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# On the Dispersion of Geodesic Acoustic Modes[1]

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**Abstract**—The problem of dispersion of geodesic acoustic modes is revisited with two different methods for the solution of the kinetic equation. The dispersive corrections to the mode frequency are calculated by including the $m = 2$ poloidal harmonics. Our obtained results agree with some earlier results but differ in various ways with other previous works. Limitations and advantages of different approaches are discussed.

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## 1. INTRODUCTION

Geodesic acoustic modes (GAMs) are the linear modes supported by plasma compressibility in toroidal geometry. GAMs derive their name from the effect of the averaged geodesic curvature providing the restoring force. Within the ideal MHD model [1], GAMs dispersion relation has a simple form $\omega^2 = 2c_s^2/R^2 + c_s^2/\left(q^2R^2\right)$, where $c_s^2 = \gamma p_0/\rho_0$ is the ideal MHD sound velocity. Over the last decade, it has gradually been realized that together with zonal flows, GAMs represent an important ingredient of drift wave turbulence in a tokamak. Drift-wave fluctuations are coupled to low-frequency zonal flows and finite frequency GAMs via toroidal effects and nonlinear Reynolds stress [2, 3]. GAMs and zonal flows rotational modes suppress the small scale fluctuations via shearing and energy sink thus creating complex interactions between drift wave turbulence, zonal flows, and GAMs [4–6]. The direct effect of GAMs on anomalous transport is not clear at the moment; however, there are significant experimental evidence indicating coupling and mutual effects of turbulent fluctuations and GAMs. In addition to their role in regulation of anomalous transport, it has been suggested that GAM can also be useful for plasma diagnostic purposes [7].

Large body of the current work investigates GAM coupling to Alfvén and drift modes (see, e.g., [8–12] and references therein). Yet, already in 1973, A.B. Mikhailovskii pointed out that the theory of drift waves in toroidal systems has to be developed taking into account the averaged curvature, magnetic well, and electromagnetic effects [13]. Using two-fluid theory, in that paper, he developed the theory of drift instabilities that include GAMs, Alfvén effects, magnetic well, and magnetic shear, as well as ion finite-Larmor-radius (FLR) effects. Such general electromagnetic perturbations in toroidal systems with averaged magnetic well are described by the dispersion equation given by Eq (3.17) in [13]. The basic GAM dispersion follows from Eq. (3.17) in the limit

$$c_\perp + q^2c_\parallel = 0, \tag{1}$$

where, in notations of [13] and neglecting the drift effects, $c_\perp = \omega^2q^2R^2/c_s^2$ and $c_\parallel = \omega\left(\omega^2 - c_s^2/\left(q^2R^2\right)\right)$. It is easy to see that, in this limit, relation (1) describes the simplest GAM. The full theory developed in [13] lays foundations for many phenomena involving GAMS, such as coupling to drift waves and temperature gradient driven modes, Alfvén effects, averaged magnetic well, and ion FLR and resonant kinetic effects, that are now being studied in numerous papers (see extensive publications list in [10, 12]).

Later, it was realized that GAM are closely related to poloidal plasma rotation [14–17]. At that time, the discrepancy between the kinetic theory $\omega^2 = 7v_{Ti}^2/\left(4R^2\right)$ [14] and ideal MHD [1] $\omega^2 = c_s^2/R^2$ results became apparent, where $v_{Ti}^2 = 2T_i/m_i$ and $c_s^2 = \gamma p_0/\rho_0$, with $\gamma = 5/3$. It was shown later that the perturbed pressure in GAM is in fact anisotropic [18] and the MHD system with anisotropic pressure allows to reconcile the difference between kinetic and fluid

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approaches. The correct kinetic result with a 7/4 factor was earlier obtained by V.A. Mazur and A.B. Mikhailovskii [19].

Numerical simulations, both in tokamak [5, 6, 20–22] and stellarator geometries [23, 24], have clearly shown the presence of oscillations in the range of GAM frequencies. The modes with similar frequencies are ubiquitously observed in tokamaks [25–34]. Despite active experimental studies of GAM frequencies, radial localization, propagation, and correlation length, general eigenmode properties of GAMs have not been firmly established, possibly due to multitude manifestations of similar, but not the same phenomena/modes. Although the frequency of the modes detected experimentally in general shows the temperature scaling consistent with GAMs, there is no exact match with available theoretical expressions and radial profiles of the observed mode frequency and amplitudes have not been explained theoretically. The GAM eigenmode structure is crucially dependent on the mode dispersion properties. It is generally understood that the dispersion of GAM is determined by the second-order effects related to the second poloidal harmonics, while the main-order GAM frequency can be calculated retaining only the first-order terms. In plasmas with finite ion temperature, the ion pressure perturbations due to toroidal compressibility are anisotropic already in the first order [18]. Anisotropic hydrodynamics with higher order finite ion Larmor radius [35] and toroidal curvature terms is not easily available, so the second-order calculations are most easily done using the kinetic theory.

There are several sources of GAM dispersion which are different in their physics. Standard sources of GAM dispersion are finite ion Larmor radius effects $k_r^2\rho_i^2$ due to a finite ion temperature, $\rho_i^2 = v_{Ti}^2/\omega_{ci}^2$ (with $v_{Ti}^2 = 2T_i/m_i$), and effects of the so-called ion-sound Larmor radius $k_r^2\rho_s^2$ (with $\rho_s^2 = T_e/\left(m_i\omega_{ci}^2\right)$) due to a finite electron temperature, $\omega_{ci} = eB_0/m_ic$. The ion-sound Larmor radius effects can be described by standard fluid equations, while the ion Larmor radius effects, in general, require kinetic theory, or advanced higher order hydrodynamics that includes second-order oscillations [35]. Another source of dispersion are the effects of finite magnetic drift, which provide the order of $\omega_d^2/\omega^2$ dispersive corrections to the main-order GAM frequency. The GAM frequency itself is a result of the balance of the inertial and diamagnetic currents, which can be expressed in the form $k_r^2\rho_i^2 \simeq \omega_d^2/\omega^2$, so that the GAM frequency is $\omega^2 \simeq \omega_d^2/\left(k_r^2\rho_i^2\right)$. Therefore, the dispersive corrections $\omega_d^2/\omega^2$, though have different physics origin, formally appear as an expansion in ion FLR parameter, $\omega_d^2/\omega^2 \simeq k_r^2\rho_i^2$. Sometimes, these terms are called finite orbit effects [36], because they are related to the particle displacement from the magnetic surface due to the geodesic curvature drift (in radial direction); however, they do not depend on the value of the safety factor $q$: the relevant expansion parameter is $\omega_d^2/\omega^2$ and not $\omega_d^2q^2R^2/v_{Ti}^2\omega^2$ (see further discussion of this in Section 5). Formally, all three different effects $\omega_d^2/\omega^2$, $k_r^2\rho_i^2$, and $k_r^2\rho_s^2$ are of the same order for $T_e \simeq T_i$. Note however, that $\omega_d^2/\omega^2$ corrections actually come from the averaging of $\omega_d^4$ terms, which have large numerical coefficients (see, e.g., Eq. (70)).

