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RESEARCH ARTICLE

Jordan theorems for embeddings and immersions in codimension one

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Abstract

Let $f\colon M \to N$ be a map where M, N are connected, topological manifolds, not necessarily orientable, of dimension $n-1$ and n, respectively. Assume that f is either an embedding or an immersion having only double points so that the set of such double points is a submanifold of dimension $n-2$ which is a strong deformation retract of a neighborhood of M. In case f is an embedding, we show that the complement $N - f(M)$ is either connected or has two connected components and therefore, the number of components is never bigger than two. In the case of an immersion, we set sharp lower bounds for the number of connected components of the complement (one or two, depending on the characteristics of the spaces and of the map f involved) and we show that in many cases the number of connected components can be realized for any integer greater than the corresponding lower bound . We point out that the fact that we are not assuming orientability of the manifolds M and N made possible to lower the least number of connected components of the complement $N - f(M)$.

Keywords Jordan theorem · Orientation and non-orientation true maps · Embedding · Immersion · Čech cohomology · Poincaré duality

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1 Introduction

The well-known Jordan's Separation Theorem states that any embedded closed curve γ into $\mathbb{R}^2$ or S^2 separates the plane into exactly two connected components. In other words, the complement of the image of a continuous injective map from the circle S^1 into S^2 has two connected components.

One possible way of extending this result is to start with a pair of closed, connected manifolds (M, N) with $\dim(N) = \dim(M) + 1 = n + 1$, and an embedding $f\colon M \longrightarrow N$. It would be natural to ask how often a pair of such manifolds (M, N) behaves as (S^1, S^2)? What can we say about the pairs (M, N) that do not behave as the pair (S^1, S^2)?

The answer to these questions will depend on some (co)homological features, including orientability hypothesis and properties involving induced homomorphisms by f. We provide a complete answer about the number of connected components of the complement $N - f(M)$. It turns out that a large family of embeddings does behave as the pair (S^1, S^2), but there will be situations where the image of an imbedding does not separate N (see Theorem 2.2).

Another way of extending Jordan's Theorem is to look at immersions, instead of embeddings. In this case, for any given integer $k \geqslant 3$, it is easy to construct an immersion $f\colon S^1 \to S^2$, where all multiple points are double points, and the number of components of $S^2 - f(S^1)$ is k.

The case of immersions, which are not embeddings, is more subtle and we will impose certain conditions. We let M and N be connected, closed manifolds of dimension $n-1$ and n, respectively. Consider $f\colon M \longrightarrow N$ an immersion, where the multiple points are all double points. We want to determine the number of connected components of $N - f(M)$ that may occur, as we vary M, N and f. It is important to mention that the immersion case has been treated in various works, and we may cite [2, 3], where this problem was analyzed for completely regular immersions where the manifolds involved were smooth and orientable or satisfying certain conditions that imply orientability. For instance, in case M is orientable and $H_1(N; \mathbb{Z}_2) = 0$, the number of components of $N - f(M)$ was found to be at least three. Observe that the hypothesis $H_1(N; \mathbb{Z}_2) = 0$ implies that N is orientable.

In 1995, Saeki presented results that generalize the ones mentioned above, in the topological context for oriented manifolds. He showed that, under some topological hypothesis, the existence of a normal crossing point of multiplicity m guaranties that the number of components of $N - f(M)$ is at least $m + 1$, see [11].

Making use of local coefficients, we are able to treat the case where the manifolds are not necessarily orientable. We found lower bounds, which are contained in the statement of Theorem 3.5, for the number of components of the complement which are smaller than three and we show that these lower bounds are realizable. Further, any number greater than the lower bound is, in fact, realizable.

The immersion case has one feature distinct from the embedding case. In many situations there is no upper bound for the number of connect components of the complement. So in this case we present a large family of examples realizing a number of components larger than the minimum number of components.

This work consists of four sections, including this one. We treat the embedding case in the second one and the immersion case in the third one. The fourth section is devoted to provide examples which realize all possible number of components above the lower bounds established in Theorem 3.5.

In this work we use Čech cohomology with local coefficients. The material on this subject in [4–6, 13] led us to treat Čech cohomology with local coefficients as a cohomology theory in the sense that it satisfies the Steenrod axioms.

2 Embeddings in codimension one

Let N be a connected, closed manifold of dimension n.

