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Chaotic behavior of a spin-glass model on a Cayley tree

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Abstract. We investigate the phase diagram of a spin-1 Ising spin-glass model on a Cayley tree. According to early work of Thompson and collaborators, this problem can be formulated in terms of a set of nonlinear discrete recursion relations along the branches of the tree. Physically relevant solutions correspond to the attractors of these mapping equations. In the limit of infinite coordination of the tree, and for some choices of the model parameters, we make contact with findings for the phase diagram of more recently investigated versions of the Blume–Emery–Griffiths spin-glass model. In addition to the anticipated phases, we numerically characterize the existence of modulated and chaotic structures.

Keywords: phase diagrams (theory), disordered systems (theory), connections between chaos and statistical physics

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Contents

In the beginning of the 1980s, Inawashiro, Frankel, and Thompson [1], used a formalism of distribution functions, which had been developed by a number of authors [2], to analyze the effects of disorder on the phase diagram of a magnetic lattice gas model. This pioneering work remained almost unknown, although the disordered magnetic lattice gas (DMLG) is similar to some versions of the spin-1 Blume–Emery–Griffiths spin-glass (BEGsg) model, which have been the subject of several investigations, both using the replica method, at a mean-field level [3], and using real-space renormalization-group techniques [4].

Thompson and collaborators [5] have also formulated the DMLG model on a Cayley tree of ramification r. The physical solutions, in the 'deep interior' of the tree, come from the analysis of the attractors of a set of coupled discrete nonlinear recursion relations. In the infinite-coordination limit, $r \to \infty$, this mapping is considerably simplified. Thompson and collaborators established the connections between the solutions on the Cayley tree and previous calculations on the basis of distribution functions. However, as far as we know, there has been no attempt to carry out a more detailed analysis of the nonlinear recursion relations, for some representative sets of model parameters, and to make contact with more recent findings for the BEGsg model.

We then decided to revisit the BEGsg model on the Cayley tree, in the limit of infinite coordination, using the recursion relations obtained by Thompson and collaborators [5]. In these spin-1 models, the competition between spin-glass and uniform quadrupolar terms may lead to fixed points and limit cycles, as has been suggested by some investigations for infinite-range spin-glass models [6,7]. Therefore, we consider a particular version of the original model, which is chosen to include this type of competition. In the infinite-coordination limit, the discrete non-linear mapping associated with this particular version is two-dimensional and becomes amenable to a detailed analysis. In the low-temperature regime, and for sufficiently repulsive biquadratic spin interactions, we find a rich behavior, including chaotic structures, characterized by positive Lyapunov exponents.

Consider a random spin-1 Hamiltonian with bilinear and biquadratic terms,

$$\mathcal{H} = -\sum_{(i,j)} J_{ij} S_i S_j - \sum_{(i,j)} U_{ij} S_i^2 S_j^2, \tag{1}$$

where $S_i = +1, 0, -1$, for all lattice sites, and the sums are over nearest-neighbor pairs of spin variables on a Cayley tree of coordination $r+1$. We assume that $\{J_{ij}\}$ and $\{U_{ij}\}$ are independent, identically distributed quenched random variables, of Gaussian form, with mean values $\langle J_{ij} \rangle = J_0/r$ and $\langle U_{ij} \rangle = K/r$, and mean square deviations, $\left\langle (\Delta J_{ij})^2 \right\rangle = J^2/r$ and $\left\langle (\Delta U_{ij})^2 \right\rangle = U^2/r$. In the limit of infinite coordination, $r \to \infty$, the recursion relations

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have already been obtained by Thompson and collaborators (see equations (3.30)–(3.35) of [5]).

We now assume a particular case, with $J_0 = 0$ (pure spin-glass) and $U = 0$ (randomness is restricted to bilinear spin interactions), and sketch a derivation of the recursion relations according to our own earlier work (see da Costa and Salinas [8]). Consider the sucessive generations of a Cayley tree of ramification r. A spin-1 variable S_0, on a site belonging to a certain generation, is connected to the spins on the r sites of the previous generation of the tree (S_j, with $j = 1, 2, ..., r$). Due to the cycle-free structure of this tree, it is possible to write the partition function in terms of partial sums over the set of variables of each generation. We then introduce some effective parameters L_0, Δ_0, $\{L_j\}$, and $\{\Delta_j\}$, and write

