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New relationships relating acoustical and electromagnetic beam shape coefficients

Gérard Gouesbet [a,*], Jianqi Shen [b], Leonardo A. Ambrosio [c]

[a] CORIA-UMR 6614- Normandie Université, CNRS-Université et INSA de Rouen, Campus Universitaire du Madrillet, 76800, Saint-Etienne-du Rouvray, France
[b] College of Science, University of Shanghai for Science and Technology, 516 Jungong Road, Shanghai, 200093, China
[c] Department of Electrical and Computer Engineering São Carlos School of Engineering, University of São Paulo, 400 Trabalhador São-carlense Ave., São Carlos, SP 13566-590, Brazil

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ABSTRACT

It has been recently demonstrated that the electromagnetic beam shape coefficients $g_{n,X}^m$ ($X = TM$ or TE) which encode the structure of structured light beams may be expressed in terms of scalar, more specifically acoustical, beam shape coefficients. Because the technique used to obtain the relevant expressions relied on the properties of what is known as the finite series method, the aforementioned expressions were different, depending on whether $(n-m)$ is even or odd. For a reason discussed in the bulk of the paper, it became obvious that the expressions obtained for different parities of $(n-m)$ could be unified. Proceeding to such an unification, the expressions previously published were not only unified, but furthermore simplified, then allowing for an easier and less time-consuming numerical implementation.

1. Introduction

In some light scattering theories dealing with the case when the illuminating beam is a structured beam, such as the analytical Generalized Lorenz-Mie theory (GLMT), e.g. [1,2], or the semi-analytical Extended Boundary Condition Method (EBCM), e.g. [3–5], both being T-matrix methods [6,7], the illuminating beam is encoded in a set of coefficients known as Beam Shape Coefficients (BSCs) allowing one to express it in terms of an expansion over Vector Spherical Wave Functions (VSWFs). Exhaustive entries to the corresponding literature is available from recent review papers [8,9], and references therein.

There exists an arsenal of different methods to evaluate the BSCs, including the original quadrature method, e.g. [10,11], the localized approximations, e.g. [12,13] and references therein dating back to [14, 15], including a variant known as the integral localized approximation, e.g. [16] dating back to [17], the finite series technique, e.g. [18, 19] and references therein dating back to [20,21], the angular spectrum decomposition [22–27], and the recently introduced R-quadrature method [28–30].

Another way of evaluating electromagnetic BSCs is the consequence of a study devoted to acoustical fields scatterings in which acoustical BSCs, similar to electromagnetic BSCs, have been introduced. The arsenal used for electromagnetic BSCs can be adapted to the case of acoustical BSCs, including the use of localized approximations [31–34] and of finite series [35]. But, furthermore, this provides a new method to the evaluation of electromagnetic BSCs by expressing them in terms of acoustical BSCs, such as expounded in [36]. In this method, two different sets of electromagnetic BSCs denoted $g_{n,TM}^m$ and $g_{n,TE}^m$ are expressed in terms of one single set of acoustical BSCs denoted $g_{n,A}^m$ (with n from 1 to ∞, m from $(-n)$ to $(+n)$, TM standing for "Transverse Magnetic" and TE standing for "Transverse Electric). This means that, if we neglect the computational time required to express the electromagnetic BSCs in terms of acoustical BSCs, the computational time required to compute the electromagnetic BSCs is divided roughly by a factor of 2.

Therefore the conversion time required to convert the acoustical BSCs to electromagnetic BSCs is an important factor which should be optimized. Independently of the present line of research in which electromagnetic fields are expressed in terms of one single vector potential (let us call it the VP1 approach), there exist another approach in which the electromagnetic fields are expressed in terms of two vector potentials (let us call it the VP2 approach), e.g. [37]. In investigating the relationship between the VP1 and the VP2 approaches, it has been observed that the expressions used to convert the BSCs in the VP1 approach could be drastically simplified. Furthermore, the VP1 approach provides two different kinds of expressions depending on whether $(n-m)$ is even or odd. An element of the simplified scheme is that these expressions, which are different depending on the parity

* Corresponding author.
E-mail address: gouesbet@coria.fr (G. Gouesbet).

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of $(n-m)$, could be unified. As a result, the coding of the expressions will be easier.

The aim of this paper is then to provide new, unified and simplified, expressions to deal with the VP1 approach. The paper is organized as follows. Section 2 recalls the original expressions of the VP1 approach. Section 3 deals with the unification and simplification process and results. Section 4 discusses a possible alternative approach. Section 5 is a summary. Section 6 is a conclusion.

2. Original VP1 expressions

2.1. Transverse electric BSCs

In the VP1 approach [36], the electromagnetic TE-BSCs are found to read as:

$$g_{n,TE}^m = \frac{k\psi_{A0}}{2\mu H_0}\frac{1}{P_n^{|m|}(0)}[g_{n,A}^{m+1}\tau_n^{|m+1|}(0) - g_{n,A}^{m-1}\tau_n^{|m-1|}(0)],\ (n-m)\text{ even} \quad (1)$$

$$\begin{aligned} g_{n,TE}^m =& \frac{k\psi_{A0}}{2\mu H_0}\frac{1}{[\frac{dP_n^{|m|}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2}} \\ &\times\{g_{n,A}^{m+1}[\tau_n'^{|m+1|}(0)+(m+1)\pi_n^{|m+1|}(0)] \\ &- g_{n,A}^{m-1}[\tau_n'^{|m-1|}(0)-(m-1)\pi_n^{|m-1|}(0)]\},\ (n-m)\text{ odd}\end{aligned} \quad (2)$$

in which ψ_{A0} is an acoustical field strength, H_0 is a magnetic field strength, k is the wavenumber of the electromagnetic fields in the medium in which they propagate, μ is the magnetic permeability of this medium, $P_n^m(\cos\theta)$ are associated Legendre functions using Hobson's convention [38], $\tau_n^m(\cos\theta)$ and $\pi_n^m(\cos\theta)$ are generalized Legendre functions reading as:

$$\tau_n^m(\cos\theta) = \frac{dP_n^m(\cos\theta)}{d\theta} \quad (3)$$

$$\pi_n^m(\cos\theta) = \frac{P_n^m(\cos\theta)}{\sin\theta} \quad (4)$$

and a prime denotes a derivative with respect to the argument.

