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Quantum Dynamics in a Comb Geometry: Green Function Solutions with Nonlocal and Fractional Potentials

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Abstract

We investigate a generalized quantum Schrödinger equation in a comb-like structure that imposes geometric constraints on spatial variables. The model is extended by the introduction of nonlocal and fractional potentials to capture memory effects in both space and time. We consider four distinct scenarios: (i) a time-dependent nonlocal potential, (ii) a spatially nonlocal potential, (iii) a combined space–time nonlocal interaction with memory kernels, and (iv) a fractional spatial derivative, which is related to distributions asymptotically governed by power laws and to a position-dependent effective mass. For each scenario, we propose solutions based on the Green's function for arbitrary initial conditions and analyze the resulting quantum dynamics. Our results reveal distinct spreading regimes, depending on the type of non-locality and the fractional operator applied to the spatial variable. These findings contribute to the broader generalization of comb models and open new questions for exploring quantum dynamics in backbone-like structures.

Keywords: comb models; quantum dynamics; Green's function

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1. Introduction

The Schrödinger equation represents an important breakthrough in describing non-relativistic quantum systems across various physical scenarios. It supports our understanding of microscopic phenomena and has become a cornerstone of modern quantum theory [1]. Initially formulated by Schrödinger in a series of seminal papers [2,3], the equation was derived from classical mechanics through a judicious choice of the action variable, leading to a variational principle that yields a partial differential equation: the Schrödinger equation. Its solutions, known as wave functions, describe the temporal evolution of quantum wave packets. However, obtaining analytical solutions is often challenging, especially for systems with complex or non-trivial potentials [4]. Once the wave function is determined, it enables the extraction of physical properties such as probability densities [5] and the diffusion behavior of quantum particles [6].

Despite its success, the Schrödinger equation has been extended in several directions to encompass experimental results. These extensions include anomalous relaxation processes [7,8], in which the wave function may exhibit stretched-exponential or power-law decay, instead of the exponential decay predicted by the traditional formalism [9,10]. An important class of generalizations introduces fractional derivatives into the Schrödinger equation [4,6,11–14], providing a compact and elegant framework to model memory effects, nonlocal correlations, and dissipative dynamics [7]. Another direction involves non-linear modifications, often motivated by connections to porous media equations and non-extensive statistical mechanics [15,16], particularly those based on Tsallis entropy [17–20]. The comb model is another intriguing extension to describe anomalous diffusion [21]. This model features a branched geometry that resembles a backbone with perpendicular fingers, where a particle that undergoes a random walk may enter and become temporarily trapped in the fingers before returning to the main axis. This structure gives rise to an anomalous diffusion, where the mean square displacement (MSD) scales as $\langle x^2(t)\rangle \sim t^\mu$, with $0 < \mu < 1$ (subdiffusive regime) or $\mu > 1$ (superdiffusive regime), depending on the configuration [22].

Further generalizations of the comb model [23] incorporate fractional time derivatives [24] and fractal geometries, as in the Refs. [25,26], which extended these ideas by incorporating linear reactions and stochastic resetting within a fractional comb framework [27,28]. From a biological perspective, comb-like geometries have been employed to model transport along spiny dendrites, which exhibit subdiffusive dynamics [29]. This diffusion regime arises from trapping mechanisms in the finger regions and is effectively described by time-fractional operators. Although comb structures have been extensively studied in classical diffusion, their quantum counterparts remain relatively unexplored [30–32]. In a quantum comb model, particle motion along the central axis (x) is confined to the line $y = 0$ due to a delta-function potential in the transverse (y) direction [30]. Notably, the fractional-time Schrödinger equation (FTSE) of order $1/2$ emerges naturally as a special case within this framework.

Here, we analyze a generalized Schrödinger equation that incorporates a backbone structure with branches (comb-like structure), fractional derivative in space [33,34], and a generic time-dependent external potential. To account these features, we write the Schrödinger equation in the following form

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\left[\delta\left(\frac{y}{l_y}\right)D_x^{\mu,\eta}\psi(\mathbf{r},t) + \frac{\partial^2}{\partial y^2}\psi(\mathbf{r},t)\right] + \mathcal{V}[\psi(\mathbf{r},t)], \tag{1}$$

where $\mathcal{V}[\psi(\mathbf{r},t)]$ it is given by

$$\begin{aligned}\mathcal{V}[\psi(\mathbf{r},t)] = &\int_0^t dt' V_{xy}^{(1)}(x,y;t-t')\psi(\mathbf{r},t') + \int_0^t dt' V_{xy}^{(2)}(y;t-t')\psi(\mathbf{r},t')\\ &+ \int_0^t dt' \int_{-\infty}^{\infty} dx' V_{xy}^{(3)}(x-x',y,t-t')\psi(x',y,t')\ .\end{aligned} \tag{2}$$

We consider four configurations in Equation (2). First, we address a nonlocal dependence on time, i.e., the external potential is time-dependent and is expressed as a product of spatial delta functions. The second setup simultaneously incorporates nonlocal dependencies in both space and time. For this, we introduce two external potentials: one that has dependence on space variables and the other that is time-dependent. In the third scenario, we consider nonlocal dependence and memory kernels, i.e., we mix the previous scenarios by incorporating memory on time and a fractional derivative in space [33,34]. For the last configuration, we include a fractional spatial operator and nonlocal terms. For the posed problems, we obtain the corresponding solutions through Green's function and show that the diffusion is anomalous, exhibiting a super-diffusive regime.

This paper is organized as follows. Section 2 introduces the generalized Schrödinger equation of the comb, and the subsequent subsections present the particular problems. In Section 3, we present our discussions, findings, and outline potential future directions.

2. Schrödinger Equation in a Comb-Model

Let us now investigate Equation (1) in connection with Equation (2) by considering different cases.

2.1. Nonlocal Dependence on Time

First, let us consider the Schrödinger equation with the geometric constraints between the directions x and y subjected to external potentials, which may have a time dependence. For this case, we consider integer operators in spatial variables, where Equation (1) results in

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\left[\delta\left(\frac{y}{l_y}\right)\frac{\partial^2}{\partial x^2}\psi(\mathbf{r},t) + \frac{\partial^2}{\partial y^2}\psi(\mathbf{r},t)\right] + \int_0^t dt' V_{xy}^{(1)}(x,y;t-t')\psi(\mathbf{r},t') + \int_0^t dt' V_{xy}^{(2)}(y;t-t')\psi(\mathbf{r},t'), \tag{3}$$

where $V_{xy}^{(1)}(x,y,t) = \mathfrak{K}_1(t)\delta(x/l_x)\delta(y/l_y)$ and $V_{xy}^{(2)}(y,t) = \mathfrak{K}_2(t)\delta(y/l_y)$ with the initial condition $\psi(\mathbf{r},0) = \varphi(\mathbf{r})$. Note that these terms have a nonlocal dependence on time, which results in a Schrödinger equation with nonlocal terms [35,36] or nonlocal potential [37,38].

