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Research Paper

(Co)homology of partial smash products

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A B S T R A C T

Given a cocommutative Hopf algebra $\mathcal{H}$ over a commutative ring K and a symmetric partial action of $\mathcal{H}$ on a K-algebra A, we obtain a first quadrant Grothendieck spectral sequence converging to the Hochschild homology of the smash product $A\#\mathcal{H}$, involving the Hochschild homology of A and the partial homology of $\mathcal{H}$. An analogous third quadrant cohomological spectral sequence is also obtained. The definition of the partial (co)homology of $\mathcal{H}$ under consideration is based on the category of the partial representations of $\mathcal{H}$. A specific partial representation of $\mathcal{H}$ on a subalgebra $\mathcal{B}$ of the partial "Hopf" algebra $\mathcal{H}_{par}$ is involved in the definition and we construct a projective resolution of $\mathcal{B}$.

1. Introduction

The notion of a partial group action on a C^*-algebra, gradually introduced in [28], [37] and [29], and its successful use (see, in particular, [31]) motivated a series of algebraic developments (see the surveys [13] and [22]). In particular, a Galois theory based on partial group actions was initiated in [25], which inspired its treatment from the point

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of view of Galois corings in [17] and the definition of a partial (co)action of a Hopf algebra on an algebra in [18]. In the latter paper related concepts were also considered, duality results were obtained and a partial Hopf-Galois theory was introduced. This was a starting point for rich and interesting Hopf theoretic developments around partial actions [4], [2], [10], [5], [20], [9], [16], [11], [8], [15], [32], [6], [11], [12], [19], [36], [40], [39], [41], [14].

In particular, partial representations of a Hopf algebra $\mathcal{H}$ were introduced in [4], extending the notion of a partial group representation (see [30] and [24]). Moreover, an algebra $\mathcal{H}_{par}$ was associated with $\mathcal{H}$ in [4], called the *partial "Hopf"* algebra, which has the universal property that each partial representation of $\mathcal{H}$ can be factorized by an algebra morphism from $\mathcal{H}_{par}$, being thus the Hopf analogue of the partial group algebra (see [24]). It was also shown in [4] that there is a partial action of $\mathcal{H}$ on a subalgebra $\mathcal{B}$ of $\mathcal{H}_{par}$, such that $\mathcal{H}_{par}$ is isomorphic to the smash product $\mathcal{B}\#\mathcal{H}$, generalizing an earlier result from [23], established in the case of groups. In addition, if $\mathcal{H}$ possesses an invertible antipode, then $\mathcal{H}_{par}$ has the structure of a Hopf algebroid [4]. Dual concepts were defined and investigated in [6].

In the present article we study the Hochschild homology and cohomology of the partial smash product $A\#\mathcal{H}$ by means of spectral sequences, where $\mathcal{H}$ is a cocommutative Hopf algebra, whose partial action on the algebra A is symmetric (see Section 2.1 for definitions). Earlier, in [1], group cohomology based on partial representations was introduced and a Grothendieck spectral sequence was produced, relating the Hochschild cohomology of the skew group ring $A \rtimes G$ by a unital partial action of G on an algebra A with the Hochschild cohomology of A and the partial group cohomology of G. This was extended in [26] to the crossed product $A * G$ by a unital twisted partial action of G on A, whose twist takes values in the base field, establishing also a similar result for the Hochschild homology. The treatment in [26] was based on the theory of partial projective group representations and the related novel concept of a twisted partial group algebra.

We begin by giving some preliminaries around partial actions and partial representations of Hopf algebras and Hochschild (co)homology in Section 2. Section 3 is dedicated to the Hochschild homology of the smash product $A\#\mathcal{H}$, where $\mathcal{H}$ is a cocommutative Hopf algebra over a commutative ring K, whose partial action on a unital algebra A is symmetric. For the main result we also assume that $\mathcal{H}$ is projective over K. The idea is to use Grothendieck's Theorem [38, Theorem 10.48] to obtain a first quadrant spectral sequence E^r converging to the Hochschild homology of $A\#\mathcal{H}$ with values in a $A\#\mathcal{H}$-bimodule M, and such that

$$E^2_{p,q} = \operatorname{Tor}_p^{\mathcal{H}_{par}}(\mathcal{B}, H_q(A, M)),$$

where $\mathcal{B}$ is the above mentioned subalgebra of $\mathcal{H}_{par}$.

In order to prepare the ingredients for the use of Grothendieck's Theorem, we work with the right exact functors of the form

$$F(-) := A\#\mathcal{H} \otimes_{(A\#\mathcal{H})^e} -, \quad F_1(-) := A \otimes_{A^e} - \quad \text{and} \quad F_2(-) := \mathcal{B} \otimes_{\mathcal{H}_{par}} -,$$

where A^e stands for the enveloping algebra $A \otimes A^{op}$ of A, with A^{op} denoting the opposite algebra of A, and the meaning of $(A\#\mathcal{H})^e$ is similar. Observe that the left derived functors of F_1 and F compute the Hochschild homology of A and $A\#\mathcal{H}$, respectively. One of the main steps is to show that the functors F_2F_1 and F are naturally isomorphic, when applied to $A\#\mathcal{H}$-bimodules. It is obtained in Corollary 3.19 as a consequence of a more general fact, Proposition 3.17, which states that the bifunctors

$$- \otimes_{\mathcal{H}_{par}} (A \otimes_{A^e} -) \quad \text{and} \quad (- \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} -,$$

defined on Mod-$\mathcal{H}_{par} \times (A\#\mathcal{H})^e$-Mod, are naturally isomorphic. In fact, Proposition 3.17 is a crucial technical tool, which is also used to make the second main step towards the use of Grothendieck's Theorem, namely Proposition 3.24, which says that F_1 sends projective $A\#\mathcal{H}$-bimodules to left F_2-acyclic modules.

A considerable point is to show in Lemma 3.12 that if $\mathcal{H}$ is projective over K, then $(A\#\mathcal{H})^e$ is projective over A^e. Consequently, any projective $(A\#\mathcal{H})$-bimodule is a projective A-bimodule, so that applying the left derived functor of F_1 to M via taking a projective resolution of M in the category of $A\#\mathcal{H}$-bimodules computes the usual Hochschild homology of A with values in the A-bimodule M (see Remark 3.13).

These facts are used to obtain the main result of the section, Theorem 3.25, which states the existence of the above mentioned spectral sequence. As an application, if the algebra A under the symmetric partial action of the cocommutative Hopf algebra $\mathcal{H}$ is separable, then we obtain an isomorphism

$$H_n(A\#\mathcal{H}, M) \cong \operatorname{Tor}_n^{\mathcal{H}_{par}}(\mathcal{B}, M/[A, M]),$$

(see Example 3.26). Furthermore, if $\mathcal{H}$ is the group algebra KG of a group G, then the spectral sequence of Theorem 4.6 takes the form

$$E^2_{p,q} = H_p^{par}(G, H_q(A, M)) \Rightarrow H_{p+q}(A \rtimes G, M),$$

where $H_\bullet^{par}(G, -) := \operatorname{Tor}_\bullet^{K_{par}G}(B, -)$, the partial homology of G introduced in [3] (see Example 3.27).

A dual work is done in Section 4 to deal with cohomology. The main functors under consideration are of the form

$$G_1 := \operatorname{Hom}_{A^e}(A, -), \quad G_2 := \operatorname{Hom}_{\mathcal{H}_{par}}(\mathcal{B}, -) \quad \text{and} \quad G := \operatorname{Hom}_{(A\#\mathcal{H})^e}(A\#\mathcal{H}, -).$$

Note that G_1 is used to compute the Hochschild cohomology of A, whereas for the cohomology of $A\#\mathcal{H}$ the functor G is employed. These functors are used to apply a variation of the Grothendieck spectral sequence [38, Theorem 10.47]. The crucial steps are Corollary 4.4, stating that the functor G_2G_1 and G are naturally isomorphic, and

Proposition 4.5, which says that G_1 sends injective $A\#\mathcal{H}$-bimodules to right G_2-acyclic modules. Both of them are obtained applying an important technical tool, Proposition 4.3, which is a result dual to the above mentioned Proposition 3.17 on bifunctor isomorphism. The main fact is Theorem 4.6, which asserts the existence of a third quadrant cohomological spectral sequence E_r such that

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{H}_{par}}(\mathcal{B}, H^q(A, M)) \Rightarrow H^{p+q}(A\#\mathcal{H}, M),$$

where $\mathcal{H}$ is a cocommutative Hopf K-algebra, projective as a K-module, whose partial action on A is symmetric, and M is an arbitrary $A\#\mathcal{H}$-bimodule. Dually to the homological case, in the proof of Theorem 4.6 we use that any injective resolution of the $A\#\mathcal{H}$-bimodule M is also an injective resolution of M as an A-bimodule (see Remark 4.2).

In Section 5, for a cocommutative Hopf algebra $\mathcal{H}$ and a left $\mathcal{H}_{par}$-module M, we define the partial Hopf homology and cohomology of $\mathcal{H}$ with coefficients in M by

$$H^{par}_\bullet(\mathcal{H}, M) := \operatorname{Tor}^{\mathcal{H}_{par}}_\bullet(\mathcal{B}, M) \quad \text{and} \quad H^\bullet_{par}(\mathcal{H}, M) := \operatorname{Ext}^\bullet_{\mathcal{H}_{par}}(\mathcal{B}, M),$$

respectively. Then the above mentioned spectral sequences take the following forms (see Theorem 5.2):

$$E^2_{p,q} = H^{par}_p(\mathcal{H}, H_q(A, M)) \Rightarrow H_{p+q}(A\#\mathcal{H}, M),$$

and

$$E_2^{p,q} = H^p_{par}(\mathcal{H}, H^q(A, M)) \Rightarrow H^{p+q}(A\#\mathcal{H}, M).$$

Note that the latter sequence extends for the Hopf theoretic setting the third quadrant cohomological spectral sequence obtained for the case of unital partial group actions in [1, Theorem 4.1]. It is also shown in Section 5 that if the action of $\mathcal{H}$ on A is global, then we obtain the following global versions of our spectral sequences (see Corollary 5.9):

$$E^2_{p,q} = \operatorname{Tor}^{\mathcal{H}}_p(K, H_q(A, M)) \Rightarrow H_{p+q}(A\#\mathcal{H}, M),$$

and

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{H}}(K, H^q(A, M)) \Rightarrow H^{p+q}(A\#\mathcal{H}, M).$$

In the final Section 6 we construct a projective resolution of $\mathcal{B}$ (see Proposition 6.3) by means of a simplicial module which gives rise to an acyclic complex. Note that a projective resolution of $\mathcal{B}$ for the case of groups was obtained in [27].

In all what follows K will stand for a commutative (associative) unital ring.

2. Preliminaries

In this section, for the reader's convenience, we recall some facts on partial Hopf representations, partial Hopf actions, and Hochschild (co)homology.

2.1. Partial representations and partial actions of Hopf algebras

The first definition of the concept of a partial representation of a Hopf algebra was given in [4] in an asymmetric way, justified by the definition of a partial group action with one-sided ideals considered in [18] and some constructions. Nevertheless, it became clear that a symmetric definition introduced in [7] gives additional advantages. According to [6], Paolo Saracco observed that some of the axioms on the definition in [7] are redundant, so the final definition is as follows:

Definition 2.1. [6, Definition 2.10] Let $\mathcal{H}$ be a Hopf K-algebra, and let A be a unital K-algebra. A *partial representation* of $\mathcal{H}$ in A is a linear map $\pi : \mathcal{H} \to A$ such that

(PR1) $\pi(1_{\mathcal{H}}) = 1_A$,
(PR2) $\pi(h)\pi(k_{(1)})\pi(S(k_{(2)})) = \pi(hk_{(1)})\pi(S(k_{(2)}))$,
(PR3) $\pi(h_{(1)})\pi(S(h_{(2)}))\pi(k) = \pi(h_{(1)})\pi(S(h_{(2)})k)$,

for all $h, k \in H$.

Lemma 2.2. *[6, Lemma 2.11] Let $\pi : \mathcal{H} \to A$ be a partial representation. Then the following axioms are satisfied as well*

(PR4) $\pi(h)\pi(S(k_{(1)}))\pi(k_{(2)}) = \pi(hS(k_{(1)}))\pi(k_{(2)})$,
(PR5) $\pi(S(h_{(1)}))\pi(h_{(2)})\pi(k) = \pi(S(h_{(1)}))\pi(h_{(2)}k)$.

Conversely, if a linear map $\pi : \mathcal{H} \to A$ satisfies (PR1), (PR4) and (PR5), then it is a partial representation.

As in the case of partial group representations, morphisms of partial representations of a fixed Hopf algebra $\mathcal{H}$ are defined in most natural way: if the pair $\pi : \mathcal{H} \to A$ and $\pi' : \mathcal{H} \to A'$ are partial representations, then by a *morphism* $\pi \to \pi'$ we understand an algebra homomorphism $f : A \to A'$ such that $\pi' = f \circ \pi$. Following [7] we denote by $\text{ParRep}_{\mathcal{H}}$ the category of the partial representations of $\mathcal{H}$ and their morphisms.

Remark 2.3. As pointed out in [7, Remark 3.2], if the Hopf algebra $\mathcal{H}$ is cocommutative, then a linear map $\pi : \mathcal{H} \to A$ satisfying (PR1), (PR2), and (PR5) also satisfies axioms (PR3) and (PR4), making π a partial representation.

Another crucial for us concept is that of a partial Hopf action first defined in [18].

Definition 2.4. (see [4]). A left partial action of a Hopf algebra $\mathcal{H}$ on a unital algebra A is a linear map

$$\begin{aligned} \cdot : \mathcal{H} \otimes A &\to A \\ h \otimes a &\mapsto h \cdot a, \end{aligned}$$

such that

(PA1) $1_{\mathcal{H}} \cdot a = a$ for all $a \in A$;
(PA2) $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$, for all $h \in \mathcal{H}$, $a, b \in A$;
(PA3) $h \cdot (k \cdot a) = (h_{(1)} \cdot 1_A)(h_{(2)}k \cdot a)$ for all $h, k \in \mathcal{H}$, $a \in A$.

The algebra A, on which $\mathcal{H}$ acts partially is called a partial left $\mathcal{H}$-module algebra. Recall also that a partial action of $\mathcal{H}$ on A is said to be *symmetric* if in addition it satisfies

(PA4) $h \cdot (k \cdot a) = (h_{(1)}k \cdot a)(h_{(2)} \cdot 1_A)$ for all $h, k \in \mathcal{H}$, $a \in A$.

Given a partial action of a Hopf algebra $\mathcal{H}$ on a unital algebra A, an associative product on $A \otimes \mathcal{H}$ is defined by

$$(a \otimes h)(b \otimes k) = a(h_{(1)} \cdot b) \otimes h_{(2)}k.$$

Then the *partial smash product* (see [18]) is the unital algebra

$$A\#\mathcal{H} := (A \otimes \mathcal{H})(1_A \otimes 1_{\mathcal{H}}) := \{x(1_A \otimes 1_{\mathcal{H}}) : x \in A \otimes \mathcal{H}\}.$$

This algebra is generated by the elements of the form

$$a\#h = a(h_{(1)} \cdot 1_A) \otimes h_{(2)}.$$

Lemma 2.5. *(see [4]). Let $\mathcal{H}$ be a Hopf algebra acting partially on a unital algebra A. Then in the partial smash product $A\#\mathcal{H}$ we have:*

(i) $(a\#h)(b\#k) = a(h_{(1)} \cdot b)\#h_{(2)}k$;
(ii) $a\#h = a(h_{(1)} \cdot 1_A)\#h_{(2)}$;
(iii) *the map $\phi_0 : A \to A\#\mathcal{H}$ given by $\phi_0(a) = a\#1_{\mathcal{H}}$ is an algebra homomorphism.*

Lemma 2.6. *The partial smash product $A\#\mathcal{H}$ is a direct summand of $A \otimes \mathcal{H}$ as A-modules.*

Proof. Consider the following maps of algebras

$$\begin{aligned} \iota : A\#\mathcal{H} &\to A \otimes \mathcal{H} & \qquad \# : A \otimes \mathcal{H} &\to A\#\mathcal{H} \\ z &\mapsto z, & x &\mapsto x(1_A \otimes 1_{\mathcal{H}}). \end{aligned}$$

Then, the above maps are morphisms of left A-modules such that $\# \circ \iota = id_{A\#\mathcal{H}}$. Whence we obtain that $A\#\mathcal{H}$ is a direct summand of $A \otimes \mathcal{H}$. $\square$

It is convenient for us to single out the following fact:

Lemma 2.7. *Suppose that the partial action of a Hopf algebra $\mathcal{H}$ on a unital algebra A is symmetric. Then*

$$(b\#1_{\mathcal{H}})(1_A\#S(h)) = (1_A\#S(h_{(1)}))(h_{(2)} \cdot b\#1_{\mathcal{H}}),$$

for all $h \in \mathcal{H}$ and $b \in A$.

Proof. Since our partial action is symmetric, in view of Lemma 2.5 we have:

$$\begin{aligned}
(1_A\#S(h_{(1)}))(h_{(2)} \cdot b\#1_{\mathcal{H}}) &= S(h_{(2)})(h_{(3)} \cdot b)\#S(h_{(1)}) \\
&= (S(h_{(3)})h_{(4)}) \cdot b)(S(h_{(2)}) \cdot 1_A)\#S(h_{(1)}) \\
&= (\varepsilon(h_{(3)})1_{\mathcal{H}}) \cdot b)(S(h_{(2)}) \cdot 1_A)\#S(h_{(1)}) \\
&= b(S(h_{(2)}) \cdot 1_A)\#S(h_{(1)}) \\
&= b\#S(h) = (b\#1_{\mathcal{H}})(1_A\#S(h)). \quad \square
\end{aligned}$$

We proceed by recalling the definition of the associative algebra $\mathcal{H}_{par}$ which governs the partial representations of a Hopf algebra H.