2.2. Transverse magnetic BSCs

In the VP1 approach [36], the electromagnetic TM-BSCs are found to read as:

$$\begin{aligned} g_{n,TM}^m =& \frac{i\omega\psi_{A0}}{2(2n+1)E_0P_n^{|m|}(0)}(T_{n-1,m}^{(2)} - T_{n+1,m}^{(2)} - nT_{n-1,m}^{(1)} \\ &- (n+1)T_{n+1,m}^{(1)}),\ (n-m)\text{ even}\end{aligned} \quad (5)$$

in which:

$$T_{nm}^{(1)} = (m-1)P_n^{|m+1|}(0)g_{n,A}^{m+1} - (m+1)P_n^{|m-1|}(0)g_{n,A}^{m-1} \quad (6)$$

$$\begin{aligned} T_{nm}^{(2)} =& [m(m+1)P_n^{|m+1|}(0)+\tau_n'^{|m+1|}(0)]g_{n,A}^{m+1} \\ &+[m(m-1)P_n^{|m-1|}(0)+\tau_n'^{|m-1|}(0)]g_{n,A}^{m-1}\end{aligned} \quad (7)$$

and:

$$\begin{aligned} g_{n,TM}^m =& \frac{i\omega\psi_{A0}}{2(2n+1)E_0\tau_n^{|m|}(0)}(V_{n-1,m}^{(2)} - V_{n+1,m}^{(2)} - nV_{n-1,m}^{(1)} \\ &- (n+1)V_{n+1,m}^{(1)}),\ (n-m)\text{ odd}\end{aligned} \quad (8)$$

in which:

$$V_{nm}^{(1)} = (m-2)\tau_n^{|m+1|}(0)g_{n,A}^{m+1} - (m+2)\tau_n^{|m-1|}(0)g_{n,A}^{m-1} \quad (9)$$

$$\begin{aligned} V_{nm}^{(2)} =& [(m^2+m+2)\tau_n^{|m+1|}(0)-\tau_n''^{|m+1|}(0)]g_{n,A}^{m+1} \\ &+[(m^2-m+2)\tau_n^{|m-1|}(0)-\tau_n''^{|m-1|}(0)]g_{n,A}^{m-1}\end{aligned} \quad (10)$$

in which ω is the angular frequency of the waves and E_0 is an electric field strength. The introduction of ω is an opportunity to specify that the time-dependence used is of the form $\exp(i\omega t)$.

3. Unified and simplified expressions

3.1. TE-BSCs, $(n-m)$ even

We deal with Eq. (1). In this equation, we have:

$$\begin{aligned} \tau_n^{|m|}(\theta) =& \frac{dP_n^{|m|}(\cos\theta)}{d\theta} = \frac{dP_n^{|m|}(\cos\theta)}{d\cos\theta}\frac{d\cos\theta}{d\theta} \\ =& -\sin\theta\frac{dP_n^{|m|}(\cos\theta)}{d\cos\theta} = -\sqrt{1-\mu^2}\frac{dP_n^{|m|}(\mu)}{d\mu}\end{aligned} \quad (11)$$

But we have, e.g. [38], tome 1, p.102, with a typo corrected:

$$(\mu^2-1)\frac{dP_n^m(\mu)}{d\mu} = \sqrt{1-\mu^2}P_n^{m+1}(\mu) + m\mu P_n^m(\mu) \quad (12)$$

and, see [39], Eqs. (15)–(91), (15)–(92):

$$(\mu^2-1)\frac{dP_n^m(\mu)}{d\mu} = -(n+m)(n-m+1)\sqrt{1-\mu^2}P_n^{m-1}(\mu) - m\mu P_n^m(\mu) \quad (13)$$

that is to say:

$$(\mu^2-1)\frac{dP_n^{|m|}(\mu)}{d\mu} = \sqrt{1-\mu^2}P_n^{|m|+1}(\mu) + |m|\,\mu P_n^{|m|}(\mu) \quad (14)$$

$$(\mu^2-1)\frac{dP_n^{|m|}(\mu)}{d\mu} = -(n+|m|)(n-|m|+1)\sqrt{1-\mu^2}P_n^{|m|-1}(\mu) - |m|\,\mu P_n^{|m|}(\mu) \quad (15)$$

From Eq. (11), we obtain:

$$\tau_n^{|m|}(0) = -[\frac{dP_n^{|m|}(\mu)}{d\mu}]_{\theta=\pi/2} \quad (16)$$

which, using Eqs. (14)–(15), becomes:

$$\tau_n^{|m|}(0) = -(n+|m|)(n-|m|+1)P_n^{|m|-1}(0) = P_n^{|m|+1}(0) \quad (17)$$

from which we have:

$$\tau_n^{|m+1|}(0) = -(n+|m+1|)(n-|m+1|+1)P_n^{|m+1|-1}(0) \quad (18)$$

$$\tau_n^{|m-1|}(0) = P_n^{|m-1|+1}(0) \quad (19)$$

We then have to consider three different cases, as follows.

(i) EM BSCs with $m>0$, so that $|m|=m$, $|m+1|=m+1$ and $|m-1|=m-1$, so that Eqs. (18)–(19) become:

$$\tau_n^{|m+1|}(0) = -(n+m+1)(n-m)P_n^m(0) \quad (20)$$

$$\tau_n^{|m-1|}(0) = P_n^m(0) \quad (21)$$

which, once inserted in Eq. (1), leads to:

$$g_{n,TE}^m = \frac{-k\psi_{A0}}{2\mu H_0}[(n+m+1)(n-m)g_{n,A}^{m+1} + g_{n,A}^{m-1}],\ (n-m)\text{ even} \quad (22)$$

(ii) EM BSCs with $m=0$. From Eq. (18), we have:

$$[\tau_n^{|m+1|}(0)]_{m=0} = [\tau_n^{|m-1|}(0)]_{m=0} = \tau_n^1(0) = -n(n+1)P_n^0(0) \quad (23)$$

so that Eq. (1) becomes:

$$g_{n,TE}^0 = \frac{-k\psi_{A0}}{2\mu H_0}n(n+1)[g_{n,A}^1 - g_{n,A}^{-1}],\ (n-m)\text{ even} \quad (24)$$

(iii) EM BSCs with $m<0$, i.e. with $|m|=-m$, $|m+1|=|m|-1$, $|m-1|=|m|+1$.

