Chaos, Solitons and Fractals 207 (2026) 117997

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

Limit cycles of planar discontinuous piecewise linear Hamiltonian systems in three regions of Y-type

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ARTICLE INFO

MSC:
primary 34C07
34C05
37G15

Keywords:
Crossing limit cycle
Linear Hamiltonian system
Discontinuous piecewise differential system

ABSTRACT

In recent years there has been a significant interest in studying discontinuous piecewise differential systems, mainly due to the wide range of applications in modeling natural phenomena. To understand the dynamics of these systems in the plane one challenge is to control their number of limit cycles. In this paper we study the existence of limit cycles in planar discontinuous piecewise linear Hamiltonian systems with three zones separated by the line $Y = \{(x,y) : x \geq 0 \text{and} y = 0\} \cup \{(x,y) : x = 0 \text{and} y \geq 0\} \cup \{(x,y) : x \leq 0 \text{and} y = 0\}$. We provide the maximum number of crossing limit cycles intersecting each branch of Y in one point, and intersecting two branches of the Y each one in two points. So we have solved the extension of the 16th Hilbert problem to this class of differential systems.

1. Introduction and statement of the main results

The study of planar discontinuous piecewise smooth vector fields has been a subject of great interest over the last years, specially because of their great applicability as mathematical models of applied phenomena as relay systems, mechanics, electrical circuits, among others. The study of them is a problem that started with Andronov, Vitt and Khainkin [1] in the 1920's, and nowadays is an important problem in the qualitative theory of the differential systems. Since then, these equations have been extensively studied due to a new range of applications in other areas such as electronics and economics, to cite a few, for a further discussion see the books of Simpson [2], di Bernardo et al. [3], the survey of Makarenkov and Lamb [4], and the references therein.

A discontinuous piecewise smooth vector field $\mathcal{W}$ with three zones in the plane is composed of three C^r, $r \geq 1$, vector fields $\mathcal{W} = (X, Y, Z)$ where X, Y and Z are defined in the regions $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$, respectively, separated by a curve Σ, and the vector field $\mathcal{W}$ follows the Filippov rules [5] on Σ. Of course, here $\mathbb{R}^2 = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \Sigma$.

We are interested in studying the crossing limit cycles of discontinuous piecewise smooth differential systems. A *limit cycle* of a differential system is a periodic orbit of that system for which there is no other periodic orbit in some sufficiently small neighborhood containing it. And a *crossing limit cycle* in a discontinuous piecewise differential system is a limit cycle whose intersection with the separation line are isolated points.

The unsolved XVI Hilbert problem asked for an upper bound on the maximum number of limit cycles that a polynomial differential systems of degree n can exhibit, see [6]. The extended version of this problem to distinct classes of differential systems is considered in several papers, in particular, for families of piecewise differential systems with two or more zones. There are several papers which focus on to determine the maximum number of crossing limit cycles in piecewise systems with two or more regions,

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https://doi.org/10.1016/j.chaos.2026.117997
Received 19 November 2025; Received in revised form 17 January 2026; Accepted 27 January 2026
Available online 30 January 2026
0960-0779/

like one straight line [7], two parallel straight lines [8–11], or conics [12,13], or reducible cubics [14], or irreducible cubics [15], or two circles [16], or non-regular discontinuous line as in [17–20].

Concerning the piecewise linear Hamiltonian systems in the plane with three zones it is important to mention that they can be of three distinct classes: the ones without equilibria, the ones with a saddle and the ones with a center. So the combination of these three classes in the three pieces of the discontinuous piecewise differential systems produce 27 different types of such planar piecewise differential systems. In this paper we shall investigate all these 27 classes.

