \frac{\eta\delta^2 L^2}{\lambda_1} - \frac{\delta\mu_0}{\lambda_1} - \frac{\delta^2 L}{\lambda_1}\right)\|v\|_{H_0^1}^2,$$

where $\eta := \frac{1+k_0}{k_0} > \frac{1}{k_0}$. It is clear that there exists $0 < \delta < \delta_0$ sufficiently small such that $3\delta + \delta^2 \leqslant \frac{k_0\eta - 1}{\eta L}$ and $\delta \leqslant \frac{\lambda_1 - \mu_0}{k_1 + \eta L + L}$. Then,

$$3\delta\varepsilon_s(t) + \frac{1}{\eta} + \delta^2\varepsilon_s(t) \leqslant 3\delta L + \frac{1}{\eta} + \delta^2 L \leqslant \left(\frac{k_0\eta - 1}{\eta L}\right)L + \frac{1}{\eta} \leqslant k_0 \leqslant k_s(t) \leqslant k_s(t) - \varepsilon_s'(t),$$

and, consequently,

$$\left(k_s(t) - \varepsilon_s'(t) - 3\delta\varepsilon_s(t) - \frac{1}{\eta} - \delta^2\varepsilon_s(t)\right)\|v_t\|_{L^2}^2 \geqslant 0.$$

On the other hand, because $\delta \leqslant \frac{\lambda_1 - \mu_0}{k_1 + \eta L + L}$, we conclude that $\left(\delta - \frac{\delta^2 k_1}{\lambda_1} - \frac{\eta\delta^2 L^2}{\lambda_1} - \frac{\delta\mu_0}{\lambda_1} - \frac{\delta^2 L}{\lambda_1}\right)\|v\|_{H_0^1}^2 \geqslant 0.$ □

Proposition 3.20. *It holds that*

$$\frac{d}{dt}\Psi_\delta^s(t) + \delta\Psi_\delta^s(t) + k_s(t)\|v_t\|_{L^2}^2 + \Gamma_\delta^s(t) \leqslant 2\delta q_0,$$

where Γ_δ^s is the function defined in the previous proposition.

Proof. Note that for any $\eta > 0$ fixed we have

$$2\delta\varepsilon_s'(t)\langle v_t, v\rangle \leqslant 2\delta L\|v_t\|_{L^2}\|v\|_{L^2} = 2\left(\frac{1}{\sqrt{\eta}}\|v_t\|_{L^2}\right)\left(\sqrt{\eta}\delta L\|v\|_{L^2}\right) \leqslant \frac{1}{\eta}\|v_t\|_{L^2}^2 + \frac{\eta\delta^2 L^2}{\lambda_1}\|v\|_{H_0^1}^2$$

which, together with Proposition 3.18, implies that

$$\frac{d}{dt}\Psi_\delta^s(t) \leqslant \left(\varepsilon_s'(t) + 2\delta\varepsilon_s(t) + \frac{1}{\eta} - 2k_s(t)\right)\|v_t\|_{L^2}^2 + \left(\frac{\eta\delta^2 L^2}{\lambda_1} - 2\delta\right)\|v\|_{H_0^1}^2 - 2\delta\langle f(v), v\rangle + 2\delta\langle h, v\rangle$$

and then, by Proposition 3.2, we have

$$\begin{aligned}
&\frac{d}{dt}\Psi_\delta^s(t) + \delta\Psi_\delta^s(t) \\
&\leqslant \delta E_s(t) + \delta^2 k_s(t)\|v\|_{L^2}^2 + 2\delta^2\varepsilon_s(t)\langle v_t, v\rangle + 2\delta\langle F(v), 1\rangle \\
&\quad - 2\delta\langle h, v\rangle + \left(\varepsilon_s'(t) + 2\delta\varepsilon_s(t) + \frac{1}{\eta} - 2k_s(t)\right)\|v_t\|_{L^2}^2 \\
&\quad + \left(\frac{\eta\delta^2 L^2}{\lambda_1} - 2\delta\right)\|v\|_{H_0^1}^2 - 2\delta\langle f(v), v\rangle + 2\delta\langle h, v\rangle \\
&\leqslant \delta E_s(t) + \delta^2 k_s(t)\|v\|_{L^2}^2 + 2\delta^2\varepsilon_s(t)\langle v_t, v\rangle + 2\delta\langle f(v), v\rangle \\
&\quad + \frac{\delta\mu_0}{\lambda_1}\|v\|_{H_0^1}^2 + 2\delta q_0 + \left(\varepsilon_s'(t) + 2\delta\varepsilon_s(t) + \frac{1}{\eta} - 2k_s(t)\right)\|v_t\|_{L^2}^2 \\
&\quad + \left(\frac{\eta\delta^2 L^2}{\lambda_1} - 2\delta\right)\|v\|_{H_0^1}^2 - 2\delta\langle f(v), v\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\Psi_\delta^s(t) + \delta\Psi_\delta^s(t) + k_s(t)\|v_t\|_{L^2}^2 \\
&\leqslant \delta E_s(t) + \delta^2 k_s(t)\|v\|_{L^2}^2 + 2\delta^2\varepsilon_s(t)\langle v_t, v\rangle \\
&\quad + \frac{\delta\mu_0}{\lambda_1}\|v\|_{H_0^1}^2 + 2\delta q_0 + \left(\varepsilon_s'(t) + 3\delta\varepsilon_s(t) + \frac{1}{\eta} - k_s(t)\right)\|v_t\|_{L^2}^2 \\
&\quad + \left(\frac{\eta\delta^2 L^2}{\lambda_1} - \delta\right)\|v\|_{H_0^1}^2 - \delta\varepsilon_s(t)\|v_t\|_{L^2}^2 - \delta\|v\|_{H_0^1}^2,
\end{aligned} \tag{3.30}$$

which can be written as

$$\begin{aligned}
&\frac{d}{dt}\Psi_\delta^s(t) + \delta\Psi_\delta^s(t) + k_s(t)\|v_t\|_{L^2}^2 + \Gamma_\delta^s(t) \\
&\leqslant \delta E_s(t) + 2\delta q_0 - \delta\varepsilon_s(t)\|v_t\|_{L^2}^2 - \delta\|v\|_{H_0^1}^2 = 2\delta q_0.
\end{aligned}$$

□

It is a consequence of Propositions 3.19 and 3.20 that for a sufficiently small $0 < \delta < \delta_0$ fixed we have for all $t \geqslant 0$ that $\frac{d}{dt}\Psi_\delta^s(t) + \delta\Psi_\delta^s(t) \leqslant 2\delta q_0$. Therefore, for a fixed $\delta > 0$ sufficiently small,

$$\Psi_\delta^s(t) \leqslant \Psi_\delta^s(0)e^{-\delta t} + 2q_0 \tag{3.31}$$

for all $t \geqslant 0$.