We start again from Eq. (1). But, from Eqs. (18)–(19), we have:

$$\begin{aligned} \tau_n^{|m+1|}(0) =& -(n+|m+1|)(n-|m+1|+1)P_n^{|m+1|-1}(0) \\ =& -(n+|m|-1)(n-|m|+2)P_n^{|m|-2}(0) \\ =& -(n-m-1)(n+m+2)P_n^{|m|-2}(0)\end{aligned} \quad (25)$$

$$\tau_n^{|m-1|}(0) = P_n^{|m-1|+1}(0) = P_n^{|m|+2}(0) \quad (26)$$

Inserting Eqs. (25)–(26) into Eq. (1) leads to:

$$g_{n,TE}^{m} = \frac{k\psi_{A0}}{2\mu H_0}\frac{1}{P_n^{|m|}(0)} \times[-g_{n,A}^{m+1}(n-m-1)(n+m+2)P_n^{|m|-2}(0) - g_{n,A}^{m-1}P_n^{|m|+2}(0)], \quad (n-m) \text{ even} \tag{27}$$

Next, we evaluate the ratios $P_n^{|m|-2}(0)/P_n^{|m|}(0)$ and $P_n^{|m|+2}(0)/P_n^{|m|}(0)$ using, e.g. Eq.(5.6) in [2]:

$$P_n^{|m|}(0) = (-1)^{(n+|m|)/2}\frac{(n+|m|-1)!!}{2^{(n-|m|)/2}(\frac{n-|m|}{2})!}, \ (n-m) \text{ even} \tag{28}$$

leading to:

$$\frac{P_n^{|m|-2}(0)}{P_n^{|m|}(0)} = \frac{-1}{(n+m+2)(n-m-1)} \tag{29}$$

$$\frac{P_n^{|m|+2}(0)}{P_n^{|m|}(0)} = -(n-|m|)(n+|m|+1) \tag{30}$$

Inserting Eqs. (29)–(30) into Eq. (27) leads to:

$$g_{n,TE}^{m} = \frac{k\psi_{A0}}{2\mu H_0}[g_{n,A}^{m+1} + g_{n,A}^{m-1}(n-|m|)(n+|m|+1)], \ (n-m) \text{ even} \tag{31}$$

3.2. TE-BSCs, $(n-m)$ odd

We start from Eq. (2). But we have:

$$[\frac{dP_n^{|m|}(\cos\theta)}{d\cos\theta}]_{\theta=\pi/2} = -[\frac{dP_n^{|m|}(\cos\theta)}{d\theta}\frac{d\theta}{d\cos\theta}]_{\theta=\pi/2} = -[\frac{dP_n^{|m|}(\cos\theta)}{d\theta}]_{\theta=\pi/2} = -\tau_n^{|m|}(0), \ (n-m) \text{ odd} \tag{32}$$

We now deal with $\tau_n^{\prime|m\pm1|}(0)$ which are involved in Eq. (2). For this, we have:

$$\tau_n^{\prime|m|}(\cos\theta) = \frac{d\tau_n^{|m|}(\cos\theta)}{d\cos\theta} = \frac{-1}{\sin\theta}\frac{d\tau_n^{|m|}(\cos\theta)}{d\theta} = \frac{-1}{\sin\theta}\frac{d}{d\theta}\frac{dP_n^{|m|}(\cos\theta)}{d\theta} = \frac{-1}{\sin\theta}\frac{d^2P_n^{|m|}(\cos\theta)}{d\theta^2} \tag{33}$$

But the associated Legendre functions satisfy the equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP_n^m(\cos\theta)}{d\theta}) + [n(n+1) - \frac{m^2}{\sin^2\theta}]P_n^m(\cos\theta) = 0 \tag{34}$$

Hence:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP_n^{|m|}(\cos\theta)}{d\theta}) + [n(n+1) - \frac{|m|^2}{\sin^2\theta}]P_n^{|m|}(\cos\theta) = 0 \tag{35}$$

from which we derive:

$$\frac{d^2P_n^{|m|}(\cos\theta)}{d\theta^2} = \frac{-\cos\theta}{\sin\theta}\frac{dP_n^{|m|}(\cos\theta)}{d\theta} - [n(n+1) - \frac{|m|^2}{\sin^2\theta}]P_n^{|m|}(\cos\theta) \tag{36}$$

Inserting Eq. (36) into Eq. (33) leads to:

$$\tau_n^{\prime|m|}(\cos\theta) = \frac{\cos\theta}{\sin^2\theta}\frac{dP_n^{|m|}(\cos\theta)}{d\theta} + \frac{1}{\sin\theta}[n(n+1) - \frac{|m|^2}{\sin^2\theta}]P_n^{|m|}(\cos\theta) \tag{37}$$

that is to say, using the definition of τ_n^m in Eq. (3):

$$\tau_n^{\prime|m|}(\cos\theta) = \frac{\cos\theta}{\sin^2\theta}\tau_n^{|m|}(\cos\theta) + \frac{1}{\sin\theta}[n(n+1) - \frac{|m|^2}{\sin^2\theta}]P_n^{|m|}(\cos\theta) \tag{38}$$

Hence:

$$\tau_n^{\prime|m|}(0) = [n(n+1) - |m|^2]P_n^{|m|}(0) = (n^2 - |m|^2 + n)P_n^{|m|}(0) \tag{39}$$

We also have, e.g. Eq. (4):

$$\pi_n^{|m|}(0) = [\frac{P_n^{|m|}(\cos\theta)}{\sin\theta}]_{\theta=\pi/2} = P_n^{|m|}(0) \tag{40}$$

Taking into account for a change from m to $m\pm1$, we insert Eqs. (32), (39) and (40) into Eq. (3), leading to:

$$g_{n,TE}^{m} = \frac{-k\psi_{A0}}{2\mu H_0}\{g_{n,A}^{m+1}[n(n+1) - |m+1|^2 + m + 1]\frac{P_n^{|m+1|}(0)}{P_n^{|m|+1}(0)} - g_{n,A}^{m-1}[n(n+1) - |m-1|^2 - (m-1)]\frac{P_n^{|m-1|}(0)}{P_n^{|m|+1}(0)}\} \tag{41}$$

in which we have used as well $\tau_n^{|m|}(0) = P_n^{|m|+1}(0)$ from Eq. (17).

We now consider again three different cases.

(i) $m > 0$, i.e. $|m| = m$, $|m+1| = m+1$, $|m-1| = m-1$.

Eq. (41) then becomes:

$$g_{n,TE}^{m} = \frac{-k\psi_{A0}}{2\mu H_0}\{g_{n,A}^{m+1}[n(n+1) - (m+1)^2 + m + 1] - g_{n,A}^{m-1}[n(n+1) - (m-1)^2 - (m-1)]\frac{P_n^{m-1}(0)}{P_n^{m+1}(0)}\} \tag{42}$$

But we have:

$$n(n+1) - (m+1)^2 + m + 1 = (n+m+1)(n-m) \tag{43}$$

$$n(n+1) - (m-1)^2 - (m-1) = (n-m+1)(n+m) \tag{44}$$

and, using Eq. (28):

$$\frac{P_n^{m-1}(0)}{P_n^{m+1}(0)} = \frac{-1}{(n+m)(n+m+1)} \tag{45}$$

Inserting Eqs. (43)–(45) into Eq. (42) then leads to:

$$g_{n,TE}^{m} = \frac{-k\psi_{A0}}{2\mu H_0}[g_{n,A}^{m+1}(n+m+1)(n-m) + g_{n,A}^{m-1}], \ (n-m) \text{ odd} \tag{46}$$

which agrees with Eq. (22), therefore unifying the expressions for the TE-BSCs, whatever the parity of $(n-m)$, for the case $m > 0$.