$$\sum_{\{S_j\}} \exp\left\{\sum_{j=1}^{r}\left[\beta J_{0j} S_j S_0 + \frac{\beta K}{r} S_j^2 S_0^2 + \Delta_j S_j^2 + L_j S_j\right]\right\} = A \exp\left[L_0 S_0 + \Delta_0 S_0^2\right], \quad (2)$$

so that

$$A \exp\left[L_0 S_0 + \Delta_0 S_0^2\right] = \prod_{j=1}^{r}\left[2\exp\left(\frac{\beta K}{r} S_0^2 + \Delta_j\right)\cosh\left(\beta J_{0j} S_0 + L_j\right) + 1\right], \quad (3)$$

with $1/\beta = k_{\mathrm{B}} T$, where k_{B} is the Boltzmann constant and T the absolute temperature. If we use the more convenient variables

$$m_j = \frac{2\sinh L_j}{2\cosh L_j + \exp\left(-\Delta_j\right)}; \quad p_j = \frac{2\cosh L_j}{2\cosh L_j + \exp\left(-\Delta_j\right)}, \quad (4)$$

it is easy to obtain the relations

$$m_0 = \frac{2\sinh\left(L_0 + x\right)}{2\cosh\left(L_0 + x\right) + \exp\left(\Delta_0 - y\right)}, \quad (5)$$

$$p_0 = \frac{2\cosh\left(L_0 + x\right)}{2\cosh\left(L_0 + x\right) + \exp\left(\Delta_0 - y\right)}, \quad (6)$$

with the new random variables

$$x = \sum_{j=1}^{r} x_j; \qquad y = \sum_{j=1}^{r} y_j, \quad (7)$$

where x_j and y_j can be written in terms of m_j and p_j,

$$x_j = \frac{1}{2}\ln\frac{\left(\cosh t_j - \delta\right) p_j + m_j \sinh t_j + \delta}{\left(\cosh t_j - \delta\right) p_j - m_j \sinh t_j + \delta}, \quad (8)$$

$$y_j = \frac{1}{2}\ln\left\{\left[\left(\cosh t_j - \delta\right) p_j + m_j \sinh t_j + \delta\right]^2 - \left[m_j \sinh t_j\right]^2\right\}, \quad (9)$$

with $t_j = \beta J_{0j}$ and $\delta = \exp\left(-\beta K/r\right)$.

In order to calculate the expectation values of m_0, p_0, and their moments, and to establish the recursion relations, it is convenient to write the joint probability distribution of the random variables x an y,

$$\mathcal{P}\left(x, y\right) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}k_1}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_2}{2\pi}\exp\left(-\mathrm{i}k_1 x - \mathrm{i}k_2 y\right) F\left(k_1, k_2\right), \quad (10)$$

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so that

$$F\left(k_1,k_2\right)=\int\int\exp\left(\mathrm{i}k_1x+\mathrm{i}k_2y\right)\mathcal{P}\left(x,y\right)\mathrm{d}x\mathrm{d}y=\left\langle\exp\left(\mathrm{i}k_1x+\mathrm{i}k_2y\right)\right\rangle. \tag{11}$$

Due to the cycle-free character of the Cayley tree, and taking into account that the random variables are identically distributed, we can also write

$$F\left(k_1,k_2\right)=\prod_{j=1}^{r}\left\langle\exp\left(\mathrm{i}k_1x_j+\mathrm{i}k_2y_j\right)\right\rangle=\left[\left\langle\exp\left(\mathrm{i}k_1x_j+\mathrm{i}k_2y_j\right)\right\rangle\right]^r. \tag{12}$$

In the limit of infinite coordination, $r\to\infty$, $r\left\langle J_{oj}\right\rangle=J_0$, $r\left\langle J_{oj}^2\right\rangle=J^2$, and $r\left\langle J_{oj}^n\right\rangle=0$, for $n\geqslant 3$, and if we restrict to the special case $J_0=0$, it is easy to show that

$$F\left(k_1,k_2\right)=\exp\left[-\frac{1}{2}\beta^2J^2qk_1^2+\frac{1}{2}i\beta J^2\left(p-q\right)k_2\right], \tag{13}$$

which depends on just two remaining moment variables, $q=\left\langle m_j^2\right\rangle$ and $p=\left\langle p_j\right\rangle$. We then have