The three regions of the discontinuous piecewise differential systems that we are studying are

$$\mathcal{R}_1 = \{(x,y) : x \geq 0, y \geq 0\},$$
$$\mathcal{R}_2 = \{(x,y) : x \leq 0, y \geq 0\},$$
$$\mathcal{R}_3 = \{(x,y) : y \leq 0\},$$

which are separated by the curve $Y = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, where

$$\Sigma_1 = \{(x,y) : x > 0 \text{ and } y = 0\},$$

$$\Sigma_2 = \{(x,y) : x = 0 \text{ and } y > 0\},$$

$$\Sigma_3 = \{(x,y) : x < 0 \text{ and } y = 0\}.$$

And the vector field of the piecewise differential systems that we shall study are

$$\mathcal{W}(x,y) = \begin{cases} X(x,y) = \left(\dfrac{\partial \mathcal{H}_1}{\partial y}, -\dfrac{\partial H_1}{\partial x}\right) \text{ if } (x,y) \in \mathcal{R}_1, \\ Y(x,y) = \left(\dfrac{\partial \mathcal{H}_2}{\partial y}, -\dfrac{\partial H_2}{\partial x}\right) \text{ if } (x,y) \in \mathcal{R}_2, \\ Z(x,y) = \left(\dfrac{\partial \mathcal{H}_3}{\partial y}, -\dfrac{\partial H_3}{\partial x}\right) \text{ if } (x,y) \in \mathcal{R}_3, \end{cases} \tag{1}$$

where

$$\mathcal{H}_i(x,y) = a_i x + b_i y + c_i x^2 + d_i xy + e_i y^2, \tag{2}$$

with $a_i, b_i, c_i, d_i, e_i \in \mathbb{R}$ for $i = 1,2,3$.

For these piecewise differential systems we investigate two types of limit cycles. First, the crossing limit cycles which intersect Y in one point in each Σ_i, for $i = 1,2,3$. Second, the crossing limit cycles intersecting Σ_1 and Σ_2 in two points. Of course, it is not necessary to consider the crossing limit cycles intersecting Σ_2 and Σ_3 in two points due to the symmetry of the problem.

The main results of this paper are the following two theorems.

Theorem 1. *Consider discontinuous planar piecewise linear Hamiltonian systems with three zones of Y-type as in (1). For these systems the maximum number of crossing limit cycles intersecting Σ_i, $i = 1,2,3$ in one point is at most three, and there are systems with three limit cycles, see Fig. 1.*

Theorem 2. *Consider discontinuous planar piecewise linear Hamiltonian systems with three zones of Y-type as in (1). For these systems the maximum number of crossing limit cycles having two intersection points with Σ_1 and Σ_2 is at most one, and there are systems with one limit cycle, see Fig. 2.*

Theorems 1 and 2 are proved in Sections 2 and 3 , respectively.

Linear Hamiltonian systems can be of three distinct classes: without equilibria, with a saddle or with a center. So the combination of these three classes of linear Hamiltonian systems in the three pieces of the discontinuous piecewise differential systems that we want to study produce 27 different types of such piecewise differential systems. From these 27 types in the paper [19] it is study the type in which the three linear Hamiltonian systems forming the piecewise differential system are without equilibria. So it remains to study the other 26 types. Moreover the proof of this upper bound in the particular type of piecewise differential systems studied in the paper [19] is different from the one that we shall provide here and it is specific for the piecewise differential systems formed by three linear Hamiltonian systems without equilibria. While the proof of Theorem 1 is easier and works for all the 27 types of piecewise differential systems. So here we complete the other 26 cases did not considered until now.

We must mention that in the paper [21] the authors also study piecewise differential systems separated by the line Y and formed by linear differential systems, that do not need to be Hamiltonian. Their main result for such kind of piecewise differential is that a lower bound for the maximum number of crossing limit cycles intersecting Σ_i, $i = 1,2,3$ in one point is three. In Theorem 1 we have proved that this lower bound is also the upper bound when we restrict the class of piecewise differential systems to the ones formed by linear Hamiltonian systems.