To solve Equation (3), we use the Green function approach, leading to

$$i\hbar\frac{\partial}{\partial t}\mathcal{G}(\mathbf{r},\mathbf{r}',t) - i\hbar\delta(x-x')\delta(y-y')\delta(t) = -\frac{\hbar^2}{2m}\left[\delta\left(\frac{y}{l_y}\right)\frac{\partial^2}{\partial x^2}\mathcal{G}(\mathbf{r},\mathbf{r}',t) + \frac{\partial^2}{\partial y^2}\mathcal{G}(\mathbf{r},\mathbf{r}',t)\right] + \int_0^t dt' V_{xy}^{(1)}(x,y;t-t')\mathcal{G}(\mathbf{r},\mathbf{r}',t') + \int_0^t dt' V_{xy}^{(2)}(y;t-t')\mathcal{G}(\mathbf{r},\mathbf{r}',t'), \tag{4}$$

with $\mathbf{r}' = (x',y')$. By using the Fourier ($\mathfrak{F}\{\mathcal{G}(\mathbf{r},\mathbf{r}',t)\} = \widetilde{\mathcal{G}}(k_x,k_y,\mathbf{r}',t)$) and Laplace transforms ($\mathfrak{L}\{\mathcal{G}(\mathbf{r},\mathbf{r}',t)\} = \widehat{\mathcal{G}}(\mathbf{r},\mathbf{r}',s)$), we obtain

$$i\hbar s\widehat{\widetilde{\mathcal{G}}}(k_x,k_y,\mathbf{r}',s) - i\hbar e^{-ik_xx'}e^{-ik_yy'} = \frac{\hbar^2}{2m}\left[l_yk_x^2\widehat{\widetilde{\mathcal{G}}}(k_x,0,\mathbf{r}',s) + k_y^2\widehat{\widetilde{\mathcal{G}}}(k_x,k_y,\mathbf{r}',s)\right] + \widehat{\mathfrak{K}}_1(s)l_xl_y\widehat{\mathcal{G}}(0,0,\mathbf{r}',s) + \widehat{\mathfrak{K}}_2(s)l_y\widehat{\widetilde{\mathcal{G}}}(k_x,0,\mathbf{r}',s), \tag{5}$$

with $\widehat{\widetilde{\mathcal{G}}}(k_x,0,\mathbf{r}',s) = \widehat{\widetilde{\mathcal{G}}}(k_x,y=0,\mathbf{r}',s)$, and $\widehat{\mathfrak{K}}_i(s)$ ($i=1,2$) is the Laplace transform of the time contribution of the respective potentials $V_{xy}^i(x,y,t)$. After performing algebraic manipulation in Equation (5), we get

$$\widehat{\widetilde{\mathcal{G}}}(k_x,k_y,\mathbf{r}',s) = e^{-ik_xx'}e^{-ik_yy'}\widehat{\widetilde{\mathcal{G}}}_y(k_y,s) - \left[\frac{i\hbar}{2m}l_yk_x^2 + \frac{i}{\hbar}\widehat{\mathfrak{K}}_x(s)l_y\right]\widehat{\widetilde{\mathcal{G}}}(k_x,0,\mathbf{r}',s)\widehat{\widetilde{\mathcal{G}}}_y(k_y,s) - \frac{i}{\hbar}\widehat{\mathfrak{K}}_1(s)l_xl_y\widehat{\mathcal{G}}(0,0,\mathbf{r}',s)\widehat{\widetilde{\mathcal{G}}}_y(k_y,s), \tag{6}$$

with

$$\widehat{\widetilde{\mathcal{G}}}_y(k_y,s) = \frac{1}{s + i\hbar k_y^2/(2m)}. \tag{7}$$

The inverse Fourier transform in the y variables results in

$$\widehat{\widetilde{\mathcal{G}}}(k_x, y, \mathbf{r}', s) = e^{-ik_x x'}\widehat{\widetilde{\mathcal{G}}}_y(y-y', s) - \left[\frac{i\hbar}{2m}l_y k_x^2 + \frac{i}{\hbar}\widehat{\mathfrak{K}}_2(s)l_y\right]\widehat{\widetilde{\mathcal{G}}}(k_x, 0, \mathbf{r}', s)\widehat{\mathcal{G}}_y(y, s) - \frac{i}{\hbar}\widehat{\mathfrak{K}}_1(s)l_x l_y \widehat{\mathcal{G}}(0, 0, \mathbf{r}', s)\widehat{\mathcal{G}}_y(y, s). \tag{8}$$

From this equation, it is possible to show that

$$\widehat{\widetilde{\mathcal{G}}}(k_x, 0, \mathbf{r}', s) = e^{-ik_x x'}\widehat{\widetilde{\mathcal{G}}}_x(k_x, s)\widehat{\mathcal{G}}_y(y', s) - \frac{i}{\hbar}\widehat{\mathfrak{K}}_1(s)l_x l_y \widehat{\mathcal{G}}(0, 0, \mathbf{r}', s)\widehat{\mathcal{G}}_y(0, s)\widehat{\widetilde{\mathcal{G}}}_x^{(1)}(k_x, s), \tag{9}$$

with

$$\widehat{\widetilde{\mathcal{G}}}_x^{(1)}(k_x, s) = \frac{1}{1 + \left(i\hbar k_x^2/(2m) + (i/\hbar)\widehat{\mathfrak{K}}_2(s)\right)l_y \widehat{\mathcal{G}}_y(0, s)}\,. \tag{10}$$

By performing the inverse of the Fourier transform in the x variable in the previous equation, we obtain

$$\widehat{\mathcal{G}}(0, 0, \mathbf{r}', s) = \frac{\widehat{\mathcal{G}}_y(y', s)\widehat{\mathcal{G}}_x^{(1)}(x', s)}{1 + (i/\hbar)\widehat{\mathfrak{K}}_1(s)l_x l_y \widehat{\mathcal{G}}_y(0, s)\widehat{\mathcal{G}}_x^{(1)}(0, s)}\,. \tag{11}$$

By using these results, it is possible to show that

$$\widehat{\widetilde{\mathcal{G}}}(k_x, 0, \mathbf{r}', s) = e^{-ik_x x'}\widehat{\widetilde{\mathcal{G}}}_x^{(1)}(k_x, s)\widehat{\mathcal{G}}_y(y', s) - \frac{i}{\hbar}\frac{\widehat{\mathfrak{K}}_1(s)l_x l_y \widehat{\mathcal{G}}_y(y', s)\widehat{\mathcal{G}}_x^{(1)}(x', s)}{1 + (i/\hbar)\widehat{\mathfrak{K}}_1(s)l_x l_y \widehat{\mathcal{G}}_y(0, s)\widehat{\mathcal{G}}_x^{(1)}(0, s)}\widehat{\mathcal{G}}_y(0, s)\widehat{\widetilde{\mathcal{G}}}_x^{(1)}(k_x, s), \tag{12}$$

and, consequently,

$$\begin{aligned}\widehat{\mathcal{G}}(x, y, \mathbf{r}', s) = {} & \delta(x-x')\left[\widehat{\mathcal{G}}_y(y-y', s) - \widehat{\mathcal{G}}_y(|y|+|y'|, s)\right] \\ & + \left[\widehat{\mathcal{G}}_x^{(1)}(x-x', s) - \widehat{\mathcal{G}}^{(1)}(|x|+|x'|, s)\right]\widehat{\mathcal{G}}_y(|y|+|y'|, s) \\ & + \frac{\widehat{\mathcal{G}}_y(|y|+|y'|, s)}{1 + (i/\hbar)\widehat{\mathfrak{K}}_1(s)l_x l_y \widehat{\mathcal{G}}_y(0, s)\widehat{\mathcal{G}}_x^{(1)}(0, s)}\widehat{\mathcal{G}}_x^{(1)}(|x|+|x'|, s)\,.\end{aligned} \tag{13}$$

