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César Polcino Milies

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César Polcino Milies *

Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 66281
05315-970 - São Paulo - Brasil
polcino@ime.usp.br

Abstract

In what follows, we determine necessary and sufficient conditions for the set of torsion units of an alternative artinian algebra to be closed under multiplication.

1 Introduction

Let R be a ring with unity. We shall denote by $\mathcal{U}(R)$ the group of units of that ring and by $T\mathcal{U}(R)$ the set of torsion units (i.e., the set of elements of finite order in $\mathcal{U}(R)$). We say that R has the *torsion product property* (or, briefly, that *R has tpp*) if the product of torsion units is again a torsion unit.

Some years ago, it was a common trend to investigate group theoretical properties of $\mathcal{U}(R)$ in the case when R was a group ring over either the ring of rational integers or a given field. In particular, group rings having *tpp* were investigated in [1], [2], [9] and [10].

More recently, this property was studied also in the case of alternative loop rings over the integers [6] and then extendeded to alternative loop algebras over fields [7] (see also [5]); in the course of this study, alternative

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division rings of characteristic $p > 0$ with *tpp* were also characterized, but the case where the characteristic is equal to zero remained opened.

In what follows, we shall give a complete characterization of artinian alternative algebras having *tpp* .

2 Simple Alternative Algebras

We recall that given elements a, b, c in an algebra A, the *commutator* and *associator* of these elements are defined respectively by:

$$[a, b] = ab - ba,$$

$$(a, b, c) = (ab)c - a(bc).$$

Accordingly, the commutative and associative centers of A are defined respectively as:

$$K(A) = \{a \in A \mid [a, x] = 0, \ \forall x \in A\},$$

$$N(A) = \{a \in A \mid (a, x, y) = (x, a, y) = (x, y, a) = 0, \ \forall x, y \in A\}.$$

Then, the *center* of A is defined as:

$$\mathcal{Z}(A) = K(A) \cap N(A).$$

Alternative division rings of characteristic $p > 0$ having *tpp* , are already known.

Proposition 2.1 ([7, Proposition 2.3]) *Let D be an alternative division ring of characteristic $p > 0$. Then D has tpp if and only if any two elements of finite order in D commute. In this case, $E = T\mathcal{U}(D) \cup \{0\}$ is a central subfield of D.*

In the case where $char(D) = 0$ we obtain the following.

Theorem 2.2 *Let D be an alternative division ring of characteristic* 0. *Then D has tpp if and only if $T\mathcal{U}(D)$ is central in D.*

Proof. Given two non-zero elements $x, y \in D$, let us denote by $D(x, y)$ the largest associative subring of D containing x, y. Notice that since an alternative algebra is diassociative, the subring generated by x, y and the center of D is associative, so it is easy to show that $D(x, y)$ actually exists.

We claim that $D(x,y)$ is a division ring. In fact, take an element $a \in D(x,y)$. Then, given any two other elements $b, c \in D(x,y)$ we have that the associator of a, b, c (in any order) is equal to 0, so by [12, Lemma 10.3.8] we have that the associator of a^{-1}, b, c (in any order) is also equal to 0, showing that $a^{-1} \in D(x,y)$.

Given an element $t \in D$ of finite order, let $y \in D$ be any other non-zero element. Then $D(t,y)$ is a division ring and $\langle t \rangle$ is a periodic subgroup of $\mathcal{U}(D(t,y))$. Hence, [8, Theorem 8] shows that $\langle t \rangle$ is central in $D(t,y)$ so $ty = yt$. Since $y \in D$ is arbitrary, this implies that $T\mathcal{U}(D) \subset K(A)$.

Now, take an element $t \in T\mathcal{U}(D)$ and a pair of arbitrary elements $x, y \in A$. It follows from the proof of [12, Corollary 7.1.1, p.136] that

$$3(t,x,y) = 3(x,t,y) = 3(x,y,t) = 0.$$

Since $char(D) = 0$ we see that all three associators above are equal to 0. Hence $t \in N(A)$, showing that actually $T\mathcal{U}(D) \subset \mathcal{Z}(A)$ □

Let F be a field. Let $\mathbf{M_Z}(F)$ be the so-called *Zorn's vector matrix algebra* with entries in F; i.e. the set of 2×2 matrices of the form:

$$\begin{bmatrix} a & x \\ y & b \end{bmatrix},$$

where $a, b \in F$ and $x, y \in F^3$, the set of ordered triples of elements of F.