Consider the system of local coefficients over N, Γ_N, given by orientation, that is, the homomorphism $\pi_1(N,*) \xrightarrow{\Gamma_N} \mathrm{Aut}(\mathbb{Z})$ is defined by

$$\Gamma_N([\alpha]) = \begin{cases} \mathrm{id}_{\mathbb{Z}}, & \text{if } \alpha \text{ preserves orientation,} \\ -\mathrm{id}_{\mathbb{Z}}, & \text{if } \alpha \text{ revertes orientation.} \end{cases}$$

Given a map $g\colon X \longrightarrow N$, denote by $g^*(\Gamma_N)$ the system of local coefficients induced by g over X, which is given by the composition

$$\pi_1(X,*) \xrightarrow{g_\#} \pi_1(N,*) \xrightarrow{\Gamma_N} \mathrm{Aut}(\mathbb{Z}).$$

Let M be a connected, closed manifold of dimension $n-1$ and consider a map $f\colon M \longrightarrow N$.

Observe that using Poincaré duality with local coefficients as in [6, Theorem 3.2], we have the following isomorphism:

$$\check{H}^n(N, f(M); \Gamma_N) \approx H_0(N - f(M); \mathbb{Z}).$$

Therefore, $\check{H}^n(N, f(M); \Gamma_N)$ is free abelian and its rank corresponds to the number of connected components of $N - f(M)$.

Before stating the main result of this section, we need to recall a definition (cf. [6]).

Definition 2.1 We say that a map $f\colon M \longrightarrow N$ is *orientation true* if f maps loops that preserve (revert) orientation into loops that preserve (reverse) orientation.

Theorem 2.2 *Let M and N be connected, closed manifolds of dimension $n-1$ and n, respectively. Consider $f\colon M \longrightarrow N$ a topological embedding. The number of connected components of $N - f(M)$ is either one or two. In addition,*

(a) *If f is not orientation true, then $N - f(M)$ is connected.*
(b) *If $H_1(N;\mathbb{Z})$ is finite and f is orientation true, then $N - f(M)$ has two connected components.*
(c) *In case $H_1(N;\mathbb{Z})$ is not finite and f is orientation true, the number of connected components is either one or two, according to $f^*\colon \check{H}^{n-1}(N;\Gamma_N) \longrightarrow \check{H}^{n-1}(M; f^*(\Gamma_N))$ being trivial or not, respectively.*

Proof In order to determine the number of components of $N - f(M)$ we need to obtain the rank of $\check{H}^n(N, f(M); \Gamma_N)$.

From the long exact sequence in cohomology of the pair $(N, f(M))$

$$\cdots \longrightarrow \check{H}^{n-1}(N; \Gamma_N) \xrightarrow{i^*} \check{H}^{n-1}(f(M); i^*(\Gamma_N)) \xrightarrow{\delta} \check{H}^n(N, f(M); \Gamma_N) \longrightarrow \check{H}^n(N; \Gamma_N) \longrightarrow \check{H}^n(f(M); i^*(\Gamma_N))$$

it follows the short exact sequence

$$0 \to \mathrm{Im}(\delta) \to \check{H}^n(N, f(M); \Gamma_N) \to \check{H}^n(N; \Gamma_N) \to 0,$$

where $\delta\colon \check{H}^{n-1}(f(M); i^*(\Gamma_N)) \to \check{H}^n(N, f(M); \Gamma_N)$ is the connecting homomorphism, and $\check{H}^n(f(M); i^*(\Gamma_N)) = 0$ because $f(M)$ is a closed $(n-1)$-dimensional manifold. The short exact sequence above splits because $\check{H}^n(N; \Gamma_N) \approx \mathbb{Z}$. From the long exact sequence it also follows that

$$\mathrm{Im}(\delta) \approx \check{H}^{n-1}(f(M); i^*(\Gamma_N))/\mathrm{Im}(i^*),$$

where $\check{H}^{n-1}(f(M); i^*(\Gamma_N))$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2$, according to f being orientation true or not.

So the rank of $\check{H}^{n-1}(f(M); i^*(\Gamma_N))/\mathrm{Im}(i^*)$ is either 0 or 1 and so is the number of components of $N - f(M)$ either 1 or 2, respectively.

In particular, if f is not orientation true then $\check{H}^{n-1}(f(M); i^*(\Gamma_N)) \approx \mathbb{Z}_2$ and $N - f(M)$ will have only one connected component. So part (a) follows.