(ii) $m = 0$

Eq. (41) then readily becomes:

$$g_{n,TE}^{0} = \frac{-k\psi_{A0}}{2\mu H_0}n(n+1)(g_{n,A}^{1} - g_{n,A}^{-1}), \ (n-m) \text{ odd} \tag{47}$$

which agrees with Eq. (24), therefore unifying the expressions for the TE-BSCs, whatever the parity of $(n-m)$, for the case $m = 0$.

(iii) $m < 0$, i.e $|m| = -m$, $|m+1| = |m| - 1$, $|m-1| = |m| + 1$.

Eq. (41) becomes:

$$g_{n,TE}^{m} = \frac{-k\psi_{A0}}{2\mu H_0}\{g_{n,A}^{m+1}[n(n+1) - (|m|-1)^2 - |m| + 1]\frac{P_n^{|m|-1}(0)}{P_n^{|m|+1}(0)} - g_{n,A}^{m-1}[n(n+1) - (|m|+1)^2 + |m| + 1]\} \tag{48}$$

But we have:

$$n(n+1) - (|m|-1)^2 - |m| + 1 = n(n+1) - |m|(|m|-1) = (n+|m|)(n-|m|+1) \tag{49}$$

$$n(n+1) - (|m|+1)^2 + |m| + 1 = n(n+1) - |m|(|m|+1) = (n-|m|)(n+|m|+1) \tag{50}$$

and also, using Eq. (28), we have:

$$\frac{P_n^{|m|-1}(0)}{P_n^{|m|+1}(0)} = \frac{-1}{(n+|m|)(n-|m|+1)} \tag{51}$$

Inserting Eqs. (49)–(51) into Eq. (48), we obtain:

$$g_{n,TE}^{m} = \frac{k\psi_{A0}}{2\mu H_0}[g_{n,A}^{m+1} + (n-|m|)(n+|m|+1)g_{n,A}^{m-1}], \ (n-m) \text{ odd} \tag{52}$$

which agrees with Eq. (31), therefore unifying the expressions for the TE-BSCs, whatever the parity of $(n-m)$, for the case $m < 0$.

3.3. TM-BSCs, $(n-m)$ even

We start from Eqs. (5)–(7). But we have, from Eq. (39):

$$\tau_n'^{|m\pm1|}(0) = (n^2 - |m\pm1|^2 + n)P_n^{|m\pm1|}(0) \tag{53}$$

Inserting Eq. (53) into Eq. (7), we obtain:

$$\begin{aligned} T_{nm}^{(2)} &= [m(m+1) + (n^2 - |m+1|^2 + n)]P_n^{|m+1|}(0)g_{n,A}^{m+1} \\ &+[m(m-1) + (n^2 - |m-1|^2 + n)]P_n^{|m-1|}(0)g_{n,A}^{m-1} \end{aligned} \tag{54}$$

Inserting Eqs. (6) and (54) into Eq. (5), and rearranging, the BSCs $g_{n,TM}^m$ may be rewritten as:

$$\begin{aligned} g_{n,TM}^m &= \frac{i\omega\psi_{A0}}{2(2n+1)E_0P_n^{|m|}(0)} \\ &\times[F_{n-1}^{m+1}g_{n-1,A}^{m+1} + F_{n-1}^{m-1}g_{n-1,A}^{m-1} + F_{n+1}^{m+1}g_{n+1,A}^{m+1} + F_{n-1}^{m-1}g_{n-1,A}^{m-1}] \end{aligned} \tag{55}$$

in which:

$$F_{n-1}^{m+1} = [m^2 + m - nm + n^2 - |m+1|^2]P_{n-1}^{|m+1|}(0) \tag{56}$$

$$F_{n-1}^{m-1} = [m^2 - m + nm + n^2 - |m-1|^2]P_{n-1}^{|m-1|}(0) \tag{57}$$

$$F_{n+1}^{m+1} = [-m^2 - 2m - nm - 2n - 1 - n^2 + |m+1|^2]P_{n+1}^{|m+1|}(0) \tag{58}$$

$$F_{n+1}^{m-1} = [-m^2 + 2m + nm - 2n - 1 - n^2 + |m-1|^2]P_{n+1}^{|m-1|}(0) \tag{59}$$

We now consider again three different cases.

(i) $m > 0$, i.e. $|m+1| = m+1$, $|m-1| = m-1$.

The terms F_i^j immediately reduce to:

$$F_{n-1}^{m+1} = (n+1)(n-m-1)P_{n-1}^{m+1}(0) \tag{60}$$

$$F_{n-1}^{m-1} = (n+1)(n+m-1)P_{n-1}^{m-1}(0) \tag{61}$$

$$F_{n+1}^{m+1} = -n(n+m+2)P_{n+1}^{m+1}(0) \tag{62}$$

$$F_{n+1}^{m-1} = -n(n-m+2)P_{n+1}^{m-1}(0) \tag{63}$$

Inserting Eqs. (60)–(63) into Eq. (55), and changing $|m|$ to m, leads to:

$$\begin{aligned} g_{n,TM}^m &= \frac{i\omega\psi_{A0}}{2(2n+1)E_0P_n^m(0)} \\ &\times[(n+1)(n-m-1)P_{n-1}^{m+1}(0)g_{n-1,A}^{m+1} \\ &+(n+1)(n+m-1)P_{n-1}^{m-1}(0)g_{n-1,A}^{m-1} \\ &-n(n+m+2)P_{n+1}^{m+1}(0)g_{n+1,A}^{m+1} \\ &-n(n-m+2)P_{n+1}^{m-1}(0)g_{n+1,A}^{m-1}] \end{aligned} \tag{64}$$

Eq. (64) contains four ratios of associated Legendre functions which are evaluated using Eq. (28), leading to:

$$\frac{P_{n-1}^{m+1}(0)}{P_n^m(0)} = (n-m) \tag{65}$$

$$\frac{P_{n-1}^{m-1}(0)}{P_n^m(0)} = \frac{-1}{n+m-1} \tag{66}$$

$$\frac{P_{n+1}^{m+1}(0)}{P_n^m(0)} = -(n+m+1) \tag{67}$$

$$\frac{P_{n+1}^{m-1}(0)}{P_n^m(0)} = \frac{1}{n-m+2} \tag{68}$$

Inserting Eqs. (65)–(68) into Eq. (64), and rearranging, then leads to:

$$\begin{aligned} g_{n,TM}^m &= \frac{i\omega\psi_{A0}}{2(2n+1)E_0} \\ &\times\{n[(n+m+1)(n+m+2)g_{n+1,A}^{m+1} - g_{n+1,A}^{m-1}] \\ &+(n+1)[(n-m-1)(n-m)g_{n-1,A}^{m+1} - g_{n-1,A}^{m-1}]\} \end{aligned} \tag{69}$$