Definition 2.8. [7, Definition 4.1] Let $\mathcal{H}$ be a Hopf algebra and let $T(\mathcal{H})$ be the tensor algebra of the K-module $\mathcal{H}$. The *partial "Hopf"* algebra $\mathcal{H}_{par}$ is the quotient of $T(\mathcal{H})$ by the ideal I generated by the elements of the form

(1) $1_{\mathcal{H}} - 1_{T(\mathcal{H})}$;
(2) $h \otimes k_{(1)} \otimes S(k_{(2)}) - hk_{(1)} \otimes S(k_{(2)})$, for all $h, k \in \mathcal{H}$;
(3) $h_{(1)} \otimes S(h_{(2)}) \otimes k - h_{(1)} \otimes S(h_{(2)})k$, for all $h, k \in \mathcal{H}$;
(4) $h \otimes S(k_{(1)}) \otimes k_{(2)} - hS(k_{(1)}) \otimes k_{(2)}$, for all $h, k \in \mathcal{H}$;
(5) $S(h_{(1)}) \otimes h_{(2)} \otimes k - S(h_{(1)}) \otimes h_{(2)}k$, for all $h, k \in \mathcal{H}$.

Observe that by [7, Theorem 4.10] the algebra $\mathcal{H}_{par}$ possesses the structure of a Hopf algebroid. Denote by $[h]$ the class of $h \in \mathcal{H}$ in $\mathcal{H}_{par}$ and consider the map

$$\begin{aligned}
[_] : \mathcal{H} &\to \mathcal{H}_{par} \\
h &\mapsto [h].
\end{aligned}$$

Recall from [7] the following easily verified relations:

(1) $[\alpha h + \beta k] = \alpha[h] + \beta[k]$, for all $\alpha, \beta \in K$ and $h, k \in \mathcal{H}$;

(2) $[1_{\mathcal{H}}] = 1_{\mathcal{H}_{par}}$;
(3) $[h][k_{(1)}][S(k_{(2)})] = [hk_{(1)}][S(k_{(2)})]$, for all $h, k \in \mathcal{H}$;
(4) $[h_{(1)}][S(h_{(2)})][k] = [h_{(1)}][S(h_{(2)})k]$, for all $h, k \in \mathcal{H}$;
(5) $[h][S(k_{(1)})][k_{(2)}] = [hS(k_{(1)})][k_{(2)}]$, for all $h, k \in \mathcal{H}$;
(6) $[S(h_{(1)})][h_{(2)}][k] = [S(h_{(1)})][h_{(2)}k]$, for all $h, k \in \mathcal{H}$.

Thus, the linear map $[\_]$ is a partial representation $\mathcal{H} \to \mathcal{H}_{par}$. The following universal property of the partial "Hopf" algebra expresses the control of $\mathcal{H}_{par}$ on the partial representations of H.

Theorem 2.9. *(see [7, Theorem 4.2]) For every partial representation $\pi : \mathcal{H} \to A$ there is a unique morphism of algebras $\hat{\pi} : \mathcal{H}_{par} \to A$ such that $\pi = \hat{\pi} \circ [\_]$. Conversely, given an algebra morphism $\pi : \mathcal{H}_{par} \to A$, there exists a unique partial representation $\pi_\phi : \mathcal{H} \to A$ such that $\phi = \hat{\pi}_\phi$.*

The universal property of $\mathcal{H}_{par}$ relates the modules over $\mathcal{H}_{par}$ with the partial H-modules, the latter being defined as follows:

Definition 2.10. [7, Definition 5.1] Let $\mathcal{H}$ be a Hopf algebra. A *partial module* over $\mathcal{H}$ is a pair (M, π), where M is a K-module and $\pi : \mathcal{H} \to \mathrm{End}_K(M)$ is a left partial representation of $\mathcal{H}$.

As in [7], by a *morphism* $(M, \pi) \to (M', \pi')$ of partial $\mathcal{H}$-modules we mean a K-linear map $f : M \to M'$ such that $f \circ \pi(h) = \pi'(h) \circ f$ for all $h \in \mathcal{H}$, and by ${}_{\mathcal{H}}\mathcal{M}^{par}$ we denote the category of the left partial $\mathcal{H}$-modules and their morphisms.

The following fact is [7, Corollary 5.3].

Proposition 2.11. *There is an isomorphism of categories ${}_{\mathcal{H}}\mathcal{M}^{par} \cong \mathcal{H}_{par}$-**Mod**. Given a partial $\mathcal{H}$-module (M, π), the action of $\mathcal{H}_{par}$ on M is determined by*

$$[h] \triangleright m := \pi_h(m). \tag{2.1}$$

2.2. Some background on Hochschild (co)homology

For the reader's convenience, we proceed by recalling some well-known background on Hochschild (co)homology.

Definition 2.12. If A is a K-algebra, where K is a unital commutative ring, then its *enveloping algebra* is

$$A^e = A \otimes_K A^{op},$$

where A^{op} stands for the opposite algebra of A.

Proposition 2.13. *Let X be an A-bimodule, then X is a left A^e-module with action*

$$(a \otimes b) \cdot x := a \cdot x \cdot b$$

and is a right A^e-module with action

$$x \cdot (a \otimes b) := b \cdot x \cdot a.$$

Definition 2.14. Let A be a unital K-algebra and M an A-bimodule. The Hochschild homology of A with coefficients in M is defined by

$$H_\bullet(A, M) := \operatorname{Tor}_\bullet^{A^e}(A, M),$$

i.e., $H_\bullet(A, -)$ is the left derived functor of $A \otimes_{A^e} -$. Dually, we define the Hochschild cohomology of A with coefficients in M by

$$H^\bullet(A, M) := \operatorname{Ext}_{A^e}^\bullet(A, M).$$

The following easy property of tensor products of bimodules will be used constantly in the development of this work.

Lemma 2.15. *Let X and Y be A-bimodules. Then for every $a \in A$, $x \in X$ and $y \in Y$ we have*

$$a \cdot x \otimes_{A^e} y = x \otimes_{A^e} y \cdot a$$

and

$$x \cdot a \otimes_{A^e} y = x \otimes_{A^e} a \cdot y.$$

Proof. By direct computations we obtain

$$\begin{aligned} a \cdot x \otimes_{A^e} y &= x \cdot (1_A \otimes a) \otimes_{A^e} y \\ &= x \otimes_{A^e} (1_A \otimes a) \cdot y \\ &= x \otimes_{A^e} y \cdot a, \end{aligned} \qquad \begin{aligned} x \cdot a \otimes_{A^e} y &= x \cdot (a \otimes 1_A) \otimes_{A^e} y \\ &= x \otimes_{A^e} (a \otimes 1_A) \cdot y \\ &= x \otimes_{A^e} a \cdot y. \quad \square \end{aligned}$$

3. Homology of the partial smash product

<u>In all what follows</u> $\mathcal{H}$ will be a cocommutative Hopf K-algebra, A a unital K-algebra and

$$\begin{aligned} \cdot : \mathcal{H} \otimes A &\to A \\ h \otimes a &\mapsto h \cdot a \end{aligned}$$

a symmetric partial action of $\mathcal{H}$ on A.

Proposition 3.1. *Let R be a K-algebra, $\pi : \mathcal{H} \to R$ a partial representation and X an R-bimodule. Then the map $\pi' : \mathcal{H} \to \operatorname{End}_K(X)$, given by*

$$\pi'_h(x) := \sum \pi(h_1) \cdot x \cdot \pi(S(h_2)),$$

is a partial representation.

Proof. (PR1) $\pi'_1(x) = \pi(1) \cdot x \cdot \pi(S(1)) = \pi(1) \cdot x \cdot \pi(1) = x$.
(PR2)

$$\begin{aligned}
\pi'_h \pi'_{k_{(1)}} \pi'_{S(k_{(2)})}(x) &= \pi'_h \pi'_{k_{(1)}} \Big(\pi(S(k_{(2)})) \cdot x \cdot \pi(S^2(k_{(3)}))\Big) \\
&= \pi'_h \Big(\pi(k_{(1)})\pi(S(k_{(3)})) \cdot x \cdot \pi(S^2(k_{(4)}))\pi(S(k_{(2)}))\Big) \\
&= \pi(h_{(1)})\pi(k_{(1)})\pi(S(k_{(3)})) \cdot x \cdot \pi(S^2(k_{(4)}))\pi(S(k_{(2)}))\pi(S(h_{(2)})) \\
\text{(By the cocommutativity)} &= \pi(h_{(1)}k_{(1)})\pi(S(k_{(3)})) \cdot x \cdot \pi(S^2(k_{(4)}))\pi(S(h_{(2)}k_{(2)})) \\
&= \pi'_{hk_{(1)}} \pi'_{S(k_{(2)})}(x).
\end{aligned}$$

(PR5)

$$\begin{aligned}
\pi'_{S(h_{(1)})}\pi'_{h_{(2)}}\pi'_k(x) &= \pi(S(h_{(1)}))\pi(h_{(3)})\pi(k_{(1)}) \cdot x \cdot \pi(S(k_{(2)}))\pi(S(h_{(4)}))\pi(S^2(h_{(2)})) \\
\text{(By the cocommutativity)} &= \pi(S(h_{(1)}))\pi(h_{(3)}k_{(1)}) \cdot x \cdot \pi(S(h_{(4)}k_{(2)}))\pi(S^2(h_{(2)})) \\
&= \pi'_{S(h_{(1)})}\pi'_{h_{(2)}k}(x). \quad \square
\end{aligned}$$

Let M be an $A\#\mathcal{H}$-bimodule. Then, M is an A-bimodule with the actions induced by the map ϕ_0 from Lemma 2.5, i.e., the A-bimodule structure is defined by

$$a \cdot m := (a\#1_{\mathcal{H}}) \cdot m \text{ and } m \cdot a := m \cdot (a\#1_{\mathcal{H}}). \tag{3.1}$$

Since our partial action is symmetric, by [7, Example 3.7], the map $\pi_0 : \mathcal{H} \to A\#\mathcal{H}$, given by $h \mapsto 1_A\#h$, is a partial representation of $\mathcal{H}$ into the partial smash product $A\#\mathcal{H}$. Consequently, by Proposition 3.1 the map $\pi' : \mathcal{H} \to \operatorname{End}_K(M)$, such that

$$\pi'_h(m) := (1_A\#h_{(1)}) \cdot m \cdot (1_A\#S(h_{(2)})),$$

is a partial representation of $\mathcal{H}$. Thus, by Proposition 2.11, M is an $\mathcal{H}_{par}$-module with action given by

$$[h] \triangleright m := \pi'_h(m) = (1_A\#h_{(1)}) \cdot m \cdot (1_A\#S(h_{(2)})). \tag{3.2}$$

Notice that given a partial representation $\phi : \mathcal{H} \to \mathcal{R}$ into a unital algebra $\mathcal{R}$, we have that $\mathcal{R}$ becomes a left $\mathcal{H}_{par}$-module by setting

$$[h] \triangleright r := \phi(h)r, \tag{3.3}$$

in view of the isomorphism $\mathcal{R} \cong \mathrm{End}_{\mathcal{R}}(\mathcal{R})$.

It is easy to verify the next fact by a direct computation.

Proposition 3.2. *Let $f : X \to Y$ be a map of $A\#\mathcal{H}$-bimodules. Then, f is a morphism of $\mathcal{H}_{par}$-modules.*

Recall that since our partial action $\cdot : \mathcal{H} \otimes A \to A$ is symmetric, then by [7, Example 3.5] the map $h \mapsto \lambda_h \in \mathrm{End}_K(A)$, where $\lambda_h(a) := h \cdot a$, is a partial representation of $\mathcal{H}$ on A. Consequently, A is a $\mathcal{H}_{par}$-module with action

$$[h] \triangleright a := h \cdot a. \tag{3.4}$$

For each $h \in c$ define

$$e_h := [h_{(1)}][S(h_{(2)})],$$

explicitly we have that

$$e_h = \mu \circ ([\_] \otimes [\_]) \circ (1 \otimes S) \circ \Delta(h),$$

where μ denotes the product in $\mathcal{H}_{par}$. Thus, the map $h \mapsto e_h$ is K-linear, in particular $e_{\lambda h} = \lambda e_h$ for all $\lambda \in K$.

Notice that for a general Hopf algebra one also needs to consider the elements $\tilde{e}_h := [S(h_{(1)})][h_{(2)}]$, $h \in \mathcal{H}$, (see [7]), however, since our Hopf algebra is cocommutative, we have that

$$\begin{aligned}\tilde{e}_h &= [S(h_{(1)})][h_{(2)}]\\ &= [S(h_{(1)})][S(S(h_{(2)}))]\\ &= [S(h)_{(1)}][S(S(h)_{(2)})]\\ &= e_{S(h)}.\end{aligned}$$

The following is [7, Lemma 4.7] stated for our particular case of a cocommutative Hopf algebra.

Lemma 3.3. *For every $h, k \in \mathcal{H}$ the following properties hold:*

(i) $e_k[h] = [h_{(2)}]e_{S(h_{(1)})k}$, *in particular* $e_{h_{(1)}}[h_{(2)}] = [h]$;
(ii) $[h]e_k = e_{h_{(1)}k}[h_{(2)}]$, *in particular* $[h_{(1)}]e_{S(h_{(2)})} = [h]$;
(iii) $e_{h_{(1)}}e_{h_{(2)}} = e_h$;
(iv) $e_h e_k = e_k e_h$.

Definition 3.4. We define $\mathcal{B}$ as the subalgebra of $\mathcal{H}_{par}$ generated by $\{e_h \mid h \in \mathcal{H}\}$.

The following lemma is a consequence of [7, Theorem 4.8] and Proposition 2.11.

Lemma 3.5. *The algebra $\mathcal{B}$ is a left $\mathcal{H}_{par}$-module with the action*

$$[h] \triangleright b := [h_{(1)}]b[S(h_{(2)})]. \tag{3.5}$$

Since $\mathcal{H}$ is cocommutative then we can consider the antipode S as an isomorphism between $\mathcal{H}$ and $\mathcal{H}^{op}$. Furthermore, it determines a partial representation

$$\begin{aligned} \mathcal{S}' : \mathcal{H} &\to (\mathcal{H}_{par})^{op}, \\ h &\mapsto [S(h)], \end{aligned}$$

and thus there exists a morphism of algebras

$$\mathcal{S} : \mathcal{H}_{par} \to (\mathcal{H}_{par})^{op}, \tag{3.6}$$

such that $\mathcal{S}([h_1][h_2] \dots [h_n]) = [S(h_n)] \dots [S(h_2)][S(h_1)]$. Since, $S : \mathcal{H} \to \mathcal{H}$ is a K-linear isomorphism, then $\mathcal{S}$ is an algebra isomorphism. Indeed, note that $\mathcal{H}_{par} = (\mathcal{H}_{par})^{op}$ as K-modules. Then, we obtain a linear map $\mathcal{T} : (\mathcal{H}_{par})^{op} \to \mathcal{H}_{par}$ such that $\mathcal{T}([h_1][h_2] \dots [h_n]) = [S(h_n)] \dots [S(h_2)][S(h_1)]$. Note that $\mathcal{S}$ and $\mathcal{T}$ are mutually inverses since $S^2 = 1_{\mathcal{H}}$ because $\mathcal{H}$ is cocommutative. Hence, we obtain the following lemma.

Lemma 3.6. *The categories $\mathcal{H}_{par}$-**Mod** and **Mod**-$\mathcal{H}_{par}$ are isomorphic. In particular, given a left $\mathcal{H}_{par}$-module V, then V is a right $\mathcal{H}_{par}$-module with action*

$$x \triangleleft [h] := [S(h)] \triangleright x.$$

Analogously, if V is a right $\mathcal{H}_{par}$-module, then V is a left $\mathcal{H}_{par}$-module with action

$$[h] \triangleright x := x \triangleleft [S(h)].$$

Proof. Obviously, B-**Mod** $\cong$ **Mod**-B^{op} for any algebra B, and since $\mathcal{H}_{par} \overset{\mathcal{S}}{\cong} (\mathcal{H}_{par})^{op}$, then **Mod**-$(\mathcal{H}_{par})^{op} \cong$ **Mod**-$\mathcal{H}_{par}$. □

In particular, by Lemmas 3.5 and 3.6 we conclude that $\mathcal{B}$ is a right $\mathcal{H}_{par}$-module with action

$$b \triangleleft [h] := [S(h_{(1)})]b[h_{(2)}]. \tag{3.7}$$

Indeed,

$$b \triangleleft [h] := [S(h)] \triangleright b = [S(h)_{(1)}]b[S(S(h)_{(2)})] = [S(h_{(2)})]b[S^2(h_{(1)})] = [S(h_{(1)})]b[h_{(2)}],$$

thanks to the cocommutativity of $\mathcal{H}$.