Lemma 3.21. *Let $R > 0$ and $V_0 = (u_0, u_1) \in X_s$ be such that $\|V_0\|_{X_s} \leqslant R$. If $\delta > 0$ is sufficiently small, then there exist $\alpha_R > 0$ and $\beta > 0$ (which does not depend on R) such that*

$$\|V_0(t, V_0)\|_{X_{t+s}} \leqslant \alpha_R e^{-\frac{\delta}{2}t} + \beta \quad forall\, t \geqslant 0.$$

Proof. From Proposition 3.17 and inequality (3.31)

$$\begin{aligned}
&\left\|V_p(t, V_0)\right\|_{X_{t+s}}^2 = E_s(t) \\
&\leqslant \frac{4\lambda_1}{\lambda_1 - \mu_0}\Psi_\delta^s(t) + \frac{16\lambda_1 h_0^2}{(\lambda_1 - \mu_0)^2} + \frac{4C_0\lambda_1}{\lambda_1 - \mu_0} \\
&\leqslant \frac{4\lambda_1}{\lambda_1 - \mu_0}\left(\Psi_\delta^s(0)e^{-\delta t} + 2q_0\right) + \frac{16\lambda_1 h_0^2}{(\lambda_1 - \mu_0)^2} + \frac{4C_0\lambda_1}{\lambda_1 - \mu_0} \\
&\leqslant \frac{4\lambda_1}{\lambda_1 - \mu_0}\left(\left(\hat{C}_R E_s(0) + 2q_0\right)e^{-\delta t} + 2q_0\right) + \frac{16\lambda_1 h_0^2}{(\lambda_1 - \mu_0)^2} + \frac{4C_0\lambda_1}{\lambda_1 - \mu_0} \\
&\leqslant \frac{4\lambda_1}{\lambda_1 - \mu_0}\left(\hat{C}_R\|V_0\|_{X_s}^2 + 2q_0\right)e^{-\delta t} + \frac{8q_0\lambda_1}{\lambda_1 - \mu_0} \\
&\quad + \frac{16\lambda_1 h_0^2}{(\lambda_1 - \mu_0)^2} + \frac{4C_0\lambda_1}{\lambda_1 - \mu_0} \\
&\leqslant \frac{4\lambda_1}{\lambda_1 - \mu_0}\left(\hat{C}_R R^2 + 2q_0\right)e^{-\delta t} + \frac{8q_0\lambda_1}{\lambda_1 - \mu_0} + \frac{16\lambda_1 h_0^2}{(\lambda_1 - \mu_0)^2} + \frac{4C_0\lambda_1}{\lambda_1 - \mu_0} \\
&\leqslant \alpha_R^2 e^{-\delta t} + \beta^2,
\end{aligned}$$

where $\alpha_R = \sqrt{\frac{4\lambda_1}{\lambda_1-\mu_0}\left(\hat{C}_R R^2 + 2q_0\right)} > 0$ and $\beta = \sqrt{\frac{8q_0\lambda_1}{\lambda_1-\mu_0} + \frac{16\lambda_1 h_0^2}{(\lambda_1-\mu_0)^2} + \frac{4C_0\lambda_1}{\lambda_1-\mu_0}} > 0.$ □

Thus, we have proven the following.

Proposition 3.22. *There exists $r_0 > 0$ such that the family $\hat{B} = \{B_t : B_t \subset X_t, t \in \mathbb{R}\}$ where $B_t = \overline{B}^t_{r_0}$, is uniformly $\mathfrak{D}_{ub}$-pullback absorbing for the evolution process $\mathcal{S}_0$.*

Proof. Set $r_0 := 1 + \beta > 0$. Let $R > 0$. For every $t \in \mathbb{R}$, if $\tau \geqslant \tau_0(R) := \ln(\alpha_R^{2/\delta})$ and $\|V_0\|_{t-\tau} \leqslant R$, we have

$$\|S_0(t, t-\tau)V_0\|_{X_t} = \|V_0(\tau, V_0)\|_{X_t} \leqslant \alpha_R e^{-\frac{\delta}{2}\tau} + \beta \leqslant 1 + \beta = r_0.$$

□

Similarly to [13, theorem 3.21], we obtain the following result.

Proposition 3.23. *For both cases, $p = 0$ and $p > 0$, there exists a family $\hat{C} = \{C_t : C_t \subset X_t, t \in \mathbb{R}\}$ which is closed, uniformly bounded, positively invariant and uniformly $\mathfrak{D}_{ub}$-pullback absorbing for the respective process $\mathcal{S}_p$. Additionally, $C_t \subset \overline{B}^t_{r_0}$ for each t.*

With the family $\hat{C}$ obtained in the previous theorem, in the next section, we show that $\mathcal{S}_p$ is $(\varphi, \mathfrak{D}_{ub})$-pullback κ-dissipative, where φ is a polynomial decay function for $p > 0$ and an exponential decay function for $p = 0$. To that end, we verify the conditions of Theorems 2.5 and 2.6, respectively. As a consequence, we will obtain condition 2.4. Finally, Proposition 2.4 ensures the existence of a generalized $(\varphi, \mathfrak{D}_{ub})$-pullback attractor for $\mathcal{S}_p$.