$$\mathcal{P}\left(x,y\right)=\left(2\pi\beta^2J^2q\right)^{-1/2}\delta\left[y-\beta^2J^2\left(q-p\right)\right]\exp\left[-\frac{x^2}{2\beta^2J^2q}\right], \tag{14}$$

from which it is straightforward to write the following recursion relations between two successive generations of the Cayley tree, j and $j+1$,

$$q_{j+1}=\int_{-\infty}^{+\infty}M_j^2\left(x\right)\exp\left(-\frac{1}{2}x^2\right)\frac{\mathrm{d}x}{\sqrt{2\pi}}, \tag{15}$$

$$p_{j+1}=\int_{-\infty}^{+\infty}P_j\left(x\right)\exp\left(-\frac{1}{2}x^2\right)\frac{\mathrm{d}x}{\sqrt{2\pi}}, \tag{16}$$

where

$$M_j\left(x\right)=\frac{2\sinh\left(\beta Jq_j^{1/2}x\right)}{Z_j\left(x\right)}, \tag{17}$$

$$P_j\left(x\right)=\frac{2\cosh\left(\beta Jq_j^{1/2}x\right)}{Z_j\left(x\right)}, \tag{18}$$

with

$$Z_j\left(x\right)=\exp\left[-\beta Kp_j-\frac{1}{2}\left(\beta J\right)^2\left(p_j-q_j\right)\right]+2\cosh\left(\beta Jq_j^{1/2}x\right). \tag{19}$$

We then investigated the behavior of this two-dimensional mapping, given by equations (15) and (16), both analytically and by numerical methods. It is interesting to have in mind the $T-K$ phase diagram obtained by da Costa and collaborators for the corresponding fully connected two-sublattice BEGsg model [6], as we show in figure 1.

We find two kinds of fixed points of the recursion relations (15) and (16). The trivial fixed point, given by $q^*=0$, $p^*\neq 0$, corresponds to the paramagnetic **P** phase. A second, and non trivial, fixed point, given by $q^*\neq 0$, $p^*\neq 0$, corresponds to the spin-glass phase **SG**. As long as K is positive, there are only these two kinds of fixed points. However, for

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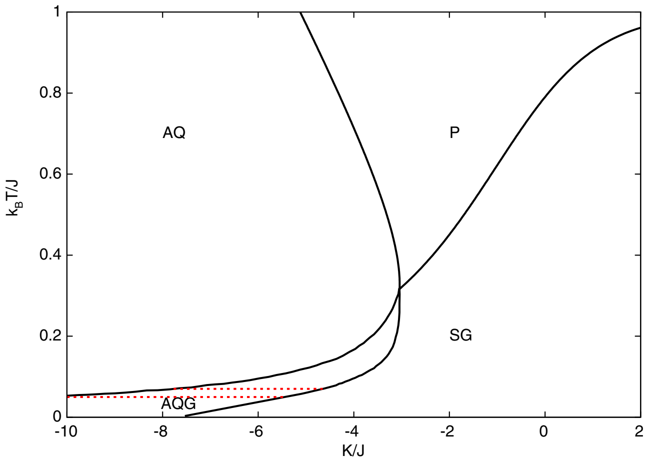

Figure 1. Phase diagram for a spin-1 spin-glass model by da Costa–Nobre–Yokoi [6]. The upper(lower) red dotted line corresponds to $k_\mathrm{B}T/J = 0.07(0.05)$.

negative values of K, the repulsive character of this parameter plays a dominant role at low temperatures. Besides the **P** and **SG** fixed points, we also find two 2-cycles, which correspond to phases already found by da Costa and collaborators [6]: (i) At sufficiently high temperatures, there appears an antiquadrupolar **AQ** solution, given by the two-cycle $p_1^* \neq p_2^*$ and $q_1^* = q_2^* = 0$; (ii) As the temperature decreases, there is an antiquadrupolar spin-glass solution **AQG** corresponding to the two-cycle $(q_1^*, p_1^*) \neq (q_2^*, p_2^*) \neq (0, 0)$.

The standard linear analysis of stability of these single fixed points and two-cycles leads essentially to the same $T - K$ phase diagram as previously found by da Costa and coauthors [6]. The regions associated with **P** and **SG** fixed points are separated by a critical line,

$$\frac{K}{J} = -\frac{J}{2k_\mathrm{B}T} - \ln\left(\frac{2J}{k_\mathrm{B}T} - 2\right). \tag{20}$$

The regions corresponding to the fixed point **P** and the **AQ** 2-cycle are separated by the critical border

$$\frac{K}{J} = -\frac{ak_\mathrm{B}T}{J} - \frac{J}{2k_\mathrm{B}T}, \tag{21}$$

with $a = 4.622...$, in agreement with da Costa and coworkers [6].