The straightforward inverse Laplace in Equation (13) is challenging due to the last term. To solve this problem, we propose an expansion in this term, allowing us to rewrite Equation (13) equal to

$$\begin{aligned}\widehat{\mathcal{G}}(x, y, \mathbf{r}', s) = {} & \delta(x-x')\left[\widehat{\mathcal{G}}_y(y-y', s) - \widehat{\mathcal{G}}_y(|y|+|y'|, s)\right] \\ & + \left[\widehat{\mathcal{G}}_x^{(1)}(x-x', s) - \widehat{\mathcal{G}}^{(1)}(|x|+|x'|, s)\right]\widehat{\mathcal{G}}_y(|y|+|y'|, s) \\ & + \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}l_x l_y\right)^n\left[\widehat{\mathfrak{K}}_1(s)\widehat{\mathcal{G}}_y(0, s)\widehat{\mathcal{G}}_x^{(1)}(0, s)\right]^n\widehat{\mathcal{G}}_y(|y|+|y'|, s)\widehat{\mathcal{G}}_x^{(1)}(|x|+|x'|, s)\,.\end{aligned} \tag{14}$$

From this expansion, we can obtain the inverse Laplace in Equation (14), which leads to

$$\begin{aligned}\mathcal{G}(x,y,\mathbf{r}',t) &= \delta(x-x')\left[\mathcal{G}_y(y-y',t)-\mathcal{G}_y(|y|+|y'|,t)\right] \\ &+\int_0^t dt'\left[\mathcal{G}_x^{(1)}(x-x',t-t')-\mathcal{G}^{(1)}(|x|+|x'|,t-t')\right]\mathcal{G}_y(|y|+|y'|,t') \\ &+\sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}l_x l_y\widehat{\mathcal{G}}_y(0,1)\widehat{\mathcal{G}}_x^{(1)}(0,1)\right)^n\int_0^t dt_n\mathcal{I}(t-t_n)\int_0^{t_n}dt_{n-1}\mathcal{I}(t_n-t_{n-1})\cdots \\ &\times\int_0^{t_2}dt_1\mathcal{I}(t_2-t_1)\int_0^{t_1}dt'\mathcal{G}_y(|y|+|y'|,t_1-t')\mathcal{G}_x^{(1)}(|x|+|x'|,t')\,,\end{aligned} \tag{15}$$

with $\mathcal{I}(t)=\big(1/\Gamma(1/4)\big)\int_0^t dt'\mathfrak{K}_1(t')/(t-t')^{3/4}$. From Equation (15), we obtain the wave function by using the following equation

$$\psi(\mathbf{r},\mathbf{r}',t)=\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy'\mathcal{G}(\mathbf{r},\mathbf{r}',t)\varphi(\mathbf{r}')\,, \tag{16}$$

subject to the initial condition previously defined.

To illustrate the behavior of solutions given by Equation (16), we set the configuration of the first problem (i) with an initial condition equal to $\varphi(\mathbf{r})=\delta(x)\delta(y)$ with $\mathfrak{K}_1(t)=(\mathfrak{K}_1/\tau)e^{-it/\tau}$ and $\mathfrak{K}_2(t)=0$. For this configuration, the behavior of the absolute value of the wave function $|\psi(x,y,t)|$ is displayed in Figure 1. The package that is initially centered in $(x,y)=(0,0)$ starts to spread along the space, exhibiting an oscillatory dynamic in one direction and a decay in another.

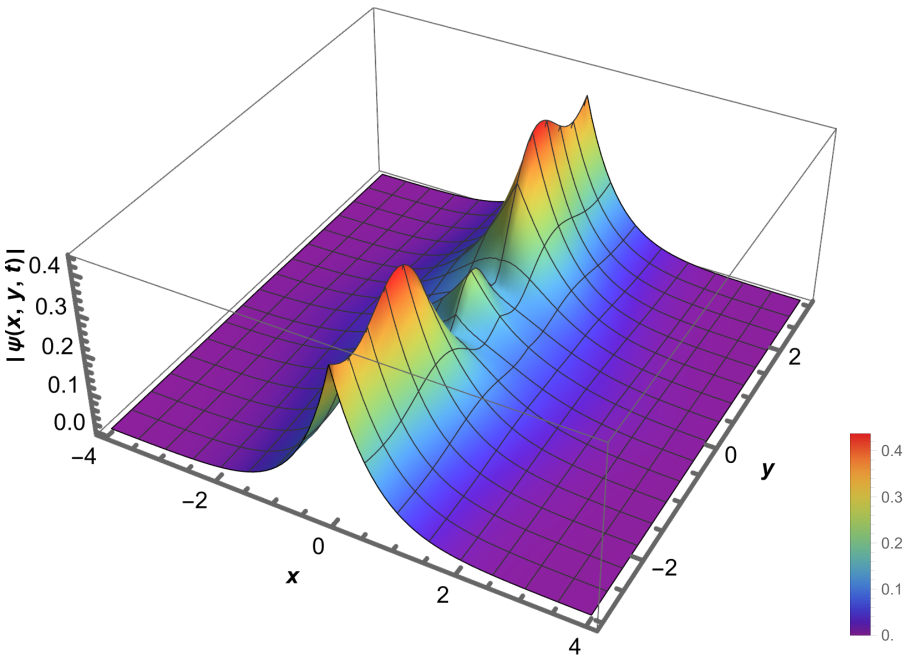

Figure 1. Behavior of the absolute value of the wave function for the initial condition $\varphi(\mathbf{r})=\delta(x)\delta(y)$ with $\mathfrak{K}_1(t)=(\mathfrak{K}_1/\tau)e^{-it/\tau}$ and $\mathfrak{K}_2(t)=0$. Without loss of generality, we consider $\tau=0.1$, $\mathfrak{K}_1/(\hbar\tau)=1$, $\hbar/m=1$, and $l_x=l_y=1$, in arbitrary unities.

To compare with the previous results, we set another parameter configuration (ii): $\varphi(\mathbf{r})=e^{-\mathbf{r}^2/2}/\sqrt{\pi}$ with $\mathfrak{K}_1(t)=\mathfrak{K}_1\delta(t)$ and $\mathfrak{K}_2(t)=0$, which represents a Gaussian package combined with a time delta potential. In the center of Figure 2, we observe a typical decay of an initial Gaussian package, which spreads along the space. However, this solution presents oscillatory waves that spread along the plane (x,y) due to the extra terms.