3.3 | Generalized $(\varphi, \mathfrak{D}_{ub})$-Pullback Attractors

Case $p > 0$. If $s \in \mathbb{R}$ and $V_1, V_2 \in C_s \subset X_s$, we know there exists a constant $c_{r_0} > 0$ such that $\left\|V_p(t, V_1)\right\|_{X_{t+s}} = \|(v(t), v_t(t))\|_{X_{t+s}} \leqslant c_{r_0}$ and $\left\|V_p(t, V_2)\right\|_{X_{t+s}} = \|(w(t), w_t(t))\|_{X_{t+s}} \leqslant c_{r_0}$ for all $t \geqslant 0$. Setting $Z(t) = (z(t), z_t(t)) = V_p(t, V_1) - V_p(t, V_2)$, we have $z(t) = v(t) - w(t)$ and $z_t(t) = v_t(t) - w_t(t)$ and, formally proceeding, it is clear that

$$\varepsilon_s(t) z_{tt} - \Delta z + k_s(t)\left(\|v_t\|^p_{L^2} v_t - \|w_t\|^p_{L^2} w_t\right) + f(v) - f(w) = 0 \tag{3.32}$$

which, when formally multiplied by z_t and integrated over Ω, results in

$$\varepsilon_s(t)\langle z_t, z_{tt}\rangle_{-1} - \langle \Delta z, z_t\rangle + k_s(t)\|v_t\|^p_{L^2}\langle v_t, z_t\rangle - k_s(t)\|w_t\|^p_{L^2}\langle w_t, z_t\rangle + \langle f(v) - f(w), z_t\rangle = 0.$$

For a fixed $T > 0$ (which will be chosen later), integrating from t to T, we obtain

$$\int_t^T \varepsilon_s(\tau)\langle z_t, z_{tt}\rangle_{-1} d\tau - \int_t^T \langle \Delta z, z_t\rangle d\tau + \int_t^T k_s(\tau)\langle \|v_t\|^p_{L^2} v_t - \|w_t\|^p_{L^2} w_t, z_t\rangle d\tau + \int_t^T \langle f(v) - f(w), z_t\rangle d\tau = 0.$$

Defining the function $E_s : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$E_s(t) = \frac{1}{2}\|Z(t)\|^2_{X_{t+s}} = \frac{1}{2}\left(\|z\|^2_{H^1_0} + \varepsilon_s(t)\|z_t\|^2_{L^2}\right) \tag{3.33}$$

and noting that

$$E_s(T) - E_s(t) = \int_t^T \frac{d}{d\tau}E_s(\tau)d\tau = \int_t^T \varepsilon_s(\tau)\langle z_t, z_{tt}\rangle d\tau - \int_t^T \langle \Delta z, z_t\rangle d\tau + \int_t^T \varepsilon'_s(\tau)\|z_t\|^2_{L^2} d\tau,$$

we have

$$E_s(t) = E_s(T) + \int_t^T k_s(\tau)\langle \|v_t\|^p_{L^2} v_t - \|w_t\|^p_{L^2} w_t, z_t\rangle d\tau + \int_t^T \langle f(v) - f(w), z_t\rangle d\tau - \int_t^T \varepsilon'_s(\tau)\|z_t\|^2_{L^2} d\tau. \tag{3.34}$$

Now, an integration of this previous equation from 0 to T leads us to the equation

$$TE_s(T) = \int_0^T E_s(t)dt - \int_0^T \int_t^T k_s(\tau)\langle \|v_t\|^p_{L^2} v_t - \|w_t\|^p_{L^2} w_t, z_t\rangle d\tau dt - \int_0^T \int_t^T \langle f(v) - f(w), z_t\rangle d\tau dt + \int_0^T \int_t^T \varepsilon'_s(\tau)\|z_t\|^2_{L^2} d\tau dt. \tag{3.35}$$

On the other hand, multiplying (3.32) by z and integrating over Ω, we obtain

$$\int_0^T \varepsilon_s(t)\langle z, z_{tt}\rangle_{-1} dt - \int_0^T \langle \Delta z, z\rangle dt + \int_0^T k_s(t)\langle \|v_t\|^p_{L^2} v_t - \|w_t\|^p_{L^2} w_t, z\rangle dt + \int_0^T \langle f(v) - f(w), z\rangle dt = 0. \tag{3.36}$$

Plotting

$$\varepsilon_s(t)\langle z, z_{tt}\rangle_{-1} - \langle \Delta z, z\rangle = \varepsilon_s(t)\frac{d}{dt}\langle z_t, z\rangle - 2\varepsilon_s(t)\|z_t\|^2_{L^2} + 2E_s(t)$$

into (3.36) results in

$$\int_0^T E_s(t)dt = -\frac{1}{2}\int_0^T \varepsilon_s(t)\frac{d}{dt}\langle z_t, z\rangle dt + \int_0^T \varepsilon_s(t)\|z_t\|^2_{L^2} dt - \frac{1}{2}\int_0^T k_s(t)\langle \|v_t\|^p_{L^2} v_t - \|w_t\|^p_{L^2} w_t, z\rangle dt - \frac{1}{2}\int_0^T \langle f(v) - f(w), z\rangle dt. \tag{3.37}$$

Joining (3.35) and (3.37), we obtain the following result.

Lemma 3.24. *Let $T > 0$ fixed. Then,*

$$
\begin{aligned}
TE_s(T) = &-\frac{1}{2}\int_0^T \varepsilon_s(t)\frac{d}{dt}\langle z_t, z\rangle dt + \int_0^T \varepsilon_s(t)\|z_t\|_{L^2}^2 dt \\
&-\frac{1}{2}\int_0^T k_s(t)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z\rangle dt \\
&-\frac{1}{2}\int_0^T \langle f(v) - f(w), z\rangle dt \\
&-\int_0^T\int_t^T k_s(\tau)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle d\tau dt \\
&-\int_0^T\int_t^T \langle f(v) - f(w), z_t\rangle d\tau dt \\
&+\int_0^T\int_t^T \varepsilon_s'(\tau)\|z_t\|_{L^2}^2 d\tau dt.
\end{aligned} \tag{3.38}
$$

Proposition 3.25. *Let $T > 0$ fixed. Given $t \in \mathbb{R}$, there exist $\Gamma = \Gamma(t, T) > 0$ and $C > 0$ such that*

$$
\begin{aligned}
E_s(T) \leqslant\ & \Gamma(t,T)\sup_{t\in[0,T]}\|z(t)\|_{L^2} \\
&+ C\left(E_s(0) - E_s(T) + \left|\int_0^T \langle f(v) - f(w), z_t\rangle d\tau\right|\right)^{\frac{2}{p+2}} \\
&+ \frac{1}{T}\left|\int_0^T\int_t^T \langle f(v) - f(w), z_t\rangle d\tau dt\right|.
\end{aligned} \tag{3.39}
$$

Proof. It follows from [13, proposition A.4] that $\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle \geqslant 0$, which implies that

$$
-\int_0^T\int_t^T k_s(\tau)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle d\tau dt \leqslant 0.
$$