We also find a critical boundary between the regions of stability of the **AQ** and **AQG** 2-cycles, as well as between the regions of stability of the **SG** fixed point and the **AQG** 2-cycle. These critical boundaries are identical to the respective critical lines for the **AQ**–**AQG** and **SG**–**AQG** phase transitions, as obtained for the two-sublattice infinite-range spin-glass model, at the replica-symmetric solution level, as it is shown in figure 1 [6].

At this point, our findings can be summarized as follows. We have found **P SG** and **AQ** solutions, which reproduce the known results for the corresponding infinite-range model at the replica-symmetric level (see figure 1). However, in the $T - K$ phase diagram, the region in which one anticipates a single **AQG** structure is much more complicated. Instead of a stable 2-cycle solution representing the **AQG** phase, we find a number of periodic limit cycles (of finite length) and even chaotic trajectories. The limit cycles can be understood as modulated structures, which are analogous to results already obtained for an Ising

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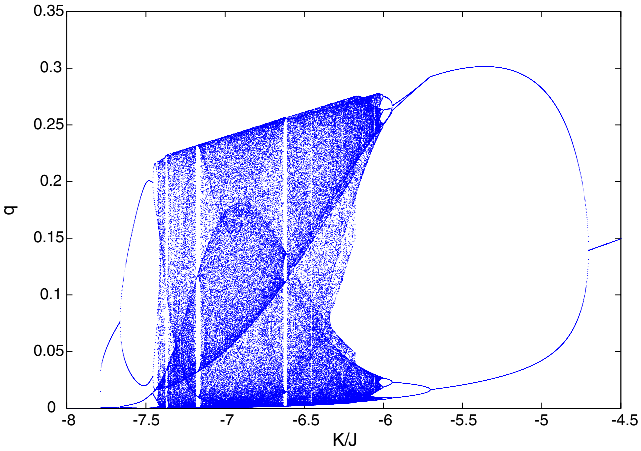

Figure 2. Behavior of the spin-glass order parameter q as function of K/J, for $k_{\mathrm{B}}T/J = 0.07$.

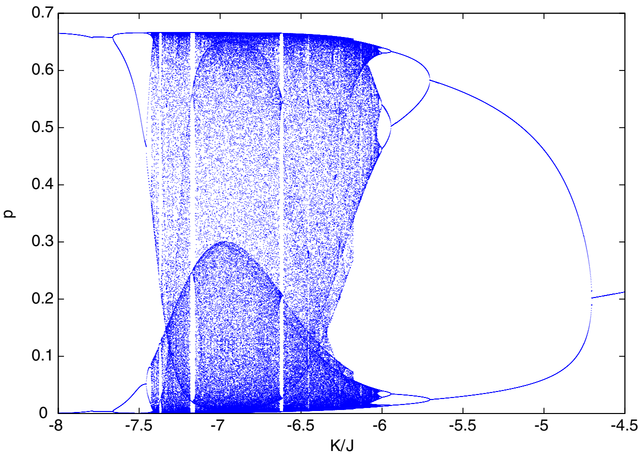

Figure 3. Behavior of the order parameter p as function of K/J, for $k_{\mathrm{B}}T/J = 0.07$.

model on a Cayley tree with competing ferro and antiferromagnetic interactions between first and second neighbors along the branches of successive generations [9]. It should be remarked, however, that neither modulated nor chaotic phases have been obtained in the replica treatments of the analogous infinite-range models, even with the introduction of two and three distinct sublattices. Indeed, in some previous investigations, one finds at most either 2-cycle [6] or 3-cycle [7] solutions.

In the present calculation, the occurrence of limit cycles and chaotic trajectories has been fully characterized at low temperatures, in the **AQG** region of the phase diagram. In figure 1, the red lines indicate two values of (low) temperatures. We have analyzed the behavior of the system at these temperatures, for K/J values inside the **AQG** region. In figures 2 and 3 we show, respectively, the order parameters q and p as a function of K/J for $k_{\mathrm{B}}T/J = 0.07$. In figure 4, we draw the largest Lyapunov exponent at this temperature. The chaotic behavior, which is associated with a positive Lyapunov exponent, takes place at a small range of values of K/J (roughly for $-7.5 < K/J < -6.0$).