Lemma 3.7. *For any $w, u \in \mathcal{B}$ we have that $w \triangleright u = wu$.*

Proof. Let $h \in \mathcal{H}$. Then, keeping in mind the cocommutativity of $\mathcal{H}$ and Lemma 3.3, we have

$$\begin{aligned} e_h \triangleright u &= \sum [h_{(1)}][S(h_{(2)})] \triangleright u \\ &= \sum [h_{(1)}] \triangleright ([S(h_{(2)})]u[h_{(3)}]) \\ &= \sum [h_{(1)}][S(h_{(3)})]u[h_{(4)}][S(h_{(2)})] \\ &= \sum [h_{(1)}][S(h_{(2)})]u[h_{(3)}][S(h_{(4)})] \\ &= \sum e_{h(1)} u e_{h(2)} \\ &= \sum e_{h(1)} e_{h(2)} u \\ &= e_h u. \end{aligned}$$

The statement with an arbitrary $w \in \mathcal{B}$ follows immediately. □

Proposition 3.8. *Let M be an $A\#\mathcal{H}$-bimodule. Then the map $\pi : \mathcal{H} \to \mathrm{End}_K(A \otimes_{A^e} M)$ such that*

$$\pi_h(a \otimes_{A^e} m) := [h_{(1)}] \triangleright a \otimes_{A^e} [h_{(2)}] \triangleright m, \tag{3.8}$$

is a partial representation of $\mathcal{H}$. In particular $A \otimes_{A^e} M$ is a $\mathcal{H}_{par}$-module with the action

$$[h] \triangleright (a \otimes_{A^e} m) := [h_{(1)}] \triangleright a \otimes_{A^e} [h_{(2)}] \triangleright m. \tag{3.9}$$

Proof. First, we have to verify that the map

$$\begin{aligned} \pi_h : A \otimes_{A^e} M &\to A \otimes_{A^e} M \\ a \otimes_{A^e} m &\mapsto [h_{(1)}] \triangleright a \otimes_{A^e} [h_{(2)}] \triangleright m \end{aligned}$$

is well-defined. Indeed, for all $b, c \in A$ we obtain using Lemma 2.15 and the cocommutativity of $\mathcal{H}$ that

$$\begin{aligned} &[h_{(1)}] \triangleright (a \cdot (c \otimes b)) \otimes_{A^e} [h_{(2)}] \triangleright m \\ &\quad = [h_{(1)}] \triangleright (bac) \otimes_{A^e} [h_{(2)}] \triangleright m \\ &\quad = h_{(1)} \cdot (bac) \otimes_{A^e} [h_{(2)}] \triangleright m \end{aligned}$$

$$
\begin{aligned}
&= (h_{(1)} \cdot b)(h_{(2)} \cdot a)(h_{(3)} \cdot c) \otimes_{A^e} [h_{(4)}] \triangleright m \\
&= (h_{(4)} \cdot b)(h_{(1)} \cdot a)(h_{(2)} \cdot c) \otimes_{A^e} [h_{(3)}] \triangleright m \\
&= (h_{(1)} \cdot a) \otimes_{A^e} (h_{(2)} \cdot c)([h_{(3)}] \triangleright m)(h_{(4)} \cdot b) \\
&= (h_{(1)} \cdot a) \otimes_{A^e} ((h_{(2)} \cdot c)\#1_{\mathcal{H}})(1_A\#h_{(3)}) \cdot m \cdot (1_A\#S(h_{(4)}))((h_{(5)} \cdot b)\#1_{\mathcal{H}}) \\
\text{(by Lemma 2.7)} &= (h_{(1)} \cdot a) \otimes_{A^e} ((h_{(2)} \cdot c)\#h_{(3)}) \cdot m \cdot (b\#1_{\mathcal{H}})(1_A\#S(h_{(4)})) \\
&= (h_{(1)} \cdot a) \otimes_{A^e} (1_A\#h_{(2)})(c\#1_{\mathcal{H}}) \cdot m \cdot (b\#1_{\mathcal{H}})(1_A\#S(h_{(3)})) \\
&= (h_{(1)} \cdot a) \otimes_{A^e} (1_A\#h_{(2)})(c \cdot m \cdot b)(1_A\#S(h_{(3)})) \\
&= ([h_{(1)}] \triangleright a) \otimes_{A^e} [h_{(2)}] \triangleright ((c \otimes b) \cdot m).
\end{aligned}
$$

Now observe that

$$\pi_h(a \otimes_{A^e} m) = \otimes_{A^e} \circ (- \triangleright a \otimes - \triangleright m) \circ [\_] \otimes [\_] \circ \Delta(h),$$

where $(- \triangleright a \otimes - \triangleright m)(x \otimes y) := x \triangleright a \otimes y \triangleright m$, for all $a \in A$, $m \in M$ and $x, y \in \mathcal{H}_{par}$. Therefore, π is a K-linear map. Furthermore, since we defined π using the $\mathcal{H}_{par}$-module (partial representation of $\mathcal{H}$) structures of A and M we have that π is a partial representation of $\mathcal{H}$. □

Remark 3.9. Let $f : X \to Y$ be a map of $A\#\mathcal{H}$-bimodules. Then, by Proposition 3.2, f is a map of $\mathcal{H}_{par}$-modules, and, therefore, $1_A \otimes_{A^e} f : A \otimes_{A^e} X \to A \otimes_{A^e} Y$ is a map of $\mathcal{H}_{par}$-modules.

In what follows M will be an $A\#\mathcal{H}$-bimodule, and we shall only consider the left $\mathcal{H}_{par}$-module structure on $A \otimes_{A^e} M$ defined by (3.8).

By Proposition 3.8 we can define the (covariant) right exact functor

$$F_1(-) := A \otimes_{A^e} - : (A\#\mathcal{H})^e\text{-Mod} \to \mathcal{H}_{par}\text{-Mod},$$

and by (3.7) the right exact functor

$$F_2(-) := \mathcal{B} \otimes_{\mathcal{H}_{par}} - := \mathcal{H}_{par}\text{-Mod} \to K\text{-Mod}\,.$$

Recall that the Hochschild homology of $A\#\mathcal{H}$ with coefficients in M is the left derived functor of

$$F(-) := A\#\mathcal{H} \otimes_{(A\#\mathcal{H})^e} - : (A\#\mathcal{H})^e\text{-Mod} \to K\text{-Mod}\,.$$

The following is Proposition 1.4 of [21].

Proposition 3.10. *Let R and S be rings and $f : R \to S$ a homomorphism of rings such that S is a projective R-module. Then, any projective S-module is a projective R-module.*

Using a straightforward dual basis argument one easily obtains the next:

Lemma 3.11. *Let R and S be unital K-algebras. Suppose that X is a projective left R-module and Y is a projective right S-module. Then $X \otimes_K Y$ is a projective left $R \otimes_K S^{op}$-module.*

Lemma 3.12. *Suppose that $\mathcal{H}$ is projective over K. Then $(A\#\mathcal{H})^e$ is a projective left A^e-module.*

Proof. Since $\mathcal{H}$ is projective over K, it is a direct summand of a free module, and tensoring by A we readily see that $A \otimes \mathcal{H}$ is a projective left A-module. Then, $A\#\mathcal{H} = (A \otimes \mathcal{H})(1_A \otimes 1_{\mathcal{H}})$ is projective over A as a left module, being a direct summand of $A \otimes \mathcal{H}$, by Lemma 2.6. In view of Lemma 3.11 it remains to show that $A\#\mathcal{H}$ is projective as a right A-module.

Let $\{h_i, f_i\}_{i\in I}$ be a dual basis for $\mathcal{H}$ over K, where I is some index set. Then $h = \sum_{i\in I} f_i(h)h_i$, for each $h \in \mathcal{H}$. Define the mappings $g_i : A \otimes \mathcal{H} \to A$, $i \in I$, by

$$g_i(a \otimes h) = f_i(h_{(1)})(S(h_{(2)}) \cdot a).$$

Then for all $a, a' \in A, h \in \mathcal{H}$ and $i \in I$ we have, using cocommutativity of $\mathcal{H}$, that

$$\begin{aligned} g_i((a \otimes h)(a' \otimes 1_H)) &= g_i(a(h_{(1)} \cdot a') \otimes h_{(2)}) \\ &= f_i(h_{(1)})S(h_{(2)}) \cdot (a(h_{(3)} \cdot a')) \\ &= f_i(h_{(1)})(S(h_{(2)}) \cdot a)(S(h_{(3)}) \cdot (h_{(4)} \cdot a')) \\ &= f_i(h_{(1)}) \underbrace{(S(h_{(2)}) \cdot a)(S(h_{(3)}) \cdot 1_A)}((\underbrace{S(h_{(4)})h_{(5)}}) \cdot a') \\ &= f_i(h_{(1)})(S(h_{(2)}) \cdot a)a' \\ &= g_i(a \otimes h)a'. \end{aligned}$$

Thus, each g_i is a map of right A-modules. In particular,

$$g_i(a\#h) = g_i((a \otimes h)(1_A \otimes 1_H)) = g_i(a \otimes h)1_A = g_i(a \otimes h),$$

for all $a \in A, h \in \mathcal{H}$. Next we show that $\{1_A\#h_i, g_i\}_{i\in I}$ is a dual basis for the right A-module $A\#\mathcal{H}$, which will complete our proof. Indeed, for each $a \in A, h \in \mathcal{H}$, using that our partial action is symmetric and the cocommutativity of $\mathcal{H}$, we see that

$$\begin{aligned} \sum_{i\in I}(1_A\#h_i)g_i(a\#h) &= \sum_{i\in I}(1_A\#h_i)f_i(h_{(1)})(S(h_{(2)}) \cdot a) \\ &= \sum_{i\in I}(1_A\#h_i)(f_i(h_{(1)})(S(h_{(2)}) \cdot a) \otimes 1_{\mathcal{H}}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in I}(1_A \otimes h_i)(1_A \otimes 1_{\mathcal{H}})(f_i(h_{(1)})(S(h_{(2)})\cdot a)\otimes 1_{\mathcal{H}})\\
&= \sum_{i\in I}(1_A \otimes h_i)(f_i(h_{(1)})(S(h_{(2)})\cdot a)\otimes 1_{\mathcal{H}})\\
&= (1_A \otimes \sum_{i\in I} f_i(h_{(1)})h_i)((S(h_{(2)})\cdot a)\otimes 1_{\mathcal{H}})\\
&= (1_A \otimes h_{(1)})((S(h_{(2)})\cdot a)\otimes 1_{\mathcal{H}})\\
&= h_{(1)}\cdot (S(h_{(2)})\cdot a)\otimes h_{(3)}\\
&= (h_{(1)}S(h_{(2)})\cdot a)(h_{(3)}\cdot 1_A)\otimes h_{(4)}\\
&= a(h_{(1)}\cdot 1_A)\otimes h_{(2)}\\
&= a\#h,
\end{aligned}
$$

as desired. □

From Proposition 3.10 and Lemma 3.12 we obtain the following:

Remark 3.13. If $\mathcal{H}$ is projective over K, then any projective $(A\#\mathcal{H})$-bimodule is a projective A-bimodule. Therefore, any projective resolution of M in $(A\#\mathcal{H})^e$-Mod is a projective resolution of M in A^e-Mod. Thus, considering M as an A-bimodule, the left derived functor of F_1 computes the Hochschild homology of M, i.e.,

$$H_\bullet(A, M) \cong L_\bullet F_1(M).$$

Lemma 3.14. *Let X be an $A\#\mathcal{H}$-bimodule and Y a $\mathcal{H}_{par}$-module, then*

(i) *For all $h \in \mathcal{H}$ we have that $h \cdot 1_A$ is central in A.*
(ii) *$e_h \triangleright a = (h\cdot 1_A)a$, for all $h \in \mathcal{H}$ and $a \in A$, considering A with the $\mathcal{H}_{par}$-module structure given by (3.4). Then, $e_h \triangleright 1_A = h\cdot 1_A$ and $w \triangleright a = (w \triangleright 1_A)a$ for all $a \in A$ and $w \in \mathcal{B}$;*
(iii) *$e_h \triangleright x = (h_{(1)}\cdot 1_A)\cdot x\cdot (h_{(2)}\cdot 1_A)$ $h \in \mathcal{H}$ and $x \in X$, considering X with the $\mathcal{H}_{par}$-module structure given by (3.2);*
(iv) *$e_h \otimes_{\mathcal{H}_{par}} y = 1_{\mathcal{B}} \otimes_{\mathcal{H}_{par}} [S(h)]\cdot y = 1_{\mathcal{B}} \otimes_{\mathcal{H}_{par}} e_h \cdot y$, as elements of $\mathcal{B}\otimes_{\mathcal{H}_{par}} Y$, for all $h \in \mathcal{H}$ and $y \in Y$,*
(v) *$e_h \triangleright (a \otimes_{A^e} x) = a \otimes_{A^e} e_h \triangleright x = e_h \triangleright a \otimes_{A^e} x$, as elements of $A\otimes_{A^e} X$; so that $w \triangleright (a\otimes_{A^e} x) = w \triangleright a \otimes_{A^e} x = a \otimes_{A^e} w \triangleright x$, for all $w \in \mathcal{B}$.*

Proof. (i) By direct computations using the cocommutativity and the symmetry of the partial action we obtain

$$(h\cdot 1_A)a = \varepsilon(h_{(1)})(h_{(2)}\cdot 1_A)a$$

$$
\begin{aligned}
&= (h_{(2)} \cdot 1_A)((\varepsilon(h_{(1)})1_{\mathcal{H}}) \cdot a) \\
&= (h_{(1)} \cdot 1_A)((h_{(2)}S(h_{(3)})) \cdot a) \\
&= h_{(1)} \cdot (S(h_{(2)}) \cdot a) \\
&= (h_{(1)}S(h_{(3)}) \cdot a)(h_{(2)} \cdot 1_A) \\
&= (h_{(1)}S(h_{(2)}) \cdot a)(h_{(3)} \cdot 1_A) \\
&= ((\varepsilon(h_{(1)})1_{\mathcal{H}}) \cdot a)(h_{(2)} \cdot 1_A) \\
&= a(\varepsilon(h_{(1)})h_{(2)} \cdot 1_A) \\
&= a(h \cdot 1_A).
\end{aligned}
$$

(ii) For any $h \in \mathcal{H}$ and $a \in A$ we obtain

$$
\begin{aligned}
e_h \triangleright a &= [h_{(1)}][S(h_{(2)})] \triangleright a \\
&= h_{(1)} \cdot (S(h_{(2)}) \cdot a) \\
&= (h_{(1)} \cdot 1_A)(h_{(2)}S(h_{(3)}) \cdot a) \\
&= (h_{(1)} \cdot 1_A)(\varepsilon(h_{(2)})1_{\mathcal{H}} \cdot a) \\
&= (h \cdot 1_A)a.
\end{aligned}
$$

(iii) Let $h \in \mathcal{H}$ and $x \in X$. Then,

$$
\begin{aligned}
e_h \triangleright x &= (1_A \# h_{(1)})(1_A \# S(h_{(3)})) \cdot x \cdot (1_A \# h_{(4)})(1_A \# S(h_{(2)})) \\
&= (h_{(1)} \cdot 1_A \# h_{(2)}S(h_{(4)})) \cdot x \cdot (h_{(5)} \cdot 1_A \# h_{(6)}S(h_{(3)})) \\
&= (h_{(1)} \cdot 1_A \# h_{(2)}S(h_{(3)})) \cdot x \cdot (h_{(4)} \cdot 1_A \# h_{(5)}S(h_{(6)})) \\
&= (h_{(1)} \cdot 1_A \# \varepsilon(h_{(2)})1_{\mathcal{H}}) \cdot x \cdot (h_{(3)} \cdot 1_A \# \varepsilon(h_{(4)})1_{\mathcal{H}}) \\
&= (h_{(1)} \cdot 1_A \# 1_{\mathcal{H}}) \cdot x \cdot (h_{(2)} \cdot 1_A \# 1_{\mathcal{H}}) \\
&= (h_{(1)} \cdot 1_A) \cdot x \cdot (h_{(2)} \cdot 1_A).
\end{aligned}
$$

(iv) For $h \in \mathcal{H}$ and $y \in Y$ we have that

$$
e_h \otimes_{\mathcal{H}_{par}} y = [h_{(1)}][S(h_{(2)})] \otimes_{\mathcal{H}_{par}} y = 1_{\mathcal{B}} \triangleleft [S(h)] \otimes_{\mathcal{H}_{par}} y = 1_{\mathcal{B}} \otimes_{\mathcal{H}_{par}} [S(h)] \cdot y
$$

and

$$
\begin{aligned}
e_h \otimes_{\mathcal{H}_{par}} y &= e_{h_{(1)}} e_{h_{(2)}} \otimes_{\mathcal{H}_{par}} y \\
&= [h_{(1)}][S(h_{(2)})]1_{\mathcal{B}}[h_{(3)}][S(h_{(4)})] \otimes_{\mathcal{H}_{par}} y \\
&= [h_{(1)}](1_{\mathcal{B}} \triangleleft [h_{(2)}])[S(h_{(3)})] \otimes_{\mathcal{H}_{par}} y \\
&= [h_{(1)}](1_{\mathcal{B}} \triangleleft [h_{(3)}])[S(h_{(2)})] \otimes_{\mathcal{H}_{par}} y
\end{aligned}
$$

$$
\begin{aligned}
&= (1_{\mathcal{B}} \triangleleft [h_{(2)}]) \triangleleft [S(h_1)] \otimes_{\mathcal{H}_{par}} y \\
&= (1_{\mathcal{B}} \triangleleft [h_{(1)}][S(h_2)]) \otimes_{\mathcal{H}_{par}} y \\
&= (1_{\mathcal{B}} \triangleleft e_h) \otimes_{\mathcal{H}_{par}} y \\
&= 1_{\mathcal{B}} \otimes_{\mathcal{H}_{par}} e_h \cdot y.
\end{aligned}
$$

(v) For $a \in A$ and $x \in X$ we obtain

$$
\begin{aligned}
e_h \triangleright (a \otimes_{A^e} x) &= [h_{(1)}] \triangleright ([S(h_{(2)})] \triangleright a \otimes_{A^e} [S(h_{(3)})] \triangleright x) \\
&= [h_{(1)}][S(h_{(3)})] \triangleright a \otimes_{A^e} [h_{(2)}][S(h_{(4)})] \triangleright x \\
&= [h_{(1)}][S(h_{(2)})] \triangleright a \otimes_{A^e} [h_{(3)}][S(h_{(4)})] \triangleright x \\
&= e_{h_{(1)}} \triangleright a \otimes_{A^e} e_{h_{(2)}} \triangleright x \\
\text{(by } (ii) \text{ and } (iii)) &= (h_{(1)} \cdot 1_A)a \otimes_{A^e} (h_{(2)} \cdot 1_A) \cdot x \cdot (h_{(3)} \cdot 1_A)
\end{aligned}
$$

Now we have that

$$
\begin{aligned}
(h_{(1)} \cdot 1_A)a \otimes_{A^e} (h_{(2)} \cdot 1_A) \cdot x \cdot (h_{(3)} \cdot 1_A) &= a \otimes_{A^e} (h_{(1)} \cdot 1_A) \cdot x \cdot (h_{(2)} \cdot 1_A)(h_{(3)} \cdot 1_A) \\
&= a \otimes_{A^e} (h_{(1)} \cdot 1_A) \cdot x \cdot (h_{(2)} \cdot 1_A) \\
&= a \otimes_{A^e} e_h \triangleright x,
\end{aligned}
$$

and on the other hand

$$
\begin{aligned}
(h_{(1)} \cdot 1_A)a \otimes_{A^e} (h_{(2)} \cdot 1_A) \cdot x \cdot (h_{(3)} \cdot 1_A) &= (h_{(3)} \cdot 1_A)(h_{(1)} \cdot 1_A)a(h_{(2)} \cdot 1_A) \otimes_{A^e} x \\
\text{(by } (i)) &= (h_{(1)} \cdot 1_A)(h_{(2)} \cdot 1_A)(h_{(3)} \cdot 1_A)a \otimes_{A^e} x \\
&= (h \cdot 1_A)a \otimes_{A^e} x \\
\text{(by } (ii)) &= e_h \triangleright a \otimes_{A^e} x. \quad \square
\end{aligned}
$$

Proposition 3.15. *The algebra $A\#\mathcal{H}$ is a $\mathcal{H}_{par}$-module with action, given by*

$$[h] \triangleright (a\#k) := (1_A\#h)(a\#k). \tag{3.10}$$

In particular

$$e_h \triangleright (a\#k) = (h \cdot 1_A\#1_{\mathcal{H}})(a\#k) = ((h \cdot 1_A)a\#k) = (e_h \triangleright a)\#k. \tag{3.11}$$

Proof. We know that $\pi_0 : \mathcal{H} \to A\#\mathcal{H}$ such that $\pi_0(h) := 1_A\#h$ is a partial representation of $\mathcal{H}$, whence we conclude that $A\#\mathcal{H}$ is a $\mathcal{H}_{par}$-module with action (3.10). Finally, by direct computation we obtain

$$\begin{aligned}
e_h \triangleright (a\#k) &= (1_A\#h_{(1)})(1_A\#S(h_{(2)}))(a\#k) \\
&= (h_{(1)} \cdot 1_A\#h_{(2)}S(h_{(3)}))(a\#k) \\
&= (h_{(1)} \cdot 1_A\#\varepsilon(h_{(2)})1_{\mathcal{H}})(a\#k) \\
&= (h \cdot 1_A\#1_{\mathcal{H}})(a\#k) \\
&= (h \cdot 1_A)a\#k \\
&= (e_h \triangleright a)\#k,
\end{aligned}$$

where the final equality is due to item (ii) of Lemma 3.14. □

Proposition 3.16. *Let X be a right $\mathcal{H}_{par}$-module. Then, $X\otimes_{\mathcal{B}}(A\#\mathcal{H})$ is an $A\#\mathcal{H}$-bimodule with actions:*

$$a\#h \cdot (x \otimes_{\mathcal{B}} c\#t) := x \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} (a\#h_{(2)})(c\#t)$$

and

$$(x \otimes_{\mathcal{B}} c\#t) \cdot a\#h := x \otimes_{\mathcal{B}} (c\#t)(a\#h).$$

Thus, we obtain a functor $- \otimes_{\mathcal{B}} A\#\mathcal{H} : \text{Mod-}\mathcal{H}_{par} \to (A\#\mathcal{H})^e\text{-Mod}$.