Now let f orientation true, then we have

$$\check{H}^{n-1}(f(M); i^*(\Gamma_N)) \approx \check{H}^{n-1}(M; f^*(\Gamma_N)) \approx Z.$$

If $H_1(N; \mathbb{Z})$ is finite, then $\check{H}^{n-1}(N; \Gamma_N)$ is also finite. Therefore i^* is the trivial homomorphism, and it follows that $N - f(M)$ has exactly two connected components. This proves part (b).

In case $H_1(N; \mathbb{Z})$ is not finite, we have that $\check{H}^{n-1}(N; \Gamma_N) \approx H_1(N; \mathbb{Z})$ is also infinite. Since f is orientation true then $\check{H}^{n-1}(f(M); i^*(\Gamma_N)) \approx \mathbb{Z}$ and therefore $i^*\colon \check{H}^{n-1}(N; \Gamma_N) \longrightarrow \check{H}^{n-1}(f(M); i^*(\Gamma_N))$ is either trivial or has image of rank 1. This completes the proof of part (c). □

Observe that Theorem 2.2 describes the number of components of the complement in a such way that we may look at three families of embeddings. Each family is defined in terms of the embedding being orientation true or non orientation true and in terms of the finiteness of the first homology group of the target space. These families can be refined if one also takes into account the orientability of the manifolds involved.

We end this section by presenting examples of topological embeddings $f\colon M \longrightarrow N$, showing that each of these more refined families of embeddings is, in fact, not empty, in the sense that there are examples for each of these refined families that realize what is established by Theorem 2.2 in terms of the number of components of the complement $N - f(M)$.

2.1 Examples

For $n > 1$, any embedding of $\mathbb{R}P^n$ into $\mathbb{R}P^{n+1}$ is not orientation true. This fact is clear for n even. For n odd, if f were orientation true then it would necessarily lift to an embedding in S^{n+1}, and so in R^{n+1}, which is known not to be possible, see [7–10]. In order to produce examples where both manifolds are not orientable, we simply consider the Cartesian product of any embedding of $\mathbb{R}P^n$ into $\mathbb{R}P^{n+1}$, $n > 1$, with the identity map in $\mathbb{R}P^{2n}$. This completes the set of examples that illustrate part (a) of Theorem 2.2 taking into account the orientability of the manifolds.

The usual embedding of S^1 into $\mathbb{S}^2$ and any embedding of S^1 into $\mathbb{R}P^2$ whose image lies in $\mathbb{R}P^2 - \mathbb{R}P^1$, constitute examples satisfying the conditions in part (b) of Theorem 2.2. As before, the Cartesian product of any of these embeddings with the identity in $\mathbb{R}P^{2n}$ provides examples where both manifolds are not orientable. Observe that those examples cover all possibilities since it is not possible to have an orientation true map from a nonorientable manifold to an orientable one.

The following examples correspond to part (c) of the above theorem.

For the case where M is orientable, consider the imbedding of the circle into the torus, or into the Klein bottle, as one of the meridian circles; in both situations, the complement will be connected. On the other hand, if the circle is imbedded into these surfaces as the boundary of a small disk, the complement will have two components.

For M non-orientable, since f is orientation true, N must be non-orientable. In this case, the usual embedding of the Klein bottle K in $K \times S^1$ is an example where the complement is connected. For an example where the complement has two components, consider the natural embedding of K into the double of the three-dimensional manifold X that has K as its boundary.

3 Immersions in codimension one

Before dealing with immersions, we will establish two general lemmas that will be relevant for our problem and might be of interest by themselves.

Lemma 3.1 *Let X and Y be compact ENR's* (Euclidean Neighborhood Retracts) (*thus Hausdorff spaces*), *and let $A \subset X$ and $B \subset Y$ be closed subspaces. Assume that A is a strong deformation retract of a closed neighborhood of A in X. Consider $f: (X, A) \longrightarrow (Y, B)$ a relative homeomorphism and a system of local coefficients in Y, Γ_Y. Then $f^*: \check{H}^*(Y, B; \Gamma_Y) \to \check{H}^*(X, A; f^*(\Gamma_Y))$ is an isomorphism.*