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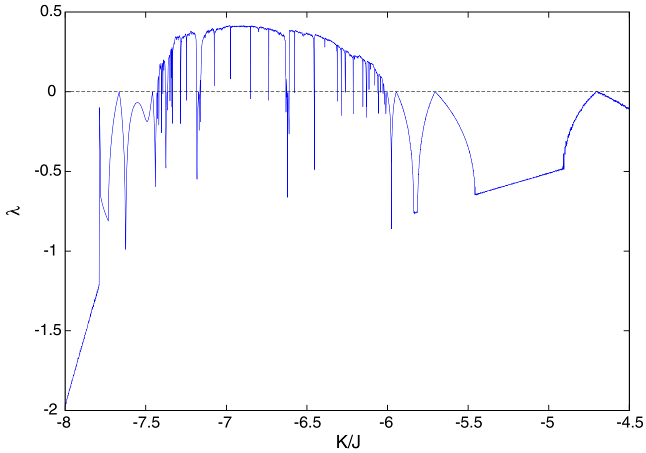

Figure 4. Largest Lyapunov exponent as a function of K/J, for $k_{\mathrm{B}}T/J = 0.07$.

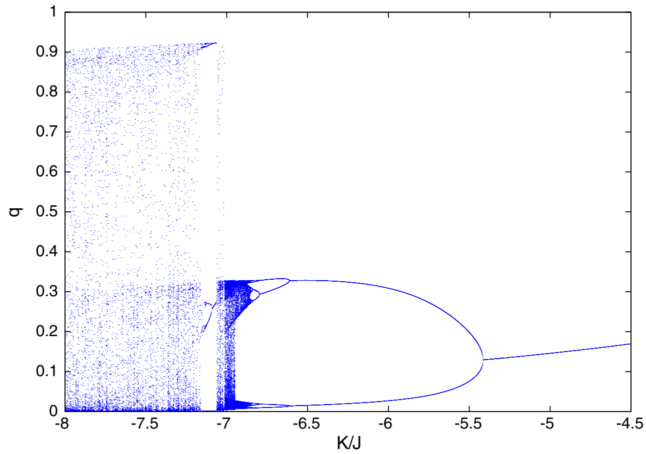

Figure 5. Behavior of the spin-glass order parameter q as function of K/J, for $k_{\mathrm{B}}T/J = 0.05$.

In figures 5–7 we illustrate our numerical findings for $k_{\mathrm{B}}T/J = 0.05$. At this temperature, as it can be seen from the largest Lyapunov exponent, the chaotic behavior persists for very large values of $-K/J$. Also, the limit cycles tend to be much less numerous. According to some preliminary investigations, this peculiar behavior is not restricted to $k_{\mathrm{B}}T/J = 0.05$, but it does hold at lower temperatures, down to absolute zero. It is remarkable that such a complex behavior has been observed for a model on a Cayley tree with first-neighbor interactions.

This surprising behavior of the BEGsg on the Cayley tree is an indication that the more subtle features of the phase diagrams, as modulated phases and chaotic structures, will be very difficult to be obtained by using the standard replica methods to solve the corresponding infinite-range models [3, 6, 7]. These findings can also provide a useful guide for the interpretation of numerical simulations, as in some recent Monte Carlo calculations [10, 11], and for the modeling of real systems, as the ferroelastic alloy $Ti_{50}\,(Pd_{50-x}Cr_x)$ in the presence of disorder [12].

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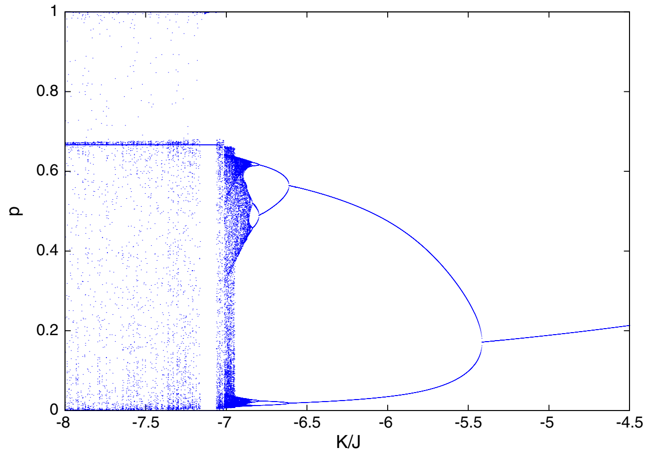

Figure 6. Behavior of the order parameter p as function of K/J, for $k_{\mathrm{B}}T/J = 0.05$.