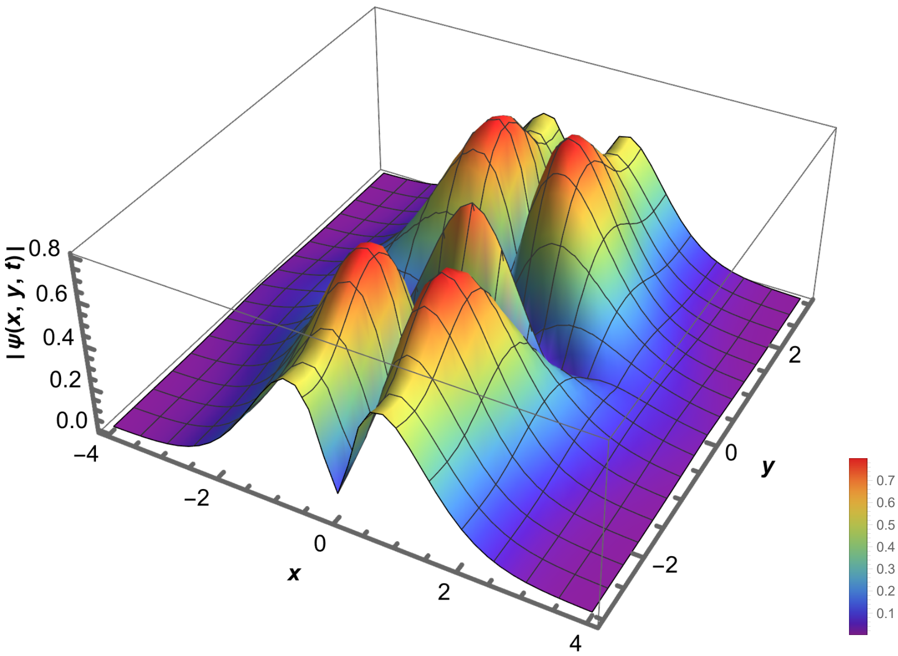

Figure 2. Behavior of the Green function the absolute value of the wave function for the initial condition $\varphi(\mathbf{r}) = e^{-\mathbf{r}^2/2}/\sqrt{\pi}$ with $\mathfrak{K}_1(t) = \mathfrak{K}_1\delta(t)$ and $\mathfrak{K}_2(t) = 0$. Without loss of generality, we consider $\mathfrak{K}_{xy}/\hbar = 1$, $\hbar/m = 1$, and $l_x = l_y = 1$, in arbitrary unities.

Additionally, we study the relaxation process for different scenarios: both the previously considered and the standard comb for two dimensions. The results are presented in Figure 3, characterized by the nonlocal terms for the initial condition and nonlocal dependence on time. By analyzing Figure 3, we verify different behaviors of the relaxation process for the wave function when memory effects are considered. Comparing our results (blue and black lines) with the standard comb model in two dimensions (orange line), we observe that both cases present a super-diffusive regime, where the former leads to $|\psi(0,0,t)|^{-2} \sim t^{2.5}$, while the latter leads to $|\psi(0,0,t)|^{-2} \sim t^{1.5}$. In this case, the additional terms, i.e., the nonlocal effects on time, make the relaxation process faster.

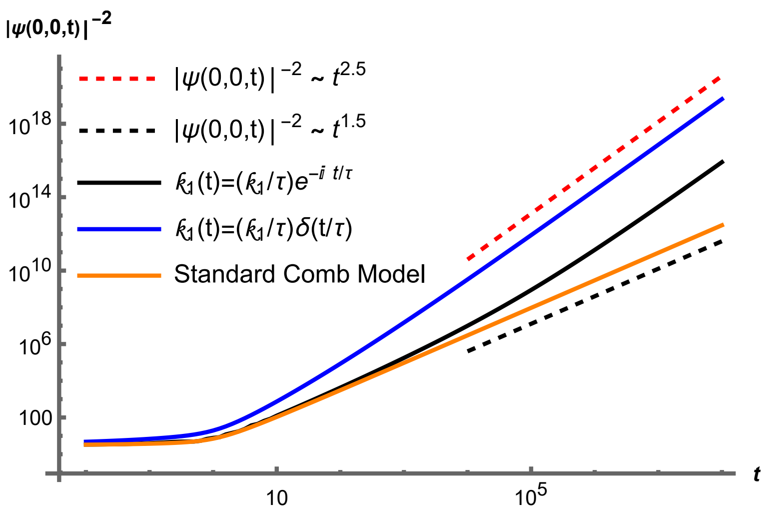

Figure 3. Behavior of the $|\psi(0,0,t)|^{-2}$ for the cases (*i*) $\varphi(\mathbf{r}) = \delta(x)\delta(y)$ with $\mathfrak{K}_1(t) = (\mathfrak{K}_1/\tau)e^{-it/\tau}$ and $\mathfrak{K}_x(t) = 0$) in the blue line; (*ii*) $\varphi(\mathbf{r}) = e^{-\mathbf{r}^2/2}/\sqrt{\pi}$ with $\mathfrak{K}_1(t) = (\mathfrak{K}_1/\tau)\delta(t/\tau)$ and $\mathfrak{K}_2(t) = 0$ in black line; and the standard two-dimensional comb model in orange line. Without loss of generality, we consider, for simplicity, $\mathfrak{K}_1/\hbar = 1$, $\hbar/m = 1$, and $l_x = l_y = 1$, in arbitrary unities.

2.2. Nonlocal Dependence on Space and Time

Now, we consider the nonlocal term with a spatial dependence on the variable x, instead of the nonlocal term at the origin with a nonlocal dependence on time, i.e., we consider the following Schrödinger equation:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) &= -\frac{\hbar^2}{2m}\left[\delta\left(\frac{y}{l_y}\right)\frac{\partial^2}{\partial x^2}\psi(\mathbf{r},t)+\frac{\partial^2}{\partial y^2}\psi(\mathbf{r},t)\right] \\ &+\int_0^t dt'\int_{-\infty}^{\infty}dx' V_{xy}^{(3)}(x-x',y,t-t')\psi(x',y,t')+\int_0^t dt' V_{xy}^{(2)}(y;t-t')\psi(\mathbf{r},t'), \end{aligned} \tag{17}$$

where $V_{xy}^{(3)}(x,y)=\mathfrak{K}_3(x,t)\delta(y/l_y)$ and $V_{xy}^{(2)}(y,t)=\mathfrak{K}_2(t)\delta(y/l_y)$ with the initial condition $\psi(\mathbf{r},0)=\varphi(\mathbf{r})$. In Equation (17), the kernel $\mathfrak{K}_3(x,t)$ introduces a nonlocal dependence on space and time, different from the previous case, which only considered a time dependence. It is worth noting that depending on the choice of the $\mathfrak{K}_3(x,t)$, we can relate this term with the fractional derivative in space such as the ones discussed in Refs. [39,40], which can be related to the Lévy distributions. One of them corresponds to the choice $\mathfrak{K}_3(x,t)=\mathfrak{K}(x)\delta(t)$ with $\mathfrak{F}\{\mathfrak{K}(x);k_x\}=-|k_x|^{\mu_x}$.

Analogously to the presented previous case, we solve Equation (17) through the Green function approach, yielding the following equation

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\mathcal{G}(\mathbf{r},\mathbf{r}',t)-i\hbar\delta(x-x')\delta(y-y')\delta(t) &= -\frac{\hbar^2}{2m}\left[\delta\left(\frac{y}{l_y}\right)\frac{\partial^2}{\partial x^2}\mathcal{G}(\mathbf{r},\mathbf{r}',t)+\frac{\partial^2}{\partial y^2}\mathcal{G}(\mathbf{r},\mathbf{r}',t)\right] \\ &+\int_0^t dt'\int_{-\infty}^{\infty}dx' V_{xy}^{(3)}(x-x',y,t-t')\mathcal{G}(\mathbf{r},\mathbf{r}',t')+\int_0^t dt' V_{xy}^{(2)}(y;t-t')\mathcal{G}(\mathbf{r},\mathbf{r}',t')\,. \end{aligned} \tag{18}$$