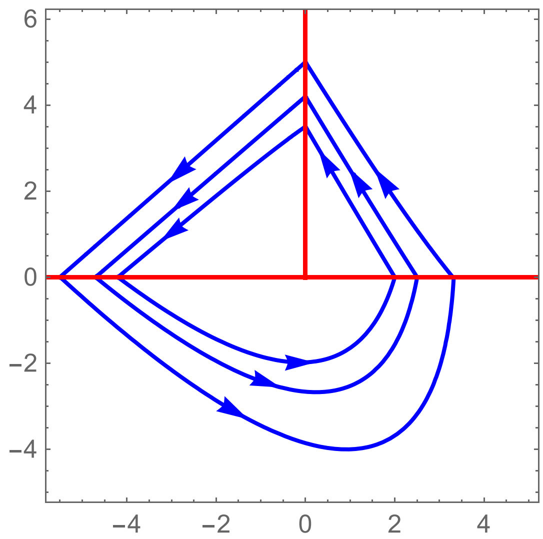

Fig. 1. Example of differential system (1) having 3 crossing limit cycles intersecting Σ_i, $i = 1, 2, 3$ in one point.

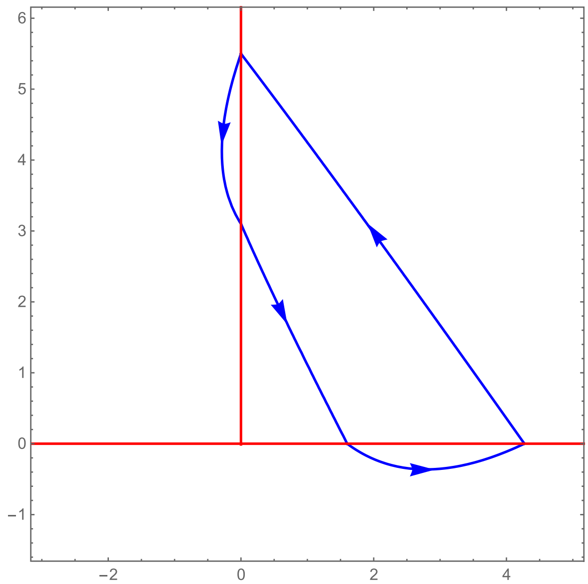

Fig. 2. Example of differential system (1) having one crossing limit cycle with two intersection points with Σ_1 and Σ_2.

2. Proof of Theorem 1

In what follows instead of saying crossing limit cycle we simple say limit cycle.

We consider the piecewise differential systems $\mathcal{W}$, see (1), formed by three linear Hamiltonian systems with Hamiltonians $\mathcal{H}_i$, for $i = 1, 2, 3$ given in (2).

If the piecewise differential system $\mathcal{W}$ has a limit cycle interesting the discontinuity line $Y = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ at the three points of the form $(x_1, 0)$, $(0, y_1)$ and $(x_2, 0)$ where $x_1 > 0$, $y_1 > 0$ and $x_2 < 0$, then the coordinates x_1, y_1 and x_2 must satisfy the following three equations

$$\begin{aligned}
E_1 &= \mathcal{H}_1(x_1, 0) - \mathcal{H}_1(0, y_1) = a_1 x_1 + c_1 x_1^2 - b_1 y_1 - e_1 y_1^2 = 0,\\
E_2 &= \mathcal{H}_2(0, y_1) - \mathcal{H}_2(x_2, 0) = -a_2 x_2 - c_2 x_2^2 + b_2 y_1 + e_2 y_1^2 = 0,\\
E_3 &= \mathcal{H}_3(x_2, 0) - \mathcal{H}_3(x_1, 0) = (x_2 - x_1)(a_3 + c_3(x_1 + x_2)) = 0.
\end{aligned}$$

As $x_1 \neq x_2$, from $E_3 = 0$ we have $F = a_3 + c_3(x_1 + x_2) = 0$.