The kinetic theory of GAMs has been studied in a number of papers [11, 36–41]. Specifically, the explicit second-order dispersive corrections have been calculated kinetically both for the mode frequency and for the mode damping in [36, 38, 41–47]. Dispersive corrections and radial group velocity have been also calculated by different method in [21]. In principle, such calculations are straightforward, however the actual expressions are cumbersome and, in some cases, are given by complex integral expressions [36]. Unfortunately, many expressions for the GAM dispersion available in the literature are not consistent with each other, a number of terms have different coefficients, and some terms are missing. Comparison of these different expressions is also difficult due to omitted details of actual calculations. The goal of this paper is to establish the common basis for the calculation of the dispersive corrections to the local GAM frequency in plasmas with finite electron and ion temperatures and compare results from different approaches.

In order to resolve numerous contradictions in the literature, we systematically present in sufficient details the calculations of the GAM dispersion equation to the second-order of accuracy that is required for the calculation of the radial group velocity and discuss different approaches for the description of the parallel and toroidal resonances.

## 2. PERTURBATIVE SOLUTION OF THE KINETIC EQUATION

We use the standard gyrokinetic equation for a homogeneous electrostatic mode in the form [48, 49]

$$f = -\frac{q_s}{T_s}F_m\phi + g, \quad (2)$$

$$\left(\omega - \hat{\omega}_d + \frac{iv_\parallel}{qR}\frac{\partial}{\partial\theta}\right)g = \omega\frac{q_s\phi}{T_s}J_0^2(k_\perp v_\perp/\omega_{ci})F_m, \quad (3)$$

where the harmonic time dependence $\exp(-i\omega t)$ was assumed and

$$\hat{\omega}_d = i\frac{v_\perp^2/2 + v_\parallel^2}{\omega_{ci}}(\mathbf{b}\times\nabla\ln B)\cdot\nabla. \quad (4)$$

As standard in GAM theory, we neglect the normal magnetic curvature, assuming that the radial gradients are large compared to the poloidal, $\nabla_r > \nabla_\theta$. Then, for $\nabla \to i\mathbf{k}$ one has

$$\begin{aligned}\hat{\omega}_d &\simeq -\frac{v_\perp^2/2 + v_\parallel^2}{\omega_{ci} R} k_r \sin\theta \\ &= -\omega_d \frac{v_\perp^2/2 + v_\parallel^2}{v_T^2} \sin\theta = -\bar{\omega}_d \sin\theta,\end{aligned} \tag{5}$$

where $\omega_d = k_r v_T^2/(R\omega_c)$.

Consider the perturbed electrostatic potential in the form

$$\begin{aligned}\phi &= \phi_0 + \phi_s \sin\theta + \phi_{2c}\cos 2\theta \\ &+ \phi_{3s}\sin 3\theta + \phi_{4c}\cos 4\theta + \ldots.\end{aligned} \tag{6}$$

The perturbed distribution function is presented in a similar form

$$\begin{aligned}g &= g_0 + g_c\cos\theta + g_s\sin\theta + g_{2c}\cos 2\theta \\ &+ g_{3s}\sin 3\theta + g_{4c}\cos 4\theta + \ldots.\end{aligned} \tag{7}$$

The GAM dispersion appears as the fourth-order terms in the perturbed density expression, $n \simeq (e\phi_0/T)\omega_d^4/\omega^4$ so we include the higher order terms in Eqs. (6) and (7).

Equation (3) can be solved by two different methods. In one approach [39, 40, 50, 51], solution is obtained perturbatively by separating different harmonics. For simplicity, we neglected the parallel ion-sound dynamics in Eq. (2) by taking the limit $q \to \infty$; however, the general case is considered in [40]. From Eqs. (2), (6), and (7), one finds the following set of equations

$$\omega g_0 + \frac{1}{2}\bar{\omega}_d g_s = \omega J_0^2(k_\perp v_\perp/\omega_{ci})\frac{e}{T_i}F_0\phi_0, \tag{8}$$

$$\bar{\omega}_d g_0 + \omega g_s - \frac{1}{2}\bar{\omega}_d g_{2c} = \omega J_0^2(k_\perp v_\perp/\omega_{ci})\frac{e}{T_i}F_0\phi_s, \tag{9}$$

$$\begin{aligned}&-\frac{1}{2}\bar{\omega}_d g_s + \omega g_{2c} + \frac{1}{2}\bar{\omega}_d g_{3s} \\ &= \omega J_0^2(k_\perp v_\perp/\omega_{ci})\frac{e}{T_i}F_0\phi_{2c},\end{aligned} \tag{10}$$

$$\begin{aligned}&\frac{1}{2}\bar{\omega}_d g_{2c} + \omega g_{3s} - \frac{1}{2}\bar{\omega}_d g_{4c} \\ &= \omega J_0^2(k_\perp v_\perp/\omega_{ci})\frac{e}{T_i}F_0\phi_{3s},\end{aligned} \tag{11}$$

$$-\frac{1}{2}\bar{\omega}_d g_{3s} + \omega g_{4c} = \omega J_0^2(k_\perp v_\perp/\omega_{ci})\frac{e}{T_i}F_0\phi_{4c}. \tag{12}$$

Solving these set of equations by using the expansion in $\omega_d / \omega < 1$, one finds

$$\begin{aligned}g_0 &= J_0^2(k_\perp v_\perp/\omega_{ci})\frac{e}{T_i}F_0\left(\phi_0 + \frac{1}{2}\frac{\bar{\omega}_d^2}{\omega^2}\phi_0\right. \\ &\left.+ \frac{3}{8}\frac{\bar{\omega}_d^4}{\omega^4}\phi_0 - \frac{1}{2}\frac{\bar{\omega}_d}{\omega}\phi_s - \frac{3}{8}\frac{\bar{\omega}_d^3}{\omega^3}\phi_s - \frac{1}{4}\frac{\bar{\omega}_d^2}{\omega^2}\phi_{2c}\right),\end{aligned} \tag{13}$$

$$\begin{aligned}g_s &= J_0^2(k_\perp v_\perp/\omega_{ci})\frac{e}{T_i}F_0\left(-\frac{\bar{\omega}_d}{\omega}\phi_0\right. \\ &\left.- \frac{3}{4}\frac{\bar{\omega}_d^3}{\omega^3}\phi_0 + \phi_s + \frac{3}{4}\frac{\bar{\omega}_d^2}{\omega^2}\phi_s + \frac{1}{2}\frac{\bar{\omega}_d}{\omega}\phi_{2c}\right),\end{aligned} \tag{14}$$

$$\begin{aligned}g_{2c} &= J_0^2(k_\perp v_\perp/\omega_{ci})\frac{e}{T_i}F_0 \\ &\times\left(-\frac{1}{2}\frac{\bar{\omega}_d^2}{\omega^2}\phi_0 + \frac{1}{2}\frac{\bar{\omega}_d}{\omega}\phi_s + \phi_{2c}\right).\end{aligned} \tag{15}$$

In fact, it can be readily shown that $g_{3s}$ and $g_{4c}$ do not contribute in our ordering and $\phi_{2c}$ should be calculated only to accuracy of $\bar{\omega}_d^2/\omega^2$ terms.