Now, using the Fourier and Laplace transforms, we obtain

$$\begin{aligned} i\hbar s\hat{\hat{\mathcal{G}}}(k_x,k_y,\mathbf{r}',s)-i\hbar e^{-ik_x x'}e^{-ik_y y'} &= \frac{\hbar^2}{2m}\left[l_y k_x^2\hat{\hat{\mathcal{G}}}(k_x,0,\mathbf{r}',s)+k_y^2\hat{\hat{\mathcal{G}}}(k_x,k_y,\mathbf{r}',s)\right] \\ &+\hat{\hat{\mathfrak{K}}}_3(k_x,s)l_y\hat{\hat{\mathcal{G}}}(k_x,0,\mathbf{r}',s)+\hat{\mathfrak{K}}_2(s)l_y\hat{\hat{\mathcal{G}}}(k_x,0,\mathbf{r}',s), \end{aligned} \tag{19}$$

which can be written as

$$\begin{aligned} \hat{\hat{\mathcal{G}}}(k_x,k_y,\mathbf{r}',s) &= e^{-ik_x x'}e^{-ik_y y'}\hat{\hat{\mathcal{G}}}_y(k_y,s)-\left[\frac{i\hbar}{2m}l_y k_x^2+\frac{i}{\hbar}\hat{\mathfrak{K}}_2(s)l_y\right]\hat{\hat{\mathcal{G}}}(k_x,0,\mathbf{r}',s)\hat{\hat{\mathcal{G}}}_y(k_y,s) \\ &-\frac{i}{\hbar}\hat{\hat{\mathfrak{K}}}_3(k_x,s)l_y\hat{\hat{\mathcal{G}}}(k_x,0,\mathbf{r}',s)\hat{\hat{\mathcal{G}}}_y(k_y,s)\,. \end{aligned} \tag{20}$$

By applying the inverse of the Fourier transform in the y variable and performing some calculations, we obtain that

$$\begin{aligned} \hat{\hat{\mathcal{G}}}(k_x,y,\mathbf{r}',s) &= e^{-k_x x'}\hat{\hat{\mathcal{G}}}_y(y-y',s) \\ &-\left[\frac{i\hbar}{2m}l_y k_x^2+\frac{i}{\hbar}l_y\left(\hat{\mathfrak{K}}_2(s)+\hat{\hat{\mathfrak{K}}}_3(k_x,s)\right)\right]\hat{\hat{\mathcal{G}}}(k_x,0,\mathbf{r}',s)\hat{\mathcal{G}}_y(y,s)\,. \end{aligned} \tag{21}$$

From this equation, it is possible to show that

$$\hat{\hat{\mathcal{G}}}(k_x,0,\mathbf{r}',s)=e^{-k_x x'}\hat{\hat{\mathcal{G}}}_x^{(2)}(k_x,s)\hat{\mathcal{G}}_y(y',s), \tag{22}$$

with

$$\hat{\hat{\mathcal{G}}}_x^{(2)}(k_x,s)=\frac{1}{1+\left[i\hbar k_x^2/(2m)+(i/\hbar)\left(\hat{\mathfrak{K}}_2(s)+\hat{\hat{\mathfrak{K}}}_3(k_x,s)\right)\right]l_y\hat{\mathcal{G}}_y(0,s)}\,. \tag{23}$$

By using these results, it is possible to show that

$$\widehat{\widetilde{\mathcal{G}}}(k_x, y, \mathbf{r}', s) = e^{-k_x x'}\widehat{\widetilde{\mathcal{G}}}_y(y-y', s) - \left[\frac{i\hbar}{2m}l_y k_x^2 + \frac{i}{\hbar}l_y\left(\widehat{\mathfrak{K}}_2(s) + \widehat{\widetilde{\mathfrak{K}}}_3(k_x, s)\right)\right]e^{-k_x x'}\widehat{\widetilde{\mathcal{G}}}_x^{(2)}(k_x, s)\widehat{\mathcal{G}}_y(y', s)\widehat{\mathcal{G}}_y(y, s) \tag{24}$$

and, consequently,

$$\widehat{\mathcal{G}}(x, y, \mathbf{r}', s) = \delta(x-x')\left[\widehat{\mathcal{G}}_y(y-y', s) - \widehat{\mathcal{G}}_y(|y|+|y'|, s)\right] + \widehat{\mathcal{G}}_x^{(2)}(x-x', s)\widehat{\mathcal{G}}_y(|y|+|y'|, s)\ . \tag{25}$$

The inverse Laplace transform applied in Equation (25) formally results in

$$\mathcal{G}(x, y, \mathbf{r}', t) = \delta(x-x')\left[\mathcal{G}_y(y-y', t) - \mathcal{G}_y(|y|+|y'|, t)\right] + \int_0^t dt'\mathcal{G}_x^{(2)}(x-x', t-t')\mathcal{G}_y(|y|+|y'|, t')\ , \tag{26}$$

which can be combined with Equation (16) to obtain the wave function related to this system.

The absolute value of the wave function, for $\varphi(\mathbf{r}) = e^{-\mathbf{r}^2/2}/\sqrt{\pi}$ for the case worked out in this section, is displayed in Figure 4. Now, observe that the nonlocal dependence on space and time leads the solution to a spread of a Gaussian package, without oscillation in the plane (x, y).

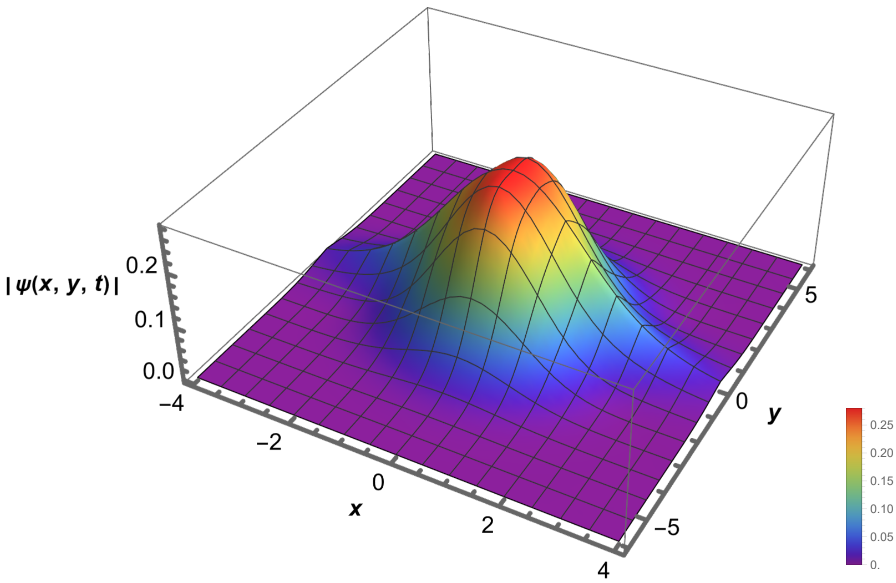

Figure 4. Behavior of the absolute value of the wave function for the initial condition $\varphi(\mathbf{r}) = e^{-\mathbf{r}^2/2}/\sqrt{\pi}$ with $\mathfrak{K}_3(t) = (\mathfrak{K}_3/\tau)\delta(t/\tau)\delta(x)$ and $\mathfrak{K}_2(t) = 0$. Without loss of generality, we consider $\mathfrak{K}_3/\hbar = 1$, $\hbar/m = 1$, $\tau = 1$, and $l_x = l_y = 1$, in arbitrary unities.