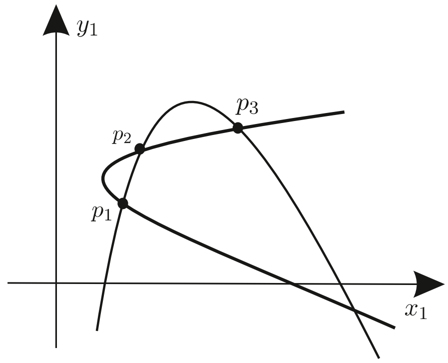

Fig. 3. A configuration that can produce three limit cycles.

Note that $c_3 \neq 0$, otherwise the equation $E_3 = 0$ has no solution if $a_3 \neq 0$, or the system $E_1 = E_2 = E_3 = 0$ has infinitely many solutions if $a_3 = 0$, and consequently the differential system cannot have limit cycles.

Since $c_3 \neq 0$ we get $x_2 = -(a_3 + c_3 x_1)/c_3$. So the equations $E_1 = 0$ and $c_3^2 E_2 = 0$ reduce to

$$\begin{aligned} E_{11} &= a_1 x_1 + c_1 x_1^2 - b_1 y_1 - e_1 y_1^2 = 0, \\ E_{21} &= -a_3(a_3 c_2 - a_2 c_3) - c_3(2a_3 c_2 - a_2 c_3)x_1 - c_2 c_3^2 x_1^2 + b_2 c_3^2 y_1 + c_3^2 e_2 y_1^2 = 0. \end{aligned} \tag{3}$$

The system $E_{11} = E_{21} = 0$ admits until four zeros (intersection of two conics) what means that there are at most four crossing limit cycles, but we shall prove that these piecewise differential systems have at most three limit cycles. We shall consider different cases.

If $a_3(a_3 c_2 - a_2 c_3) = 0$, then system (3) has the solution $(x_1, y_1) = (0, 0)$. Consequently under this assumption the differential system has at most three limit cycles.

In what follows we assume that $a_3(a_3 c_2 - a_2 c_3) \neq 0$, and we distinguish some cases.

Case 1: $c_1 = c_2 = 0$. Then system (3) has at most two positive solutions for y_1 because the resultant of the polynomial E_{11} and E_{21} with respect to the variable x_1 is the following polynomial in the variable y_1 of degree 2

$$a_1 a_2 a_3 c_3 + (a_2 b_1 + a_1 b_2) c_3^2 y_1 + c_3^2 (a_2 e_1 + a_1 e_2) y_1^2.$$

Consequently at most two limit cycles in this case.

Case 2: $c_1 = 0$ and $c_2 \neq 0$. Now we have that $E_{11} = 0$ reduces to a parabola symmetric with respect to some horizontal straight line, passing through the origin. The conic $E_{21} = 0$ can intersect the parabola only in two points with coordinates y_1 positive. So the piecewise differential Eq. (1) has at most two limit cycles in this case.

Case 3: $c_1 \neq 0$ and $c_2 = 0$. This case follows using the same arguments than in Case 2 but now the parabola is $E_{21} = 0$.

Case 4: $c_1 c_2 \neq 0$. From $E_{11} = E_{21} = 0$ it follows that $\tilde{E}_1 = c_2 c_3^2 E_{11} + c_1 E_{21} = 0$ and $\tilde{E}_2 = c_3^2 e_2 E_{11} + e_1 E_{21} = 0$ become

$$\begin{aligned} \tilde{E}_1 = \; & -a_3 c_1 (a_3 c_2 - a_2 c_3) - c_3(2a_3 c_1 c_2 - a_2 c_1 c_3 - a_1 c_2 c_3)x_1 + \\ & c_3^2 (b_2 c_1 - b_1 c_2) y_1 - c_3^2 (c_2 e_1 - c_1 e_2) y_1^2 = 0, \\ \tilde{E}_2 = \; & -a_3(a_3 c_2 - a_2 c_3)e_1 - c_3(2a_3 e_1 c_2 - a_2 e_1 c_3 - a_1 e_2 c_3)x_1 - \\ & c_3^2 (c_2 e_1 - c_1 e_2) x_1^2 + c_3^2 (b_2 e_1 - b_1 e_2) y_1 = 0. \end{aligned} \tag{4}$$

If $c_2 e_1 - c_1 e_2 = 0$, then equations $E_{11} = E_{21} = 0$ are straight lines, and so they have either at most one solution implying that the piecewise differential system (1) has at most one limit cycle, or a continuum of solutions and the piecewise differential system (1) has no limit cycles. So in what follows we assume that $c_2 e_1 - c_1 e_2 \neq 0$.