Using Eqs. (13)–(15), the ion density equations can be written in the form

$$\begin{aligned}n_0 &= \frac{e\phi_0}{T_i}\left(-\frac{1}{2}I_2 + \frac{3}{32}I_4 - \frac{1}{4}I_{4d} + \frac{1}{2}K_2 + \frac{3}{8}K_4\right) \\ &+ \frac{e\phi_s}{T_i}\left(-\frac{1}{2}K_1 - \frac{3}{8}K_3 + \frac{1}{4}I_{3d}\right) - \frac{1}{4}\frac{e\phi_{2c}}{T_i}K_2,\end{aligned} \tag{16}$$

$$\begin{aligned}n_s &= \frac{e\phi_0}{T_i}\left(-K_1 + \frac{1}{2}I_{3d} - \frac{3}{4}K_3\right) \\ &+ \frac{e\phi_s}{T_i}\left(-\frac{1}{2}I_2 + \frac{3}{4}K_2\right) + \frac{1}{2}\frac{e\phi_{2c}}{T_i}K_1,\end{aligned} \tag{17}$$

$$\begin{aligned}n_{2c} &= \frac{e\phi_0}{T_i}\left(-\frac{1}{2}K_2 + \frac{1}{4}I_{4d}\right) + \frac{e\phi_s}{T_i}\left(\frac{1}{2}K_1 - \frac{1}{4}I_{3d}\right) \\ &+ \frac{e\phi_{2c}}{T_i}\left(-\frac{1}{2}I_2 + \frac{1}{2}K_2\right).\end{aligned} \tag{18}$$

Here, the following expansion was used $J_0^2 = 1 - k_\perp^2 v_\perp^2/(2\omega_c^2) + 3k_\perp^4 v_\perp^4/(32\omega_c^4)$. Definitions of relevant integrals are given in the Appendix A. Note that the perturbed density is normalized to the equilibrium value.

The perturbed density for the electron components can be found either from electron kinetic equation or from simple fluid equations [39]. The perturbation of the electron density in $m = 0$ component is absent, $n_0 = 0$, and the first and second poloidal harmonics follow Boltzmann distribution,

$$n_\mu = \frac{e}{T_e}\phi_\mu, \tag{19}$$

where $\mu = (s, 2c)$.

## 3. DISPERSION EQUATION

Using quasineutrality conditions for $n_0$, $n_s$, and $n_{2c}$, one obtains the following equations for $\phi_0$, $\phi_s$, and $\phi_{2c}$

$$\left(-\frac{1}{2}I_2+\frac{3}{32}I_4+\frac{1}{2}K_2-\frac{1}{4}I_{4d}+\frac{3}{8}K_4\right)\phi_0 + \left(-\frac{1}{2}K_1-\frac{3}{8}K_3+\frac{1}{4}I_{3d}\right)\phi_s-\frac{1}{4}K_2\phi_{2c}=0, \quad (20)$$

$$\left(\frac{1}{\tau_e}+\frac{1}{2}I_2-\frac{3}{4}K_2\right)\phi_s = \left(-K_1-\frac{3}{4}K_3+\frac{1}{2}I_{3d}\right)\phi_0+\frac{1}{2}K_1\phi_{2c}, \quad (21)$$

$$\left(\frac{1}{\tau_e}+\frac{1}{2}I_2-\frac{1}{2}K_2\right)\phi_{2c}=\left(-\frac{1}{2}K_2+\frac{1}{4}I_{4d}-\frac{3}{8}K_4\right)\phi_0 + \left(\frac{1}{2}K_1+\frac{3}{8}K_3-\frac{1}{4}I_{3d}\right)\phi_s, \quad (22)$$

where $\tau_e = T_e/T_i$. Excluding $\phi_{2c}$ from Eqs. (20)–(22), one has

$$\left(-\frac{1}{2}I_2+\frac{1}{2}K_2+\frac{3}{32}I_4-\frac{1}{4}I_{4d}+\frac{3}{8}K_4 + \frac{1}{8}\frac{K_2(K_2+3K_4/4-I_{4d}/2)}{\tau_e^{-1}+I_2/2-K_2/2}\right)\phi_0 + \left(-\frac{1}{2}K_1-\frac{3}{8}K_3+\frac{1}{4}I_{3d} - \frac{1}{8}\frac{K_2(K_1+3K_3/4-I_{3d}/2)}{\tau_e^{-1}+I_2/2-K_2/2}\right)\phi_s=0, \quad (23)$$

$$\left(\frac{1}{\tau_e}+\frac{1}{2}I_2-\frac{3}{4}K_2 - \frac{1}{4}\frac{K_1(K_1+3K_3/4-I_{3d}/2)}{\tau_e^{-1}+I_2/2-K_2/2}\right)\phi_s = \left(-K_1-\frac{3}{4}K_3 + \frac{1}{2}I_{3d}-\frac{1}{4}\frac{K_1(K_2+3K_4/4-I_{4d}/2)}{\tau_e^{-1}+I_2/2-K_2/2}\right)\phi_0. \quad (24)$$

The combination $\tau_e^{-1}+I_2/2$ corresponds to the ion-sound Larmor radius dispersion factor $\left(1+k_r^2\rho_s^2\right)$, $\rho_s^2 = T_e/\left(m_i\omega_{ci}^2\right)$, and, in general, $k_r^2\rho_s^2$ does not have to be small compared to unity.

The standard (dispersionless) expression for GAMs is obtained by neglecting all fourth-order terms and assuming that $\tau_e^{-1} > \left(I_2, K_2, K_1^2\right)$, which gives $I_2 = K_2 + K_1^2\tau_e$, or

$$\omega_0^2 = \frac{\omega_d^2}{k_r^2\rho_i^2}\left(\frac{7}{4}+\tau_e\right) = \frac{v_{Ti}^2}{R^2}\left(\frac{7}{4}+\tau_e\right). \quad (25)$$

The next-order dispersion corrections are obtained by expansion in small parameters $\left(I_2, K_2, K_1^2\right) < \tau_e \simeq 1$ and retaining the fourth-order terms,

$$-I_2+K_2+\tau_eK_1^2+\frac{3}{16}I_4-\frac{1}{2}I_{4d}+\frac{3}{2}K_4 + \tau_e\left[-K_1I_{3d}+\frac{3}{2}K_1K_3+\frac{1}{4}K_2^2\right] + \tau_e^2K_1^2\left(-\frac{1}{2}I_2+\frac{5}{4}K_2\right)+\frac{1}{4}\tau_e^3K_1^4=0. \quad (26)$$

Using the definitions of the integrals in the Appendix A, this dispersion equation becomes identical to the one obtained previously [38, 41],

$$-1+\frac{v_{Ti}^2}{\omega^2R^2}\left(\frac{7}{4}+\tau_e\right)+\frac{1}{2}k_r^2\rho_i^2\left[\frac{3}{4}-\frac{v_{Ti}^2}{R^2\omega^2} \times\left(\frac{13}{4}+3\tau_e+\tau_e^2\right)+\frac{v_{Ti}^4}{R^4\omega^4}\left(\frac{747}{32}+\frac{481}{32}\tau_e + \frac{35}{8}\tau_e^2+\frac{1}{2}\tau_e^3\right)\right]=0. \quad (27)$$