Adding this matrices entry by entry and multiplying according to the following rule

$$\begin{bmatrix} a_1 & x_1 \\ y_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ y_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + x_1.y_2 & a_1x_2 + b_2x_1 - y_1 \times y_2 \\ a_2y_1 + b_1x_2 + x_1 \times x_2 & b_1b_2 + y_1x_2 \end{bmatrix},$$

where $x.y$ and $x \times y$ denote the dot and cross product of $x, y \in F^3$ respectively, $\mathbf{M_Z}(F)$ becomes an alternative algebra.

We recall that a Theorem due to E. Kleinfeld shows that a simple alternative algebra over a field F which is not associative is either a division algebra or is isomorphic to Zorn's vector matrix algebra $\mathbf{M_Z}(F)$ (see [12, p.46 and Corollary 7.3.1] or [5, Corollary I.4.7 and Proposition VI.4.6]).

Algebras of this kind over fields of characteristic $p > 0$, having *tpp* are known.

Proposition 2.3 ([7, Theorem 2.5]) *Let F be a field of characteristic $p > 0$. Then Zorn's vector matrix algebra $\mathbf{M_Z}(F)$ over F has tpp if and only if F is algebraic over its prime field P.*

Now, let F be any field of characteristic 0 and consider the following elements.

$$a = \begin{bmatrix} 0 & (-1,0,0) \\ (1,0,0) & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & (1,0,0) \\ (-1,0,0) & -1 \end{bmatrix} \in \mathrm{M}(R).$$

Then, it is easy to see that $a^4 = b^3 = I$ and

$$ab = \begin{bmatrix} 1 & (1,0,0) \\ 0 & 1 \end{bmatrix}, \quad \text{so} \quad (ab)^n = \begin{bmatrix} 1 & (n,0,0) \\ 0 & 1 \end{bmatrix} \neq I, \ \forall n > 1.$$

Hence, we get the following.

Proposition 2.4 *Let R be a commutative ring with unity, of characteristic 0. Then, $\mathrm{M}(R)$ does not have tpp.*

Let A be a semisimple alternative artinian algebra over a field F. Then, A is the direct sum of simple alternative algebras, each of which is either a division algebra of finite dimension over F, or a simple associative algebra of the form $M_n(D)$ for some $n > 1$ and some division algebra D finite dimensional over F, or isomorphic to Zorn's vector matrix algebra over a field which is a finite extension of F (see [12, Theorem 12.2.3]). Hence, an examination of its simple components in the light of the results just above, makes it possible to determine whether or not A has *tpp* . The case where $char(F) = p > 0$ is already implicit in [7].

Theorem 2.5 *Let A be a semisimple alternative artinian algebra over a field K of characteristic $p > 0$. Then A has tpp if and only if its simple componentes are either division rings whose torsion units are central, or isomorphic to full matrix rings or to Zorn's vector matrix algebras over fields which are algebraic over their corresponding prime fields.*

Theorem 2.6 *Let A be a semisimple alternative artinian algebra over a field K of characteristic 0. Then A has tpp if and only if all its torision units are central. In this case A is a direct sum of division rings with this property.*

Now, we deal with alternative artinian algebras in general, showing how to reduce the problem to the semisimple case. The proofs of our results are very similar to those given for the associative case in [4]; we include them here only to show that they work also in the alternative case, with only minor changes.

We recall that an element a in an alternative algebra A is called *quasi-regular* if there exists $a' \in A$ such that $a + a' + aa' = 0$. An ideal is called *quasi-regular* is all its elements are quasi-regular. The *Jacobson radical* of A, denoted $J(A)$, is the largest quasi-regular two-sided ideal of A. This radical is also known in the literature as the *Smiley, Zhevlakov* or *Kleinfeld radical.* It can be shown that, if A is artinian, then $J(A)$ is nilpotent (see [12, Theorem 12.2.2]).

Lemma 2.7 *Let A be an artinian algebra over a field K. Then $L = 1+J(A)$ is a multiplicative Moufang loop. If $char(K) = p > 0$ then L is a torsion loop of exponent a power of p and, if $char(K) = 0$ then L is torsion-free.*

Proof. It is clear that $1+J(A)$ is closed under multiplication, Also, since $J(A)$ is nilpotent it is easy to see that given an element $\mu = 1+\alpha \in 1+J(A)$, there exists a positive integer n such that $\alpha^n = 0$ so that $\mu^{-1} = 1 - \alpha + \cdots + (-\alpha)^{n-1} \in 1 + J(A)$ is the inverse of μ. As L is contained in an alternative algebra, it is clear that it is a Moufang loop.

If $char(K) = p > 0$ we can choose a positive integer m such that $p^m > n$ so that $\alpha^{p^m} = 0$ for all $\alpha \in J(A)$ and thus $\mu^{p^m} = (1+\alpha)^{p^m} = 1 + a^{p^m} = 1$, showing that L is torsion, of exponent p^m.