If $b_2 e_1 - b_1 e_2 = 0$, from the equation $\tilde{E}_2 = 0$, it follows that the system $\tilde{E}_{11} = \tilde{E}_{21} = 0$ only can have at most two positive solutions for x_1, so at most two limit cycles. Therefore in what follows we assume that $b_2 e_1 - b_1 e_2 \neq 0$.

If $2a_3 c_1 c_2 - a_2 c_1 c_3 - a_1 c_2 c_3 = 0$, from the equation $\tilde{E}_1 = 0$, it follows that the system $\tilde{E}_{11} = \tilde{E}_{21} = 0$ only can have at most two positive solutions for y_1, so at most two limit cycles. Hence in what follows we assume that $2a_3 c_1 c_2 - a_2 c_1 c_3 - a_1 c_2 c_3 \neq 0$.

Since $(2a_3 c_1 c_2 - a_2 c_1 c_3 - a_1 c_2 c_3)(b_2 e_1 - b_1 e_2) \neq 0$, $\tilde{E}_1 = 0$ is a parabola symmetric with respect to some horizontal straight line. While $\tilde{E}_2 = 0$ is a parabola symmetric with respect to some vertical straight line. See for instance Fig. 3, where the three points of intersection, (x^i, y^i) for $i = 1, 2, 3$, between the two parabolas contained in the positive quadrant of the plane $p_i = (x_1, y_1)$ satisfy $0 < x^1 < x^2 < x^3$ and $0 < y^1 < y^2 < y^3$, so in principle these three solutions of the system (7) could provide three limit cycles of the

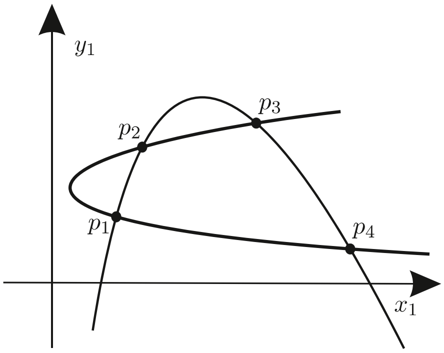

Fig. 4. Another configuration that at most can produce three limit cycles.

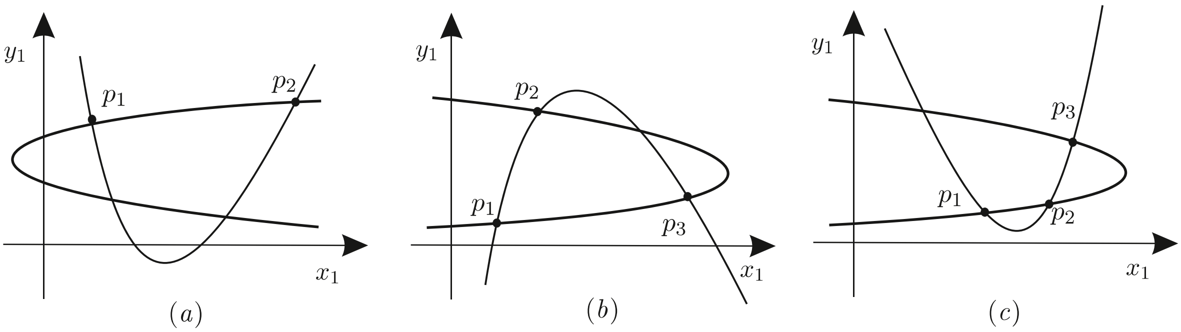

Fig. 5. (a) At most can produce two limit cycles due to the points p_1 and p_2. (b) At most can produce two limit cycles due to the points p_1 and p_2 or p_1 and p_3. (c) At most can produce three limit cycles due to the points p_1, p_2 and p_3.