Additionally, because $\varepsilon_s'(\tau) \leqslant 0$ for every $\tau \in \mathbb{R}$, we have $\int_0^T\int_t^T \varepsilon_s'(\tau)\|z_t\|_{L^2}^2 d\tau dt \leqslant 0$. In what follows we explore the other terms appearing on the right side of Equation (3.38).

$$
\begin{aligned}
-\frac{1}{2}\int_0^T \varepsilon_s(t)\frac{d}{dt}\langle z_t, z\rangle dt &= -\frac{1}{2}\varepsilon_s(t)\langle z_t, z\rangle|_0^T + \frac{1}{2}\int_0^T \varepsilon_s'(t)\langle z_t, z\rangle dt \\
&\leqslant \sqrt{L}c_{r_0}\sup_{t\in[0,T]}\|z(t)\|_{L^2} \\
&\quad + \frac{L^{\frac{3}{2}}c_{r_0}T}{\varepsilon(t+T)}\sup_{t\in[0,T]}\|z(t)\|_{L^2},
\end{aligned}
$$

noting that $\varepsilon_s(\tau) \geqslant \varepsilon_s(T) > 0$ for $\tau \in [0, T]$, and then

$$
\begin{aligned}
\frac{1}{2}\int_0^T \varepsilon_s'(\tau)\langle z_t, z\rangle d\tau &\leqslant \frac{L}{2}\int_0^T \frac{\varepsilon_s(\tau)}{\varepsilon_s(T)}\|z_t(\tau)\|_{L^2}\|z(\tau)\|_{L^2} d\tau \\
&\leqslant \frac{L}{2\varepsilon(t+T)}\int_0^T \varepsilon_s(\tau)\|z_t(\tau)\|_{L^2}\|z(\tau)\|_{L^2} d\tau \\
&\leqslant \frac{L\sqrt{L}}{2\varepsilon(t+T)}\int_0^T \sqrt{\varepsilon_s(\tau)}\|z_t(\tau)\|_{L^2}\|z(\tau)\|_{L^2} d\tau \\
&\leqslant \frac{L^{\frac{3}{2}}c_{r_0}T}{\varepsilon(t+T)}\sup_{\tau\in[0,T]}\|z(\tau)\|_{L^2}.
\end{aligned}
$$

In the previous inequalities we used that $\varepsilon_s(T) = \varepsilon(s+T) \geqslant \varepsilon(t+T)$, because $s \leqslant t$ and ε is decreasing, the fact that $\varepsilon_s(T) \leqslant \sqrt{L}\sqrt{\varepsilon_s(t)}$ and $\sqrt{\varepsilon_s(t)}\|z_t\|_{L^2} \leqslant 2c_{r_0}$.

From [13, proposition 3.22, inequality (3.21)],

$$
\begin{aligned}
0 &\leqslant \int_0^T \varepsilon_s(t)\|z_t\|_{L^2}^2 dt \leqslant L\int_0^T \|z_t\|_{L^2}^2 dt \\
&\leqslant L(4T)^{\frac{p}{p+2}} k_0^{\frac{-2}{p+2}} \\
&\quad \left(\int_0^T k_s(t)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle dt\right)^{\frac{2}{p+2}}.
\end{aligned} \tag{3.40}
$$

In addition, (3.34) implies that

$$
\begin{aligned}
0 &\leqslant \int_0^T k_s(t)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z_t\rangle dt \\
&= E_s(0) - E_s(T) - \int_0^T \langle f(v) - f(w), z_t\rangle d\tau \\
&\quad + \int_0^T \varepsilon_s'(\tau)\|z_t\|_{L^2} d\tau \\
&\leqslant E_s(0) - E_s(T) + \left|\int_0^T \langle f(v) - f(w), z_t\rangle d\tau\right|.
\end{aligned} \tag{3.41}
$$

Therefore, from (3.40) and (3.41), we obtain

$$
\begin{aligned}
&\int_0^T \varepsilon_s(t)\|z_t\|_{L^2}^2 dt \\
&\leqslant L(4T)^{\frac{p}{p+2}} k_0^{\frac{-2}{p+2}}\Bigg(E_s(0) - E_s(T) \\
&\quad + \left|\int_0^T \langle f(v) - f(w), z_t\rangle d\tau\right|\Bigg)^{\frac{2}{p+2}}.
\end{aligned}
$$

Considering that $\varepsilon_s(t)\|v_t\|_{L^2}^2 \leqslant c_{r_0}^2$ and $\varepsilon_s(t)\|w_t\|_{L^2}^2 \leqslant c_{r_0}^2$,

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T k_s(\tau)\langle \|v_t\|_{L^2}^p v_t - \|w_t\|_{L^2}^p w_t, z\rangle d\tau \\
&\leqslant \frac{1}{2}k_1\int_0^T \left(\|v_t\|_{L^2}^{p+1} + \|w_t\|_{L^2}^{p+1}\right)\|z\|_{L^2} d\tau \\
&\leqslant \frac{k_1}{2\varepsilon(t+T)^{\frac{p+1}{2}}}\int_0^T \\
&\left(\varepsilon_s(\tau)^{\frac{p+1}{2}}\|v_t\|_{L^2}^{p+1} + \varepsilon_s(\tau)^{\frac{p+1}{2}}\|w_t\|_{L^2}^{p+1}\right)\|z\|_{L^2} d\tau \\
&\leqslant \frac{k_1}{2\varepsilon(t+T)^{\frac{p+1}{2}}}\int_0^T \\
&\left(\left(\varepsilon_s(\tau)\|v_t\|_{L^2}^2\right)^{\frac{p+1}{2}} + \left(\varepsilon_s(\tau)\|w_t\|_{L^2}^2\right)^{\frac{p+1}{2}}\right)\|z\|_{L^2} d\tau \\
&\leqslant \frac{k_1 c_{r_0}^{p+1}T}{\varepsilon(t+T)^{\frac{p+1}{2}}}\sup_{\tau\in[0,T]}\|z(\tau)\|_{L^2}.
\end{aligned} \tag{3.42}
$$

Finally, from Lemma 3.3, there exists a constant $C_{r_0} \geqslant 0$ such that

$$-\frac{1}{2}\int_0^T \langle f(v)-f(w),z\rangle dt \leqslant \frac{1}{2}\int_0^T \|f(v)-f(w)\|_{L^2}\|z\|_{L^2}dt \leqslant \frac{C_{r_0}}{2}\int_0^T \|z(t)\|_{H_0^1}\|z(t)\|_{L^2}dt \leqslant C_{r_0}c_{r_0}T\sup_{t\in[0,T]}\|z(t)\|_{L^2}, \tag{3.43}$$

because $\|z(t)\|_{H_0^1}\leqslant 2c_{r_0}$.