Proof. The right action is well-defined since it is just the action induced by the right multiplication on $A\#\mathcal{H}$. For the left action we want to verify that $a\#h \cdot (x \cdot e_k \otimes_{\mathcal{B}} c\#t) = a\#h \cdot (x \otimes_{\mathcal{B}} e_k \triangleright (c\#t))$, i.e.,

$$(x \cdot e_k) \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} (a\#h_{(2)})(c\#t) = x \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} (a\#h_{(2)})(e_k \triangleright (c\#t)).$$

Starting the computations with the right part of the above equation we have

$$\begin{aligned}
&x \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} (a\#h_{(2)})(e_k \triangleright (c\#t)) \\
\text{(by (3.11))} &= x \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} (a\#h_{(2)})(k \cdot 1_A\#1_{\mathcal{H}})(c\#t) \\
&= x \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} (a(h_{(2)} \cdot (k \cdot 1_A))\#h_{(3)})(c\#t) \\
&= x \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} \big(a(h_{(2)} \cdot 1_A)(h_{(3)}k \cdot 1_A)\#h_{(4)}\big)(c\#t) \\
\text{(by (3.11))} &= x \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} e_{h_{(2)}} \triangleright \big((a(h_{(3)}k \cdot 1_A)\#h_{(4)})(c\#t)\big) \\
&= x \cdot [S(h_{(1)})]e_{h_{(2)}} \otimes_{\mathcal{B}} \big((a(h_{(3)}k \cdot 1_A)\#h_{(4)})(c\#t)\big) \\
\text{(by Lemma 3.3 } (ii)) &= x \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} \big((a(h_{(2)}k \cdot 1_A)\#h_{(3)})(c\#t)\big) \\
\text{(by (3.11))} &= x \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} e_{h_{(2)}k} \triangleright \big((a\#h_{(3)})(c\#t)\big) \\
&= x \cdot [S(h_{(1)})]e_{h_{(2)}k} \otimes_{\mathcal{B}} \big((a\#h_{(3)})(c\#t)\big) \\
\text{(by Lemma 3.3 } (i)) &= x \cdot e_k[S(h_{(1)})] \otimes_{\mathcal{B}} \big((a\#h_{(2)})(c\#t)\big) \\
&= (x \cdot e_k) \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} (a\#h_{(2)})(c\#t).
\end{aligned}$$

Then, $a\#h\cdot(x\otimes_{\mathcal{B}}c\#t)=x\cdot[S(h_{(1)})]\otimes_{\mathcal{B}}(a\#h_{(2)})(c\#t)$ is well-defined. Now observe that

$$\begin{aligned}
(b\#k)\cdot\big(a\#h\cdot(x\otimes_{\mathcal{B}}c\#t)\big) &= (b\#k)\cdot\big(x\cdot[S(h_{(1)})]\otimes_{\mathcal{B}}(a\#h_{(2)})(c\#t)\big)\\
&= x\cdot[S(h_{(1)})][S(k_{(1)})]\otimes_{\mathcal{B}}(b\#k_{(2)})(a\#h_{(2)})(c\#t)\\
\text{(By Lemma 3.3 }(ii)) &= x\cdot[S(h_{(1)})][S(k_{(1)})]e_{k_{(2)}}\otimes_{\mathcal{B}}(b\#k_{(3)})(a\#h_{(2)})(c\#t)\\
&= x\cdot[S(h_{(1)})S(k_{(1)})]e_{k_{(2)}}\otimes_{\mathcal{B}}(b\#k_{(3)})(a\#h_{(2)})(c\#t)\\
&= x\cdot[S(h_{(1)})S(k_{(1)})]\otimes_{\mathcal{B}}e_{k_{(2)}}\triangleright\big((b\#k_{(3)})(a\#h_{(2)})(c\#t)\big)\\
&= x\cdot[S(k_{(1)}h_{(1)})]\otimes_{\mathcal{B}}e_{k_{(2)}}\triangleright\big((b\#k_{(3)})(a\#h_{(2)})(c\#t)\big)\\
&= x\cdot[S(k_{(1)}h_{(1)})]\otimes_{\mathcal{B}}((k_{(2)}\cdot 1_A)b\#k_{(3)})(a\#h_{(2)})(c\#t)\\
\text{(by Lemma 2.5 }(ii)) &= x\cdot[S(k_{(1)}h_{(1)})]\otimes_{\mathcal{B}}(b\#k_{(2)})(a\#h_{(2)})(c\#t)\\
&= x\cdot[S(k_{(2)}h_{(1)})]\otimes_{\mathcal{B}}(b\#k_{(1)})(a\#h_{(2)})(c\#t)\\
&= x\cdot[S(k_{(2)}h_{(1)})]\otimes_{\mathcal{B}}(b(k_{(1)}\cdot a)\#k_{(3)}h_{(2)})(c\#t)\\
&= (b(k_{(1)}\cdot a)\#k_{(2)}h)\cdot(x\otimes_{\mathcal{B}}c\#t)\\
&= (b\#k)(a\#h)\cdot(x\otimes_{\mathcal{B}}c\#t).
\end{aligned}$$

Therefore, the left action is well-defined, and since

$$\big(a\#h\cdot(x\otimes_{\mathcal{B}}c\#t)\big)\cdot b\#k=x\cdot[S(h_{(1)})]\otimes_{\mathcal{B}}(a\#h_{(2)})(c\#t)(b\#k)=a\#h\cdot\big((x\otimes_{\mathcal{B}}c\#t)\cdot b\#k\big),$$

we have that $X\otimes_{\mathcal{B}}A\#\mathcal{H}$ is a $A\#\mathcal{H}$-bimodule. Finally, if $f:X\to Y$ is a map of right $\mathcal{H}_{par}$-modules then $f\otimes_{\mathcal{B}}1:X\otimes_{\mathcal{B}}A\#\mathcal{H}\to Y\otimes_{\mathcal{B}}A\#\mathcal{H}$ is a map of $A\#\mathcal{H}$-bimodules. Indeed, it is clear that $f\otimes_{\mathcal{B}}1$ is a map of right $A\#\mathcal{H}$-modules. On the other hand, observe that for any $x\in X$ and $z\in A\#\mathcal{H}$ we have

$$\begin{aligned}
(a\#h)\cdot\big((f\otimes_{\mathcal{B}}1)(x\otimes_{\mathcal{B}}z)\big) &= (a\#h)\cdot(f(x)\otimes_{\mathcal{B}}z)\\
&= f(x)\cdot[S(h_{(1)})]\otimes_{\mathcal{B}}(a\#h_{(2)})z\\
&= f(x\cdot[S(h_{(1)})])\otimes_{\mathcal{B}}(a\#h_{(2)})z\\
&= (f\otimes_{\mathcal{B}}1)\big(x\cdot[S(h_{(1)})]\otimes_{\mathcal{B}}(a\#h_{(2)})z\big)\\
&= (f\otimes_{\mathcal{B}}1)\big((a\#h)\cdot(x\otimes_{\mathcal{B}}z)\big). \quad \square
\end{aligned}$$

Proposition 3.17. *The functors*

$$-\otimes_{\mathcal{H}_{par}}(A\otimes_{A^e}-):\text{Mod-}\mathcal{H}_{par}\times(A\#\mathcal{H})^e\text{-Mod}\to K\text{-Mod}$$

and

$$(-\otimes_{\mathcal{B}}A\#\mathcal{H})\otimes_{(A\#\mathcal{H})^e}-:\text{Mod-}\mathcal{H}_{par}\times(A\#\mathcal{H})^e\text{-Mod}\to K\text{-Mod}$$

are naturally isomorphic.

Proof. Let X be a right $\mathcal{H}_{par}$-module and M and $A\#\mathcal{H}$-bimodule. For a fixed $x \in X$ define

$$\begin{aligned}\tilde{\gamma}_{x,M} : A \times M &\to (X \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} M\\ (a,m) &\mapsto (x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot m,\end{aligned}$$

Observe that $\tilde{\gamma}_{x,M}$ is A^e-balanced. Indeed, for $d \otimes c \in A^e$ we have

$$\begin{aligned}\tilde{\gamma}_{x,M}(a \cdot (d \otimes c), m) &= \tilde{\gamma}_{x,M}(cad, m)\\ &= (x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} cad \cdot m\\ (\text{by Lemma } 2.15) &= (x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \cdot (c\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot (d \cdot m)\\ &= (x \otimes_{\mathcal{B}} c\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot (d \cdot m)\\ &= (c\#1_{\mathcal{H}}) \cdot (x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot (d \cdot m)\\ (\text{by Lemma } 2.15) &= (x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot (d \cdot m) \cdot (c\#1_{\mathcal{H}})\\ &= (x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot (d \cdot m \cdot c)\\ &= \tilde{\gamma}_{x,M}(a, (d \otimes c) \cdot m),\end{aligned}$$

so, $\tilde{\gamma}_{x,M}$ is A^e-balanced. Therefore, the following map is well-defined

$$\begin{aligned}\gamma_{x,M} : A \otimes_{A^e} M &\to (X \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} M\\ a \otimes_{A^e} m &\mapsto (x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot m.\end{aligned}$$

Now define the function

$$\begin{aligned}\tilde{\gamma}_{(X,M)} : X \times (A \otimes_{A^e} M) &\to (X \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} M\\ (x, a \otimes_{A^e} m) &\mapsto \gamma_{x,M}(a \otimes_{A^e} m) = (x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot m.\end{aligned}$$

We want to show that $\tilde{\gamma}_{(X,M)}$ is $\mathcal{H}_{par}$-balanced. Recall the $A\#\mathcal{H}$-bimodule structure of $X \otimes_{\mathcal{B}} A\#\mathcal{H}$ given by Proposition 3.16. Let $h \in \mathcal{H}$, thus

$$\begin{aligned}&\tilde{\gamma}_{(X,M)}(x \cdot [h], a \otimes_{A^e} m)\\ &\qquad= (x \cdot [h] \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot m\\ &\qquad= (x \cdot [h_{(1)}] e_{S(h_{(2)})} \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot m\\ &\qquad= (x \cdot [h_{(1)}] \otimes_{\mathcal{B}} e_{S(h_{(2)})} \triangleright (1_A\#1_{\mathcal{H}})) \otimes_{(A\#\mathcal{H})^e} a \cdot m\\ (\text{by } (3.11)) &\qquad= \left(x \cdot [h_{(1)}] \otimes_{\mathcal{B}} (S(h_{(2)}) \cdot 1_A)\#1_{\mathcal{H}}\right) \otimes_{(A\#\mathcal{H})^e} a \cdot m\\ &\qquad= \left(x \cdot [h_{(1)}] \otimes_{\mathcal{B}} (1_A\#S(h_{(2)}))(1_A\#h_{(3)})\right) \otimes_{(A\#\mathcal{H})^e} a \cdot m\\ &\qquad= 1_A\#S(h_{(1)}) \cdot \left(x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}\right) \cdot (1_A\#h_{(2)}) \otimes_{(A\#\mathcal{H})^e} a \cdot m\end{aligned}$$

$$\begin{aligned}
\text{(by Lemma 2.15)} &= \big(x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}\big) \otimes_{(A\#\mathcal{H})^e} (1_A\#h_{(2)}) \cdot (a \cdot m) \cdot (1_A\#S(h_{(1)}))\\
&= \big(x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}\big) \otimes_{(A\#\mathcal{H})^e} (1_A\#h_{(2)})(a\#1_{\mathcal{H}}) \cdot m \cdot (1_A\#S(h_{(1)}))\\
&= \big(x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}\big) \otimes_{(A\#\mathcal{H})^e} (h_{(2)} \cdot a\#h_{(3)}) \cdot m \cdot (1_A\#S(h_{(1)}))\\
&= \big(x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}\big) \otimes_{(A\#\mathcal{H})^e} (h_{(2)} \cdot a\#1_{\mathcal{H}})(1_A\#h_{(3)}) \cdot m \cdot (1_A\#S(h_{(1)}))\\
&= \big(x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}\big) \otimes_{(A\#\mathcal{H})^e} (h_{(1)} \cdot a\#1_{\mathcal{H}})(1_A\#h_{(2)}) \cdot m \cdot 1_A\#S(h_{(3)})\\
&= \big(x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}\big) \otimes_{(A\#\mathcal{H})^e} ([h_{(1)}] \triangleright a\#1_{\mathcal{H}}) \cdot ([h_{(2)}] \triangleright m)\\
&= \tilde{\gamma}_{(X,M)}(x, ([h_{(1)}] \triangleright a) \otimes_{A^e} ([h_{(2)}] \cdot m))\\
&= \tilde{\gamma}_{(X,M)}(x, [h] \triangleright (a \otimes_{A^e} m)).
\end{aligned}$$

Then, the following map is well-defined

$$\begin{aligned}
\gamma_{(X,M)} : X \otimes_{\mathcal{H}_{par}} (A \otimes_{A^e} M) &\to (X \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} M\\
x \otimes_{\mathcal{H}_{par}} (a \otimes_{A^e} m) &\mapsto (x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot m.
\end{aligned}$$

In order to obtain its inverse map we fix an $m \in M$ and define the map

$$\begin{aligned}
\tilde{\psi}_{X,m} : X \times A\#\mathcal{H} &\to X \otimes_{\mathcal{H}_{par}} (A \otimes_{A^e} M)\\
(x, a\#h) &\mapsto x \otimes_{\mathcal{H}_{par}} (1_A \otimes_{A^e} (a\#h) \cdot m).
\end{aligned}$$

Then, $\tilde{\psi}_{X,m}$ is $\mathcal{B}$-balanced. Indeed,

$$\begin{aligned}
\tilde{\psi}_{X,m}(x \cdot e_k, a\#h) &= x \cdot e_k \otimes_{\mathcal{H}_{par}} (1_A \otimes_{A^e} (a\#h) \cdot m)\\
&= x \otimes_{\mathcal{H}_{par}} e_k \triangleright (1_A \otimes_{A^e} (a\#h) \cdot m)\\
\text{(by Lemma 3.14 } (v) \text{ and } (ii)) &= x \otimes_{\mathcal{H}_{par}} \big(k \cdot 1_A \otimes_{A^e} (a\#h) \cdot m\big)\\
&= x \otimes_{\mathcal{H}_{par}} \big(1_A \otimes_{A^e} (k \cdot 1_A\#1_{\mathcal{H}}) \cdot ((a\#h) \cdot m)\big)\\
&= x \otimes_{\mathcal{H}_{par}} \big(1_A \otimes_{A^e} ((k \cdot 1_A)a\#h) \cdot m\big)\\
&= \tilde{\psi}_{X,m}(x, (k \cdot 1_A)a\#h)\\
\text{(by (3.11))} &= \tilde{\psi}_{X,m}(x, e_k \triangleright (a\#h)).
\end{aligned}$$

Then, the following map is well-defined

$$\begin{aligned}
\psi_{X,m} : X \otimes_{\mathcal{B}} A\#\mathcal{H} &\to X \otimes_{\mathcal{H}_{par}} (A \otimes_{A^e} M)\\
x \otimes_{\mathcal{B}} (a\#h) &\mapsto x \otimes_{\mathcal{H}_{par}} (1_A \otimes_{A^e} (a\#h) \cdot m).
\end{aligned}$$