piecewise differential system (1). Moreover, we have piecewise differential system (1) having three limit cycles, the Hamiltonians

$$H_1(x,y) = -\frac{483}{218}x^2 - \sqrt{\frac{483}{109}}xy + \frac{37989}{2180}x - y^2 + \frac{20003}{2180}\,y,$$
$$H_2(x,y) = -\frac{135}{314}\,x^2 + 3\sqrt{\frac{15}{157}}\,xy - \frac{1983}{628}\,x - \frac{1}{2}y^2 + \frac{10589}{3140}\,y$$
$$H_3(x,y) = -2\,x^2 - 2\,xy - \frac{22}{5}\,x - \frac{1}{2}y^2 + 5\,y.$$

has the three limit cycles as in Fig. 1.

Now consider the Fig. 4 where the two parabolas intersect in four points, $p_i = (x^i, y^i)$ for $i = 1,2,3,4$, in the positive quadrant of the plane (x_1, y_1). It is clear that the coordinates of these four points satisfy $0 < x^1 < x^2 < x^3 < x_4$ and $0 < y^4 < y^1 < y^2 < y^3$. Consequently, these four points cannot provide four limit cycles, because the limit cycle through the points (x^4, y^4) would intersect the other limit cycles due to the fact that $0 < y^4 < y^1 < y^2 < y^3$.

Other three configurations of the two parabolas with four intersections points, (x^i, y^i) for $i = 1,2,3,4$, in the positive quadrant of the plane (x_1, y_1) are possible, see Fig. 5, but similar arguments as the ones of the previous paragraph show that never the coordinates of these four points can satisfy the inequalities $0 < x^1 < x^2 < x^3 < x_4$ and $0 < y^1 < y^2 < y^3 < y^4$, and consequently these four points cannot provide four limit cycles. This completes the proof of Theorem 1.

3. Proof of Theorem 2

Let $\mathcal{W}$ be the piecewise differential system (1). We assume that the intersection points of a limit cycle of $\mathcal{W}$ with the discontinuity set $Y = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ has four points of the form $(x_1, 0)$, $(0, y_1)$, $(0, y_2)$ and $(x_2, 0)$ where

$$0 < x_1 < x_2 \quad \text{and} \quad 0 < y_1 < y_2. \tag{5}$$

Then the coordinates of these points must satisfy the following four equations

$$\begin{aligned}
E_1 &= \mathcal{H}_1(x_2,0) - \mathcal{H}_1(0,y_2) = 0,\\
E_2 &= \mathcal{H}_2(0,y_2) - \mathcal{H}_2(0,y_1) = 0,\\
E_3 &= \mathcal{H}_1(0,y_1) - \mathcal{H}_1(x_1,0) = 0,\\
E_4 &= \mathcal{H}_3(x_1,0) - \mathcal{H}_3(x_2,0) = 0.
\end{aligned}$$

As $x_1 \neq x_2$ and $y_1 \neq y_2$ we get the following system

$$\begin{aligned}
E_1 &= a_1x_2 + c_1x_2^2 - b_1y_2 - e_1y_2^2 = 0,\\
E_{21} &= b_2 + e_2(y_1+y_2) = 0,\\
E_3 &= -a_1x_1 - c_1x_1^2 + b_1y_1 + e_1y_1^2 = 0,\\
E_{41} &= a_3 + c_3(x_1+x_2) = 0.
\end{aligned} \tag{6}$$