Plotting all these considerations into (3.38) and setting

$$\Gamma(t,T)=\frac{1}{T}\left(\sqrt{L}c_{r_0}+\frac{L^{\frac{3}{2}}c_{r_0}T}{\varepsilon(t+T)}+\frac{k_1c_{r_0}^{p+1}T}{\varepsilon(t+T)^{\frac{p+1}{2}}}+C_{r_0}c_{r_0}T\right)$$

and

$$C=\frac{L(4T)^{\frac{p}{p+2}}k_0^{-\frac{2}{p+2}}}{T}$$

we conclude the proof. □

Remark 3.26. It is an immediate consequence of (3.41) that

$$E_s(T)\leqslant E_s(0)+\left|\int_0^T\langle f(v)-f(w),z_t\rangle d\tau\right| \tag{3.44}$$

Setting $\psi_s,\theta_s : X_s\times X_s\to\mathbb{R}^+$ by

$$\psi_s(V_1,V_2)=2\left|\int_0^T\langle f(v)-f(w),z_t\rangle d\tau\right|$$

and

$$\theta_s(V_1,V_2)=\frac{2}{T}\left|\int_0^T\int_t^T\langle f(v)-f(w),z_t\rangle d\tau dt\right|,$$

we can show that $\psi_s,\theta_s\in \text{contr}(\overline{B}_{r_0}^s)$. See [13, proposition 3.25].

Additionally, setting

$$\rho^{(s)}(V_1,V_2)=2\sup_{t\in[0,T]}\|z(t)\|_{L^2},$$

we can prove that $\rho^{(s)}$ is a precompact pseudometric on $\overline{B}_{r_0}^s$.

Finally consider the functions $g,h:\mathbb{R}_+\to\mathbb{R}_+$ given by $g(\alpha)=0$ and $h(\alpha)=\Gamma(t,T)\alpha$. It is obvious that g,h are non-decreasing, $g(0)=h(0)=0$ and that they are continuous at 0.

Therefore, from (3.39) and (3.44), we get

$$\begin{aligned}
&d_{X_{T+s}}\left(S_p(T+s,s)V_1,S_p(T+s,s)V_2\right)^2\\
&=\left\|S_p(T+s,s)V_1-S_p(T+s,s)V_2\right\|_{X_{T+s}}^2\\
&=\left\|V_p(t,V_1)-V_p(t,V_2)\right\|_{X_{T+s}}^2=\|Z(T)\|_{X_{T+s}}^2=2E_s(T)\\
&\leqslant 2\Gamma(t,T)\sup_{t\in[0,T]}\|z(t)\|_{L^2}\\
&\quad+2C\left(E_s(0)-E_s(T)+\left|\int_0^T\langle f(v)-f(w),z_t\rangle d\tau\right|\right)^{\frac{2}{p+2}}\\
&\quad+\frac{2}{T}\left|\int_0^T\int_t^T\langle f(v)-f(w),z_t\rangle d\tau dt\right|\\
&\leqslant 2C\left(E_s(0)-E_s(T)+\frac{1}{2}\psi_s(V_1,V_2)\right)^{\frac{2}{p+2}}\\
&\quad+\Gamma(t,T)\rho^{(s)}(V_1,V_2)+\theta_s(V_1,V_2)\\
&=2^{\frac{p}{p+2}}C\big(2E_s(0)-2E_s(T)+g(\rho^{(s)}(V_1,V_2))\\
&\quad+\psi_s(V_1,V_2)\big)^{\frac{2}{p+2}}\\
&\quad+h(\rho^{(s)}(V_1,V_2))+\theta_s(V_1,V_2)\\
&=2^{\frac{p}{p+2}}C\big(d_{X_s}(V_1,V_2)^2\\
&\quad-d_{X_{T+s}}(S(T+s,s)V_1,S(T+s,s)V_2)^2\\
&\quad+g(\rho^{(s)}(V_1,V_2))+\psi_s(V_1,V_2)\big)^{\frac{2}{p+2}}\\
&\quad+h(\rho^{(s)}(V_1,V_2))+\theta_s(V_1,V_2),
\end{aligned}$$

and

$$\begin{aligned}
&d_{X_{T+s}}(S_p(T+s,s)V_1,S_p(T+s,s)V_2)^2\\
&=2E_s(T)\leqslant 2E_s(0)+2\left|\int_0^T\langle f(v)-f(w),z_t\rangle d\tau\right|\\
&=d_{X_s}(V_1,V_2)^2+\psi_s(V_1,V_2)\\
&=d_{X_s}(V_1,V_2)^2+g(\rho^{(s)}(V_1,V_2))+\psi_s(V_1,V_2).
\end{aligned}$$

Now, a direct application of Theorem 2.6 ensures that the process S_p has a generalized $(\varphi,\mathfrak{D}_{ub})$-pullback attractor $\hat{M}\in\mathfrak{D}_{ub}$, where

$$\varphi(s)=s^{\frac{\beta}{r(\beta-1)}}=s^{-1/p},$$

because $r=2$ and $\beta=\frac{2}{p+2}$.

Case $p=0$**.** Taking $p=0$ in (3.32), we obtain

$$\varepsilon_s(t)z_{tt}-\Delta z+k_s(t)z_t(t,x)+f(v)-f(w)=0 \tag{3.45}$$

Considering again $E_s(t)$ as in (3.33) and setting $\Psi_s(t)=E_s(t)+\frac{\delta_0}{2}\varepsilon_s(t)\langle z,z_t\rangle$, where δ_0 is a constant satisfying $0<\delta_0\leqslant\min\left\{\frac{\lambda_1}{L},1,\frac{k_0}{L}\right\}$, we reach the following result.