So, we can define the map

$$\begin{aligned}
\tilde{\psi}_{(X,M)} : (X \otimes_{\mathcal{B}} A\#\mathcal{H}) \times M &\to X \otimes_{\mathcal{H}_{par}} (A \otimes_{A^e} M)\\
(x \otimes_{\mathcal{B}} a\#h, m) &\mapsto \psi_{X,m}(x \otimes_{\mathcal{B}} a\#h) = x \otimes_{\mathcal{H}_{par}} (1_A \otimes_{A^e} (a\#h) \cdot m),
\end{aligned}$$

which is $(A\#\mathcal{H})^e$-balanced. Indeed, for the left action, we have

$$\begin{aligned}
&\tilde{\psi}_{(X,M)}((b\#k)\cdot(x\otimes_{\mathcal{B}}(a\#h)),m)\\
&=\tilde{\psi}_{(X,M)}(x\cdot[S(k_{(1)})]\otimes_{\mathcal{B}}(b\#k_{(2)})(a\#h),m)\\
&=x\cdot[S(k_{(1)})]\otimes_{\mathcal{H}_{par}}\big(1_A\otimes_{A^e}(b\#k_{(2)})(a\#h)\cdot m\big)\\
&=x\cdot[S(k_{(1)})]\otimes_{\mathcal{H}_{par}}\big(b\otimes_{A^e}(1_A\#k_{(2)})(a\#h)\cdot m\big)\\
&=x\cdot[S(k_{(1)})]\otimes_{\mathcal{H}_{par}}\big(1_A\otimes_{A^e}(1_A\#k_{(2)})(a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})\big)\\
&=x\otimes_{\mathcal{H}_{par}}[S(k_{(1)})]\triangleright\big(1_A\otimes_{A^e}(1_A\#k_{(2)})(a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})\big)\\
&=x\otimes_{\mathcal{H}_{par}}\big([S(k_{(1)})]\triangleright 1_A\otimes_{A^e}[S(k_{(2)})]\triangleright((1_A\#k_{(3)})(a\#h)\cdot m\cdot(b\#1_{\mathcal{H}}))\big)\\
&=x\otimes_{\mathcal{H}_{par}}\Big(S(k_{(1)})\cdot 1_A\otimes_{A^e}(1_A\#S(k_{(2)}))(1_A\#k_{(3)})(a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})(1_A\#k_{(4)})\Big)\\
&=x\otimes_{\mathcal{H}_{par}}\Big(S(k_{(1)})\cdot 1_A\otimes_{A^e}((S(k_{(2)})\cdot 1_A\#(S(k_{(3)})k_{(4)}))(a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})(1_A\#k_{(5)}))\Big)\\
&=x\otimes_{\mathcal{H}_{par}}\Big(S(k_{(1)})\cdot 1_A\otimes_{A^e}((S(k_{(2)})\cdot 1_A\#\varepsilon(k_{(3)})1_{\mathcal{H}})(a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})(1_A\#k_{(4)}))\Big)\\
&=x\otimes_{\mathcal{H}_{par}}\Big(S(k_{(1)})\cdot 1_A\otimes_{A^e}((S(k_{(2)})\cdot 1_A\#1_{\mathcal{H}})(a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})(1_A\#k_{(3)}))\Big)\\
&=x\otimes_{\mathcal{H}_{par}}\Big((S(k_{(1)})\cdot 1_A)(S(k_{(2)})\cdot 1_A)\otimes_{A^e}((a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})(1_A\#k_{(3)}))\Big)\\
&=x\otimes_{\mathcal{H}_{par}}\Big((S(k_{(1)})\cdot 1_A)\otimes_{A^e}((a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})(1_A\#k_{(2)}))\Big)\\
&=x\otimes_{\mathcal{H}_{par}}\Big(1_A\otimes_{A^e}((a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})(1_A\#k_{(2)})((S(k_{(1)})\cdot 1_A)\#1_{\mathcal{H}}))\Big)\\
&=x\otimes_{\mathcal{H}_{par}}\Big(1_A\otimes_{A^e}((a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})(1_A\#k_{(1)})((S(k_{(2)})\cdot 1_A)\#1_{\mathcal{H}}))\Big)\\
&=x\otimes_{\mathcal{H}_{par}}\Big(1_A\otimes_{A^e}((a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})(k_{(1)}\cdot(S(k_{(2)})\cdot 1_A)\#k_{(3)}))\Big)\\
&=x\otimes_{\mathcal{H}_{par}}\Big(1_A\otimes_{A^e}((a\#h)\cdot m\cdot(b\#1_{\mathcal{H}})(k_{(1)}\cdot 1_A\#k_{(2)}))\Big)\\
&=x\otimes_{\mathcal{H}_{par}}\Big(1_A\otimes_{A^e}((a\#h)\cdot m\cdot(b\#k))\Big)\\
&=\tilde{\psi}_{(X,M)}((x\otimes_{\mathcal{B}}(a\#h)),m\cdot(b\#k)),
\end{aligned}$$

and, for the right action

$$\begin{aligned}
\tilde{\psi}_{(X,M)}((x\otimes_{\mathcal{B}}(a\#h))\cdot(b\#k),m)&=\tilde{\psi}_{(X,M)}(x\otimes_{\mathcal{B}}(a\#h)(b\#k),m)\\
&=x\otimes_{\mathcal{H}_{par}}(1_A\otimes_{A^e}(a\#h)(b\#k)\cdot m)\\
&=x\otimes_{\mathcal{H}_{par}}(1_A\otimes_{A^e}(a\#h)\cdot((b\#k)\cdot m))\\
&=\tilde{\psi}_{(X,M)}((x\otimes_{\mathcal{B}}a\#h),(b\#k)\cdot m).
\end{aligned}$$

Thus, the map

$$\begin{aligned}\psi_{(X,M)} : (X \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} M &\to X \otimes_{\mathcal{H}_{par}} (A \otimes_{A^e} M) \\ (x \otimes_{\mathcal{B}} a\#h) \otimes_{(A\#\mathcal{H})^e} m &\mapsto x \otimes_{\mathcal{H}_{par}} (1_A \otimes_{A^e} (a\#h) \cdot m)\end{aligned}$$

is well-defined. Observe that $\gamma_{(X,M)}$ and $\psi_{(X,M)}$ are mutual inverses since

$$\begin{aligned}\psi_{(X,M)}\big(\gamma_{(X,M)}(x \otimes_{\mathcal{H}_{par}} (a \otimes_{A^e} m))\big) &= \psi_{(X,M)}\big((x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot m\big) \\ &= x \otimes_{\mathcal{H}_{par}} (1_A \otimes_{A^e} a \cdot m) \\ &= x \otimes_{\mathcal{H}_{par}} (a \otimes_{A^e} m)\end{aligned}$$

and

$$\begin{aligned}\gamma_{(X,M)}\big(\psi_{(X,M)}((x \otimes_{\mathcal{B}} a\#h) \otimes_{(A\#\mathcal{H})^e} m)\big) &= \gamma_{(X,M)}\big(x \otimes_{\mathcal{H}_{par}} (1_A \otimes_{A^e} a\#h \cdot m)\big) \\ &= (x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a\#h \cdot m \\ &= (x \otimes_{\mathcal{B}} a\#h) \otimes_{(A\#\mathcal{H})^e} m.\end{aligned}$$

Let X' be another right $\mathcal{H}_{par}$-module and M' be a $A\#\mathcal{H}$-bimodule, $f : X \to X'$ a map of right $\mathcal{H}_{par}$-modules and $g : M \to M'$ a map of $A\#\mathcal{H}$-bimodules. Then, the following diagram commutes

$$\begin{array}{ccc} X \otimes_{\mathcal{H}_{par}} (A \otimes_{A^e} M) & \xrightarrow{\overline{(f,g)}} & X' \otimes_{\mathcal{H}_{par}} (A \otimes_{A^e} M') \\ \downarrow{\scriptstyle \gamma_{(X,M)}} & & \downarrow{\scriptstyle \gamma_{(X',M')}} \\ (X \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} M & \xrightarrow{\underline{(f,g)}} & (X' \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} M' \end{array}$$

where, $\overline{(f,g)} = f \otimes_{\mathcal{H}_{par}} (1_A \otimes_{A^e} g)$ and $\underline{(f,g)} = (f \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} g$. Indeed,

$$\begin{aligned}\gamma_{(X',M')}\left(\overline{(f,g)}(x \otimes_{\mathcal{H}_{par}} (a \otimes_{A^e} m))\right) &= \gamma_{(X,M)}\left(f(x) \otimes_{\mathcal{H}_{par}} (a \otimes_{A^e} g(m))\right) \\ &= (f(x) \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot g(m) \\ &= (f(x) \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} g(a \cdot m) \\ &= \underline{(f,g)}\left((x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \otimes_{(A\#\mathcal{H})^e} a \cdot m\right) \\ &= \underline{(f,g)}\left(\gamma_{(X,M)}(x \otimes_{\mathcal{H}_{par}} (a \otimes_{A^e} m))\right).\end{aligned}$$

Thus, γ is a natural isomorphism. □

Lemma 3.18. *The isomorphism of K-modules*

$$\begin{aligned}\phi : A\#\mathcal{H} &\to \mathcal{B} \otimes_{\mathcal{B}} A\#\mathcal{H} \\ a\#h &\mapsto 1_{\mathcal{B}} \otimes_{\mathcal{B}} a\#h\end{aligned}$$

is an isomorphism of $A\#\mathcal{H}$-bimodules.

Proof. For any $b\#k, a\#h, c\#t \in A\#\mathcal{H}$ we have that

$$\begin{aligned}
\phi((b\#k)(a\#h)(c\#t)) &= 1_{\mathcal{B}} \otimes_{\mathcal{B}} (b\#k)(a\#h)(c\#t) \\
&= \big(1_{\mathcal{B}} \otimes_{\mathcal{B}} (b\#k)(a\#h)\big) \cdot (c\#t) \\
&= 1_{\mathcal{B}} \otimes_{\mathcal{B}} ((k_{(1)} \cdot 1_A)b\#k_{(2)})(a\#h)) \cdot (c\#t) \\
(\text{by } (3.11)) &= 1_{\mathcal{B}} \otimes_{\mathcal{B}} e_{k_{(1)}} \triangleright ((b\#k_{(2)})(a\#h))\big) \cdot (c\#t) \\
&= 1_{\mathcal{B}} \triangleleft e_{k_{(1)}} \otimes_{\mathcal{B}} (b\#k_{(2)})(a\#h)\big) \cdot (c\#t) \\
&= e_{k_{(1)}} 1_{\mathcal{B}} e_{k_{(2)}} \otimes_{\mathcal{B}} (b\#k_{(3)})(a\#h)\big) \cdot (c\#t) \\
&= e_{k_{(1)}} \otimes_{\mathcal{B}} (b\#k_{(2)})(a\#h)\big) \cdot (c\#t) \\
&= \big([k_{(1)}]1_{\mathcal{B}}[S(k_{(2)})] \otimes_{\mathcal{B}} (b\#k_{(3)})(a\#h)\big) \cdot (c\#t) \\
&= \big(1_{\mathcal{B}} \triangleleft [S(k_{(1)})] \otimes_{\mathcal{B}} (b\#k_{(2)})(a\#h)\big) \cdot (c\#t) \\
&= (b\#k) \cdot \big(1_{\mathcal{B}} \otimes_{\mathcal{B}} (a\#h)\big) \cdot (c\#t) \\
&= (b\#k) \cdot \phi(a\#h) \cdot (c\#t). \quad \square
\end{aligned}$$

Corollary 3.19. *The functors F_2F_1 and F are naturally isomorphic.*

Proof. Using Proposition 3.17 for the particular case $X = \mathcal{B}$ we obtain that the functors

$$\mathcal{B} \otimes_{\mathcal{H}_{par}} (A \otimes_{A^e} -) : (A\#\mathcal{H})^e\text{-Mod} \to K\text{-Mod}$$

and

$$(\mathcal{B} \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} - : (A\#\mathcal{H})^e\text{-Mod} \to K\text{-Mod}$$

are naturally isomorphic. Clearly, $F_2F_1 = \mathcal{B} \otimes_{\mathcal{H}_{par}} (A \otimes_{A^e} -)$. On the other hand, by Lemma 3.18 we know that $\mathcal{B} \otimes_{\mathcal{B}} A\#\mathcal{H} \cong A\#\mathcal{H}$ as $A\#\mathcal{H}$-bimodules. Therefore, $F \cong (\mathcal{B} \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} -$, whence we get the desired conclusion. $\square$

In all what follows in this section, we assume that the cocommutative Hopf algebra $\mathcal{H}$ is projective over K.

Lemma 3.20. *$\mathcal{H}_{par}$ is projective as a left $\mathcal{B}$-module.*

Proof. It was proved in [7, Theorem 4.8] that the partial action (3.5) of $\mathcal{H}$ on $\mathcal{B}$ is such that $\hat{\pi} : \mathcal{H}_{par} \to \mathcal{B}\#\mathcal{H}$ is an isomorphism of algebras induced by the partial representation

$$\begin{aligned}
\pi : \mathcal{H} &\to \mathcal{B}\#\mathcal{H} \\
h &\mapsto 1_{\mathcal{B}}\#h.
\end{aligned}$$

Recall that the structure of the left $\mathcal{B}$-module $\mathcal{H}_{par}$ is just the induced by the natural inclusion $\mathcal{B} \hookrightarrow \mathcal{H}_{par}$. On the other hand, the structure of $B\#\mathcal{H}$ as a left $\mathcal{B}$-module is determined by the morphism of algebras $\phi_0 : \mathcal{B} \to \mathcal{B}\#\mathcal{H}$ such that $w \mapsto w\#1_{\mathcal{H}}$. Let $h \in \mathcal{H}$ and $x \in \mathcal{H}_{par}$. Then,

$$\begin{aligned}
\hat{\pi}(e_h x) &= \hat{\pi}(e_h)\hat{\pi}(x) \\
&= (1\#h_{(1)})(1\#S(h_{(2)}))\hat{\pi}(x) \\
&= \big(h_{(1)} \cdot 1_{\mathcal{B}}\#h_{(2)}S(h_{(3)})\big)\hat{\pi}(x) \\
&= \big(h \cdot 1_{\mathcal{B}}\#1_{\mathcal{H}}\big)\hat{\pi}(x) \\
&= \big(e_h\#1_{\mathcal{H}}\big)\hat{\pi}(x) \\
&= e_h \cdot \hat{\pi}(x).
\end{aligned}$$

Therefore, $\hat{\pi}$ is an isomorphism of left $\mathcal{B}$-modules. Therefore, it is enough to prove that $\mathcal{B}\#\mathcal{H}$ is projective as a left $\mathcal{B}$-module. Using the Lemma 3.11 for $X = R = \mathcal{B}$, $Y = \mathcal{H}$ and $S = K$ we obtain that $\mathcal{B} \otimes \mathcal{H}$ is projective as a left $\mathcal{B}$-module. By Lemma 2.6 we know that $\mathcal{B}\#\mathcal{H}$ is a direct summand of $\mathcal{B} \otimes \mathcal{H}$, whence $\mathcal{B}\#\mathcal{H}$ is projective as a left $\mathcal{B}$-module. □

By Lemma 3.20 and Proposition 3.10 we obtain the following proposition

Proposition 3.21. *Any projective left $\mathcal{H}_{par}$-module is projective as a left $\mathcal{B}$-module.*

Lemma 3.22. *Any projective right $\mathcal{H}_{par}$-module is a projective right $\mathcal{B}$-module. Thus, any projective resolution of $\mathcal{B}$ in **Mod**-$\mathcal{H}_{par}$ is a projective resolution in **Mod**-$\mathcal{B}$.*

Proof. Let X be a projective right $\mathcal{H}_{par}$-module. Then, by Lemma 3.6 we have that X is a projective left $\mathcal{H}_{par}$-module with action

$$z \triangleright x := x \triangleleft \mathcal{S}(z), \ \forall z \in \mathcal{H}_{par}.$$

Observe that by the cocommutativity of $\mathcal{H}$ we have that

$$\mathcal{S}(e_h) = \mathcal{S}([h_{(1)}][S(h_{(2)})]) = [S^2(h_{(2)})][S([h_{(1)}])] = [h_{(1)}][S([h_{(2)}])] = e_h,$$

and since $\mathcal{B}$ is commutative we conclude that $\mathcal{S}|_{\mathcal{B}} = id_{\mathcal{B}}$. Now by Proposition 3.21 we have that X is projective as a left $\mathcal{B}$-module with action

$$w \triangleright x := x \triangleleft \mathcal{S}(w) = x \triangleleft w.$$

Using again that $\mathcal{B}$ is commutative we have that any right $\mathcal{B}$-module is a left $\mathcal{B}$-module with the natural action. Thus, X is projective as right $\mathcal{B}$-module. □

Lemma 3.23. *Let M be a left $\mathcal{H}_{par}$-module, and let $P_\bullet \to \mathcal{B}$ be a projective resolution of left (right) $\mathcal{H}_{par}$-modules of $\mathcal{B}$. Then, $H_n(P_\bullet \otimes_{\mathcal{B}} M) = 0$ for all $n \geq 1$.*

Proof. Notice that by Proposition 3.21 or Lemma 3.22, for the left and right case respectively, we have that $P_\bullet \to \mathcal{B}$ is also a projective resolution of $\mathcal{B}$ in **Mod**-$\mathcal{B}$. Therefore,

$$H_n(P_\bullet \otimes_{\mathcal{B}} M) = \mathrm{Tor}_n^{\mathcal{B}}(\mathcal{B}, M) = \begin{cases} 0 & \text{if } n \geq 1 \\ M & \text{if } n = 0. \end{cases} \qquad \square$$

Proposition 3.24. *F_1 sends projective $(A\#\mathcal{H})$-bimodules to left F_2-acyclic modules.*

Proof. We have to see that

$$(L_n F_2)(A \otimes_{A^e} P) = 0, \ \forall n \geq 1,$$

for any projective $(A\#\mathcal{H})^e$-module P. Now observe that

$$\begin{aligned} (L_n F_2)(A \otimes_{A^e} P) &= L_n(\mathcal{B} \otimes_{\mathcal{H}_{par}} -)(A \otimes_{A^e} P) \\ &\cong L_n(- \otimes_{\mathcal{H}_{par}} (A \otimes_{A^e} P))(\mathcal{B}) \\ \text{(by Proposition 3.17)} &\cong L_n((- \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} P)(\mathcal{B}). \end{aligned}$$

Let $Q_\bullet \to \mathcal{B}$ a projective resolution of $\mathcal{B}$ in **Mod**-$\mathcal{H}_{par}$. Then,

$$(L_n F_2)(A \otimes_{A^e} P) \cong L_n((- \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} P)(\mathcal{B}) = H_n\left((Q_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} P\right).$$

Observe that if the complex $(Q_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H})$ is exact for all $n \geq 1$ then the complex $(Q_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} P$ is exact for all $n \geq 1$ since P is projective as $(A\#\mathcal{H})^e$-module, and so

$$H_n\left((Q_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H}) \otimes_{(A\#\mathcal{H})^e} P\right) = 0, \forall n \geq 1,$$

which is exactly what we want. Therefore, it is enough to show that $(Q_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H})$ is exact for all $n \geq 1$. Recall that $(Q_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H})$ is exact in n if, and only if, $H_n(Q_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H}) = 0$. But the latter equality holds by Lemma 3.23. $\square$

Theorem 3.25. *Let $\mathcal{H}$ be a cocommutative Hopf K-algebra, such that $\mathcal{H}$ is projective as a K-module. If $\cdot : \mathcal{H} \otimes A \to A$ is a symmetric partial action of $\mathcal{H}$ on a unital algebra A, then for any $(A\#\mathcal{H})^e$-module M there exists a first quadrant homological spectral sequence E^r such that*

$$E^2_{p,q} = \mathrm{Tor}_p^{\mathcal{H}_{par}}(\mathcal{B}, H_q(A, M)) \Rightarrow H_{p+q}(A\#\mathcal{H}, M).$$

Proof. Keeping in mind Remark 3.13 we know that

$$L_qF_1(-) = H_q(A, -),\ L_pF_2(-) = \mathrm{Tor}_p^{\mathcal{H}_{par}}(\mathcal{B}, -) \text{ and } L_{p+q}F(-) = H_{p+q}(A\#\mathcal{H}, -).$$

We also have that F_1 and F_2 are right exact functors, by Proposition 3.24 the functor F_1 sends projective $(A\#\mathcal{H})$-bimodules to F_2-acyclic modules and by Corollary 3.19 we have that $F_2F_1 \cong F$. Thus, by [38, Theorem 10.48] we obtain the desired spectral sequence. □

Example 3.26. If A is a separable algebra, then

$$H_q(A, M) = \begin{cases} 0 & \text{if } q \geq 1 \\ M/[A, M] & \text{if } q = 0. \end{cases}$$

Therefore, the spectral sequence in Theorem 3.25 collapses on the p-axis, and thus we obtain the following isomorphism:

$$H_n(A\#\mathcal{H}, M) \cong \mathrm{Tor}_n^{\mathcal{H}_{par}}(\mathcal{B}, M/[A, M]).$$

Example 3.27. Let G be a group. If $\mathcal{H} = KG$ then the spectral sequence of Theorem 4.6 takes the form

$$E^2_{p,q} = H_p^{par}(G, H_q(A, M)) \Rightarrow H_{p+q}(A \rtimes G, M),$$

where $H_\bullet^{par}(G, -) := \mathrm{Tor}_\bullet^{K_{par}G}(B, -)$.