Proof The proof will follow the same strategy as the one given to [12, Theorem 9, Section 4.8] adapted to Čech cohomology with local coefficients. Let V be a closed neighborhood of A in X such that A is a strong deformation retract of V and let U be an open subset of X so that $A \subset U \subset \overline{U} \subset V$. Following exactly what is done in [12], we see that for $V' = f(V) \cup B$ and $U' = f(U) \cup B$, we have B a strong deformation retract of V' and also $B \subset U' \subset \overline{U}' \subset V'$. Consider now the following diagram:

$$\begin{array}{ccccc}
\check{H}^*(Y, B; \Gamma_Y) & \longleftarrow & \check{H}^*(Y, V'; \Gamma_Y) & \longrightarrow & \check{H}^*(Y - U', V' - U'; k^*(\Gamma_Y)) \\
\downarrow f^* & & \downarrow & & \downarrow \\
\check{H}^*(X, A; f^*(\Gamma_Y)) & \longleftarrow & \check{H}^*(X, V; f^*(\Gamma_Y)) & \longrightarrow & \check{H}^*(X - U, V - U; (f|_{X-U})^*(\Gamma_Y))
\end{array}$$

where the horizontal homomorphisms are induced by inclusions and the vertical ones by f and its restrictions.

Since A and B are strong deformation retracts of V and V', respectively, we have

$$\check{H}^*(A; (f|_A)^*(\Gamma_Y)) \approx \check{H}^*(V; (f|_V)^*(\Gamma_Y)),$$
$$\check{H}^*(B; i^*(\Gamma_Y)) \approx \check{H}^*(V'; j^*(\Gamma_Y)).$$

Therefore from the long exact sequences of the pairs,

$$\check{H}^*(X, A; f^*(\Gamma_Y)) \approx \check{H}^*(X, V; f^*(\Gamma_Y)),$$
$$\check{H}^*(Y, B; \Gamma_Y) \approx \check{H}^*(Y, V'; \Gamma_Y).$$

Moreover, by excision, we have isomorphisms, also induced by inclusions,

$$\check{H}^*(X - U, V - U; (f|_{X-U})^*(\Gamma_Y)) \approx \check{H}^*(X, V; f^*(\Gamma_Y)),$$
$$\check{H}^*(Y - U', V' - U'; k^*(\Gamma_Y)) \approx \check{H}^*(Y, V'; \Gamma_Y),$$

where $k\colon Y - U' \to Y$ is the inclusion.

Therefore, all horizontal homomorphisms are isomorphisms.

Observe that $f\colon (X - U, V - U) \to (Y - U', V' - U')$ is a homeomorphism and therefore it induces an isomorphism

$$\check{H}^*(X - U, V - U; (f|_{X-U})^*(\Gamma_Y)) \approx \check{H}^*(Y - U', V' - U'; \Gamma_Y).$$

From the commutativity of the diagram, it follows that

$$\check{H}^*(X, A; f^*(\Gamma_Y)) \approx \check{H}^*(Y, B; \Gamma_Y). \qquad \square$$

Lemma 3.2 *Let M, N be closed, connected manifolds of dimensions $n - 1$, n, respectively. Let A be a proper closed subset of M such that $M - A$ is a submanifold of the same dimension as M. Consider $f\colon (M, A) \to (N, B)$ a continuous map such that $f|_{M-A}$ is injective and $f(M - A) \subset f(M) - B$. Then $f|_{M-A}\colon M - A \to f(M) - B$ is a homeomorphism.*

Proof Let $U \subset M - A$ be an open subset of $M - A$. We want to show that $f(U)$ is open in $f(M) \subset N$. Let $y_0 \in f(U)$ be an arbitrary point of $f(U)$. Since N is a manifold and B is closed, let V' be a Euclidean neighborhood of y_0, which, up to homeomorphism, we may think of as a Euclidean neighborhood centered in the origin.