(ii) $m = 0$

From Eqs. (56)–(59), the terms F_i^j immediately reduce to:

$$F_{n-1}^{1} = (n^2-1)P_{n-1}^{+1}(0) \tag{70}$$

$$F_{n-1}^{-1} = (n^2-1)P_{n-1}^{+1}(0) \tag{71}$$

$$F_{n+1}^{+1} = -n(n+2)P_{n+1}^{+1}(0) \tag{72}$$

$$F_{n+1}^{-1} = -n(n+2)P_{n+1}^{+1}(0) \tag{73}$$

We also have to deal with a few ratios of associated Legendre functions, namely, using Eq. (28):

$$\frac{P_{n-1}^{+1}(0)}{P_n^0(0)} = n \tag{74}$$

$$\frac{P_{n+1}^{+1}(0)}{P_n^0(0)} = -(n+1) \tag{75}$$

Inserting Eqs. (70)–(75) into Eq. (55), we obtain:

$$\begin{aligned} g_{n,TM}^0 &= \frac{i\omega\psi_{A0}}{2(2n+1)E_0}n(n+1) \\ &\times[(n-1)(g_{n-1,A}^1 + g_{n-1,A}^{-1}) + (n+2)(g_{n+1,A}^1 + g_{n+1,A}^{-1})] \end{aligned} \tag{76}$$

(iii) $m < 0$, i.e. $|m| = -m$, $|m+1| = |m|-1$, $|m-1| = |m|+1$.

Eqs. (56)–(59) then lead to:

$$F_{n-1}^{m+1} = (n+1)(n+|m|-1)P_{n-1}^{|m|-1}(0) \tag{77}$$

$$F_{n-1}^{m-1} = (n+1)(n-|m|-1)P_{n-1}^{|m|+1}(0) \tag{78}$$

$$F_{n+1}^{m+1} = -n(n-|m|+2)P_{n+1}^{|m|-1}(0) \tag{79}$$

$$F_{n+1}^{m-1} = -n(n+|m|+2)P_{n+1}^{|m|+1}(0) \tag{80}$$

Inserting Eqs. (77)–(80) into Eq. (55) leads to:

$$\begin{aligned} g_{n,TM}^m &= \frac{i\omega\psi_{A0}}{2(2n+1)E_0} \\ &\times[(n+1)(n+|m|-1)\frac{P_{n-1}^{|m|-1}(0)}{P_n^{|m|}(0)}g_{n-1,A}^{m+1} \\ &+(n+1)(n-|m|-1)\frac{P_{n-1}^{|m|+1}(0)}{P_n^{|m|}(0)}g_{n-1,A}^{m-1} \\ &-n(n-|m|+2)\frac{P_{n+1}^{|m|-1}(0)}{P_n^{|m|}(0)}g_{n+1,A}^{m+1} \\ &-n(n+|m|+2)\frac{P_{n+1}^{|m|+1}(0)}{P_n^{|m|}(0)}g_{n+1,A}^{m-1}] \end{aligned} \tag{81}$$

The associated Legendre functions ratios then read as, according to Eqs. (65)–(68):

$$\frac{P_{n-1}^{|m|+1}(0)}{P_n^{|m|}(0)} = (n-|m|) \tag{82}$$

$$\frac{P_{n-1}^{|m|-1}(0)}{P_n^{|m|}(0)} = \frac{-1}{n+|m|-1} \tag{83}$$

$$\frac{P_{n+1}^{|m|+1}(0)}{P_n^{|m|}(0)} = -(n+|m|+1) \tag{84}$$

$$\frac{P_{n+1}^{|m|-1}(0)}{P_n^{|m|}(0)} = \frac{1}{n-|m|+2} \tag{85}$$

Inserting Eqs. (82)–(85) into Eq. (81) leads to:

$$g_{n,TM}^{m} = \frac{i\omega\Psi_{A0}}{2(2n+1)E_0} \times\{n[(n+|m|+1)(n+|m|+2)g_{n+1,A}^{m-1} - g_{n+1,A}^{m+1}] + (n+1)[(n-|m|-1)(n-|m|)g_{n-1,A}^{m-1} - g_{n-1,A}^{m+1}]\} \quad (86)$$

3.4. TM-BSCs, $(n-m)$ odd

We start from Eqs. (8)–(10). We next have:

$$\tau_n^{\prime\prime|m|}(\cos\theta) = \frac{d\tau_n^{\prime|m|}(\cos\theta)}{d\cos\theta} = \frac{-1}{\sin\theta}\frac{d\tau_n^{\prime|m|}(\cos\theta)}{d\theta} \quad (87)$$

Using Eq. (37), Eq. (87) becomes:

$$\tau_n^{\prime\prime|m|}(\cos\theta) = \frac{-1}{\sin\theta}\frac{d}{d\theta}\{\frac{\cos\theta}{\sin^2\theta}\tau_n^{|m|}(\cos\theta) + \frac{1}{\sin\theta}[n(n+1) - \frac{|m|^2}{\sin^2\theta}]P_n^{|m|}(\cos\theta)\} \quad (88)$$

which is next evaluated to:

$$\tau_n^{\prime\prime|m|}(\cos\theta) = \frac{1}{\sin\theta}\{\frac{1+\cos^2\theta}{\sin^3\theta}\tau_n^{|m|}(\cos\theta) - \cos\theta[\frac{3\,|m|^2}{\sin^4\theta} - \frac{n(n+1)}{\sin^2\theta}]P_n^{|m|}(\cos\theta) + \frac{\cos\theta}{\sin\theta}\tau_n^{\prime|m|}(\cos\theta) - \frac{1}{\sin\theta}[n(n+1) - \frac{|m|^2}{\sin^2\theta}]\tau_n^{|m|}(\cos\theta)\} \quad (89)$$

from which we deduce:

$$\tau_n^{\prime\prime|m|}(0) = [|m|^2 - n(n+1) + 1]\tau_n^{|m|}(0) \quad (90)$$

Inserting Eq. (90) into Eq. (10), we obtain:

$$V_{nm}^{(2)} = [m(m+1) - |m+1|^2 + n(n+1) + 1]\tau_n^{|m+1|}(0)g_{n,A}^{m+1} + [m(m-1) - |m-1|^2 + n(n+1) + 1]\tau_n^{|m-1|}(0)g_{n,A}^{m-1} \quad (91)$$