The wave function resulting from Equation (26) also leads the relaxation process in a super-diffusive regime for certain potential choices, as observed in Figure 5, where the blue line is for $\mathfrak{K}_2(t) = (\mathfrak{K}_2/\tau)\delta(t/\tau)$ and $\widetilde{\mathfrak{K}}_3(k_x, t) = \mathfrak{K}_3$ with $\varphi(\mathbf{r}) = \delta(x)\delta(y)$; and the black line is for $\mathfrak{K}_2(t) = (\mathfrak{K}_2/\tau)e^{-t/\tau}$ and $\widetilde{\mathfrak{K}}_3(k_x, t) = (\mathfrak{K}_3/\tau)k_x^2 e^{-t/\tau}$ with $\varphi(\mathbf{r}) = e^{-\mathbf{r}^2/2}/\sqrt{\pi}$. In the orange line, we show the $|\psi(0,0,t)|^{-2}$ associated with the standard comb model in two dimensions. In this case, we also verify a super-diffusive regime, which goes with $t^{1.5}$.

However, in our modification, we observe that the relaxation process is faster than in the standard case, going with $\sim t^3$.

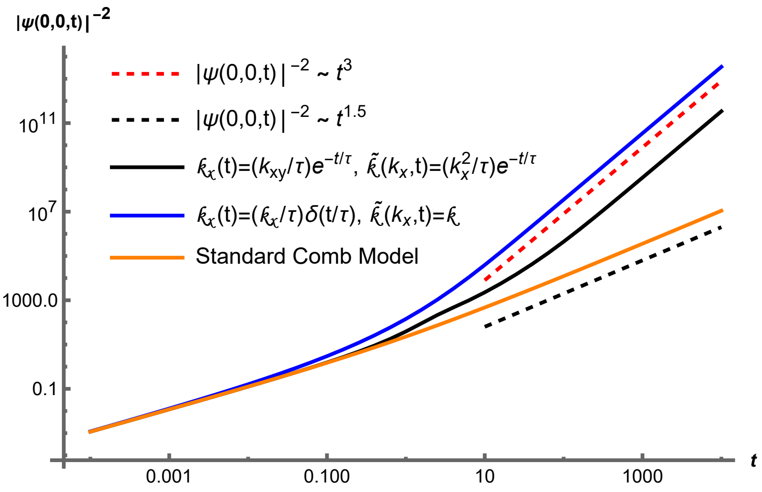

Figure 5. Behavior of the $|\psi(0,0,t)|^{-2}$ for $\mathfrak{K}_x(t) = (\mathfrak{K}_x/\tau)\delta(t/\tau)$ and $\mathfrak{K}(k_x,t) = k$ with $\varphi(\mathbf{r}) = \delta(x)\delta(y)$ in the blue line; $\mathfrak{K}_2(t) = (\mathfrak{K}_2/\tau)e^{-t/\tau}$ and $\widetilde{\mathfrak{K}}_3(k_x,t) = (\mathfrak{K}_3/\tau)k_x^2 e^{-t/\tau}$ with $\varphi(\mathbf{r}) = e^{-\mathbf{r}^2/2}/\sqrt{\pi}$ in the black line; and the standard comb model with orange line. Without loss of generality, we consider, for simplicity, $\mathfrak{K}_2/\hbar = 1$, $\mathfrak{K}_3/\hbar = 1$ $\hbar/m = 1$, and $l_x = l_y = 1$, in arbitrary unities.

2.3. Nonlocal Dependence and Memory Kernels

Following, we consider the mixing between the two previous cases, i.e., the memory kernels related to each case, in Equation (3), yielding

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\left[\delta\left(\frac{y}{l_y}\right)\frac{\partial^2}{\partial x^2}\psi(\mathbf{r},t) + \frac{\partial^2}{\partial y^2}\psi(\mathbf{r},t)\right] + \int_0^t dt' V_{xy}^{(2)}(y;t-t')\psi(\mathbf{r},t') + \int_0^t dt' V_{xy}^{(1)}(x,y;t-t')\psi(\mathbf{r},t') + \int_{-\infty}^{\infty}dx'\int_0^t dt' V_{xy}^{(3)}(x-x',y;t-t')\psi(x',y,t'). \tag{27}$$

The Green function connected with this case can be obtained by solving the following equation

$$i\hbar\frac{\partial}{\partial t}\mathcal{G}(\mathbf{r},\mathbf{r}',t) - i\hbar\delta(x-x')\delta(y-y')\delta(t) = -\frac{\hbar^2}{2m}\left[\delta\left(\frac{y}{l_y}\right)\frac{\partial^2}{\partial x^2}\mathcal{G}(\mathbf{r},\mathbf{r}',t) + \frac{\partial^2}{\partial y^2}\mathcal{G}(\mathbf{r},\mathbf{r}',t)\right] + \int_0^t dt'\int_{-\infty}^{\infty}dx'\left[V_{xy}^{(1)}(x-x',y,t-t') + V_{xy}^{(3)}(x-x',y,t-t')\right]\mathcal{G}(\mathbf{r},\mathbf{r}',t') + \int_0^t dt' V_{xy}^{(2)}(y;t-t')\mathcal{G}(\mathbf{r},\mathbf{r}',t')\,, \tag{28}$$

which in the Fourier–Laplace space can be written as follows:

$$i\hbar s\hat{\tilde{\mathcal{G}}}(k_x,k_y,\mathbf{r}',s) - i\hbar e^{-ik_x x'}e^{-ik_y y'} = \frac{\hbar^2}{2m}\left[l_y k_x^2\hat{\tilde{\mathcal{G}}}(k_x,0,\mathbf{r}',s) + k_y^2\hat{\tilde{\mathcal{G}}}(k_x,k_y,\mathbf{r}',s)\right] + \hat{\tilde{\mathfrak{K}}}_3(k_x,s)l_y\hat{\tilde{\mathcal{G}}}(k_x,0,\mathbf{r}',s) + \hat{\mathfrak{K}}_2(s)l_y\hat{\tilde{\mathcal{G}}}(k_x,0,\mathbf{r}',s) + \hat{\mathfrak{K}}_1(s)l_x l_y\hat{\mathcal{G}}(0,0,\mathbf{r}',s), \tag{29}$$

From this equation, it is possible to show that

$$\hat{\tilde{\mathcal{G}}}(k_x,0,\mathbf{r}',s) = e^{-ik_x x'}\hat{\tilde{\mathcal{G}}}_x(k_x,s)\hat{\mathcal{G}}_y(y',s) - \frac{i}{\hbar}\hat{\mathfrak{K}}_1(s)l_x l_y\hat{\mathcal{G}}(0,0,\mathbf{r}',s)\hat{\mathcal{G}}_y(0,s)\hat{\tilde{\mathcal{G}}}_x^{(2)}(k_x,s), \tag{30}$$

and, consequently,

$$\hat{\mathcal{G}}(0,0,\mathbf{r}',s) = \frac{\hat{\mathcal{G}}_y(y',s)\hat{\mathcal{G}}_x^{(2)}(x',s)}{1+(i/\hbar)\hat{\mathfrak{K}}_1(s)l_x l_y\hat{\mathcal{G}}_y(0,s)\hat{\mathcal{G}}_x^{(2)}(0,s)}\,. \tag{31}$$