Consider a sequence of neighborhoods V'_r, centered at y_0, such that $V'_{r+1} \subset V'_r$ and the "radius" of V'_r goes to zero as $r \to \infty$. Then the intersection $\bigcap_1^\infty V'_r$ is the singleton set $\{y_0\}$. Let $V_r = V'_r \cap f(M)$, which is open in $f(M)$, and consider the sequence $W_r = f^{-1}(V_r) \subset M - A$. Observe that $\bigcap_1^\infty W_r = f^{-1}(y_0)$ is a singleton set and we denote it by $f^{-1}(y_0) = z_0$. We claim that there exists r_0 such that $W_{r_0} \subset U$, which will imply that $V_r \subset f(U)$ for all $r \geqslant r_0$. This follows since if not, then we will have an infinite sequence of elements $x_r, r \in M$, such that $x_r \in W_r$, and so $f(x_r) \in V_r$, but $f(x_r) \notin f(U)$. Since M is a compact manifold, this sequence admits a subsequence $\{x_{r_i}\}_{i \in N}$ which converges to a point, say x_0, and the sequence $f(x_{r_i}) \in V_{r_i}$ converges to y_0. This implies that $f(x_0) = y_0$. Since clearly $x_0 \neq z_0$, since in fact $x_0 \notin U$ we get a contradiction. So for some r_0, $V_{r_0} \subset f(U)$ and the result follows. □

From now on, M and N will be connected, closed, topological manifolds of dimension $n-1$ and n, respectively. We begin by establishing a general result on the number of components of $N - f(M)$, for a map $f\colon M \to N$.

As in the case of embeddings, Γ_N will denote the system of coefficients in N given by orientation and for a map $g\colon X \to N$, $g^*(\Gamma_N)$ will represent the system induced by g over X.

Proposition 3.3 *For a map $f\colon M \to N$, the number of connected components of $N - f(M)$ is*

$$1 + \operatorname{rank}(\operatorname{Im}(\delta_n)),$$

where $\delta_n\colon \check{H}^{n-1}(f(M); i^*(\Gamma_N)) \longrightarrow \check{H}^n(N, f(M); \Gamma_N)$ is the connecting homomorphism of the long exact sequence of the pair $(N, f(M))$. Also

$$\operatorname{Im}(\delta_n) \approx \check{H}^{n-1}(f(M); i^*(\Gamma_N))/\operatorname{Im}(i^*_{n-1}),$$

where the homomorphism $i^*_{n-1}\colon \check{H}^{n-1}(N; \Gamma_N) \longrightarrow \check{H}^{n-1}(f(M); i^*(\Gamma_N))$ is induced by the inclusion. Therefore, the number of connected components of $N - f(M)$ is $\leqslant 1 + \operatorname{rank}(\check{H}^{n-1}(f(M); i^*(\Gamma_N)))$. If $H_1(N; \mathbb{Z})$ is finite then the equality holds.

Proof As we have done in the beginning of the previous section, we have that $\check{H}^n(N, f(M); \Gamma_N)$ is free abelian and its rank corresponds to the number of connected components of $N - f(M)$.

We look at the long exact sequence in Čech cohomology of the pair $(N, f(M))$,

$$\begin{aligned} \cdots \longrightarrow \check{H}^{n-1}(N; \Gamma_N) \xrightarrow{i^*_{n-1}} \check{H}^{n-1}(f(M); i^*(\Gamma_N)) \xrightarrow{\delta_n} \check{H}^n(N, f(M); \Gamma_N) \\ \longrightarrow \check{H}^n(N; \Gamma_N) \xrightarrow{i^*_n} \check{H}^n(f(M); i^*(\Gamma_N)) \longrightarrow \cdots \end{aligned}$$

We first observe that the homomorphism $i^*_n\colon \check{H}^n(N; \Gamma_N) \longrightarrow \check{H}^n(f(M); i^*(\Gamma_N))$ is the trivial homomorphism. To see this observe that the inclusion $i\colon f(M) \to N$ factors through $N - \{x_0\}$ for $x_0 \notin f(M)$ and $\check{H}^n(N - \{x_0\}, \Gamma_N)$ is trivial.

So we obtain a splitting short exact sequence

$$0 \to \mathrm{Im}(\delta_n) \to \check{H}^n(N, f(M); \Gamma_N) \to \check{H}^n(N; \Gamma_N) \to 0,$$

because $\check{H}^n(N; \Gamma_N) \approx \mathbb{Z}$. So the first part of the proposition follows.

Since $\mathrm{Im}(\delta_n) \approx \check{H}^{n-1}(f(M); i^*(\Gamma_N))/\mathrm{Im}(i^*_{n-1})$, the inequality also holds.

If $H_1(N; \mathbb{Z})$ is finite then $\check{H}^{n-1}(N; \Gamma_N)$ is also finite and therefore the rank of $\mathrm{Im}(i^*_{n-1})$ is trivial. □

Let M and N be manifolds closed of dimension $n-1$ and n, respectively, and $A \subset M$ and $B \subset N$ be closed subsets. For a map $f: (M, A) \to (N, B)$ denote also by f the map $(M, A) \mapsto (f(M), B)$.