Lemma 3.27. *For* $t\geqslant 0$*, we have*

$$\frac{1}{2}E_s(t)\leqslant\Psi_s(t)\leqslant\frac{3}{2}E_s(t).$$

Proof. This result follows directly from

$$\begin{aligned}
\left|\frac{\delta_0}{2}\varepsilon_s(t)\langle z,z_t\rangle\right|&\leqslant\frac{\delta_0\varepsilon_s(t)}{2}\|z_t\|_{L^2}\frac{1}{\sqrt{\lambda_1}}\|z\|_{H_0^1}\\
&\leqslant\frac{\delta_0\varepsilon_s(t)}{4}\left(\|z_t\|_{L^2}^2+\frac{1}{\lambda_1}\|z\|_{H_0^1}^2\right)\\
&\leqslant\frac{\delta_0\varepsilon_s(t)}{4}\|z_t\|_{L^2}^2+\frac{\delta_0L}{4\lambda_1}\|z\|_{H_0^1}^2
\end{aligned}$$

$$\leqslant \max\left\{\frac{\delta_0 L}{4\lambda_1},\frac{\delta_0}{4}\right\}\left(\|z\|_{H_0^1}^2+\varepsilon_s(t)\|z_t\|_{L^2}^2\right)$$
$$\leqslant \max\left\{\frac{\delta_0 L}{2\lambda_1},\frac{\delta_0}{2}\right\}.$$
$$\left[\frac{1}{2}\left(\|z\|_{H_0^1}^2+\varepsilon_s(t)\|z_t\|_{L^2}^2\right)\right]\leqslant \frac{1}{2}E_s(t).$$

□

Formally multiplying (3.45) by $z_t+\frac{\delta_0}{2}z$ in $L^2(\Omega)$,

$$\varepsilon_s(t)\langle z_{tt},z_t\rangle_{-1}-\langle z_t,\Delta z\rangle+k_s(t)\|z_t\|_{L^2}^2+\langle f(v)-f(w),z_t\rangle$$
$$+\frac{\delta_0}{2}\langle z_{tt},z\rangle_{-1}-\frac{\delta_0}{2}\langle\Delta z,z\rangle+\frac{\delta_0}{2}k_s(t)\langle z_t,z\rangle$$
$$+\frac{\delta_0}{2}\langle f(v)-f(w),z\rangle=0. \tag{3.46}$$

Because

$$\begin{aligned}\frac{d}{dt}\Psi_s(t)&=\frac{d}{dt}\left(E_s(t)+\frac{\delta_0}{2}\varepsilon_s(t)\langle z,z_t\rangle\right)\\&=\frac{d}{dt}E_s(t)+\frac{\delta_0}{2}\varepsilon_s'(t)\langle z,z_t\rangle\\&\quad+\frac{\delta_0}{2}\varepsilon_s(t)\|z_t\|_{L^2}^2+\frac{\delta_0}{2}\varepsilon_s(t)\langle z,z_{tt}\rangle_{-1},\end{aligned}\tag{3.47}$$

we have

$$\begin{aligned}\frac{d}{dt}\Psi_s(t)+\delta_0\Psi_s(t)&=\frac{d}{dt}E_s(t)+\frac{\delta_0}{2}\varepsilon_s'(t)\langle z,z_t\rangle\\&\quad+\frac{\delta_0}{2}\varepsilon_s(t)\|z_t\|_{L^2}^2+\frac{\delta_0}{2}\varepsilon_s(t)\langle z,z_{tt}\rangle_{-1}\\&\quad+\frac{\delta_0}{2}\|z\|_{H_0^1}^2+\frac{\delta_0}{2}\varepsilon_s(t)\|z_t\|_{L^2}^2+\frac{\delta_0^2}{2}\varepsilon_s(t)\langle z,z_t\rangle\\&=\varepsilon_s(t)\langle z_{tt},z_t\rangle_{-1}-\langle z_t,\Delta z\rangle+\frac{1}{2}\varepsilon_s'(t)\|z_t\|_{L^2}^2\\&\quad+\frac{\delta_0}{2}\varepsilon_s'(t)\langle z,z_t\rangle+\frac{\delta_0}{2}\varepsilon_s(t)\|z_t\|_{L^2}^2\\&\quad+\frac{\delta_0}{2}\varepsilon_s(t)\langle z,z_{tt}\rangle_{-1}+\frac{\delta_0}{2}\|z\|_{H_0^1}^2\\&\quad+\frac{\delta_0}{2}\varepsilon_s(t)\|z_t\|_{L^2}^2+\frac{\delta_0^2}{2}\varepsilon_s(t)\langle z,z_t\rangle\\&\leqslant -k_s(t)\|z_t\|_{L^2}^2-\langle f(v)-f(w),z_t\rangle\\&\quad-\frac{\delta_0}{2}k_s(t)\langle z_t,z\rangle-\frac{\delta_0}{2}\langle f(v)-f(w),z\rangle\\&\quad+\frac{\delta_0^2}{2}\varepsilon_s(t)\langle z,z_t\rangle+\delta_0\varepsilon_s(t)\|z_t\|_{L^2}^2\\&\quad+\frac{\delta_0}{2}\varepsilon_s'(t)\langle z,z_t\rangle,\end{aligned}\tag{3.48}$$

where in the last inequality of the calculation above we applied (3.46).

Note that, because $\delta_0\leqslant\frac{k_0}{L}$, we have

$$-k_s(t)\|z_t\|_{L^2}^2+\delta_0\varepsilon_s(t)\|z_t\|_{L^2}^2\leqslant 0.$$

Additionally, it follows from Lemma 3.3 that there exists $C_{r_0}>0$ such that

$$\begin{aligned}-\frac{\delta_0}{2}\langle f(v)-f(w),z\rangle&\leqslant\frac{\delta_0}{2}\|f(v)-f(w)\|_{L^2}\|z\|_{L^2}\\&\leqslant\frac{C_{r_0}\delta_0}{2}\|z\|_{H_0^1}\|z\|_{L^2}\\&\leqslant\frac{C_{r_0}\delta_0}{2}\|z\|_{H_0^1}\|z\|_{L^2}\leqslant\delta_0C_{r_0}c_{r_0}\|z\|_{L^2}.\end{aligned}$$

Also, observe that

$$\begin{aligned}\frac{\delta_0^2}{2}\varepsilon_s(t)\langle z,z_t\rangle&\leqslant\frac{\delta_0^2}{2}\varepsilon_s(t)\|z_t\|_{L^2}\|z\|_{L^2}\\&\leqslant\frac{\delta_0^2}{2}\sqrt{L}\sqrt{\varepsilon_s(t)}\|z_t\|_{L^2}\|z\|_{L^2}\leqslant\delta_0^2\sqrt{L}c_{r_0}\|z\|_{L^2}.\end{aligned}$$