Using Eqs. (9) and Eq. (91), and rearranging, Eq. (8) becomes:

$$g_{n,TM}^{m} = \frac{i\omega\Psi_{A0}}{2(2n+1)E_0\tau_n^{|m|}(0)} \times\{[m(m+1) - nm - |m+1|^2 + n(n+1) + 1]\tau_{n-1}^{|m+1|}(0)g_{n-1,A}^{m+1} + [m(m-1) + nm - |m-1|^2 + n(n+1) + 1]\tau_{n-1}^{|m-1|}(0)g_{n-1,A}^{m-1} - [m(m+2) + nm - |m+1|^2 + n(n+1) + 1]\tau_{n+1}^{|m+1|}(0)g_{n+1,A}^{m+1} - [m(m-2) - nm - |m-1|^2 + n(n+1) + 1]\tau_{n+1}^{|m-1|}(0)g_{n+1,A}^{m-1}\} \quad (92)$$

We now have again to consider different cases as follows.

(i) $m > 0$, i.e. $|m+1| = m+1$, $|m-1| = m-1$.

Eq. (92) simplifies to:

$$g_{n,TM}^{m} = \frac{i\omega\Psi_{A0}}{2(2n+1)E_0\tau_n^{m}(0)} \times[(n+1)(n-m)\tau_{n-1}^{m+1}(0)g_{n-1,A}^{m+1} + (n+1)(n+m)\tau_{n-1}^{m-1}(0)g_{n-1,A}^{m-1} - n(n+m+1)\tau_{n+1}^{m+1}(0)g_{n+1,A}^{m+1} - n(n-m+1)\tau_{n+1}^{m-1}(0)g_{n+1,A}^{m-1}] \quad (93)$$

We next use Eqs. (17) and (28), implying:

$$\tau_n^m(0) = P_n^{m+1}(0) = \frac{(-1)^{(n+m+1)/2}(n+m)!!}{2^{(n-m-1)/2}(\frac{n-m-1}{2})!} \quad (94)$$

and leading to:

$$\frac{\tau_{n-1}^{m+1}(0)}{\tau_n^m(0)} = (n-m-1) \quad (95)$$

$$\frac{\tau_{n-1}^{m-1}(0)}{\tau_n^m(0)} = \frac{-1}{n+m} \quad (96)$$

$$\frac{\tau_{n+1}^{m+1}(0)}{\tau_n^m(0)} = -(n+m+2) \quad (97)$$

$$\frac{\tau_{n+1}^{m-1}(0)}{\tau_n^m(0)} = \frac{1}{n-m+1} \quad (98)$$

Inserting Eqs. (95)–(98) into Eq. (93) leads to:

$$g_{n,TM}^{m} = \frac{i\omega\Psi_{A0}}{2(2n+1)E_0} \times\{n[(n+m+1)(n+m+2)g_{n+1,A}^{m+1} - g_{n+1,A}^{m-1}] + (n+1)[(n-m-1)(n-m)g_{n-1,A}^{m+1} - g_{n-1,A}^{m-1}]\} \quad (99)$$

which identifies with Eq. (69), unifying the expressions for the TM-BSCs, whatever the parity of $(n-m)$.

(ii) $m = 0$.

Eq. (92) becomes:

$$g_{n,TM}^{m} = \frac{i\omega\Psi_{A0}}{2(2n+1)E_0}n(n+1)\frac{1}{\tau_n^0(0)} \times[(g_{n-1,A}^{1} + g_{n-1,A}^{-1})\tau_{n-1}^{1}(0) - (g_{n+1,A}^{1} + g_{n+1,A}^{-1})\tau_{n+1}^{1}(0)] \quad (100)$$

But, from Eqs. (95) and (97), we have:

$$\frac{\tau_{n-1}^{1}(0)}{\tau_n^0(0)} = (n-1) \quad (101)$$

$$\frac{\tau_{n+1}^{1}(0)}{\tau_n^0(0)} = -(n+2) \quad (102)$$

which, when inserted into Eq. (100) leads to:

$$g_{n,TM}^{0} = \frac{i\omega\Psi_{A0}}{2(2n+1)E_0}n(n+1) \times[(n-1)(g_{n-1,A}^{1} + g_{n-1,A}^{-1}) + (n+2)(g_{n+1,A}^{1} + g_{n+1,A}^{-1})] \quad (103)$$

which is exactly Eq. (76), i.e. leading again to the unification with the case $(n-m)$ even.

(iii) $m < 0$, i.e. $|m| = -m$, $|m+1| = |m|-1$, $|m-1| = |m|+1$.

Eq. (92) leads to:

$$g_{n,TM}^{m} = \frac{i\omega\Psi_{A0}}{2(2n+1)E_0\tau_n^{|m|}(0)} \times[(n+1)(n+|m|)\tau_{n-1}^{|m|-1}g_{n-1,A}^{m+1} + (n+1)(n-|m|)\tau_{n-1}^{|m|+1}g_{n-1,A}^{m-1} - n(n-|m|+1)\tau_{n+1}^{|m|-1}g_{n+1,A}^{m+1} - n(n+|m|+1)\tau_{n+1}^{|m|+1}g_{n+1,A}^{m-1}] \quad (104)$$

Next, from Eqs. (95)–(98), we have:

$$\frac{\tau_{n-1}^{|m|+1}(0)}{\tau_n^{|m|}(0)} = (n-|m|-1) \quad (105)$$

$$\frac{\tau_{n-1}^{|m|-1}(0)}{\tau_n^{|m|}(0)} = \frac{-1}{n+|m|} \quad (106)$$

$$\frac{\tau_{n+1}^{|m|+1}(0)}{\tau_n^{|m|}(0)} = -(n+|m|+2) \quad (107)$$

$$\frac{\tau_{n+1}^{|m|-1}(0)}{\tau_n^{|m|}(0)} = \frac{1}{n-|m|+1} \quad (108)$$

Then, inserting Eqs. (105)–(108) into Eq. (104), we obtain:

$$g_{n,TM}^{m} = \frac{i\omega\Psi_{A0}}{2(2n+1)E_0} \times\{n[(n+|m|+1)(n+|m|+2)g_{n+1,A}^{m-1} - g_{n+1,A}^{m+1}] + (n+1)[(n-|m|-1)(n-|m|)g_{n-1,A}^{m-1} - g_{n-1,A}^{m+1}]\} \quad (109)$$

which exactly identifies with Eq. (86), i.e. leading again to the unification with the case $(n-m)$ even.