Consider the homomorphism, induced by $f: (M, A) \to (f(M), B)$, between the long cohomology exact sequences of the pairs, as shown in the diagram below, where $i: f(M) \hookrightarrow N$, $k: B \hookrightarrow f(M)$ and $l: A \hookrightarrow M$ are the natural inclusions,

$$\begin{array}{ccccccccc}
\longrightarrow & \check{H}^{i-1}(f(M); i^*(\Gamma_N)) & \xrightarrow{k^*} & \check{H}^{i-1}(B; k^*(i^*(\Gamma_N))) & \longrightarrow & \check{H}^{i}(f(M), B; i^*(\Gamma_N)) & \xrightarrow{j^*} & \check{H}^{i}(f(M); i^*(\Gamma_N)) & \longrightarrow \\
& \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & \\
\longrightarrow & \check{H}^{i-1}(M; f^*(\Gamma_N)) & \longrightarrow & \check{H}^{i-1}(A; l^*(f^*(\Gamma_N))) & \xrightarrow{\delta'_{i-1}} & \check{H}^{i}(M, A; f^*(\Gamma_N)) & \longrightarrow & \check{H}^{i}(M; f^*(\Gamma_N)) & \longrightarrow
\end{array}$$

The following proposition gives us a better understanding of the rank of $\check{H}^{n-1}(f(M); i^*(\Gamma_N))$.

Proposition 3.4 *In this setting, assume also that A is a closed $(n-2)$-submanifold of M which is a strong deformation retract of a closed neighborhood of A in M and assume $f: (M, A) \to (f(M), B)$ is a relative homeomorphism. Then the following sequence*

$$\check{H}^{n-2}(M; f^*(\Gamma_N)) \oplus \check{H}^{n-2}(B; k^*(i^*(\Gamma_N))) \xrightarrow{\eta_{n-2}} \check{H}^{n-2}(A; l^*(f^*(\Gamma_N)))$$
$$\xrightarrow{\theta_{n-1}} \check{H}^{n-1}(f(M); i^*(\Gamma_N)) \xrightarrow{\gamma} \check{H}^{n-1}(M; f^*(\Gamma_N)) \oplus \check{H}^{n-1}(B; k^*(i^*(\Gamma_N))) \to 0$$

is exact, where $\eta_{n-2} = l^* - (f|_A)^*$, θ_{n-1} is given by the composite of the homomorphisms $j^* \circ (f^*)^{-1} \circ \delta'_{n-1}$ and $\gamma = (f^*, k^*)$. Therefore,

$$\mathrm{rank}\, \check{H}^{n-1}(f(M); i^*(\Gamma_N)) = \mathrm{rank}\, \mathrm{Im}(f^*) + \mathrm{rank}\, \mathrm{Im}(\theta_{n-1}).$$

Proof Applying Lemma 3.1, we obtain an isomorphism $\check{H}^*(f(M), B; i^*(\Gamma_N)) \to \check{H}^*(M, A; f^*(\Gamma_N))$ induced by the relative homeomorphism $(M, A) \to (f(M), B)$. By [1, Lemma 7.4], it follows from the long exact sequences of the pairs (M, A) and $(f(M), B)$ the following long exact sequence:

$$\cdots \to \check{H}^{n-2}(M; f^*(\Gamma_N)) \oplus \check{H}^{n-2}(B; k^*(i^*(\Gamma_N))) \to \check{H}^{n-2}(A; l^*(f^*(\Gamma_N)))$$
$$\xrightarrow{\theta_{n-1}} \check{H}^{n-1}(f(M); i^*(\Gamma_N)) \xrightarrow{\gamma} \check{H}^{n-1}(M; f^*(\Gamma_N)) \oplus \check{H}^{n-1}(B; k^*(i^*(\Gamma_N)))$$
$$\to \check{H}^{n-1}(A; l^*(f^*(\Gamma_N))) \to \cdots .$$

Since A is a closed $(n-2)$-manifold we have that $\check{H}^{n-1}(A; l^*(f^*(\Gamma_N))) = 0$ and the result follows. □

We turn now our attention to the case of immersions. Let $f: M \longrightarrow N$ be an immersion, that is, a locally injective continuous map.