Inserting these estimates into (3.48),

$$\begin{aligned}\frac{d}{dt}\Psi_s(t)+\delta_0\Psi_s(t)&\leqslant\left(\delta_0C_{r_0}c_{r_0}+\delta_0^2\sqrt{L}c_{r_0}\right)\|z\|_{L^2}\\&\quad-\langle f(v)-f(w),z_t\rangle+\left(\frac{\delta_0}{2}\varepsilon_s'(t)-\frac{\delta_0}{2}k_s(t)\right)\langle z,z_t\rangle.\end{aligned}$$

Let $T>0$ fixed.

$$\begin{aligned}\frac{d}{dt}\left(e^{\delta_0t}\Psi_s(t)\right)&=\delta_0e^{\delta_0t}\Psi_s(t)+e^{\delta_0t}\frac{d}{dt}\Psi_s(t)\\&=e^{\delta_0t}\left(\frac{d}{dt}\Psi_s(t)+\delta_0\Psi_s(t)\right)\\&\leqslant e^{\delta_0t}\left(\delta_0C_{r_0}c_{r_0}+\delta_0^2\sqrt{L}c_{r_0}\right)\|z\|_{L^2}\\&\quad-e^{\delta_0t}\langle f(v)-f(w),z_t\rangle\\&\quad+\frac{\delta_0e^{\delta_0t}}{2}\left(\varepsilon_s'(t)-k_s(t)\right)\langle z,z_t\rangle.\end{aligned}$$

Integrating from 0 to T,

$$\begin{aligned}e^{\delta_0T}\Psi_s(T)-\Psi_s(0)&\leqslant e^{\delta_0T}\left(\delta_0C_{r_0}c_{r_0}+\delta_0^2\sqrt{L}c_{r_0}\right)\int_0^T\|z(t)\|_{L^2}dt\\&\quad+\left|\int_0^Te^{\delta_0t}\langle f(v)-f(w),z_t\rangle dt\right|\\&\quad+\frac{\delta_0e^{\delta_0T}}{2}\int_0^T\left|\varepsilon_s'(t)-k_s(t)\right|\left|\langle z,z_t\rangle\right|dt.\end{aligned}\tag{3.49}$$

Additionally,

$$\begin{aligned}&\frac{\delta_0e^{\delta_0T}}{2}\int_0^T\left|\varepsilon_s'(\tau)-k_s(\tau)\right|\left|\langle z,z_t\rangle\right|d\tau\\&\leqslant\frac{\delta_0e^{\delta_0T}}{2}(L+k_1)\int_0^T\frac{\varepsilon_s(\tau)}{\varepsilon_s(T)}\|z_t(\tau)\|_{L^2}\|z(\tau)\|_{L^2}d\tau\\&\leqslant\frac{\delta_0e^{\delta_0T}}{2}\frac{L+k_1}{\varepsilon(s+T)}\int_0^T\varepsilon_s(\tau)\|z_t(\tau)\|_{L^2}\|z(\tau)\|_{L^2}d\tau\\&\leqslant\frac{\delta_0e^{\delta_0T}}{2}\frac{L+k_1}{\varepsilon(t+T)}\int_0^T\sqrt{L}\sqrt{\varepsilon_s(\tau)}\|z_t(\tau)\|_{L^2}\|z(\tau)\|_{L^2}d\tau\\&\leqslant\frac{\delta_0e^{\delta_0T}}{2}\frac{L+k_1}{\varepsilon(t+T)}2c_{r_0}\sqrt{L}\int_0^T\|z(\tau)\|_{L^2}d\tau.\end{aligned}\tag{3.50}$$

From (3.49) and (3.50), and considering that $\int_0^T \|z(\tau)\| d\tau \leqslant T \sup_{t\in[0,T]} \|z(t)\|_{L^2}$, we obtain

$$\Psi_s(T) \leqslant e^{-\delta_0 T}\Psi_s(0) + \left| \int_0^T e^{\delta_0(t-T)} \langle f(v) - f(w), z_t \rangle dt \right| + \Gamma(t,T) \sup_{t\in[0,T]} \|z(t)\|_{L^2}, \tag{3.51}$$

where

$$\Gamma(t,T) = T\left(\delta_0 C_{r_0} c_{r_0} + \delta_0^2 \sqrt{L} c_{r_0}\right) + \frac{T\delta_0 (L + k_1) c_{r_0} \sqrt{L}}{\varepsilon(t+T)} > 0.$$

This implies that

$$E_s(T) \leqslant 2\Psi_s(T) \leqslant 3e^{-\delta_0 T} E_s(0) + 2\left| \int_0^T e^{\delta_0(t-T)} \langle f(v) - f(w), z_t \rangle dt \right| + 2\Gamma(t,T) \sup_{t\in[0,T]} \|z(t)\|_{L^2}. \tag{3.52}$$

Finally,

$$\begin{aligned} d_{X_{T+s}}&(S_0(T+s,s)V_1, S_0(T+s,s)V_2)^2 = \|Z(T)\|^2_{X_{T+s}} \leqslant 2E_s(T) \\ &\leqslant 3e^{-\delta_0 T} d_{X_s}(V_1,V_2)^2 \\ &\quad + 4\left| \int_0^T e^{\delta_0(t-T)} \langle f(v) - f(w), z_t \rangle dt \right| \\ &\quad + 4\Gamma(t,T) \sup_{t\in[0,T]} \|z(t)\|_{L^2} \\ &= \mu d_{X_s}(V_1,V_2)^2 + g(\rho^{(s)}(V_1,V_2)) + \psi_s(V_1,V_2) \end{aligned} \tag{3.53}$$

setting $\mu := 3e^{-\delta_0 T}$, $\rho^{(s)} : X_s \times X_s \to \mathbb{R}^+$ given by $\rho^{(s)}(V_1,V_2) = 4\sup_{t\in[0,T]} \|z(t)\|_{L^2}$, $g : \mathbb{R}^+ \to \mathbb{R}^+$ given by $g(\alpha) = \Gamma(t,T)\alpha$ and $\psi_s : X_s \times X_s \to \mathbb{R}^+$ by $\psi_s(V_1,V_2) = 4\left| \int_0^T e^{\delta_0(t-T)} \langle f(v) - f(w), z_t \rangle dt \right|$.