It is worth noting that the form of the dispersion equation given by Eq. (27) is not unique, but may depend on the way how Eqs. (20)–(22) are manipulated and expanded in small parameter $k_r^2\rho_i^2$. For instance, direct expansion of the determinant of the system of homogenous equations (20)–(22) gives the following equation:

$$-1+\frac{v_{Ti}^2}{\omega^2R^2}\left(\frac{7}{4}+\tau_e\right)+\frac{1}{2}k_r^2\rho_i^2\left[\frac{3}{4}-2\tau_e + \frac{v_{Ti}^2}{R^2\omega^2}\left(-\frac{13}{4}+\frac{39}{8}\tau_e+\frac{3}{2}\tau_e^2\right) + \frac{v_{Ti}^4}{R^4\omega^4}\left(\frac{747}{32}+\frac{59}{8}\tau_e-\frac{7}{8}\tau_e^2\right)\right]=0, \quad (28)$$

which looks quite different from Eq. (27). In fact, Eqs. (27) and (28), are equivalent to the order of the terms $k_r^2\rho_i^2$. This can be seen by noting that the dispersion factors $D_1$ and $D_2$ in these equations are different in the third term, containing the lowest order dispersion equation,

$$D_1 \equiv \left[\frac{3}{4}-\frac{v_{Ti}^2}{R^2\omega^2}\left(\frac{13}{4}+3\tau_e+\tau_e^2\right) + \frac{v_{Ti}^4}{R^4\omega^4}\left(\frac{747}{32}+\frac{481}{32}\tau_e+\frac{35}{8}\tau_e^2+\frac{1}{2}\tau_e^3\right)\right], \quad (29)$$

$$D_2 \equiv \left[\frac{3}{4}-2\tau_e+\frac{v_{Ti}^2}{R^2\omega^2}\left(-\frac{13}{4}+\frac{39}{8}\tau_e+\frac{3}{2}\tau_e^2\right) + \frac{v_{Ti}^4}{R^4\omega^4}\left(\frac{747}{32}+\frac{59}{8}\tau_e-\frac{7}{8}\tau_e^2\right)\right], \quad (30)$$

and

$$D_2 = D_1 + \tau\left[-1 + \frac{v_{Ti}^2}{\omega^2 R^2}\left(\frac{7}{4} + \tau_e\right)\right] \times \left[2 - \frac{v_{Ti}^2}{\omega^2 R^2}\left(\frac{35}{8} + \frac{\tau}{2}\right)\right]. \quad (31)$$

As a result, the difference $D_1$ and $D_2$ does not affect the first-order dispersive corrections in $k_r^2\rho_i^2$.

The explicit dispersive corrections to the GAM frequency can be written in the form

$$\omega^2 = \omega_0^2\left(1 + \frac{1}{2}k_r^2\rho_i^2 D_1'\right), \quad (32)$$

where the dispersion coefficient $D_1'$ is given by the expression

$$D_1' \equiv \left[\frac{3}{4} - \left(\frac{7}{4} + \tau_e\right)^{-1}\left(\frac{13}{4} + 3\tau_e + \tau_e^2\right) + \left(\frac{7}{4} + \tau_e\right)^{-2}\left(\frac{747}{32} + \frac{481}{32}\tau_e + \frac{35}{8}\tau_e^2 + \frac{1}{2}\tau_e^3\right)\right]. \quad (33)$$

We note that the expressions in [43, 45, 46] miss some terms and some have incorrect coefficients. It is interesting that the dispersion effects due to ion-sound Larmor radius are negative [21], while those due to ion temperature effects are positive. The latter are dominant in plasmas with $\tau_e \simeq 1$. The sign of the dispersion correction $D_1'$ changes to negative around $\tau_e \simeq 5.45$ [41]. Note that the dispersion relation obtained in [47] results in positive dispersion for any values of $\tau_e$. Equation (32) gives the following group velocity:

$$v_{\rm gr} = \frac{\partial\omega}{\partial k_r} = \frac{1}{2}\omega_0 k_r \rho_i^2 D_1'. \quad (34)$$

The GAM group velocity was also calculated in [21] as a ratio of radial free energy flux to the total free energy. We were unable to match our expression (34) with the expression in [21], possibly due to some typos.

In the limit of cold ions, $T_e \gg T_i$, the GAM dispersion equation can be found in nonperturbative form not assuming small $k_r^2\rho_s^2$. From Eqs. (23) and (24), one obtains

$$\frac{1}{\tau_e} + \frac{1}{2}I_2 - \frac{1}{4}K_1^2\left(\frac{1}{\tau_e} + \frac{1}{2}I_2\right)^{-1} - \frac{K_1^2}{I_2} = 0, \quad (35)$$

resulting in

$$\omega^2 = \frac{2c_s^2}{R^2}\frac{1 + 3k_r^2\rho_s^2/2}{\left(1 + k_r^2\rho_s^2\right)^2}, \quad (36)$$

where $\rho_s^2 = T_e/\left(m_i\omega_{ci}^2\right)$, which, in the limit of weak dispersion, gives the result $\omega^2 = 2\left(c_s^2/R^2\right)\left(1 - k_r^2\rho_s^2\right)$ consistent with Eq. (32).

## 4. PERTURBED DISTRIBUTION FUNCTION BASED ON THE EXACT SOLUTION OF KINETIC EQUATION

Alternative approach to the solution of the gyrokinetic equation

$$\left(\omega - \tilde{\omega}_{d\alpha}\sin\theta + i\omega_t\frac{\partial}{\partial\theta}\right)h = \omega\frac{q_\alpha}{T_\alpha}\phi J_0^2(k_\perp v_\perp/\omega_{c\alpha})F_m \quad (37)$$

was suggested in [44, 52], (see also [20, 36, 53]) with $\tilde{\omega}_{d\alpha} = k_r(v_\perp^2/2 + v_\parallel^2)/\omega_{cj}R$ and $\omega_t = v_\parallel/(qR)$. Note that here we are following the notations of [44, 52], where the sign of $\omega_d$ term is different: $\tilde{\omega}_{d\alpha} = -\omega_{d\alpha}$ (cf. kinetic equations (3) and (37)).

By integrating over the particle trajectories, the formal solution of Eq. (37) can be written in the form [44]

$$f_{mi} = -\frac{eF_0}{T_i}\left\{\phi_m - J_0^2(kv_\perp/\Omega_i) \times \sum_{n,l=-\infty}^{\infty} i^{-l+m}\frac{\omega J_n(\xi)J_{n-l+m}(\xi)}{\omega + (n-l)\omega_{ti}}\phi_l\right\}, \quad (38)$$

where $\xi = \tilde{\omega}_{di}/\omega_t = \tilde{\omega}_{di}qR/v_\parallel$.