We are going to consider the case where all multiple points of f are double points, that is $\#\{f^{-1}(f(p))\} \leqslant 2$, for all $p \in M$. Denote by $A = \{p \in M \mid \#\{f^{-1}(f(p))\} = 2\}$ the set of double points of f, assume it is non-empty, and set $B = f(A)$.

We observe that $f|_A: A \to B$ is a double covering. To see this, we let b in B and consider a neighborhood V of b in B. Then $f^{-1}(V)$ is open in A and contains $f^{-1}(b) = \{a_1, a_2\}$. Since A is a manifold, we can choose neighborhoods U_1 and U_2 of a_1 and a_2, respectively, so that $\overline{U}_1 \cap \overline{U}_2 = \varnothing$, $U_1 \subset \overline{U}_1 \subset f^{-1}(V)$ and $U_2 \subset \overline{U}_2 \subset f^{-1}(V)$. We may also assume, since f is an immersion, that $f|_{\overline{U}_1}: \overline{U}_1 \to f(\overline{U}_1)$ and $f|_{\overline{U}_2}: \overline{U}_2 \to f(\overline{U}_2)$ are injective, which implies that they are, in fact, homeomorphisms. Therefore $f|_{U_1}: U_1 \to f(U_1)$ and $f|_{U_2}: U_2 \to f(U_2)$ are also homeomorphisms and $V_1 = f(U_1) \cap f(U_2)$ is an open neighborhood of b in B. It is now easy to see that $f^{-1}(V_1)$ is the union of two open subsets of A, each of which homeomorphic to V_1.

We can now state our main result.

Theorem 3.5 *Let $f: M \longrightarrow N$ be a topological immersion between closed manifolds of dimension $n-1$ and n, respectively. Assume that the multiple points are all double points and the set of double points is a closed $(n-2)$-submanifold of M which is also a strong deformation retract of a neigborhood of M.*

(a) *If f is orientation true, the number of connected components of $N - f(M)$ is* $2 + \operatorname{rank} \check{H}^{n-2}(A; l^*(f^*(\Gamma_N))) - \operatorname{rank}(\operatorname{Im}(\eta_{n-2})) - \operatorname{rank}(\operatorname{Im}(i^*_{n-1}))$.
In case $H_1(M; \mathbb{Z})$ is finite, the summand $\operatorname{rank}(\operatorname{Im}(\eta_{n-2}))$ *coincides with* $\operatorname{rank}(\operatorname{Im}(f|_A)^*)$.
In case $H_1(N; \mathbb{Z})$ is finite, the summand $\operatorname{rank}(\operatorname{Im}(i^*_{n-2}))$ *vanishes.*
Therefore if both are finite then the number of connected components is given by $2 + \operatorname{rank} \check{H}^{n-2}(A; l^*(f^*(\Gamma_N))) - \operatorname{rank}(\operatorname{Im}(f|_A)^*)$.

(b) *If f is not orientation true then in all cases presented in part* (a) *the number of components of $N - f(M)$ are exactly the ones given there when we replace the number* 2 *by* 1.

Here, $i: f(M) \to N$ and $l: A \to M$ are inclusions and $\eta_{n-2} = l^* - (f|_A)^*$ as in Proposition 3.4.

Proof Since $f|_A: A \to B$ is a double covering, B is also an $(n-2)$-manifold. We then have $\check{H}^{n-1}(B; k^*(i^*(\Gamma_N))) = 0$ and the sequence given by Proposition 3.4 becomes

$$\check{H}^{n-2}(A; l^*(f^*(\Gamma_N))) \xrightarrow{\theta_{n-1}} \check{H}^{n-1}(f(M); i^*(\Gamma_N)) \xrightarrow{\gamma} \check{H}^{n-1}(M; f^*(\Gamma_N)) \to 0.$$

(a) If f is orientation true, then $\check{H}^{n-1}(M; f^*(\Gamma_N)) = \check{H}^{n-1}(M; \Gamma_M) \approx \mathbb{Z}$. By Proposition 3.3, the number of components of $N - f(M)$ is $1 + \operatorname{rank}(\operatorname{Im}(\delta_n)) =$