Example 3.28. From [7] we know that there exists a partial action of $\mathcal{H}$ on $\mathcal{B}$ such that $\mathcal{H}_{par} \cong \mathcal{B}\#\mathcal{H}$. Therefore,

$$E^2_{p,q} = \mathrm{Tor}_p^{\mathcal{H}_{par}}(\mathcal{B}, H_q(\mathcal{B}, M)) \Rightarrow H_{p+q}(\mathcal{H}_{par}, M).$$

In particular if $\mathcal{H} = KG$ then the spectral sequence takes the form

$$E^2_{p,q} = H_p^{par}(G, H_q(\mathcal{B}, M)) \Rightarrow H_{p+q}(K_{par}G, M).$$

Observe that $\mathcal{B}^e$ is generated as a K-algebra by the set of idempotent $\{e_g \otimes e_h : g, h \in G\}$. Therefore, $\mathcal{B}^e$ is a Von Neumann regular algebra, and consequently, $\mathcal{B}$ is flat as a $\mathcal{B}^e$-module. Thus, the above spectral sequence collapses on the p-axis, and we obtain the following isomorphism

$$H_n^{par}(G, M/[\mathcal{B}, M]) \cong H_n(K_{par}G, M).$$

The above isomorphism generalizes the Mac Lane isomorphism (see for example [35, 7.4.2]) in the sense that if M is a G-group. Then,

$$H_n(KG, M) = H_\bullet(K_{par}G, M) \cong H^{par}_\bullet(G, M) = H_\bullet(G, M),$$

as it is shown in [26].

4. Cohomology of the partial smash product

Now we proceed to show the existence of a dual cohomological spectral sequence for the Hochschild cohomology. Recall that $\mathcal{H}$ is a cocommutative Hopf K-algebra, A a unital K-algebra and

$$\begin{aligned} \cdot : \mathcal{H} \otimes A &\to A \\ h \otimes a &\mapsto h \cdot a \end{aligned}$$

a symmetric partial action of $\mathcal{H}$ on A.

Proposition 4.1. *Let M be an $A\#\mathcal{H}$-bimodule. Then,* $\mathrm{Hom}_{A^e}(A, M)$ *is an $\mathcal{H}_{par}$-module, with the action determined by*

$$([h] \triangleright f)(a) = [h_{(1)}] \triangleright f([S(h_{(2)})] \triangleright a). \tag{4.1}$$

Proof. For $h \in \mathcal{H}$ define $\pi_h : \mathrm{Hom}_{A^e}(A, M) \to \mathrm{Hom}_{A^e}(A, M)$, by $\pi_h(f)(a) := [h_{(1)}] \triangleright f([S(h_{(2)})] \triangleright a)$, for all $f \in \mathrm{Hom}_{A^e}(A, M)$ and $a \in A$. Then,

$$\begin{aligned} \pi : \mathcal{H} &\to \mathrm{End}_K(\mathrm{Hom}_{A^e}(A, M)) \\ h &\mapsto \pi_h \end{aligned}$$

is a partial representation. Indeed, for any $h, t \in \mathcal{H}$ we have

$$\begin{aligned} \left(\pi_t \pi_{h_{(1)}} \pi_{S(h_{(2)})}(f)\right)(a) &= [t_{(1)}] \triangleright \left(\left(\pi_{h_{(1)}} \pi_{S(h_{(2)})}(f)\right)([S(t_{(2)})] \triangleright a)\right) \\ &= [t_{(1)}][h_{(1)}] \triangleright \left(\left(\pi_{S(h_{(3)})}(f)\right)([S(h_{(2)})][S(t_{(2)})] \triangleright a)\right) \\ &= [t_{(1)}][h_{(1)}][S(h_{(3)})] \triangleright \left(f([h_{(4)}][S(h_{(2)})][S(t_{(2)})] \triangleright a)\right) \\ &= [t_{(1)}][h_{(1)}][S(h_{(2)})] \triangleright \left(f([h_{(3)}][S(h_{(4)})][S(t_{(2)})] \triangleright a)\right) \\ &= [t_{(1)}h_{(1)}][S(h_{(2)})] \triangleright \left(f([h_{(3)}][S(t_{(2)}h_{(4)})] \triangleright a)\right) \\ &= [t_{(1)}h_{(1)}] \triangleright \left(\pi_{S(h_{(2)})}(f)([S(t_{(2)}h_{(3)})] \triangleright a)\right) \\ &= [t_{(1)}h_{(1)}] \triangleright \left(\pi_{S(h_{(3)})}(f)([S(t_{(2)}h_{(2)})] \triangleright a)\right) \\ &= \pi_{th_{(1)}} \pi_{S(h_{(2)})}(f)(a). \end{aligned}$$

Analogously, we have that

$$\pi_{S(h_{(1)})}\pi_{h_{(2)}}\pi_t(f)(a) = \pi_{h_{(1)}}\pi_{S(h_{(2)})t}(f)(a).$$

It is clear that $\pi_{1_{\mathcal{H}}}(f) = f$. Then, π is a partial representation and thus $\operatorname{Hom}_{A^e}(A, M)$ is a $\mathcal{H}_{par}$-module. □

Now we can define the following functors, by Proposition 4.1

$$G_1 := \operatorname{Hom}_{A^e}(A, -) : (A\#\mathcal{H})^e\textbf{-Mod} \to \mathcal{H}_{par}\textbf{-Mod}. \tag{4.2}$$

Recall that $\mathcal{B}$ is a left $\mathcal{H}_{par}$-module with the action given by (3.5), thus we can define

$$G_2 := \operatorname{Hom}_{\mathcal{H}_{par}}(\mathcal{B}, -) : \mathcal{H}_{par}\textbf{-Mod} \to K\textbf{-Mod}. \tag{4.3}$$

The functor used to compute the Hochschild cohomology of $A\#\mathcal{H}$ with coefficients in M is

$$G := \operatorname{Hom}_{(A\#\mathcal{H})^e}(A\#\mathcal{H}, -) : (A\#\mathcal{H})^e\textbf{-Mod} \to K\textbf{-Mod}. \tag{4.4}$$

Recall that by Lemma 3.6 any left $\mathcal{H}_{par}$-module X is a right $\mathcal{H}_{par}$-module. If X is a left $\mathcal{H}_{par}$-module, then by Proposition 3.16 we have that $X \otimes_{\mathcal{B}} A\#\mathcal{H}$ is a $A\#\mathcal{H}$-bimodule with actions:

$$a\#h \cdot (x \otimes_{\mathcal{B}} c\#t) := x \cdot [S(h_{(1)})] \otimes_{\mathcal{B}} (a\#h_{(2)})(c\#t) = [h_{(1)}] \cdot x \otimes_{\mathcal{B}} (a\#h_{(2)})(c\#t) \tag{4.5}$$

and

$$(x \otimes_{\mathcal{B}} c\#t) \cdot a\#h := x \otimes_{\mathcal{B}} (c\#t)(a\#h). \tag{4.6}$$

For the cohomological setting we have the dual version of Remark 3.13.

Remark 4.2. Recall that the morphism of rings $A^e \to (A\#\mathcal{H})^e$ induced by the natural inclusion of A into $(A\#\mathcal{H})$ determines the structure of A^e-module of $(A\#\mathcal{H})^e$, and by Lemma 3.12, if $\mathcal{H}$ is projective over K, then $(A\#\mathcal{H})^e$ is projective as A^e-module. Thus, by [34, Corollary 3.6A] we conclude that any injective $(A\#\mathcal{H})^e$-module is an injective A^e-module. Consequently,

$$H^\bullet(A, M) \cong R^\bullet G_1(M),$$

for any $(A\#\mathcal{H})^e$-module M.

Proposition 4.3. *Let M be a fixed $A\#\mathcal{H}$-bimodule. Then, the functors*

$$\operatorname{Hom}_{\mathcal{H}_{par}}(-, \operatorname{Hom}_{A^e}(A, M)) : \mathcal{H}_{par}\textbf{-}\textit{\textbf{Mod}} \to K\textbf{-}\textit{\textbf{Mod}}$$

and

$$\mathrm{Hom}_{(A\#\mathcal{H})^e}(-\otimes_{\mathcal{B}} A\#\mathcal{H}, M) : \mathcal{H}_{par}\text{-}\boldsymbol{Mod} \to K\text{-}\boldsymbol{Mod}$$

are naturally isomorphic. On the other hand, if X is a fixed left $\mathcal{H}_{par}$-module, then the functors

$$\mathrm{Hom}_{\mathcal{H}_{par}}(X, \mathrm{Hom}_{A^e}(A, -)) : (A\#\mathcal{H})^e\text{-}\boldsymbol{Mod} \to K\text{-}\boldsymbol{Mod}$$

and

$$\mathrm{Hom}_{(A\#\mathcal{H})^e}(X \otimes_{\mathcal{B}} A\#\mathcal{H}, -) : (A\#\mathcal{H})^e\text{-}\boldsymbol{Mod} \to K\text{-}\boldsymbol{Mod}$$

are naturally isomorphic.

Proof. Let X be a left $\mathcal{H}_{par}$-module and M be an $(A\#\mathcal{H})$-bimodule. Define

$$\begin{aligned}\gamma_{(X,M)} : \mathrm{Hom}_{\mathcal{H}_{par}}(X, \mathrm{Hom}_{A^e}(A, M)) &\to \mathrm{Hom}_{(A\#\mathcal{H})^e}(X \otimes_{\mathcal{B}} A\#\mathcal{H}, M)\\ f &\mapsto \gamma_{(X,M)}(f),\end{aligned}$$

such that

$$\gamma_{(X,M)}(f)(x \otimes_{\mathcal{B}} (a\#h)) = f_x(1_A) \cdot (a\#h),$$

where $f \in \mathrm{Hom}_{\mathcal{H}_{par}}(X, \mathrm{Hom}_{A^e}(A, M))$ and $f_x := f(x)$. First, we have to verify that $\gamma_{(X,M)}(f)$ is well-defined. Indeed, notice that

$$\begin{aligned}&\gamma_{(X,M)}(f)(x \cdot e_h \otimes_{\mathcal{B}} (a\#h)) \\ &= f_{x\cdot e_h}(1_A) \cdot (a\#h) \\ &= f_{e_h\cdot x}(1_A) \cdot (a\#h) \\ &= (e_h \triangleright f_x)(1_A) \cdot (a\#h) \\ &= ([h_{(1)}][S(h_{(2)})] \triangleright f_x)(1_A) \cdot (a\#h) \\ &= \big([h_{(1)}] \triangleright \big([s(h_{(2)})] \triangleright f_x\big)\big)(1_a) \cdot (a\#h),\end{aligned} \tag{4.7}$$

by Equation (4.1) we have that

$$\begin{aligned}&\big([h_{(1)}]\triangleright\big([s(h_{(2)})] \triangleright f_x\big)\big)(1_a) \cdot (a\#h) \\ &= [h_{(1)}] \triangleright \Big(\big([S(h_{(3)})] \triangleright f_x\big)([S(h_{(2)})] \triangleright 1_A)\Big) \cdot (a\#h) \\ &= [h_{(1)}][S(h_{(2)})] \triangleright f_x\big([h_{(3)}][S(h_{(4)})] \triangleright 1_A\big) \cdot (a\#h) \\ (\flat) &= \big(e_{h_{(1)}} \triangleright f_x(h_{(2)} \cdot 1_A)\big) \cdot (a\#h),\end{aligned} \tag{4.8}$$

where (♭) holds by item (*ii*) of Lemma 3.14 (*ii*). Moreover, observe that:

$$
\begin{aligned}
&\Big(e_{h_{(1)}}\triangleright f_x(h_{(2)}\cdot 1_A)\Big)\cdot(a\#h) && (4.9)\\
&\quad=\big((1_A\#h_{(1)})(1_A\#S(h_{(2)}))\cdot f_x(h_{(3)}\cdot 1_A)\cdot(1_A\#h_{(4)})(1_A\#S(h_{(5)}))\big)\cdot(a\#h)\\
&\quad=\big((h_{(1)}\cdot 1_A\#h_{(2)}S(h_{(3)}))\cdot f_x(h_{(4)}\cdot 1_A)\cdot(h_{(5)}\cdot 1_A\#h_{(6)}S(h_{(7)}))\big)\cdot(a\#h)\\
&\quad=\big((h_{(1)}\cdot 1_A\#1_{\mathcal{H}})\cdot f_x(h_{(2)}\cdot 1_A)\cdot(h_{(3)}\cdot 1_A\#1_{\mathcal{H}})\big)\cdot(a\#h)\\
&\quad=f_x\left((h_{(1)}\cdot 1_A)(h_{(2)}\cdot 1_A)(h_{(3)}\cdot 1_A)\right)\cdot(a\#h)\\
&\quad=f_x(h\cdot 1_A)\cdot(a\#h)\\
&\quad=f_x(1_A)\cdot(h\cdot 1_A\#1_{\mathcal{H}})(a\#h)\\
&\quad=f_x(1_A)\cdot((h\cdot 1_A)a\#h)\\
(\flat\flat)\ &\quad=f_x(1_A)\cdot(e_h\triangleright a\#h)\\
&\quad=\gamma_{(X,M)}(f)(x\otimes_{\mathcal{B}}e_h\triangleright(a\#h)),
\end{aligned}
$$

where the equality (♭♭) holds by (*ii*) of Lemma 3.14. Thus, by Equations (4.7), (4.8) and (4.9) we get

$$\gamma_{(X,M)}(f)(x\cdot e_h\otimes_{\mathcal{B}}(a\#h))=\gamma_{(X,M)}(f)(x\otimes_{\mathcal{B}}e_h\triangleright(a\#h)).$$

Now we have to verify that $\gamma_{(X,M)}(f)$ is a morphism of $(A\#\mathcal{H})$-bimodules. It is clear that $\gamma_{(X,M)}(f)$ is a morphism of right $A\#\mathcal{H}$-modules, so we only have to see that it is a morphism of left $A\#\mathcal{H}$-modules. Observe that

$$
\begin{aligned}
\gamma_{(X,M)}(f)(b\#t\cdot(x\otimes_{\mathcal{B}}(a\#h)))&=\gamma_{(X,M)}(f)([t_{(1)}]\cdot x\otimes_{\mathcal{B}}(b\#t_{(2)})(a\#h))\\
&=f_{[t_{(1)}]\cdot x}(1_A)\cdot(b\#t_{(2)})(a\#h)\\
&=([t_{(1)}]\triangleright f_x)(1_A)\cdot(b\#t_{(2)})(a\#h)\\
&=\big([t_{(1)}]\triangleright f_x([S(t_{(2)})]\triangleright 1_A)\big)\cdot(b\#t_{(3)})(a\#h)\\
&=(1_A\#t_{(1)})\cdot f_x(S(t_{(3)})\cdot 1_A)\cdot(1_A\#S(t_{(2)}))(b\#t_{(4)})(a\#h)\\
&=(1_A\#t_{(1)})\cdot f_x(S(t_{(5)})\cdot 1_A)\cdot(S(t_{(2)})\cdot b\#S(t_{(3)})t_{(4)})(a\#h)\\
&=(1_A\#t_{(1)})\cdot f_x(S(t_{(4)})\cdot 1_A)\cdot(S(t_{(2)})\cdot b\#\varepsilon(t_{(3)})1_{\mathcal{H}})(a\#h)\\
&=(1_A\#t_{(1)})\cdot f_x(S(t_{(3)})\cdot 1_A)\cdot(S(t_{(2)})\cdot b\#1_{\mathcal{H}})(a\#h)\\
&=(1_A\#t_{(1)})\cdot f_x\big((S(t_{(3)})\cdot 1_A)(S(t_{(2)})\cdot b)\big)\cdot(a\#h)\\
&=(1_A\#t_{(1)})\cdot f_x\big((S(t_{(2)})\cdot b)\big)\cdot(a\#h)\\
&=(1_A\#t_{(1)})((S(t_{(2)})\cdot b)\#1_{\mathcal{H}})\cdot f_x(1_A)\cdot(a\#h)\\
\text{(by Lemma 2.7)}\ &=(b\#t)\cdot f_x(1_A)\cdot(a\#h)\\
&=(b\#t)\cdot\gamma_{(X,M)}(f)(x\otimes_{\mathcal{B}}(a\#h)).
\end{aligned}
$$