To compare this solution with our expressions (13)–(15), we separate the $m = 0$, $m = \pm1$, and $m = \pm2$ harmonics. The $m = 0$ harmonic has the form

$$f_0 = -\frac{eF_0}{T_i}\left\{\phi_0 - J_0^2(kv_\perp/\Omega_i) \times \sum_{n,l=-\infty}^{\infty} i^{-l}\frac{\omega J_n(\xi)J_{n-l}(\xi)}{\omega + (n-l)\omega_{ti}}\phi_l\right\}, \quad (39)$$

introducing the shorthand notations

$$f_0 = \frac{eF_0}{T_i}\left\{\phi_0 L_0^0 + \phi_1 L_0^1 + \phi_{-1}L_0^{-1} + \phi_2 L_0^2 + \phi_{-2}L_0^{-2}\right\}, \quad (40)$$

where

$$L_0^0 = -1 + J_0^2(kv_\perp/\Omega_i)\sum_{n=-\infty}^{\infty}\frac{\omega J_n(\xi)J_n(\xi)}{\omega + n\omega_{ti}}, \quad (41)$$

$$L_0^1 = J_0^2(kv_\perp/\Omega_i)\sum_{n=-\infty}^{\infty} i^{-1}\frac{\omega J_n(\xi)J_{n-1}(\xi)}{\omega + (n-1)\omega_{ti}}, \quad (42)$$

$$L_0^{-1} = J_0^2(kv_\perp/\Omega_i)\sum_{n=-\infty}^{\infty} i^{1}\frac{\omega J_n(\xi)J_{n+1}(\xi)}{\omega + (n+1)\omega_{ti}}, \quad (43)$$

$$L_0^2 = -J_0^2(kv_\perp/\Omega_i)\sum_{n=-\infty}^{\infty}\frac{\omega J_n(\xi)J_{n-2}(\xi)}{\omega + (n-2)\omega_{ti}}, \quad (44)$$

$$L_0^{-2} = -J_0^2(kv_\perp/\Omega_i)\sum_{n=-\infty}^{\infty}\frac{\omega J_n(\xi)J_{n+2}(\xi)}{\omega+(n+2)\omega_{ti}}. \quad (45)$$

Practical applications of these expressions require the expansion in small parameter $\xi$ and truncating the infinite series to several lowest order terms. Note that such truncation is only possible for $\xi \ll 1$, so that higher order terms $J_n(\xi)J_{n+2}(\xi)$ can be neglected. The formal expansion assuming $\xi \ll 1$ gives

$$L_0^0 = -1 + J_0^2(kv_\perp/\Omega_i) \times\left[J_0^2(\xi) + \frac{2\omega^2 J_1^2(\xi)}{\omega^2-\omega_{ti}^2} + \frac{2\omega^2 J_2^2(\xi)}{\omega^2-4\omega_{ti}^2}\right]. \quad (46)$$

Next, the Bessel functions can be approximated giving the form

$$L_0^0 = -1 + J_0^2(kv_\perp/\Omega_i)\left[\left(1-\frac{1}{2}\xi^2+\frac{3}{32}\xi^4\right) + \frac{2\omega^2}{\omega^2-\omega_{ti}^2}\frac{\xi^2}{4}\left(1-\frac{\xi^2}{4}\right) + \frac{2\omega^2}{\omega^2-4\omega_{ti}^2}\frac{\xi^4}{64}\right]. \quad (47)$$

Note that each term in Eq. (47) will diverge at $v_\parallel = 0$ due to the factors with $\xi \propto 1/v_\parallel$. In fact, the singular contributions cancel exactly when all terms in each order of $\xi$ are retained in $J_0^2(\xi)$, $J_1^2(\xi)$, and $J_2^2(\xi)$. By combining these terms, one gets the expression

$$L_0^0 = -1 + J_0^2(kv_\perp/\Omega_i) + J_0^2(kv_\perp/\Omega_i) \times\left[\frac{1}{2}\frac{\omega_{ti}^2\xi^2}{\left(\omega^2-\omega_{ti}^2\right)} + \frac{3}{8}\frac{\omega_{ti}^4\xi^4}{\left(\omega^2-\omega_{ti}^2\right)\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (48)$$

which does not contain any singularities at $v_\parallel \to 0$, since $\omega_{ti}\xi = \tilde{\omega}_d/\omega$.

Similarly, one gets the following expressions for other coefficients

$$L_0^1 = iJ_0^2(kv_\perp/\Omega_i) \times\left[\frac{\xi\omega_{ti}}{2(\omega-\omega_{ti})} + \frac{3}{8}\frac{\omega_{ti}^3\xi^3}{\left(\omega^2-\omega_{ti}^2\right)(\omega-2\omega_{ti})}\right], \quad (49)$$

$$L_0^{-1} = -iJ_0^2(kv_\perp/\Omega_i) \times\left[\frac{\omega_{ti}\xi}{2(\omega+\omega_{ti})} + \frac{3}{8}\frac{\omega_{ti}^3\xi^3}{\left(\omega^2-\omega_{ti}^2\right)(\omega+2\omega_{ti})}\right], \quad (50)$$

$$L_0^2 = -J_0^2(kv_\perp/\Omega_i)\left[\frac{\omega_{ti}^2\xi^2}{4(\omega-\omega_{ti})(\omega-2\omega_{ti})}\right], \quad (51)$$

$$L_0^{-2} = -J_0^2(kv_\perp/\Omega_i)\left[\frac{\omega_{ti}^2\xi^2}{4(\omega+\omega_{ti})(\omega+2\omega_{ti})}\right]. \quad (52)$$

According to Eqs. (6) and (7), it is more convenient to write the perturbed distribution function in terms of the $\cos\theta$ and $\sin\theta$ components of $\phi$. Introducing

$$\chi_{js} = i\left(\chi_j - \chi_{-j}\right), \qquad \chi_{jc} = \chi_j + \chi_{-j},$$

for $\chi_j = \left(\phi_j, f_j\right)$, where $j = 1, 2$, the principal component takes the form

$$f_0 = \frac{eF_0}{T_i}\left(\phi_0 M_0^0 + \phi_{1s}M_0^{1s} + \phi_{1c}M_0^{1c} + \phi_{2c}M_0^{2c} + \phi_{2s}M_0^{2s}\right), \quad (53)$$

where

$$M_0^0 = L_0^0 = -1 + J_0^2(kv_\perp/\Omega_i) + J_0^2(kv_\perp/\Omega_i) \times\left[\frac{1}{2}\frac{\omega_{ti}^2\xi^2}{\left(\omega^2-\omega_{ti}^2\right)} + \frac{3}{8}\frac{\omega_{ti}^4\xi^4}{\left(\omega^2-\omega_{ti}^2\right)\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (54)$$

$$M_0^{1c} = \frac{1}{2}\left(L_0^1 + L_0^{-1}\right) = iJ_0^2(kv_\perp/\Omega_i)\frac{\omega_{ti}^2\xi}{2\left(\omega^2-\omega_{ti}^2\right)}\left[1+\frac{3}{2}\frac{\omega_{ti}^2\xi^2}{\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (55)$$

$$M_0^{1s} = \frac{1}{2i}\left(L_0^1 - L_0^{-1}\right) = J_0^2(kv_\perp/\Omega_i)\frac{\omega\omega_{ti}\xi}{2\left(\omega^2-\omega_{ti}^2\right)}\left[1+\frac{3}{4}\frac{\omega_{ti}^2\xi^2}{\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (56)$$

$$M_0^{2c} = \frac{1}{2}\left(L_0^2 + L_0^{-2}\right) = -J_0^2(kv_\perp/\Omega_i)\left[\frac{\omega_{ti}^2\xi^2\left(\omega^2+2\omega_{it}^2\right)}{4\left(\omega^2-\omega_{ti}^2\right)\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (57)$$

$$M_0^{2s} = \frac{1}{2i}\left(L_0^2 - L_0^{-2}\right) = \frac{3i}{4}J_0^2(kv_\perp/\Omega_i)\left[\frac{\omega\omega_{ti}^3\xi^2}{\left(\omega^2-\omega_{ti}^2\right)\left(\omega^2-4\omega_{ti}^2\right)}\right]. \quad (58)$$