4. A possible alternative approach

The approach discussed in [36] and in the present paper has been motivated by the finite series technique, e.g. [18,19] and references therein dating back to [20,21]. As a consequence, the cases $(n-m)$ even and $(n-m)$ odd have been considered separately, and the angle θ in the expressions of the BSCs has been set to $\pi/2$. An alternative method is to take advantage of the fact that the BSCs are constant complex numbers, and therefore actually do not depend on θ. This means that the dependences of BSCs with respect to θ are apparent. To draw the consequences of this fact, let us consider Eq.(40) of [36] rewritten below:

$$g_{n,TE}^m = \frac{k\psi_{A0}}{2\mu H_0}\frac{1}{P_n^{|m|}(\cos\theta)} \times\{g_{n,A}^{m+1}[\tau_n^{|m+1|}(\cos\theta)+(m+1)\cos\theta\pi_n^{|m+1|}(\cos\theta)] - g_{n,A}^{m-1}[\tau_n^{|m-1|}(\cos\theta)-(m-1)\cos\theta\pi_n^{|m-1|}(\cos\theta)]\} \tag{110}$$

The fact that $g_{n,TE}^m$ does not actually depend on θ implies that the ratios written below are complex numbers:

$$T_+ = \frac{\tau_n^{|m+1|}(\cos\theta)+(m+1)\cos\theta\pi_n^{|m+1|}(\cos\theta)}{P_n^{|m|}(\cos\theta)} \tag{111}$$

$$T_- = \frac{\tau_n^{|m-1|}(\cos\theta)-(m-1)\cos\theta\pi_n^{|m-1|}(\cos\theta)}{P_n^{|m|}(\cos\theta)} \tag{112}$$

Using Eqs. (12) and (13), we obtain:

$$\frac{dP_n^m(\cos\theta)}{d\theta} - m\frac{\cos\theta}{\sin\theta}P_n^m(\cos\theta) = P_n^{m+1}(\cos\theta) \tag{113}$$

$$\frac{dP_n^m(\cos\theta)}{d\theta} + m\frac{\cos\theta}{\sin\theta}P_n^m(\cos\theta) = -(n+m)(n-m+1)P_n^{m-1}(\cos\theta) \tag{114}$$

Recalling the definition of the generalized Legendre functions $\tau_n^m(\cos\theta)$ and $\pi_n^m(\cos\theta)$, Eqs. (113) and (114) may be rewritten as:

$$\tau_n^m(\cos\theta) - m\cos\theta\pi_n^m(\cos\theta) = P_n^{m+1}(\cos\theta) \tag{115}$$

$$\tau_n^m(\cos\theta) + m\cos\theta\pi_n^m(\cos\theta) = -(n+m)(n-m+1)P_n^{m-1}(\cos\theta) \tag{116}$$

From Eq. (115), we may express $\tau_n^{|m-1|}(\cos\theta)$ and, from Eq. (116), we may express $\tau_n^{|m+1|}(\cos\theta)$, leading to:

$$\tau_n^{m-1}(\cos\theta) - (m-1)\cos\theta\pi_n^{m-1}(\cos\theta) = P_n^m(\cos\theta) \tag{117}$$

$$\tau_n^{m+1}(\cos\theta) + (m+1)\cos\theta\pi_n^{m+1}(\cos\theta) = -(n+m+1)(n-m)P_n^m(\cos\theta) \tag{118}$$

For $m>0$, we therefore simply obtain:

$$T_+ = \frac{\tau_n^{m+1}(\cos\theta)+(m+1)\cos\theta\pi_n^{m+1}(\cos\theta)}{P_n^m(\cos\theta)} = -(n+m+1)(n-m) \tag{119}$$

$$T_- = \frac{\tau_n^{m-1}(\cos\theta)-(m-1)\cos\theta\pi_n^{m-1}(\cos\theta)}{P_n^m(\cos\theta)} = 1 \tag{120}$$

which do not depend on θ as it should, and agree with Eq. (22)

For $m=0$, Eqs. (111) and (112) become:

$$T_+ = T_- = \frac{\tau_n^1(\cos\theta)+\cos\theta\pi_n^1(\cos\theta)}{P_n^0(\cos\theta)} \tag{121}$$

while Eq. (118) become:

$$\tau_n^1(\cos\theta)+\cos\theta\pi_n^1(\cos\theta) = -n(n+1)P_n^0(\cos\theta) \tag{122}$$

so that Eq. (121) leads to:

$$T_+ = T_- = -n(n+1) \tag{123}$$

which, again does not depend on θ, as it should, and agrees with Eq. (24).

Finally, for $m<0$, Eqs. (111) and (112) become:

$$T_+ = \frac{\tau_n^{|m|-1}(\cos\theta)-(|m|-1)\cos\theta\pi_n^{|m|-1}(\cos\theta)}{P_n^{|m|}(\cos\theta)} \tag{124}$$

$$T_- = \frac{\tau_n^{|m|+1}(\cos\theta)+(|m|+1)\cos\theta\pi_n^{|m|+1}(\cos\theta)}{P_n^{|m|}(\cos\theta)} \tag{125}$$

From Eqs. (117) and (118), we then obtain:

$$T_+ = 1 \tag{126}$$

$$T_- = -(n+|m|+1)(n-|m|) \tag{127}$$

which do not depend on θ, again as it should, and agree with Eq. (31).

However, Eqs. (22), (24) and (31) were valid for $(n-m)$ even. The equations obtained above in the present section do not depend on the parity of $(n-m)$. This implies that the expressions for $(n-m)$ even or odd have been unified, in a way different from the one used in Section 3. A similar approach may be used in principle for the TM-BSCs but dealing with Eq.(50) of [36], which depends on both r and θ, is far more complicated than dealing with Eq.(40) of [36], as it can be seen below where it is repeated for convenience:

$$2E_0\sum_{n=0}^{\infty}c_n^{pw}g_{n,TM}^m n(n+1)\frac{j_n(kr)}{r}P_n^{|m|}(\cos\theta) \tag{128}$$

$$\begin{aligned}
&= \frac{\psi_{A0}}{i\omega\mu\varepsilon}\sum_{n=0}^{\infty}c_{n,A}^{pw}\{\frac{dj_n(kr)}{rdr}[-2\sin\theta P_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}\\
&-2\sin\theta P_n^{|m-1|}(\cos\theta)g_{n,A}^{m-1}\\
&+\cos\theta\tau_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}+\cos\theta\tau_n^{|m-1|}(\cos\theta)g_{n,A}^{m-1}\\
&+\frac{1}{\sin\theta}(m+1)P_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}-\frac{1}{\sin\theta}(m-1)P_n^{|m-1|}(\cos\theta)g_{n,A}^{m-+1}]\\
&+\frac{j_n(kr)}{r^2}[-2\cos\theta\tau_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}-2\cos\theta\tau_n^{|m-1|}(\cos\theta)g_{n,A}^{m-1}\\
&-\frac{1}{\sin\theta}(m+1)P_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}+\frac{1}{\sin\theta}(m-1)P_n^{|m-1|}(\cos\theta)g_{n,A}^{m-1}\\
&-\sin\theta\frac{d\tau_n^{|m+1|}(\cos\theta)}{d\theta}g_{n,A}^{m+1}-\sin\theta\frac{d\tau_n^{|m-1|}(\cos\theta)}{d\theta}g_{n,A}^{m-1}\\
&+\frac{1}{\sin\theta}(m+1)^2P_n^{|m+1|}(\cos\theta)g_{n,A}^{m+1}+\frac{1}{\sin\theta}(m-1)^2P_n^{|m-1|}(\cos\theta)g_{n,A}^{m-1}]\}
\end{aligned}$$