To show that $\gamma_{(X,M)}$ is an isomorphism of K-modules for any $X \in \mathcal{H}_{par}$-**Mod** and $M \in (A\#\mathcal{H})^e$-**Mod** we will define its inverse map

$$\Lambda_{(X,M)} : \mathrm{Hom}_{(A\#\mathcal{H})^e}(X \otimes_{\mathcal{B}} A\#\mathcal{H}, M) \to \mathrm{Hom}_{\mathcal{H}_{par}}(X, \mathrm{Hom}_{A^e}(A, M))$$

such that

$$\big(\Lambda_{(X,M)}(f)\big)_x(a) := f(x \otimes_{\mathcal{B}} a\#1_{\mathcal{H}}),$$

for all $f \in \mathrm{Hom}_{(A\#\mathcal{H})^e}(X \otimes_{\mathcal{B}} A\#\mathcal{H}, M)$. Observe that $\big(\Lambda_{(X,M)}(f)\big)_x$ is a morphism of A^e-modules. Indeed,

$$\begin{aligned}
\big(\Lambda_{(X,M)}(f)\big)_x(bac) &= f(x \otimes_{\mathcal{B}} bac\#1_{\mathcal{H}}) \\
&= f(x \otimes_{\mathcal{B}} (b\#1_{\mathcal{H}})(a\#1_{\mathcal{H}})(c\#1_{\mathcal{H}})) \\
&= f((b\#1_{\mathcal{H}}) \cdot (x \otimes_{\mathcal{B}} (a\#1_{\mathcal{H}})) \cdot (c\#1_{\mathcal{H}})) \\
&= (b\#1_{\mathcal{H}}) \cdot f((x \otimes_{\mathcal{B}} (a\#1_{\mathcal{H}}))) \cdot (c\#1_{\mathcal{H}}) \\
&= b \cdot \big(\Lambda_{(X,M)}(f)\big)_x(a) \cdot c.
\end{aligned}$$

Now observe that $\Lambda_{(X,M)}(f)$ is a morphism of $\mathcal{H}_{par}$-modules.

$$\begin{aligned}
\big(\Lambda_{(X,M)}(f)\big)_{[h]\cdot x}(a) &= f([h] \cdot x \otimes_{\mathcal{B}} a\#1_{\mathcal{H}}) \\
&= f(e_{h_{(1)}}[h_{(2)}] \cdot x \otimes_{\mathcal{B}} a\#1_{\mathcal{H}}) \\
&= f(e_{h_{(1)}} \cdot ([h_{(2)}] \cdot x) \otimes_{\mathcal{B}} a\#1_{\mathcal{H}}) \\
&= f(([h_{(2)}] \cdot x) \cdot e_{h_{(1)}} \otimes_{\mathcal{B}} a\#1_{\mathcal{H}}) \\
&= f([h_{(2)}] \cdot x \otimes_{\mathcal{B}} e_{h_{(1)}} \triangleright (a\#1_{\mathcal{H}})) \\
&= f([h_{(3)}] \cdot x \otimes_{\mathcal{B}} (1_A\#h_{(1)})(1_A\#S(h_{(2)}))(a\#1_{\mathcal{H}})) \\
&= f\left((1_A\#h_{(1)}) \cdot (x \otimes_{\mathcal{B}} (1_A\#S(h_{(2)}))(a\#1_{\mathcal{H}}))\right) \\
&= f\left((1_A\#h_{(1)}) \cdot (x \otimes_{\mathcal{B}} (S(h_{(2)}) \cdot a\#S(h_{(3)})))\right) \\
&= f\left((1_A\#h_{(1)}) \cdot (x \otimes_{\mathcal{B}} (S(h_{(2)}) \cdot a\#1_{\mathcal{H}})) \cdot (1_A\#S(h_{(3)}))\right) \\
&= (1_A\#h_{(1)}) \cdot f\left((x \otimes_{\mathcal{B}} (S(h_{(3)}) \cdot a\#1_{\mathcal{H}}))\right) \cdot (1_A\#S(h_{(2)})) \\
&= [h_{(1)}] \triangleright \big(f(x \otimes_{\mathcal{B}} ([S(h_{(2)})] \triangleright a)\#1_{\mathcal{H}})\big) \\
\text{(by (4.1))} &= [h_{(1)}] \triangleright \big(\Lambda_{(X,M)}(f)\big)_x([S(h_{(2)})] \triangleright a) \\
&= ([h] \triangleright \big(\Lambda_{(X,M)}(f)\big)_x)(a).
\end{aligned}$$

Thus, $\Lambda_{(X,M)}$ is well-defined. Finally, by direct computations we obtain that $\gamma_{(X,M)}$ and $\Lambda_{(X,M)}$ are mutually inverses:

$$
\begin{aligned}
(\Lambda_{(X,M)}(\gamma_{(X,M)}(f)))_x(a) &= \gamma_{(X,M)}(f)(x \otimes_{\mathcal{B}} a\#1_{\mathcal{H}}) \\
&= f_x(1_A) \cdot (a\#1_{\mathcal{H}}) \\
&= f_x(a)
\end{aligned}
$$

and

$$
\begin{aligned}
(\gamma_{(X,M)}(\Lambda_{(X,M)}(f)))(x \otimes_{\mathcal{B}} (a\#h)) &= \big(\Lambda_{(X,M)}(f)\big)_x(1_A) \cdot a\#h \\
&= f(x \otimes_{\mathcal{B}} 1_A\#1_{\mathcal{H}}) \cdot a\#h \\
&= f(x \otimes_{\mathcal{B}} a\#h).
\end{aligned}
$$

For a fixed M define $\gamma_M(X) = \gamma_{(X,M)}$. Then,

$$
\gamma_M : \mathrm{Hom}_{\mathcal{H}_{par}}(-, \mathrm{Hom}_{A^e}(A, M)) \to \mathrm{Hom}_{(A\#\mathcal{H})^e}(- \otimes_{\mathcal{B}} A\#\mathcal{H}, M)
$$

is a natural transformation. Indeed, let $\xi : X \to X'$ be a map of left $\mathcal{H}_{par}$-modules. Then, we want to verify that the following diagram commutes

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{H}_{par}}(X, \mathrm{Hom}_{A^e}(A, M)) & \xleftarrow{\xi^*} & \mathrm{Hom}_{\mathcal{H}_{par}}(X', \mathrm{Hom}_{A^e}(A, M)) \\
\gamma_{(X,M)} \downarrow & & \gamma_{(X',M)} \downarrow \\
\mathrm{Hom}_{(A\#\mathcal{H})^e}(X \otimes_{\mathcal{B}} A\#\mathcal{H}, M) & \xleftarrow{\xi_*} & \mathrm{Hom}_{(A\#\mathcal{H})^e}(X' \otimes_{\mathcal{B}} A\#\mathcal{H}, M)
\end{array}
$$

where

$$
\xi^* := \mathrm{Hom}_{\mathcal{H}_{par}}(\xi, \mathrm{Hom}_{A^e}(A, M)),
$$

and

$$
\xi_* := \mathrm{Hom}_{(A\#\mathcal{H})^e}(\xi \otimes_{\mathcal{B}} A\#\mathcal{H}, M).
$$

Indeed,

$$
\begin{aligned}
(\gamma_{(X,M)}(\xi^*(f)))(x \otimes_{\mathcal{B}} a\#h) &= (\xi^*(f))_x(1_A) \cdot a\#h \\
&= f_{\xi(x)}(1_A) \cdot a\#h
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_*(\gamma_{(X',M)}(f))(x \otimes_{\mathcal{B}} a\#h) &= \gamma_{(X',M)}(f)(\xi(x) \otimes_{\mathcal{B}} a\#h) \\
&= f_{\xi(x)}(1_A) \cdot a\#h.
\end{aligned}
$$

Thus, the above diagram commutes. Finally, for a fixed $\mathcal{H}_{par}$-module X we define

$$\gamma_X : \mathrm{Hom}_{\mathcal{H}_{par}}(X, \mathrm{Hom}_{A^e}(A, -)) \to \mathrm{Hom}_{(A\#\mathcal{H})^e}(X \otimes_{\mathcal{B}} A\#\mathcal{H}, -),$$

such that $\gamma_X(M) := \gamma_{(M,X)}$. Analogously we have that γ_X is a natural transformation. Indeed, we have to see that for any $\zeta : M \to M'$ morphism of $A\#\mathcal{H}$-bimodules the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{H}_{par}}(X, \mathrm{Hom}_{A^e}(A, M)) & \xrightarrow{\zeta^*} & \mathrm{Hom}_{\mathcal{H}_{par}}(X, \mathrm{Hom}_{A^e}(A, M')) \\ {\scriptstyle \gamma_{(X,M)}} \downarrow & & {\scriptstyle \gamma_{(X,M')}} \downarrow \\ \mathrm{Hom}_{(A\#\mathcal{H})^e}(X \otimes_{\mathcal{B}} A\#\mathcal{H}, M) & \xrightarrow{\zeta_*} & \mathrm{Hom}_{(A\#\mathcal{H})^e}(X \otimes_{\mathcal{B}} A\#\mathcal{H}, M') \end{array}$$

where

$$\zeta^* := \mathrm{Hom}_{\mathcal{H}_{par}}(X, \mathrm{Hom}_{A^e}(A, \zeta)),$$

and

$$\zeta_* := \mathrm{Hom}_{(A\#\mathcal{H})^e}(X \otimes_{\mathcal{B}} A\#\mathcal{H}, \zeta).$$

Observe that

$$\begin{aligned} (\gamma_{(X,M')}(\zeta^*(f)))(x \otimes_{\mathcal{B}} a\#h) &= (\zeta^*(f))_x(1_A) \cdot (a\#h) \\ &= \zeta(f_x(1_A)) \cdot (a\#h) \end{aligned}$$

and

$$\begin{aligned} \zeta_*(\gamma_{(X,M)}(f))(x \otimes_{\mathcal{B}} a\#h) &= \zeta(\gamma_{(X,M)}(f)(x \otimes_{\mathcal{B}} a\#h)) \\ &= \zeta\left(f_x(1_A) \cdot (a\#h)\right) \\ &= \zeta\left(f_x(1_A)\right) \cdot (a\#h). \end{aligned}$$

Thus, the above diagram commutes. □

Corollary 4.4. *The functors G_2G_1 and G are naturally isomorphic.*

Proof. By Proposition 4.3, taking $X = \mathcal{B}$ we have that the functors

$$\mathrm{Hom}_{\mathcal{H}_{par}}(\mathcal{B}, \mathrm{Hom}_{A^e}(A, -)) : (A\#\mathcal{H})^e\text{-}\mathbf{Mod} \to K\text{-}\mathbf{Mod}$$

and

$$\mathrm{Hom}_{(A\#\mathcal{H})^e}(\mathcal{B} \otimes_{\mathcal{B}} A\#\mathcal{H}, -) : (A\#\mathcal{H})^e\text{-}\mathbf{Mod} \to K\text{-}\mathbf{Mod}$$

are naturally isomorphic. It is clear that $G_2G_1 = \mathrm{Hom}_{\mathcal{H}_{par}}(\mathcal{B}, \mathrm{Hom}_{A^e}(A, -))$. On the other hand by Lemma 3.18 we have that $\mathcal{B} \otimes_{\mathcal{B}} A\#\mathcal{H} \cong A\#\mathcal{H}$ as $A\#\mathcal{H}$-bimodules. Thus, $G \cong \mathrm{Hom}_{(A\#\mathcal{H})^e}(\mathcal{B} \otimes_{\mathcal{B}} A\#\mathcal{H}, -)$. □

In all what follows, we assume that $\mathcal{H}$ is projective over K.

Proposition 4.5. *G_1 sends injective $(A\#\mathcal{H})$-bimodules to right G_2-acyclic modules.*

Proof. We have to see that

$$(R^nG_2)(\mathrm{Hom}_{A^e}(A, Q)) = 0\, \forall n \geq 1,$$

for any injective $(A\#\mathcal{H})^e$-module Q. Now observe that

$$\begin{aligned}(R^nG_2)(\mathrm{Hom}_{A^e}(A, Q)) &= R^n(\mathrm{Hom}_{\mathcal{H}_{par}}(\mathcal{B}, -))(\mathrm{Hom}_{A^e}(A, Q)) \\ &\cong R^n(\mathrm{Hom}_{\mathcal{H}_{par}}(-, \mathrm{Hom}_{A^e}(A, Q)))(\mathcal{B}) \\ (\text{by Proposition 4.3}) &\cong R^n(\mathrm{Hom}_{(A\#\mathcal{H})^e}(- \otimes_{\mathcal{B}} A\#\mathcal{H}, Q))(\mathcal{B}).\end{aligned}$$

Let $P_\bullet \to \mathcal{B}$ a projective resolution of $\mathcal{B}$ in $\mathcal{H}_{par}$-**Mod**. Then,

$$(R^nG_2)(\mathrm{Hom}_{A^e}(A, Q)) \cong H^n\left(\mathrm{Hom}_{(A\#\mathcal{H})^e}(P_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H}, Q)\right).$$

Observe that if the complex $P_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H}$ is exact for all $n \geq 1$ then the chain complex $\mathrm{Hom}_{(A\#\mathcal{H})^e}(P_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H}, Q)$ is exact for all $n \geq 1$ since Q is injective as $(A\#\mathcal{H})^e$-module, and so

$$H^n\left(\mathrm{Hom}_{(A\#\mathcal{H})^e}(P_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H}, Q)\right) = 0, \forall n \geq 1,$$

which is exactly what we want. Therefore, it is enough to show that $P_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H}$ is exact for all $n \geq 1$, but this follows from Lemma 3.23 since $P_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H}$ is exact in n if, and only if, $H_n(P_\bullet \otimes_{\mathcal{B}} A\#\mathcal{H}) = 0$. □

Theorem 4.6. *Let $\mathcal{H}$ be a cocommutative Hopf K-algebra, which is projective as a K-module, and $\cdot : \mathcal{H} \otimes A \to A$ be a symmetric partial action of $\mathcal{H}$ on A. Then, for any $(A\#\mathcal{H})^e$-module M there exists a cohomological spectral sequence E_r such that*

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{H}_{par}}(\mathcal{B}, H^q(A, M)) \Rightarrow H^{p+q}(A\#\mathcal{H}, M).$$

Proof. Recalling Remark 4.2 we know that

$$R^qG_1(-) = H^q(A, -),\ R^pG_2(-) = \mathrm{Ext}^p_{\mathcal{H}_{par}}(\mathcal{B}, -) \text{ and } R^{p+q}G(-) = H^{p+q}(A\#\mathcal{H}, -).$$

We also have that G_1 and G_2 are left exact functors, by Proposition 4.5 the functor G_1 sends injective $(A\#\mathcal{H})$-bimodules to right G_2-acyclic modules and by Corollary 4.4

we have that $G_2G_1 \cong G$. Thus, by [38, Theorem 10.47] we obtain the desired spectral sequence. □

5. Hopf (co)homology based on partial representations

In analogy with the case of groups (see [1] and [3]) we give the following definition:

Definition 5.1. Let $\mathcal{H}$ be a cocommutative Hopf algebra. For a left $\mathcal{H}_{par}$-module M, we define the partial Hopf homology of $\mathcal{H}$ with coefficients in M by

$$H_{\bullet}^{par}(\mathcal{H}, M) := \operatorname{Tor}_{\bullet}^{\mathcal{H}_{par}}(\mathcal{B}, M). \tag{5.1}$$

Analogously, we define the partial Hopf cohomology of $\mathcal{H}$ with coefficients in M by

$$H_{par}^{\bullet}(\mathcal{H}, M) := \operatorname{Ext}_{\mathcal{H}_{par}}^{\bullet}(\mathcal{B}, M). \tag{5.2}$$

Observe that as in the case of groups, the above-defined cohomology differs from the cohomology based on partial actions introduced in [15].

In view of Definition 5.1 we can reformulate Theorem 3.25 and Theorem 4.6 as follows:

Theorem 5.2. *Let $\cdot : \mathcal{H} \otimes A \to A$ be a symmetric partial action of a cocommutative Hopf algebra on A, such that $\mathcal{H}$ is projective over K. Then for any $(A\#\mathcal{H})^e$-module M there exists a first quadrant homological spectral sequence E^r such that*

$$E^2_{p,q} = H_p^{par}(\mathcal{H}, H_q(A, M)) \Rightarrow H_{p+q}(A\#\mathcal{H}, M),$$

and a third quadrant cohomological spectral sequence E_r such that

$$E_2^{p,q} = H_{par}^p(\mathcal{H}, H^q(A, M)) \Rightarrow H^{p+q}(A\#\mathcal{H}, M).$$

We proceed by showing that the above-defined (co)homology is a generalization of the usual (co)homology for $\mathcal{H}$-modules.

Remark 5.3. It is easy to see that the identity map $id_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ is a partial representation. Thus, by Theorem 2.9 we have a surjective homomorphism of algebras $\mathcal{H}_{par} \to \mathcal{H}$ such that $[h] \mapsto h$. Consequently, any left (right) $\mathcal{H}$-module will be a left (right) $\mathcal{H}_{par}$-module.

Items (i) and (ii) of [38, Lemma 10.69] directly lead to the following lemma.

Lemma 5.4. *Let P be a projective left (right) $\mathcal{H}_{par}$-module. Then, $\mathcal{H} \otimes_{\mathcal{H}_{par}} P$ ($P \otimes_{\mathcal{H}_{par}} \mathcal{H}$) is projective as left (right) $\mathcal{H}$-module.*

Recall that the co-unit $\varepsilon : \mathcal{H} \to K$ is a homomorphism of algebras, thus we have a homomorphism of algebras $\mathcal{H}_{par} \to K$ such that $[h] \mapsto \varepsilon(h)$.