Note that upon integration over $v_\parallel$, $M_0^{1c}$ and $M_0^{2s}$ do not give any contribution to density moments. The remaining term, in the limit $\omega \gg \omega_{ti}$ simplify to the expressions

$$M_0^0 = -1 + J_0^2(kv_\perp/\Omega_i)\left[1+\frac{1}{2}\frac{\tilde{\omega}_{di}^2}{\omega^2}+\frac{3}{8}\frac{\tilde{\omega}_{di}^4}{\omega^4}\right],$$

$$M_0^{1s} = J_0^2(kv_\perp/\Omega_i)\frac{\tilde{\omega}_{di}}{2\omega}\left[1+\frac{3}{4}\frac{\tilde{\omega}_{di}^2}{\omega^2}\right],$$

$$M_0^{2c} = -J_0^2(kv_\perp/\Omega_i)\frac{\tilde{\omega}_{di}^2}{4\omega^2}.$$

They are identical to Eq. (13) taking into account that $\tilde{\omega}_d = -\omega_d$. A similar procedure gives the following expression for the $\sin\theta$ parity of the first side band:

$$f_{1s} = \frac{eF_0}{T_i}\left\{\phi_0 M_{1s}^0 + \phi_{1s} M_{1s}^{1s} + \phi_{2c} M_{1s}^{2c}\right\}. \quad (59)$$

The full expressions for the components of the **M** matrix are given in the Appendix B. In the limit $\omega \gg \omega_t$, the relevant components are

$$M_{1s}^0 = J_0^2(kv_\perp/\Omega_i)\left[\frac{\tilde{\omega}_{di}}{\omega} + \frac{3}{4}\frac{\tilde{\omega}_{di}^3}{\omega^3}\right], \quad (60)$$

$$M_{1s}^{1s} = -1 + J_0^2(kv_\perp/\Omega_i)\left[1 + \frac{3}{4}\frac{\tilde{\omega}_{di}^2}{\omega^2}\right], \quad (61)$$

$$M_{1s}^{2c} = -J_0^2(kv_\perp/\Omega_i)\frac{\tilde{\omega}_{di}}{2\omega}. \quad (62)$$

Correspondingly,

$$f_{2c} = \frac{eF_0}{T_i}\left(\phi_0 M_{2c}^0 + \phi_{1s} M_{2c}^{1s} + \phi_{2c} M_{2c}^{2c}\right), \quad (63)$$

with the expressions

$$M_{2c}^0 = -J_0^2(kv_\perp/\Omega_i)\frac{\tilde{\omega}_{di}^2}{2\omega^2}, \quad (64)$$

$$M_{2c}^{1s} = -J_0^2(kv_\perp/\Omega_i)\frac{\tilde{\omega}_{di}}{2\omega}, \quad (65)$$

$$M_{2c}^{2c} = -1 + J_0^2(kv_\perp/\Omega_i). \quad (66)$$

Note that expressions (59) and (63) with formulas (60)–(66) are identical to expressions (14) and (15). The full expressions in the Appendix C clearly show the coupling of the first and second harmonic resonances $\omega^2 = v_\parallel^2/q^2R^2$ and $\omega^2 = 4v_\parallel^2/q^2R^2$. One should note, however, that the infinite sum of the products of the Bessel function in Eqs. (41)–(45) can only be truncated by assuming that $\xi = \tilde{\omega}_{di}/\omega_t = \tilde{\omega}_{di}qR/v_\parallel$ is a small parameter. In fact, this is not required for GAMs, not even for weak dispersion case $\left(k_r^2\rho_i^2, k_r^2\rho_s^2\right) < 1$. In the GAM frequency ordering, we have $\omega \simeq \omega_{\text{GAM}} \simeq v_{Ti}/R \gg (\omega_d, v_{Ti}/qR)$, $q > 1$. Somewhat artificial truncation of the infinite sum of the Bessel function has some effect on the resonant damping, which is discussed in the Summary.

## 5. SUMMARY

The problem of dispersion of GAMS has been revisited in this paper. There are several methods for calculation of GAM dispersion in the literature. In this paper, we have recalculated the dispersive corrections to the standard GAM frequency by accurate expansion in small parameters $\tau_e\left(I_2, K_2, K_1^2\right) < 1$. It is shown that our expression is identical to the one obtained in [38] by using the kinetic theory for ballooning Alfvén waves [11]. The same result was also obtained by using very different variational approach in [44]. Explicit expressions for GAM dispersion given in [43, 45–47] are inaccurate in various ways.

We have also analyzed the details of the alternative approach based on the exact solution of the drift-kinetic equation as in [20, 36, 44, 52, 53]. An inherent problem of this methods is a representation by the infinite sum of Bessel functions. Practical application of this series requires a truncation which is possible for small values of the Bessel function argument $\xi = \tilde{\omega}_{di}/\omega_t = \tilde{\omega}_{di}qR/v_\parallel$. In general, this is not a small parameter for the GAM frequency. Another problem is that the formal expansion generates expressions which are diverging in calculations of the density perturbation, due to the arguments containing $1/v_\parallel^{2n}$ contributions. These divergences are superficial; as we have shown in Section 4 above, the divergent terms are cancelled out when all terms of the same order are included in the expansion of the Bessel functions. This problem was also recognized and treated in a different way in [20]. It was pointed out in [44] that Landau damping of GAMs is enhanced when the second-order resonances $\omega^2 = 4v_\parallel^2/q^2R^2$ are taken into account; however, only the resonant parts of the second-order terms were included to avoid divergences (see, e.g., Eq. (2.6) in [44]). Another inherent difficulty in using exact solution (39) appears when one consider the resonant contributions for high values of $q$. In this case, the argument of the Bessel functions $\xi = \tilde{\omega}_{di}/\omega_t = \tilde{\omega}_{di}qR/v_\parallel$ is not small, formally, series (39) cannot be truncated and large number of terms have to be included with full Bessel functions, as was shown by the comparison with direct numerical simulations in [54]. In fact, GAMs still will be damped even in the limit $q \to \infty$ due to the toroidal resonances. Such a damping will be especially large for short wavelength modes [46].

We would like to note that coupling of the parallel and toroidal resonances exists even for the first-order terms. This is apparent when one uses the approach used in [39, 40, 51]. Using this method, one obtains the perturbed distribution function as given by Eqs. (113) and (114). These expressions clearly show the contribution of the toroidal resonances $\omega^2 = \omega_d^2/2$, which remain active even in the limit of $q \to \infty$. The resonances are equivalent to the contributions of infinite series (39) in the high $q$ limit. The authors of [46] have used a perturbative method similar to that of [39, 40, 51] and have considered the contribution of the toroidal resonance separately. However, they have not included the coupling of the higher harmonics to the lower order terms given by the last terms on the left-hand sides of Eqs. (8) and (9) (cf. Eqs. (8) and (9) with Eq. (5)–(7) in [46]). As result, the dispersive corrections to the real frequency given in [46] are not accurate (cf. our Eq. (28) with Eq. (9) of [46]). The calculations of the GAM damping due to toroidal resonances based on the method of [39, 40, 51] will be reported in a separate publication.