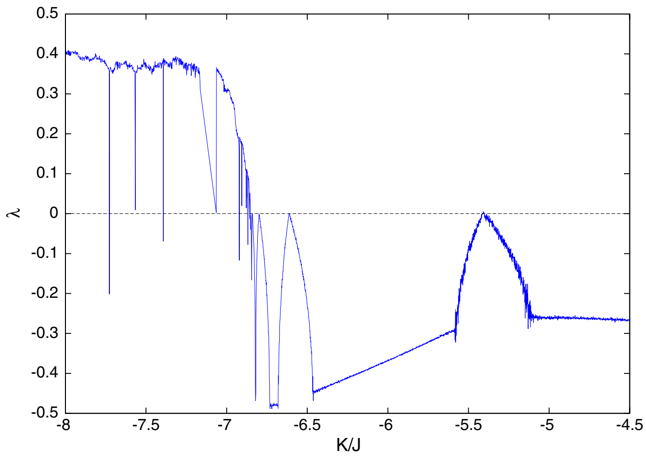

Figure 7. Largest Lyapunov exponent as a function of K, for $k_{\mathrm{B}}T/J = 0.05$.

In conclusion, the Cayley tree is useful to investigate the spin-glass properties of a short-range spin-1 Ising model. In the ferromagnetic case, it is well know that the solutions deep inside the tree correspond to the Bethe approximation. In this note we report a numerical analysis, for an adequately chosen version of the BEGsg model, and using the recursion relations obtained by Thompson and collaborators many years ago. The subtle details of our findings, including long-period cycles and chaotic behavior, are related to the high degree of frustration of the BEGsg model. All of these points, as well as some possible connections with more realistic finite-range spin-glass models at finite temperature, are planned to be further investigated.

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References

[1] Inawashiro S, Frankel N E and Thompson C J 1981 *Phys. Rev.* B **24** 6532
[2] Katsura S and Fujiki S 1979 *J. Phys.* C **12** 1087
Morita T 1979 *Physica* A **98** 566
[3] Sellitto M, Nicodemi M and Arenzon J J 1997 *J. Phys.* I **7** 945
Schreiber G R 1999 *Eur. Phys. J.* B **9** 479
Crisanti A and Leuzzi L 2004 *Phys. Rev.* B **70** 014409
Crisanti A and Leuzzi L 2005 *Phys. Rev. Lett.* **95** 087201
[4] Falicov A and Berker A N 1996 *Phys. Rev. Lett.* **76** 4380
Ozçelik V O and Berker A N 2008 *Phys. Rev.* E **78** 031104
Antenucci F, Crisanti A, and Leuzzi L 2014 *J. Stat. Phys.* **155** 909
Antenucci F, Crisanti A, and Leuzzi L 2014 *Phys. Rev.* E **90** 012112
[5] Thompson C J, Inawashiro S and Frankel N E 1986 *Prog. Theor. Phys. Suppl.* **87** 155
[6] da Costa F A, Nobre F D and Yokoi C S O 1997 *J. Phys.* A **30** 2317
[7] Katayama K and Horiguchi T 2000 *Physica* A **277** 12
[8] da Costa F A and Salinas S R 1990 *Rev. Bras. Fis.* **20** 49 www.sbfisica.org.br/bjp
[9] Yokoi C S O, de Oliveira M J and Salinas S R 1985 *Phys. Rev. Lett.* **54** 163
[10] Paoluzzi M, Leuzzi L and Crisanti A 2010 *Phys. Rev. Lett.* **104** 120602
Leuzzi L, Paoluzzi M, and Crisanti A 2011 *Phys. Rev.* B **83** 014107
[11] Li L S, Chen W, Dong W and Chen X S 2011 *Eur. Phys. J.* B **80** 189
[12] Vasseur R, Xue D, Zhou Y, Ettoumi W, Ding X, Ren X and Lookman T 2012 *Phys. Rev.* B **86** 184103

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