Note that $\mu \in (0,1)$ if we fix $T > \frac{ln3}{\delta_0}$. Additionally, $\rho^{(s)}$ is a precompact pseudometric in $\overline{B}^s_{r_0}$ and $\psi_s \in \text{contr}(\overline{B}^s_{r_0})$. Finally also note that g satisfies the conditions required in Theorem 2.5. All these considerations leads us to the existence of a generalized $(\varphi, \mathfrak{D}_{ub})$-pullback attractor $\hat{M} \in \mathfrak{D}_{ub}$ for S_0, where $\varphi(s) = \mu^s$.

Author Contributions

Matheus Cheque Bortolan: conceptualization, investigation, writing – original draft, methodology, writing – review and editing. **Carlos Pecorari Neto:** conceptualization, investigation, writing – original draft, methodology, writing – review and editing. **Heraclio López Lázaro:** conceptualization, investigation, writing – original draft, methodology, writing – review and editing. **Paulo Seminario-Huertas:** conceptualization, writing – original draft, investigation, methodology, writing – review and editing.

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Conflicts of Interest

The authors declare no conflicts of interest.

Ethics Statement

The authors have nothing to report.

References

1. M. C. Bortolan, A. N. Carvalho, and J. A. Langa, Attractors Under Autonomous and Non-Autonomous Perturbations of Math. Surv. Monogr., vol. 246 (American Mathematical Society (AMS), 2020).

2. A. N. Carvalho, J. A. Langa, and J. C. Robinson, Attractors for Infinite-Dimensional Non-Autonomous Dynamical Systems of Appl. Math. Sci., vol. 182 (Springer, 2013).

3. P. E. Kloeden and M. Rasmussen, Nonautonomous Dynamical Systems of Math. Surv. Monogr., vol. 176 (American Mathematical Society (AMS), 2011).

4. F. Di Plinio, G. S. Duane, and R. Temam, "Time-Dependent Attractor for the Oscillon Equation," *Discrete and Continuous Dynamical Systems* 29, no. 1 (2011): 141–167.

5. M. Conti and V. Pata, "Asymptotic Structure of the Attractor for Processes on Time-Dependent Spaces," *Nonlinear Analysis: Real World Applications* 19 (2014): 1–10.

6. M. Conti, V. Pata, and R. Temam, "Attractors for Processes on Time-Dependent Spaces: Applications to wave equations," *Journal of Differential Equations* 255, no. 6 (2013): 1254–1277.

7. Y. Li and Z. Yang, "Continuity of the Attractors in Time-Dependent Spaces and Applications," *Journal of Mathematical Analysis and Applications* 524, no. 1 (2023): 127081.

8. F. Meng, M. Yang, and C. Zhong, "Attractors for Wave Equations With Nonlinear Damping on Time-Dependent Space," *Discrete and Continuous Dynamical Systems Series B* 21, no. 1 (2016): 205–225.

9. S. Zhang, Q. Li, and J. Zhang, "Dynamical Behavior of Nonclassical Diffusion Equations With the Driving Delay in Time-Dependent Spaces," *Discrete and Continuous Dynamical Systems-B* 30, no. 7 (2025): 2568–2585.

10. K. Zhu, Y. Xie, and F. Zhou, "Attractors for the Nonclassical Reaction–Diffusion Equations on Time-Dependent Spaces," *Boundary Value Problems* 2020, no. 1 (2020): 95.

11. C. Zhao, C. Zhong, and X. Zhu, "Existence of Compact φ-Attracting Sets and Estimate of Their Attractive Velocity for Infinite-Dimensional Dynamical Systems," *Discrete and Continuous Dynamical Systems - B* 27, no. 12 (2022): 7493–7520.

12. M. C. Bortolan, T. Caraballo, and C. P. Neto. Generalized Exponential Pullback Attractor for a Nonautonomous Wave Equation. Preprint, arXiv:2401.06631 [math. DS] (2024), 2024.

13. M. C. Bortolan, T. Caraballo, and C. P. Neto, "Generalized φ-Pullback Attractors for Evolution Processes and Application to a Nonautonomous Wave Equation," *Applied Mathematics and Optimization* 89, no. 62 (2024): 1–52.

14. J. Zhang, P. E. Kloeden, M. Yang, and C. Zhong, "Global Exponential κ-Dissipative Semigroups and Exponential Attraction," *Discrete and Continuous Dynamical Systems* 37, no. 6 (2017): 3487–3502.

15. A. N. De Carvalho and S. Sonner, "Pullback Exponential Attractors for Evolution Processes in Banach Spaces: Theoretical Results," *Communications on Pure and Applied Analysis* 12, no. 6 (2013): 3047–3071.

16. A. N. De Carvalho and S. Sonner, "Pullback Exponential Attractors for Evolution Processes in Banach Spaces: Properties and Applications," *Communications on Pure and Applied Analysis* 13, no. 3 (2014): 1141–1165.

17. C. P. Neto. "Generalized φ–Pullback Attractors for Evolution Processes and Application to a Nonautonomous Wave Equation," PhD thesis, Universidade Federal de Santa Catarina, Florianópolis, SC, 2024. Available at: https://repositorio.ufsc.br/bitstream/handle/123456789/257020/PMTM0311-T.pdf?sequence=-1isAllowed=y.

18. J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Lineaires* (Dunod, 1969).

19. L. C. Evans, *Partial Differential Equations* (American Mathematical Society, 2022).

20. J. Necas, J. Málek, M. Rokyta, and M. Ruzicka, Weak and Measure-Valued Solutions to Evolutionary PDEs, of Appl. Math. Math. Comput., vol. 13 (Chapman & Hall, 1996).

21. C. Zhao, C. Zhao, and C. Zhong, "The Global Attractor for a Class of Extensible Beams With Nonlocal Weak Damping," *Discrete and Continuous Dynamical Systems Series B* 35, no. 3 (2020): 935–955.

22. C. Zhao, C. Zhao, and C. Zhong, "Asymptotic Behaviour of the Wave Equation With Nonlocal Weak Damping and Anti-Damping," *Journal of Mathematical Analysis and Applications* 490, no. 1 (2020): 124186.