*APPENDIX A*

## AUXILIARY INTEGRALS

$$K_1 \equiv \left\langle \frac{\overline{\omega}_{di}}{\omega} \right\rangle = \frac{\omega_{di}}{\omega}, \tag{67}$$

$$K_2 \equiv \left\langle \frac{\overline{\omega}_{di}^2}{\omega^2} \right\rangle = \frac{7}{4}\frac{\omega_{di}^2}{\omega^2}, \tag{68}$$

$$K_3 \equiv \left\langle \frac{\overline{\omega}_{di}^3}{\omega^3} \right\rangle = \frac{9}{2}\frac{\omega_{di}^3}{\omega^3}, \tag{69}$$

$$K_4 \equiv \left\langle \frac{\overline{\omega}_{di}^4}{\omega^4} \right\rangle = \frac{249}{16}\frac{\omega_{di}^4}{\omega^4}, \tag{70}$$

$$I_2 \equiv \left\langle \frac{k_r^2 v_\perp^2}{\omega_{ci}^2} \right\rangle = k_r^2\rho_i^2, \tag{71}$$

$$I_4 \equiv \left\langle \frac{k_r^4 v_\perp^4}{\omega_{ci}^4} \right\rangle = 2k_r^4\rho_i^4, \tag{72}$$

$$I_{3d} \equiv \left\langle \frac{\overline{\omega}_{di}}{\omega}\frac{k_r^2 v_\perp^2}{\omega_{ci}^2} \right\rangle = \frac{3}{2}\frac{\omega_{di}}{\omega}k_r^2\rho_i^2, \tag{73}$$

$$I_{4d} \equiv \left\langle \frac{\overline{\omega}_{di}^2}{\omega^2}\frac{k_r^2 v_\perp^2}{\omega_{ci}^2} \right\rangle = \frac{13}{4}\frac{\omega_{di}^2}{\omega^2}k_r^2\rho_i^2, \tag{74}$$

$$\rho_i^2 = v_{Ti}^2/\omega_{ci}^2.$$

The angle brackets $\langle ... \rangle$ mean the averaging over the Maxwellian distribution function.

*APPENDIX B*

## FINAL EXPRESSIONS FOR THE FIRST AND SECOND HARMONICS OF THE PERTURBED DISTRIBUTION FUNCTION

The $m = \pm 1$ components of the perturbed distribution function can be written from Eq. (38) in the form

$$f_1 = \frac{eF_0}{T_i}\left\{\phi_0 L_1^0 + \phi_1 L_1^1 + \phi_{-1} L_1^{-1} + \phi_2 L_1^2 + \phi_{-2} L_1^{-2}\right\}, \tag{75}$$

$$f_{-1} = \frac{eF_0}{T_i} \times\left\{\phi_0 L_{-1}^0 + \phi_1 L_{-1}^1 + \phi_{-1} L_1^{-1} + \phi_2 L_{-1}^2 + \phi_{-2} L_{-1}^{-2}\right\}, \tag{76}$$

where

$$L_1^0 = -iJ_0^2(kv_\perp/\Omega_i) \times\left[\frac{\omega_{ti}\xi}{2(\omega-\omega_{ti})} + \frac{3}{8}\frac{\omega_{ti}^3\xi^3}{\left(\omega^2-\omega_{ti}^2\right)(\omega-2\omega_{ti})}\right], \tag{77}$$

$$L_1^1 = -1 + J_0^2(kv_\perp/\Omega_i) \times\left[\frac{\omega}{\omega-\omega_{ti}} + \frac{1}{2}\frac{\omega_{ti}^2\xi^2}{(\omega-\omega_{ti})(\omega-2\omega_{ti})}\right], \tag{78}$$

$$L_1^{-1} = -\frac{1}{4}J_0^2(kv_\perp/\Omega_i)\left[\frac{\omega_{ti}^2\xi^2}{\omega^2-\omega_{ti}^2}\right], \tag{79}$$

$$L_1^2 = iJ_0^2(kv_\perp/\Omega_i)\left[\frac{1}{2}\frac{\omega\omega_{ti}\xi}{(\omega-\omega_{ti})(\omega-2\omega_{ti})}\right], \tag{80}$$

$$L_1^{-2} = 0, \tag{81}$$

$$L_{-1}^0 = iJ_0^2(kv_\perp/\Omega_i) \times\left[\frac{\omega_{ti}\xi}{2(\omega+\omega_{ti})} + \frac{3}{8}\frac{\omega_{ti}^3\xi^3}{\left(\omega^2-\omega_{ti}^2\right)(\omega+2\omega_{ti})}\right], \tag{82}$$

$$L_{-1}^1 = -\frac{1}{4}J_0^2(kv_\perp/\Omega_i)\left[\frac{\omega_{ti}^2\xi^2}{\omega^2-\omega_{ti}^2}\right], \tag{83}$$

$$L_{-1}^{-1} = -1 + J_0^2(kv_\perp/\Omega_i) \times\left[\frac{\omega}{\omega+\omega_{ti}} + \frac{1}{2}\frac{\omega_{ti}^2\xi^2}{(\omega+\omega_{ti})(\omega+2\omega_{ti})}\right], \tag{84}$$

$$L_{-1}^2 = 0, \tag{85}$$

$$L_{-1}^{-2} = -iJ_0^2(kv_\perp/\Omega_i)\left[\frac{1}{2}\frac{\omega\omega_{ti}\xi}{(\omega+\omega_{ti})(\omega+2\omega_{ti})}\right]. \tag{86}$$

The second-order harmonics $m = \pm 2$ are given by

$$f_2 = \frac{eF_0}{T_i}\left\{\phi_0 L_2^0 + \phi_1 L_2^1 + \phi_{-1} L_2^{-1} + \phi_2 L_2^2 + \phi_{-2} L_2^{-2}\right\}, \tag{87}$$

$$f_{-2} = \frac{eF_0}{T_i}\left\{\phi_0 L_{-2}^0 + \phi_1 L_{-2}^1 + \phi_{-1} L_{-2}^{-1} + \phi_2 L_{-2}^2 + \phi_{-2} L_{-2}^{-2}\right\}, \tag{88}$$

$$L_2^0 = -\frac{1}{4}J_0^2(kv_\perp/\Omega_i)\left[\frac{\omega_{ti}^2\xi^2}{(\omega-\omega_{ti})(\omega-2\omega_{ti})}\right], \tag{89}$$

$$L_2^1 = -iJ_0^2(kv_\perp/\Omega_i)\left[\frac{1}{2}\frac{\omega\omega_{ti}\xi}{(\omega-\omega_{ti})(\omega-2\omega_{ti})}\right], \tag{90}$$

$$L_2^{-1} = 0, \tag{91}$$

$$L_2^2 = -1 + J_0^2(kv_\perp/\Omega_i)\frac{\omega}{\omega-2\omega_{ti}}, \tag{92}$$

$$L_2^{-2} = 0, \tag{93}$$

$$L_{-2}^{0} = -\frac{1}{4}J_0^2(kv_\perp/\Omega_i)\left[\frac{\omega_{ti}^2\xi^2}{(\omega+\omega_{ti})(\omega+2\omega_{ti})}\right], \quad (94)$$