5. Summary and discussion

It is certainly convenient to summarize in this section the results which have been obtained in the present paper. They read as:

$$g_{n,TE}^m = \frac{-k\psi_{A0}}{2\mu H_0}[(n+m+1)(n-m)g_{n,A}^{m+1}+g_{n,A}^{m-1}],\ m>0 \tag{129}$$

$$g_{n,TE}^0 = \frac{-k\psi_{A0}}{2\mu H_0}n(n+1)[g_{n,A}^1-g_{n,A}^{-1}],\ m=0 \tag{130}$$

$$g_{n,TE}^m = \frac{k\psi_{A0}}{2\mu H_0}[g_{n,A}^{m+1}+g_{n,A}^{m-1}(n-|m|)(n+|m|+1)],\ m<0 \tag{131}$$

$$g_{n,TM}^m = \frac{i\omega\psi_{A0}}{2(2n+1)E_0} \times\{n[(n+m+1)(n+m+2)g_{n+1,A}^{m+1}-g_{n+1,A}^{m-1}] +(n+1)[(n-m-1)(n-m)g_{n-1,A}^{m+1}-g_{n-1,A}^{m-1}]\},\ m>0 \tag{132}$$

$$g_{n,TM}^0 = \frac{i\omega\psi_{A0}}{2(2n+1)E_0}n(n+1) \times[(n-1)(g_{n-1,A}^1+g_{n-1,A}^{-1})+(n+2)(g_{n+1,A}^1+g_{n+1,A}^{-1})],\ m=0 \tag{133}$$

$$g_{n,TM}^m = \frac{i\omega\psi_{A0}}{2(2n+1)E_0} \times\{n[(n+|m|+1)(n+|m|+2)g_{n+1,A}^{m-1}-g_{n+1,A}^{m+1}] +(n+1)[(n-|m|-1)(n-|m|)g_{n-1,A}^{m-1}-g_{n-1,A}^{m+1}]\},\ m<0 \tag{134}$$

To use these equations, we must furthermore remember that the electromagnetic BSCs are defined for $n > 0$, although the case $n = 0$ is valid for acoustical fields, e.g. Eqs.(4), (5) and (10) in [36]. Also, it is interesting to remark that Eqs. (129)–(131) on one hand, and Eqs. (132)–(134) on the other hand, can be merged and then read as, with $m > 0$:

$$g_{n,TE}^{\pm m} = \mp \frac{k\psi_{A0}}{2\mu H_0}[g_{n,A}^{\pm(m-1)} + g_{n,A}^{\pm(m+1)}(n-m)(n+m+1)] \tag{135}$$

$$\begin{aligned} g_{n,TM}^{\pm m} &= \frac{-i\omega\psi_{A0}}{2(2n+1)E_0}\{n[g_{n+1,A}^{\pm(m-1)} - g_{n+1,A}^{\pm(m+1)}(n+m+1)(n+m+2)] \\ &\quad + (n+1)g_{n-1,A}^{\pm(m-1)} - g_{n-1,A}^{\pm(m+1)}(n-m-1)(n-m)\} \end{aligned} \tag{136}$$

To better appreciate the improvements obtained in the present paper, let us compare the old and the new equations allowing one to express the electromagnetic BSCs in terms of the scalar BSCs, omitting the case $m = 0$ which does not lead to significant improvements. Then, the original Transverse Electric BSCs $g_{n,TE}^m$ are provided by Eq. (1) for $(n - m)$ even where they are expressed using special functions, namely the generalized Legendre functions $\tau_n^m(\cos\theta)$. For $(n - m)$ odd, original expressions of the BSCs $g_{n,TE}^m$ are given by Eq. (2) where they are expressed using again special functions, namely the derivative of the Legendre functions $\tau_n^m(\cos\theta)$ and the Legendre functions $\pi_n^m(\cos\theta)$. All these complicated expressions are simplified and unified to a simple equation, namely Eq. (135) for both $(n-m)$ even and odd, furthermore without using any special functions.

Similarly, the original Transverse Magnetic BSCs $g_{n,TM}^m$ are provided by Eqs. (5)–(7) for $(n - m)$ even where they are expressed using special functions, namely associated Legendre functions $P_n^m(\cos\theta)$ and the derivative of the Legendre functions $\tau_n^m(\cos\theta)$. For $(n - m)$ odd, original expressions of the BSCs $g_{n,TM}^m$ are given by Eqs. (8)–(10) where they are expressed using again special functions, namely the Legendre functions $\tau_n^m(\cos\theta)$ and their second derivatives. All these complicated expressions are simplified and unified as well to a simple equation, namely Eq. (136) for both $(n-m)$ even and odd, furthermore without using any special functions.

It is then obvious that the modifications of the original expressions provided in the present paper lead to a drastic simplification of the original expressions, furthermore dramatically simplifying their coding.

6. Conclusion

This paper presents a drastic simplification of equations previously published expressing electromagnetic BSCs in terms of acoustical BSCs [36], in which the equations were different depending on the parity of $(n - m)$ in which n is a partial wave order and m is an azimuthal order. One important element of the simplification is due to the fact that it has been possible to unify the different expressions corresponding to different parities of $(n - m)$. Furthermore, it has been possible to get rid of the many special functions and derivatives involved in the original expressions.

Furthermore, while the formulation discussed in the present paper relies on a VP1 approach (in which electromagnetic fields depend on one kind of potential vector), there also exist another approach, named the VP2 approach, in which the electromagnetic fields are expressed in terms of two kinds of potential vectors. We shall explain elsewhere that a successful relationship between the VP1 and VP2 approaches is only possible using the simplified relationships established in the present paper.

CRediT authorship contribution statement

Gérard Gouesbet: Writing – original draft, Formal analysis, Conceptualization. **Jianqi Shen:** Writing – review & editing, Formal analysis, Conceptualization. **Leonardo A. Ambrosio:** Writing – review & editing, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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