By using these results, it is possible to show that

$$\hat{\tilde{\mathcal{G}}}(k_x,0,\mathbf{r}',s) = e^{-ik_x x'}\hat{\tilde{\mathcal{G}}}_x^{(2)}(k_x,s)\hat{\mathcal{G}}_y(y',s) - \frac{i}{\hbar}\frac{\hat{\mathfrak{K}}_1(s)l_x l_y \hat{\mathcal{G}}_y(y',s)\hat{\mathcal{G}}_x^{(2)}(x',s)}{1+(i/\hbar)\hat{\mathfrak{K}}_1(s)l_x l_y\hat{\mathcal{G}}_y(0,s)\hat{\mathcal{G}}_x^{(2)}(0,s)}\hat{\mathcal{G}}_y(0,s)\hat{\tilde{\mathcal{G}}}_x^{(2)}(k_x,s), \tag{32}$$

and, therefore,

$$\begin{aligned}\hat{\mathcal{G}}(x,y,\mathbf{r}',s) &= \delta(x-x')\left[\hat{\mathcal{G}}_y(y-y',s)-\hat{\mathcal{G}}_y(|y|+|y'|,s)\right]\\ &+\left[\hat{\mathcal{G}}_x^{(2)}(x-x',s)-\hat{\mathfrak{G}}^{(2)}(x,x',s)\right]\hat{\mathcal{G}}_y(|y|+|y'|,s)\\ &+\frac{\hat{\mathcal{G}}_y(|y|+|y'|,s)}{1+(i/\hbar)\hat{\mathfrak{K}}_1(s)l_x l_y\hat{\mathcal{G}}_y(0,s)\hat{\mathcal{G}}_x^{(2)}(0,s)}\hat{\mathfrak{G}}^{(2)}(x,x',s)\,,\end{aligned} \tag{33}$$

with

$$\hat{\mathfrak{G}}^{(2)}(x,x',s) = \hat{\mathcal{G}}_x^{(2)}(|x|,s)\hat{\mathcal{G}}_x^{(2)}(|x'|,s)/\hat{\mathcal{G}}_x^{(2)}(0,s) \tag{34}$$

Equation (33) can be formally written as Equation (13) when the equation $\hat{\mathfrak{G}}^{(2)}(x,x',s) = \hat{\mathcal{G}}_x^{(2)}(|x|+|x'|,s)$ is verified.

2.4. Fractional Spatial Operator and Nonlocal Terms

Another possibility considers a fractional operator applied to the spatial variable. In this case, we consider the following fractional operator [33,34] applied to the x variable:

$$\frac{1}{2}\int_{-\infty}^{\infty} dx\zeta_{\pm,\eta}(x,k_x)\left(D_x^{\mu,\eta}\psi(\mathbf{r},t)\right) \equiv -|k_x|^{\mu+\eta}\tilde{\psi}_\pm(k_x,y,t), \tag{35}$$

with the integral transform given by:

$$\frac{1}{2}\int_{-\infty}^{\infty} dx\zeta_{\pm,\eta}(x,k)\psi(\mathbf{r},t) = \tilde{\psi}_\pm(k_x,y,t)\,, \tag{36}$$

$$\frac{1}{2}\int_{-\infty}^{\infty} dk_x\zeta_{\pm,\eta}(x,k)\tilde{\psi}_\pm(k_x,y,t) = \psi(\mathbf{r},t), \tag{37}$$

where

$$\zeta_{+,\eta}(x,k_x) = (|k_x||x|)^{\frac{1}{2}(1+\eta)}\mathrm{J}_{-\nu}\left(2(|k_x||x|)^{\frac{1}{2}(2+\eta)}/(2+\eta)\right) \quad \text{and} \tag{38}$$

$$\zeta_{-,\eta}(x,k_x) = xk_x(|k_x||x|)^{\frac{1}{2}(1+\eta)-1}\mathrm{J}_{\nu}\left(2(|k_x||x|)^{\frac{1}{2}(2+\eta)}/(2+\eta)\right)\,, \tag{39}$$

where the sub-indexes $+$ and $-$ refer to the odd and even solutions, $\nu = (1+\eta)/(2+\eta)$, and $\mathrm{J}_\nu(x)$ is the Bessel function [7]. We stress that Equations (36) and (37) may be related to a generalized Hankel transform [41–44]. The Green function connected with this case can be obtained by solving the following equation

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\mathcal{G}(\mathbf{r},\mathbf{r}',t) - i\hbar\delta(x-x')\delta(y-y')\delta(t) &= -\frac{\hbar^2}{2m}\left[\delta\left(\frac{y}{l_y}\right)D_x^{\mu,\eta}\mathcal{G}(\mathbf{r},\mathbf{r}',t)+\frac{\partial^2}{\partial y^2}\mathcal{G}(\mathbf{r},\mathbf{r}',t)\right]\\ &+\int_0^t dt' V_{xy}^{(1)}(x,y;t-t')\mathcal{G}(\mathbf{r},\mathbf{r}',t')+\int_0^t dt' V_{xy}^{(2)}(y;t-t')\mathcal{G}(\mathbf{r},\mathbf{r}',t')\,,\end{aligned} \tag{40}$$

One noticeable point regarding this extension of the Schrödinger equation is that the behavior of the solutions can be characterized by power laws and stretched exponential. Additionally, the Schrödinger equation with an effective-position-dependent mass can directly relate with Equation (35), for example, $\mu = 2$ results in the standard differential operators: $D_x^{2,\eta}(\cdots) \equiv \partial_x\left[|x|^{-\eta}\partial_x(\cdots)\right]$. This case allows us to relate the x - direction of Equation (40) with a Schrödinger equation with an effective-position dependent mass, i.e., $m(x) = m|x|^\eta$ [45–47], which has been analyzed by taking several situations such as hetero-structures [48] and/or heterogeneous media [49–51], into account. To solve Equation (40), we write the Green's as follows:

$$\mathcal{G}(\mathbf{r},\mathbf{r}',t) = \int_0^\infty dk_x k_x \left[\zeta_+(x,k_x)\widetilde{\mathcal{G}}_+(k_x,y,\mathbf{r}',t) + \zeta_-(x,k_x)\widetilde{\mathcal{G}}_-(k_x,y,\mathbf{r}',t)\right] \tag{41}$$

with

$$\widetilde{\mathcal{G}}_\pm(k_x,y,\mathbf{r}',t) = \frac{1}{2}\int_{-\infty}^{\infty} dx\zeta_\pm(x,k_x)\mathcal{G}(\mathbf{r},\mathbf{r}',t)\ , \tag{42}$$

where $\widetilde{\mathcal{G}}_\pm(k_x,y,\mathbf{r}',t)$ is determined by the Equation (40). By substituting Equation (41) in Equation (40) and using the orthogonality of the eigenfunction and the Fourier transform for the y variable, it is possible to show that

$$\begin{aligned}\left(i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}|k_y|^2\right)\widetilde{\mathcal{G}}_\pm(k_x,k_y,\mathbf{r}',t) - i\hbar\zeta_\pm(x',k_x)e^{-ik_y y'}\delta(t) &= \frac{\hbar^2}{2m}l_y|k_x|^{\mu+\eta}\widetilde{\mathcal{G}}_\pm(k_x,0,\mathbf{r}',t)\\ + l_y\int_0^t dt' k_2(t-t')\widetilde{\mathcal{G}}_\pm(k_x,0,\mathbf{r}',t') + l_y l_x \int_0^t dt' k_1(t-t')\mathcal{G}_\pm(0,0,\mathbf{r}',t')\ ,\end{aligned} \tag{43}$$