$$L_{-2}^{1} = 0, \quad (95)$$

$$L_{-2}^{-1} = iJ_0^2(kv_\perp/\Omega_i)\left[\frac{1}{2}\frac{\omega\omega_{ti}\xi}{(\omega+\omega_{ti})(\omega+2\omega_{ti})}\right], \quad (96)$$

$$L_{-2}^{2} = 0, \quad (97)$$

$$L_{-2}^{-2} = -1 + J_0^2(kv_\perp/\Omega_i)\frac{\omega}{\omega+2\omega_{ti}}, \quad (98)$$

with

$$M_{1s}^{0} = i\left(L_1^0 - L_{-1}^0\right) = J_0^2(kv_\perp/\Omega_i) \times\left[\frac{\omega_{ti}\xi\omega}{\left(\omega^2-\omega_{ti}^2\right)} + \frac{3}{4}\frac{\omega_{ti}^3\xi^3\omega}{\left(\omega^2-\omega_{ti}^2\right)\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (99)$$

$$M_{1s}^{1c} = \frac{i}{2}\left(L_1^1 - L_{-1}^{-1}\right) = iJ_0^2(kv_\perp/\Omega_i)\frac{\omega\omega_{ti}}{\omega^2-\omega_{ti}^2}\left[1+\frac{3}{2}\frac{\omega_{ti}^2\xi^2}{\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (100)$$

$$M_{1s}^{1s} = \frac{1}{2}\left[L_1^1 - L_{-1}^1 - \left(L_1^{-1} - L_{-1}^{-1}\right)\right] = -1 + J_0^2(kv_\perp/\Omega_i)\left[\frac{\omega^2}{\omega^2-\omega_{ti}^2} + \frac{1}{4}\frac{\omega_{ti}^2\xi^2}{\omega^2-\omega_{ti}^2} + \frac{1}{2}\frac{\omega_{ti}^2\xi^2\left(\omega^2+2\omega_{it}^2\right)}{\left(\omega^2-\omega_{ti}^2\right)\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (101)$$

$$M_{1s}^{2c} = \frac{i}{2}\left(L_1^2 - L_{-1}^{-2}\right) = -J_0^2(kv_\perp/\Omega_i)\frac{1}{2}\left[\frac{\omega\omega_{ti}\xi\left(\omega^2+2\omega_{it}^2\right)}{\left(\omega^2-\omega_{ti}^2\right)\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (102)$$

$$M_{1s}^{2s} = \frac{1}{2}\left[L_1^2 + L_{-1}^{-2}\right] = iJ_0^2(kv_\perp/\Omega_i)\frac{3}{2}\left[\frac{\omega^2\omega_{ti}^2\xi}{\left(\omega^2-\omega_{ti}^2\right)\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (103)$$

$$M_{2c}^{0} = -\frac{1}{2}J_0^2(kv_\perp/\Omega_i)\omega_{ti}^2\xi^2 \times\left[\frac{\omega^2+2\omega_{it}^2}{\left(\omega^2-\omega_{ti}^2\right)\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (104)$$

$$M_{2c}^{1c} = iJ_0^2(kv_\perp/\Omega_i)\left[\frac{3}{2}\frac{\omega^2\omega_{ti}^2\xi}{\left(\omega^2-\omega_{ti}^2\right)\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (105)$$

$$M_{2c}^{1s} = -J_0^2(kv_\perp/\Omega_i)\left[\frac{1}{2}\frac{\omega\omega_{ti}\xi\left(\omega^2+2\omega_{it}^2\right)}{\left(\omega^2-\omega_{ti}^2\right)\left(\omega^2-4\omega_{ti}^2\right)}\right], \quad (106)$$

$$M_{2c}^{2c} = -1 + J_0^2(kv_\perp/\Omega_i)\frac{\omega^2}{\left(\omega^2-4\omega_{ti}^2\right)}, \quad (107)$$

$$M_{2c}^{2s} = iJ_0^2(kv_\perp/\Omega_i)\frac{2\omega\omega_{ti}}{\left(\omega^2-4\omega_{ti}^2\right)}. \quad (108)$$

*APPENDIX C*

## CONTRIBUTIONS OF TOROIDAL RESONANCES

As it was shown in [39], the parallel Landau $\omega = v_\parallel/qR$ and toroidal resonances $\omega = \omega_d$ are coupled already for the first harmonics. For completeness we give relevant expressions for the harmonics of the perturbed distribution function here. The corresponding equations can be written in the form

$$\mathbf{D}\cdot\mathbf{X} = \Psi, \quad (109)$$

where the vector $\mathbf{X} = (g_0, g_{1s})$ and the vector $\Psi$ is given by

$$\Psi = \omega\frac{e_\alpha F_{M\alpha}}{T_\alpha}J_0^2(z_\alpha)\begin{pmatrix}\phi_0\\ \phi_{1s}\end{pmatrix}. \quad (110)$$

The corresponding solutions for $f_0$ and $f_{1s}$ are [39]

$$f_0 = -\frac{e_\alpha F_{M\alpha}}{T_\alpha}\left[\phi_0 + \frac{J_0^2(z_\alpha)}{W} \times\left[\frac{i\omega_{d\alpha}\omega}{2}\phi_{1s} - \left(\omega^2 - \frac{v_\parallel^2}{q^2R^2}\right)\phi_0\right]\right], \quad (111)$$

$$f_{1s} = -\frac{e_\alpha F_{M\alpha}}{T_\alpha}\left[\phi_{1s} + \frac{J_0^2(z_\alpha)}{W}\left[-\omega^2\phi_{1s} - i\omega\omega_{d\alpha}\phi_0\right]\right]. \quad (112)$$

Here, $W = \omega^2 - v_\parallel^2/q^2R^2 - \omega_{d\alpha}^2/2$ clearly shows the coupling of the toroidal and ion-sound (parallel) resonances. To compare with Eq. (39), we write Eqs. (111) and (112) in the form

$$f_0 = \frac{eF_0}{T_i}\left(\phi_0 G_0^0 + \phi_{1s}G_0^{1s}\right), \quad (113)$$

$$f_{1s} = \frac{eF_0}{T_i}\left(\phi_0 G_{1s}^0 + \phi_{1s}G_{1s}^{1s}\right), \quad (114)$$

where

$$G_0^0 = -1 + J_0^2(z_i)\frac{\left[\omega^2-\omega_{ti}^2\right]}{\omega^2-\omega_{ti}^2-\omega_{di}^2/2}, \quad (115)$$

$$G_0^{1s} = -iJ_0^2(z_i)\frac{\omega\omega_{ti}\xi}{2\left(\omega^2-\omega_{ti}^2-\omega_{di}^2/2\right)}, \quad (116)$$

$$G_{1s}^{0} = iJ_0^2(z_i)\frac{\omega\omega_{ti}\xi}{\omega^2 - \omega_{ti}^2 - \omega_{di}^2/2}, \quad (117)$$

$$G_{1s}^{1s} = -1 + J_0^2(z_i)\frac{\omega^2}{\omega^2 - \omega_{ti}^2 - \omega_{di}^2/2}. \quad (118)$$

It is straightforward to generalize this approach to include the second harmonics [51].

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