If $char(K) = 0$, to see that L torsion-free, assume that there exists an element $x \in J(A)$ such that $1 + x$ is of finite order. Then, there exists a positive integer m such that:

$$(1+x)^m = 1 + mx + \sum_{i=2}^{m} \binom{m}{i} x^i = 1.$$

Thus:

$$mx \left[1 + \frac{1}{m} \sum_{i=2}^{m} \binom{m}{i} x^{i-1} \right] = 0.$$

Since $(1/m) \sum_{i=2}^{m} \binom{m}{i} x^{i-1} \in J(A)$ we have that $1 + (\frac{1}{m}) \sum_{i=2}^{m} \binom{m}{i} x^{i-1} \in 1 + J(A)$ is invertible, so $x = 0$ □

The case when the characteristic of the ground field is $p > 0$ admits a direct reduction.

Theorem 2.8 *Let A be an semisimple alternative artinian algebra over a field K of characteristic $p > 0$. Then A has tpp if and only if $A/J(A)$ has tpp.*

Proof. Assume A has *tpp* and let μ_1, $\mu_2 \in A$ be elements such that the corresponding elements under the natural epimorphism $\overline{\mu_1}$, $\overline{\mu_2} \in A/J(A)$ are units of finite order in $A/J(A)$. Since A is artinian, we have that $J(A)$ is nilpotent so it follows easily that both μ_1 and μ_2 are units in A. We claim that they are also of finite order. In fact, since $\overline{\alpha_i}$ is of finite order, there exists a positive integer n_i such that $\overline{\mu_i}^{n_i} = 1$, so $\mu_i^{n_i} = 1+\alpha_i$ with $\alpha_i \in J(A)$, $i = 1, 2$ and Lemma 2.7 shows that $\overline{\mu_i}^{n_i}$ is of finite order so $\overline{\mu_i}$ itself is also of finite order,$i = 1, 2$. Since A has *tpp* , it follows that the product $\mu_1\mu_2$ is of finite order and thus also $\overline{\mu_1\mu_2}$ is torsion.

Conversely, assume that $A/J(A)$ has *tpp* and let α, β be arbitrary elements of $\mathcal{TU}(A)$. Then we have that $\overline{\alpha}$, $\overline{\beta} \in \mathcal{TU}(A/J(A))$ so $\overline{\alpha\beta}$ is of finite order and, by the argument above, it follows that also $\alpha\beta$ is torsion, completing the proof. □

Theorem 2.9 *Let A be an alternative artinian algebra over a field K of characteristic 0. Then, A has tpp if and only if $\mathcal{TU}(A)$ is central. In this case, all nilpotent elements of A belong to $J(A)$.*

Proof. Let $J(A)$ denote the Jacobson radical of A. Since $char(K) = 0$, we have, from Wedderburn's Principar Theorem (see [11, Theorem iii.3.13]) that we can write A as a direct sum (of vector spaces) $A = S \oplus J(A)$, where S is a semisimple subalgebra of A, though not a two-sided ideal.

Since A has *tpp* we have that S also has *tpp* so, by Theorem 2.6, we have that $\mathcal{TU}(S)$ is central in S. We claim that the elements of $\mathcal{TU}(S)$ also commute with the elements of $J(A)$ and are thus central in A itself.

In fact, set $x \in \mathcal{TU}(S)$ and $\alpha \in J(A)$. Since alternative algebras are diassociative, the subalgebra generated by x and α is associative. Then $1+\alpha$ is a unit and $(1+\alpha)x(1+\alpha)^{-1}$ is a torsion unit of A. Since A has *tpp* , it follows that $(1+\alpha)x(1+\alpha)^{-1}x^{-1} \in T(A) \cap (1 + J(A))$, so Lemma 2.7 shows that $(1+\alpha)x(1+\alpha)^{-1}x^{-1} = 1$. Hence, x commutes with $1+\alpha$ and thus also with α, as desired.

To complete the proof, it will suffice to show that $\mathcal{TU}(A) = \mathcal{TU}(S)$. To do so, take $u \in \mathcal{TU}(A)$ and write it as $u = s + n$ with $s \in S$ and $n \in J(A)$. Notice that, if m is a positive integer, it is easy to see that u^m is of the form $u^m = s^m + n_1$, with $n_1 \in J(A)$, so it follows that $s \in \mathcal{TU}(S)$. Since s is central, we have that $s^{-1}u = 1 + s^{-1}n \in 1 + J(A)$ is a torsion unit, thus Lemma 2.7 shows again that $s^{-1}u = 1$ and we get that $u = s \in S$. $\square$

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