If $x_1^2y_2^2 - x_2^2y_1^2 \neq 0$ from system (6) we obtain that

$$\begin{aligned}
a_3 &= -c_3(x_1+x_2), \quad & c_1 &= \frac{b_1y_1y_2(y_2-y_1) + a_1(x_2y_1^2 - x_1y_2^2)}{x_1^2y_2^2 - x_2^2y_1^2},\\
b_2 &= -e_2(y_1+y_2), \quad & e_1 &= \frac{a_1x_1(x_1-x_2)x_2 + b_1(x_2^2y_1 - x_1^2y_2)}{x_1^2y_2^2 - x_2^2y_1^2}.
\end{aligned} \tag{7}$$

Putting these coefficients into the differential system (1), this system has a limit cycle passing through the points $(x_1,0), (x_2,0), (0,y_1), (0,y_2)$.

Now we shall prove that system (1) can have at most one limit cycle intersecting the discontinuity line Y in two points in the half positive x-axis and two points in the half positive y-axis. Indeed, assume that there is a second limit cycle intersecting the discontinuity line in the points $(X_1,0),(X_2,0),(0,Y_1),(0,Y_2)$. Then the coordinates of these points must satisfy the following equations:

$$\begin{aligned}
\tilde{E}_1 &= \mathcal{H}_1(X_2,0) - \mathcal{H}_1(0,Y_2) = 0,\\
\tilde{E}_2 &= \mathcal{H}_2(0,Y_2) - \mathcal{H}_2(0,Y_1) = 0,\\
\tilde{E}_3 &= \mathcal{H}_1(0,Y_1) - \mathcal{H}_1(X_1,0) = 0,\\
\tilde{E}_4 &= \mathcal{H}_3(X_1,0) - \mathcal{H}_3(X_2,0) = 0.
\end{aligned} \tag{8}$$

As $X_1 \neq X_2$ and $Y_1 \neq Y_2$ we get the following system

$$\begin{aligned}
\tilde{E}_1 =\ & a_1x_2^2X_2y_1^2 - a_1x_2X_2^2y_1^2 + b_1X_2^2y_1^2y_2 - a_1x_1^2X_2y_2^2 + a_1x_1X_2^2y_2^2 - b_1X_2^2y_1y_2^2\\
& - b_1x_2^2y_1^2Y_2 + b_1x_1^2y_2^2Y_2 + a_1x_1^2x_2Y_2^2 - a_1x_1x_2^2Y_2^2 + b_1x_2^2y_1Y_2^2 - b_1x_1^2y_2Y_2^2 = 0,\\
\tilde{E}_{21} =\ & e_2(y_1 - Y_1 + y_2 - Y_2) = 0,\\
\tilde{E}_3 =\ & b_1(x_2^2y_1(y_1-Y_1)Y_1 + x_1^2Y_1(Y_1-y_2)y_2 + X_1^2y_1y_2(y_2-y_1))\\
& + a_1(x_1x_2(x_2-x_1)Y_1^2 + X_1^2(x_2y_1^2 - x_1y_2^2) + X_1(-x_2^2y_1^2 + x_1^2y_2^2)) = 0,\\
\tilde{E}_{41} =\ & -c_3(x_1 - X_1 + x_2 - X_2) = 0.
\end{aligned} \tag{9}$$

taking into account that $x_1^2y_2^2 - x_2^2y_1^2 \neq 0$. After some easy but tedious computations the solution of system (9) with respect the variables X_1, X_2, Y_1, Y_2 is

$$X_1 = x_1,\ X_2 = x_2,\ Y_1 = y_1,\ Y_2 = y_2 \text{ and } X_1 = x_2,\ X_2 = x_1,\ Y_1 = y_2,\ Y_2 = y_1.$$

Therefore the second possible limit cycle coincides with the first one.