By performing some calculations, it is possible to show that

$$\begin{aligned}\widehat{\widetilde{\mathcal{G}}}_\pm(k_x,k_y,\mathbf{r}',s) = \zeta_\pm(x',k_x)e^{-ik_y y'}\widehat{\widetilde{\mathcal{G}}}_y(k_y,s) - \frac{i}{\hbar}l_y l_x \widehat{k}_1(s)\widehat{\widetilde{\mathcal{G}}}_y(k_y,s)\widehat{\mathcal{G}}_\pm(0,0,\mathbf{r}',t)\\ - \frac{i}{\hbar}\left[\frac{\hbar^2}{2m}l_y|k_x|^{\mu+\eta} + \widehat{k}_2(s)\right]\widehat{\widetilde{\mathcal{G}}}_y(k_y,s)\widehat{\widetilde{\mathcal{G}}}_\pm(k_x,0,\mathbf{r}',t)\end{aligned} \tag{44}$$

with

$$\begin{aligned}\widehat{\widetilde{\mathcal{G}}}_\pm(k_x,0,\mathbf{r}',s) = \zeta_\pm(x',k_x)\widehat{\widetilde{\mathcal{G}}}^{(3)}_{\pm,x}(k_x,s)\widehat{\mathcal{G}}_y(y',s)\\ - \frac{i}{\hbar}\widehat{\mathfrak{K}}_1(s)l_x l_y\widehat{\mathcal{G}}_\pm(0,0,\mathbf{r}',s)\widehat{\mathcal{G}}_y(0,s)\widehat{\widetilde{\mathcal{G}}}^{(3)}_{\pm,x}(k_x,s),\end{aligned} \tag{45}$$

and, as a consequence,

$$\widehat{\widetilde{\mathcal{G}}}^{(3)}_{\pm,x}(k_x,s) = \frac{1}{1+\left(i\hbar|k|_x^{\mu+\eta}/(2m) + (i/\hbar)\widehat{\mathfrak{K}}_2(s)\right)l_y\widehat{\mathcal{G}}_y(0,s)}\ . \tag{46}$$

By performing additional calculations, we can show that

$$\widehat{\mathcal{G}}_+(0,0,\mathbf{r}',s) = \frac{\widehat{\mathcal{G}}_y(y',s)\widehat{\mathfrak{G}}^{(3)}_{+,x}(0,x',s)}{1+(i/\hbar)\widehat{\mathfrak{K}}_1(s)l_x l_y\widehat{\mathcal{G}}_y(0,s)[\widehat{\mathfrak{G}}^{(3)}_{+,x}(0,0,s)/\zeta_+(0,1)]}\ , \tag{47}$$

where

$$\widehat{\mathfrak{G}}^{(3)}_{\pm,x}(x,x',s) = \int_0^\infty dk_x\zeta_\pm(x',k_x)\zeta_\pm(x,k_x)\widehat{\widetilde{\mathcal{G}}}^{(3)}_{\pm,x}(k_x,s)\ . \tag{48}$$

For the initial condition $\varphi(x,y) = \delta(x)\delta(y)$, the evolution of the absolute Green function is displayed in Figure 6. For the sake of simplicity, we consider the absence of nonlocal terms to show the changes produced by the spatial operator incorporated in the Schrödinger equation with the backbone structure with branches. We observe that the presence of these operators changes the shape of the Green function when compared with the case of Figure 1 by introducing a different behavior connected with the spatial operator.

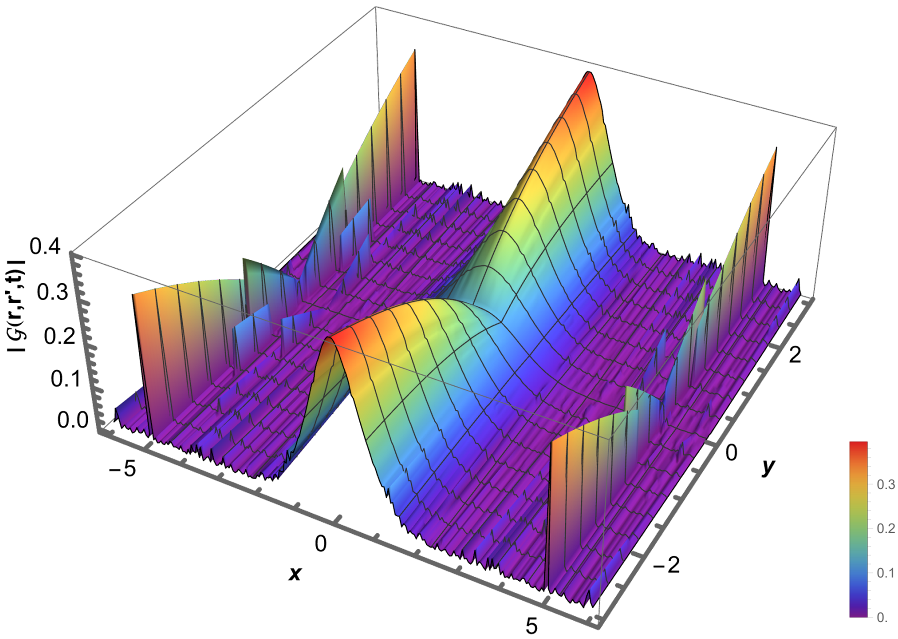

Figure 6. Behavior of the absolute Green function given by Equation (41) in absence of nonlocal terms, for $\mu = 1.8$ and $\eta = 1$. Without loss of generality, we consider $\hbar/m = 1$, $l_x = l_y = 1$, $\mathbf{r}' = 0$, and the initial condition $\varphi(x, y) = \delta(x)\delta(y)$, in arbitrary unities.

3. Discussion and Conclusions

We have analyzed a Schrödinger model with geometric constraints that couple the spatial variables x and y, under arbitrary time-dependent potentials. Our study considered four distinct cases: (i) a model with nonlocal dependence on time, (ii) a model with nonlocal spatial dependence, (iii) a mixed case that incorporates memory effects in both space and time through external potential kernels, and (iv) a formulation involving a fractional spatial derivative linked to a position-dependent effective mass. For each configuration, we constructed solutions using Green's function techniques, allowing us to determine wave functions for arbitrary initial conditions.

For the proposed solutions, we examined their behavior, which shows the evolution and spreading of wave packets under different nonlocal regimes. Our results show distinct scaling laws and dynamics depending on the nature of memory and spatial coupling. In particular, we observed that nonlocal terms and fractional operators can modify relaxation behavior. Furthermore, the cases explored here may be physically realized in engineered quantum systems such as optical lattices, photonic waveguides, and mesoscopic devices. For instance, comb-shaped waveguide arrays fabricated using femtosecond laser writing can emulate the geometry of a quantum comb, where injected light propagates similarly to a quantum particle constrained within the comb structure. As shown by Longhi [9], photonic lattices with comb-shaped configurations can simulate anomalous transport phenomena within such systems. Additionally, temporal modulation of the refractive index can be utilized to replicate time-dependent or memory-like potentials. Comb-like optical lattices can also be engineered by interfering laser beams to trap ultra-cold atoms in a backbone-finger arrangement, as demonstrated by the experimental work of Salger et al. [45,47]. We hope that the theoretical results presented here can provide insights into quantum dynamics in backbone-structured media and stimulate further experimental studies involving branched quantum systems that feature memory and spatial heterogeneity. Finally, in future works, we will explore how different forms of nonlocality, i.e., fractional operators, influence the thermal behavior of the wave function.

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