Lemma 5.5. *The functors $- \otimes_{\mathcal{H}_{par}} \mathcal{H}$ and $- \otimes_{\mathcal{B}} K$ are naturally isomorphic. Analogously, the functors $\mathcal{H} \otimes_{\mathcal{H}_{par}} -$ and $K \otimes_{\mathcal{B}} -$ are naturally isomorphic.*

Proof. Let X be a right $\mathcal{H}_{par}$-module.

$$\begin{aligned} \tilde{\eta}_X : X \times \mathcal{H} &\to X \otimes_{\mathcal{B}} K \\ (x, h) &\mapsto x \cdot [h] \otimes_{\mathcal{B}} 1. \end{aligned}$$

Notice that $\tilde{f}$ is $\mathcal{H}_{par}$-balanced. Indeed,

$$\begin{aligned} \tilde{\eta}_X(x \cdot [t], h) &= x \cdot [t][h] \otimes_{\mathcal{B}} 1 \\ &= x \cdot [t][h_{(1)}] e_{S(h_{(2)})} \otimes_{\mathcal{B}} 1 \\ &= x \cdot [th_{(1)}] e_{S(h_{(2)})} \otimes_{\mathcal{B}} 1 \\ &= x \cdot [th_{(1)}] \otimes_{\mathcal{B}} e_{S(h_{(2)})} \triangleright 1 \\ &= x \cdot [th_{(1)}] \otimes_{\mathcal{B}} \varepsilon(S(h_{(2)})) \\ &= x \cdot [th_{(1)}] \otimes_{\mathcal{B}} \varepsilon(h_{(2)}) \\ &= x \cdot [th_{(1)} \varepsilon(h_{(2)})] \otimes_{\mathcal{B}} 1 \\ &= x \cdot [th] \otimes_{\mathcal{B}} 1 \\ &= \tilde{\eta}_X(x, th) = \tilde{\eta}_X(x, [t] \triangleright h). \end{aligned}$$

Therefore, the map

$$\begin{aligned} \eta_X : X \otimes_{\mathcal{H}_{par}} \mathcal{H} &\to X \otimes_{\mathcal{B}} K \\ x \otimes_{\mathcal{H}_{par}} h &\mapsto x \cdot [h] \otimes_{\mathcal{B}} 1 \end{aligned}$$

is well-defined. Let $f : X \to X'$ be a morphism of $\mathcal{H}_{par}$-modules. Then, the following diagram commutes

$$\begin{array}{ccc} X \otimes_{\mathcal{H}_{par}} \mathcal{H} & \xrightarrow{f \otimes_{\mathcal{H}_{par}} 1_{\mathcal{H}}} & X' \otimes_{\mathcal{H}_{par}} \mathcal{H} \\ \eta_X \downarrow & & \eta_{X'} \downarrow \\ X \otimes_{\mathcal{B}} K & \xrightarrow{f \otimes_{\mathcal{B}} 1} & X' \otimes_{\mathcal{B}} K \end{array}$$

Indeed,

$$\begin{aligned}\eta_{X'} \circ (f \otimes_{H_{par}} 1_{\mathcal{H}})(x \otimes_{\mathcal{H}_{par}} h) &= \eta_{X'}(f(x) \otimes_{\mathcal{H}_{par}} h) \\ &= f(x) \cdot [h] \otimes_{\mathcal{B}} 1 \\ &= f(x \cdot [h]) \otimes_{\mathcal{B}} 1 \\ &= (f \otimes_{\mathcal{B}} 1)(x \cdot [h] \otimes_{\mathcal{B}} 1) \\ &= (f \otimes_{\mathcal{B}} 1) \circ \eta_X(x \otimes_{\mathcal{H}_{par}} h).\end{aligned}$$

Thus, $\eta : (- \otimes_{\mathcal{H}_{par}} \mathcal{H}) \to (- \otimes_{\mathcal{B}} K)$ is a natural transformation. Finally, consider the map

$$\begin{aligned}\tilde{\Phi}_X : X \times K &\to X \otimes_{\mathcal{H}_{par}} \mathcal{H} \\ (x, r) &\mapsto x \otimes_{\mathcal{H}_{par}} r1_{\mathcal{H}}.\end{aligned}$$

Observe that $\tilde{\Phi}_X$ is $\mathcal{B}$-balanced. Indeed,

$$\begin{aligned}\tilde{\Phi}_X(x \cdot e_h, r) &= x \cdot e_h \otimes_{\mathcal{B}} r1_{\mathcal{H}} \\ &= x \otimes_{\mathcal{B}} e_h \triangleright (r1_{\mathcal{H}}) \\ &= x \otimes_{\mathcal{B}} \varepsilon(h) r1_{\mathcal{H}} \\ &= x \otimes_{\mathcal{B}} (e_h \triangleright r)1_{\mathcal{H}} \\ &= \tilde{\Phi}_X(x, e_h \triangleright r).\end{aligned}$$

Thus, the map

$$\begin{aligned}\Phi_X : X \otimes_{\mathcal{B}} K &\to X \otimes_{\mathcal{H}_{par}} \mathcal{H} \\ x \otimes_{\mathcal{B}} r &\mapsto x \otimes_{\mathcal{H}_{par}} r1_{\mathcal{H}}\end{aligned}$$

is well-defined. The maps η_X and Φ_X are mutual inverses:

$$\eta_X \circ \Phi_X(x \otimes_{\mathcal{B}} r) = \eta_X(x \otimes_{\mathcal{H}_{par}} r1_{\mathcal{H}}) = x \cdot (r1_{\mathcal{H}}) \otimes_{\mathcal{B}} 1 = x \otimes_{\mathcal{B}} r,$$

and

$$\Phi_X \circ \eta_X(x \otimes_{\mathcal{H}_{par}} h) = \Phi_X(x \cdot [h] \otimes_{\mathcal{B}} 1) = x \cdot [h] \otimes_{\mathcal{H}_{par}} 1_{\mathcal{H}} = x \otimes_{\mathcal{H}_{par}} h. \quad \square$$

Proposition 5.6. *Let $P_\bullet \to \mathcal{B}$ be a projective resolution of right $\mathcal{H}_{par}$-modules of $\mathcal{B}$. Then, $P_\bullet \otimes_{\mathcal{H}_{par}} \mathcal{H}$ is a projective resolution of right $\mathcal{H}$-modules of K.*

Proof. By Lemma 5.4 we know that $P_\bullet \otimes_{\mathcal{H}_{par}} \mathcal{H}$ is a complex of projective right $\mathcal{H}$-modules. Then, $P_\bullet \otimes_{\mathcal{H}_{par}} \mathcal{H}$ is a resolution of K if, and only if,

$$H_n(P_\bullet \otimes_{\mathcal{H}_{par}} \mathcal{H}) = \begin{cases} 0 & \text{if } n \geq 1 \\ K & \text{if } n = 0. \end{cases}$$

Observe that by Lemma 5.5 and Lemma 3.22 we have that

$$\begin{aligned} H_n(P_\bullet \otimes_{\mathcal{H}_{par}} \mathcal{H}) &= H_n(P_\bullet \otimes_{\mathcal{B}} K) \\ &= \mathrm{Tor}_n^{\mathcal{B}}(\mathcal{B}, K) \cong \begin{cases} 0 & \text{if } n \geq 1 \\ K & \text{if } n = 0. \end{cases} \quad \square \end{aligned}$$

Remark 5.7. It follows from the proof of Proposition 5.6 that

$$H_n^{par}(\mathcal{H}, \mathcal{H}) = \mathrm{Tor}_n^{\mathcal{H}_{par}}(\mathcal{B}, \mathcal{H}) = \mathrm{Tor}_n^{\mathcal{B}}(\mathcal{B}, K) \cong \begin{cases} 0 & \text{if } n \geq 1 \\ K & \text{if } n = 0. \end{cases}$$

Proposition 5.8. *Let X be a left (right) $\mathcal{H}$-module. Then,*

$$\mathrm{Tor}_\bullet^{\mathcal{H}_{par}}(\mathcal{B}, X) = \mathrm{Tor}_\bullet^{\mathcal{H}}(K, X) \text{ and } \mathrm{Ext}^\bullet_{\mathcal{H}_{par}}(\mathcal{B}, X) = \mathrm{Ext}^\bullet_{\mathcal{H}}(K, X).$$

Proof. Let $P_\bullet$ be a projective resolution of $\mathcal{B}$ in **Mod**-$\mathcal{H}_{par}$. Then, if X is a left $\mathcal{H}$-module we have

$$\begin{aligned} \mathrm{Tor}_n^{\mathcal{H}_{par}}(\mathcal{B}, X) &= H_n(P_\bullet \otimes_{\mathcal{H}_{par}} X) \\ &= H_n(P_\bullet \otimes_{\mathcal{H}_{par}} (\mathcal{H} \otimes_{\mathcal{H}} X)) \\ &= H_n((P_\bullet \otimes_{\mathcal{H}_{par}} \mathcal{H}) \otimes_{\mathcal{H}} X) \\ (\text{by Proposition 5.6}) &= \mathrm{Tor}_\bullet^{\mathcal{H}}(K, X). \end{aligned}$$

In the dual settings, if X is a right $\mathcal{H}$-module, we have

$$\begin{aligned} \mathrm{Ext}^n_{\mathcal{H}_{par}}(\mathcal{B}, X) &= H_n(\mathrm{Hom}_{\mathcal{H}_{par}}(P_\bullet, X)) \\ &= H_n\left(\mathrm{Hom}_{\mathcal{H}_{par}}(P_\bullet, \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}, X))\right) \\ &= H_n\left(\mathrm{Hom}_{\mathcal{H}_{par}}(P_\bullet, \mathrm{Hom}_{\mathcal{H}_{par}}(\mathcal{H}, X))\right) \\ &= H_n\left(\mathrm{Hom}_{\mathcal{H}_{par}}(P_\bullet \otimes_{\mathcal{H}_{par}} \mathcal{H}, X)\right) \\ (\text{by Proposition 5.6}) &= \mathrm{Ext}^n_{\mathcal{H}}(K, X). \quad \square \end{aligned}$$

Assume that we have a global action $\cdot : \mathcal{H} \otimes A \to A$. Then, any $A\#\mathcal{H}$-bimodule M is an $\mathcal{H}$-module, consequently, $H_n(A, M)$ and $H^n(A, M)$ are $\mathcal{H}$-modules. Therefore, by Proposition 5.8 the spectral sequences of Theorems 3.25 and 4.6 take the global forms:

Corollary 5.9. *Let $\mathcal{H}$ be a cocommutative Hopf algebra, which is projective as K-module, and let $\cdot : \mathcal{H} \otimes A \to A$ be a global action. Then there exist spectral sequences*

$$E^2_{p,q} = \mathrm{Tor}_p^{\mathcal{H}}(K, H_q(A, M)) \Rightarrow H_{p+q}(A\#\mathcal{H}, M),$$

and

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{H}}(K, H^q(A, M)) \Rightarrow H^{p+q}(A\#\mathcal{H}, M).$$

6. A projective resolution of $\mathcal{B}$

Finally, we construct a projective resolution of $\mathcal{B}$. Define the map

$$\begin{aligned}\psi : \mathcal{H}_{par} &\to \mathcal{B} \\ x &\mapsto x \triangleright 1_{\mathcal{B}}.\end{aligned}$$

Notice that ψ is a morphism of $\mathcal{H}_{par}$-modules. Indeed, let $x, z \in \mathcal{H}_{par}$. Then,

$$\psi(xz) = xz \triangleright 1_{\mathcal{B}} = x \triangleright (z \triangleright 1_{\mathcal{B}}) = x \triangleright \psi(z).$$

We define the modules of our resolution by

$$C'_n(\mathcal{H}) := \mathcal{H}_{par}^{\otimes_{\mathcal{B}} n+1}, \tag{6.1}$$

as the tensor product of $n+1$ copies of $\mathcal{H}_{par}$ over $\mathcal{B}$. Henceforth, for the sake of simplicity, since there is no ambiguity we will denote $C'_n(\mathcal{H})$ just by C'_n.

Proposition 6.1. *C'_n is projective as left (right) $\mathcal{H}_{par}$-module.*

Proof. By induction, obviously $C'_0 = \mathcal{H}_{par}$ is free as left (right) $\mathcal{H}_{par}$-module. Assume that C'_n is projective as left (right) $\mathcal{H}_{par}$-module, by [33, Proposition 3.9] we know that C'_{n+1} is projective if, and only if, $\operatorname{Hom}_{\mathcal{H}_{par}}(C'_{n+1}, -)$ is exact. First, we consider the right module case. Then, by [38, Theorem 2.75] we have that

$$\begin{aligned}\operatorname{Hom}_{\mathcal{H}_{par}}(C'_{n+1}, -) &= \operatorname{Hom}_{\mathcal{H}_{par}}(\mathcal{H}_{par} \otimes_{\mathcal{B}} C'_n, -) \\ &\cong \operatorname{Hom}_{\mathcal{B}}(\mathcal{H}_{par}, \operatorname{Hom}_{\mathcal{H}_{par}}(C'_n, -)).\end{aligned}$$

Thus, $\operatorname{Hom}_{\mathcal{H}_{par}}(C'_{n+1}, -)$ is exact since C'_n is projective as a right $\mathcal{H}_{par}$-module by hypothesis and $\mathcal{H}_{par}$ is projective as a right $\mathcal{B}$-module. Analogously for the left module case by [38, Theorem 2.76] we have that

$$\begin{aligned}\operatorname{Hom}_{\mathcal{H}_{par}}(C'_{n+1}, -) &= \operatorname{Hom}_{\mathcal{H}_{par}}(C'_n \otimes_{\mathcal{B}} \mathcal{H}_{par}, -) \\ &\cong \operatorname{Hom}_{\mathcal{B}}(\mathcal{H}_{par}, \operatorname{Hom}_{\mathcal{H}_{par}}(C'_n, -)),\end{aligned}$$

from which we conclude that C'_{n+1} also is projective as a left $\mathcal{H}_{par}$-module. □

Observe that product in $\mathcal{H}_{par}$ is a well-defined morphism of $\mathcal{H}_{par}$-bimodules.

$$\begin{aligned}\mu : \mathcal{H}_{par} \otimes_{\mathcal{B}} \mathcal{H}_{par} &\to \mathcal{H}_{par} \\ x \otimes_{\mathcal{B}} y &\mapsto xy.\end{aligned}$$

Using this map we are able to define the following face maps of left $\mathcal{H}_{par}$-modules for $n \geq 1$. If $0 \leq i \leq n-1$ we set $d_i : C'_n \to C'_{n-1}$ such that

$$d_i := id_{\mathcal{H}_{par}} \otimes_{\mathcal{B}} \ldots \underbrace{\otimes_{\mathcal{B}} \mu \otimes_{\mathcal{B}}}_{i\text{-position}} \ldots \otimes_{\mathcal{B}} id_{\mathcal{H}_{par}},$$

In particular, for a basic element of C'_n, this formula is determined by

$$d_i(x_0 \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} x_n) = x_0 \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} x_i x_{i+1} \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} x_n. \tag{6.2}$$

For $i = n$ we define

$$d_n := id_{\mathcal{H}_{par}} \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} \mu(id_{\mathcal{H}_{par}} \otimes_{\mathcal{B}} \psi).$$

Thus, for the basic elements of C'_n, we obtain

$$d_n(x_0 \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} x_n) = x_0 \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} x_{n-1}\psi(x_n) \tag{6.3}$$

Analogously, we define the maps $s_i : C'_n \to C'_{n+1}$, $0 \leq i \leq n$ and $n \geq 0$, by

$$s_i(x_0 \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} x_n) := x_0 \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} x_i \otimes_{\mathcal{B}} 1_{\mathcal{H}_{par}} \otimes_{\mathcal{B}} x_{n+1} \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} x_n. \tag{6.4}$$

Proposition 6.2. *$(C'_\bullet, d_i, s_i)$ is a simplicial module.*

Proof. Direct computations. □

Consequently, $(C'_\bullet, \partial_\bullet)$ is a complex where

$$\partial_n := \sum_{i=0}^{n} (-1)^i d_i.$$

Proposition 6.3. *$C'_\bullet \xrightarrow{\psi} \mathcal{B}$ is a projective resolution of $\mathcal{B}$ in $\mathcal{H}_{par}$-**Mod**.*

Proof. For $n \geq 0$ consider the map $s : C'_n \to C'_{n+1}$ such that

$$s(x_0 \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} x_n) := 1_{\mathcal{H}_{par}} \otimes_{\mathcal{B}} x_0 \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} x_n. \tag{6.5}$$

Observe that such a map satisfies

(a) $d_0 s = id_{\mathcal{C}'_n}$,
(b) $d_i s = s d_{i-1}$, for all $i \geq 1$.

Thus,

$$\partial_{n+1}s = \sum_{i=0}^{n+1}(-1)^i d_i s = id_{C'_n} + \sum_{i=1}^{n+1}(-1)^i s d_{i-1}$$
$$= id_{C'_n} - \sum_{i=0}^{n}(-1)^i s d_i = id_{C'_n} - s\partial_n.$$

Therefore,

$$s\partial_n + \partial_{n+1}s = id_{C'_n}, \forall n \geq 1.$$

Whence $H_n(C'_\bullet, \partial_\bullet) = 0$ for all $n \geq 1$. Now consider the map ∂_1. Explicitly this map takes the form

$$\partial_1(x_0 \otimes_{\mathcal{B}} x_1) = x_0x_1 - x_0\psi(x_1).$$

Thus,

$$\begin{aligned} \psi\partial_0(x_0 \otimes_{\mathcal{B}} x_1) &= \psi(x_0x_1) - \psi(x_0\psi(x_1)) \\ &= x_0x_1 \triangleright 1_{\mathcal{B}} - x_0\psi(x_1) \triangleright 1_{\mathcal{B}} \\ &= x_0 \triangleright (x_1 \triangleright 1_{\mathcal{B}}) - x_0 \triangleright (\psi(x_1) \triangleright 1_{\mathcal{B}}) \\ \text{(by Lemma 3.7)} &= x_0 \triangleright \psi(x_1) - x_0 \triangleright \psi(x_1) = 0. \end{aligned}$$

Then, $C'_\bullet \xrightarrow{\psi} \mathcal{B}$ is a complex. Now, let z be in the kernel of ψ, then

$$\partial_1 s(z) = \partial_1(1_{\mathcal{H}_{par}} \otimes_{\mathcal{B}} z) = z - \psi(z) = z.$$

Whence, $z \in \operatorname{im}\partial_1$. Thus, $\ker\psi = \operatorname{im}\partial_1$. Thus, $C'_\bullet \xrightarrow{\psi} \mathcal{B}$ is an exact sequence. Finally, by Proposition 6.1 each C'_n is projective. □

Data availability

No data was used for the research described in the article.

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