Assume now that $x_1^2y_2^2 - x_2^2y_1^2 = 0$. Then, from (5) we have that $y_2 = x_2y_1/x_1$, and the solutions (7) become

$$a_3 = -c_3(x_1+x_2), \quad b_1 = a_1x_1/y_1, \quad b_2 = -e_2(x_1+x_2)y_1/x_1, \quad e_1 = c_1x_1^2/y_1^2.$$

Substituting these coefficients into the piecewise differential system (1), such a system has one limit cycle passing through the points $(x_1,0),(x_2,0),(0,y_1),(0,y_2)$. We shall prove that this piecewise differential system has at most one limit cycle.

Indeed, assume that there is a second limit cycle intersecting the discontinuity line Y in the points $(X_1,0),(X_2,0),(0,Y_1),(0,Y_2)$. Then the coordinates of these points must satisfy system (8). As $X_1 \neq X_2$ and $Y_1 \neq Y_2$ system (8) reduces to the system

$$\begin{aligned}
\tilde{E}_1 &= (X_2y_1 - x_1Y_2)(a_1y_1 + c_1X_2y_1 + c_1x_1Y_2) = 0,\\
\tilde{E}_{21} &= e_2(x_1y_1 + x_2y_1 - x_1Y_1 - x_1Y_2) = 0,\\
\tilde{E}_3 &= (X_1y_1 - x_1Y_1)(a_1y_1 + c_1X_1y_1 + c_1x_1Y_1) = 0,\\
\tilde{E}_{41} &= -c_3(x_1 - X_1 + x_2 - X_2) = 0.
\end{aligned} \tag{10}$$

taking into account that x_1, x_2, y_1, y_2 are positive. Again after some easy but tedious computations the solution of system (10) with respect the variables X_1, X_2, Y_1, Y_2 is

$$X_1,\ X_2 = x_1 - X_1 + x_2,\ Y_1 = X_1y_1/x_1,\ Y_2 = (x_1 - X_1 + x_2)y_1/x_1.$$

Therefore we have a continuum of solutions varying $X_1 \in \mathbb{R}$. Hence the piecewise differential system (1) has no a second limit cycle. In Fig. 2 we have provided one limit cycle of this kind for the piecewise differential systems here studied, this example is realized by the differential system (1) with the Hamiltonians

$$H_1(x,y) = -\frac{157135}{223671}x^2 - \sqrt{\frac{314279}{223671}}xy + \frac{17072429}{4473420}x - \frac{1}{2}y^2 + \frac{720859}{213020}y,$$

$$H_2(x,y) = \frac{5a}{935457583046}x^2 + \frac{1}{2}y^2 - \frac{18}{5}y - \frac{1}{10651}\sqrt{\frac{5a}{4123}}xy + \left(\frac{704}{133} - \frac{a}{140670313240}\right)x,$$

$$H_3(x,y) = -\frac{242}{81}x^2 + \frac{22}{9}xy + \frac{71027}{4050}x - \frac{1}{2}y^2 - y,$$

when $a = 782031760005 - 556339\sqrt{1459934861005}$.

CRediT authorship contribution statement

Jaume Llibre: Writing – review & editing, Writing – original draft, Project administration, Methodology, Formal analysis, Conceptualization. **Regilene Oliveira:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Regilene Oliveira reports financial support was provided by University of São Paulo. Jaume Llibre reports financial support was provided by Autonomous University of Barcelona. Regilene Oliveira reports a relationship with University of São Paulo that includes: employment. Jaume Llibre reports a relationship with Autonomous University of Barcelona that includes: employment. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The first author is partially supported by the Agencia Estatal de Investigación of Spain grant PID2022-136613NB-100, AGAUR (Generalitat de Catalunya) grant 2021SGR00113, and by the Reial Acadèmia de Ciències i Arts de Barcelona. The second author is partially supported by the FAPESP grant "Projeto Temático" 2019/21181-0, by the CNPq "Projeto Universal" 407454/2023-3, by the CNPq "Bolsa de Produtividade em Pesquisa" 310857/2023-6 and, by the project 101183111-DSYREKI-HORIZON-MSCA-2023-SE-01 "Dynamical Systems and Reaction Kinetics Networks".

Data availability

No data was used for the research described in the article.

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