$1+\mathrm{rank}\,(\check{H}^{n-1}(f(M); i^*(\Gamma_N))) - \mathrm{rank}\,(\mathrm{Im}\,(i^*_{n-1}))$. From the short exact sequence above we obtain that this number is

$$\begin{aligned}&1+\mathrm{rank}\,(\check{H}^{n-1}(M;\, f^*(\Gamma_N))) + \mathrm{rank}\,(\mathrm{Im}\,(\theta_{n-1})) - \mathrm{rank}\,(\mathrm{Im}\,(i^*_{n-1}))\\ &\quad = 2+\mathrm{rank}\,(\check{H}^{n-2}(A; l^*(f^*(\Gamma_N)))) - \mathrm{rank}\,(\ker(\theta_{n-1})) - \mathrm{rank}\,(\mathrm{Im}\,(i^*_{n-1})).\end{aligned}$$

Since $\ker(\theta_{n-1}) = \mathrm{Im}\,(\eta_{n-2})$, we obtain

$$2+\mathrm{rank}\,\check{H}^{n-2}(A; l^*(f^*(\Gamma_N))) - \mathrm{rank}\,(\mathrm{Im}\,(\eta_{n-2})) - \mathrm{rank}\,(\mathrm{Im}\,(i^*_{n-1}),$$

and the first part of item (a) follows.

If $H_1(M;\mathbb{Z})$ is finite, it follows by duality that $\check{H}^{n-2}(M;\, f^*(\Gamma_N))$ is also finite. Therefore the rank of the image of the homomorphism

$$\eta_{n-2}\colon\, \check{H}^{n-2}(M;\, f^*(\Gamma_N)) \oplus \check{H}^{n-2}(B; k^*(i^*(\Gamma_N))) \to \check{H}^{n-2}(A; l^*(f^*(\Gamma_N)))$$

is the same as the rank of the image of η_{n-2} restricted to the the summand $\check{H}^{n-2}(B; k^*(i^*(\Gamma_N)))$ and therefore coincides with $\mathrm{rank}\,(\mathrm{Im}\,(f|_A)^*)$.

If $H_1(N;\mathbb{Z})$ is finite again, by duality, it follows that $\check{H}^{n-1}(N;\Gamma_N)$ is also finite. Therefore $\mathrm{rank}\,(\mathrm{Im}\,(i^*_{n-1})) = 0$.

In case both groups $H_1(M;\mathbb{Z})$ and $H_1(N;\mathbb{Z})$ are finite, combining what we have obtained above, we conclude the proof of part (a).

(b) Since f is not orientation true, we first observe that $f^*(\Gamma_N)$ is not orientation system of the manifold M. Therefore $\check{H}^{n-1}(M;\, f^*(\Gamma_N)) \approx \mathbb{Z}_2$ and $\mathrm{rank}\,(\check{H}^{n-1}(M;\, f^*(\Gamma_N))) = 0$ instead of 1 as in the case of item (a). The rest of the proof is completely similar to the proof of item (a) and the result follows. □

For the case of immersions, under our hypothesis, of S^1 into the sphere S^2, the above result shows that the minimal number of components of the complement is greater than or equal to 3. It is easy to exhibit examples of immersions, where the number of components of the complement is exactly k, for any integer $k \geqslant 3$.

Inspired by the example above and by the embedding case, in the next section we will seek for the realization of all integers, greater than or equal to the lower bound, established by Theorem 3.5, as the number of connected components of the complement $N - f(M)$, for an immersion $f\colon M \longrightarrow N$. Moreover, we will try to do that for various families of immersions, taking into account the immersion being orientation true or not, the finiteness of the first homology groups and the orientability of the manifolds involved.

4 Examples

Observe that Theorem 3.5 describes the number of components under assumptions which consider not only the immersion being orientation true or non orientation true, but also the first homology groups of M and/or N being finite or not.

In this section we present examples of topological immersions $f\colon M \longrightarrow N$ that, except for two cases, realize what is established by Theorem 3.5 for the number of components for the complement $N - f(M)$, taking into account the various possibilities regarding the orientation of the manifolds involved as well as the finiteness of the first homology groups of M and/or N, on the same line used in the embedding case.

Lower bounds for the number of connected components for the complement $N - f(M)$ are promptly provided by Theorem 3.5 and we succeed realizing these lower bounds, as well as all numbers above it, except in two cases, where we could not find examples realizing exactly the lower bound, although any number greater than or equal to the lower bound plus one is realizable.

We begin this section by establishing two propositions that will be useful to construct examples of immersions.

Proposition 4.1 *Let $\overline{N}$ be an n-dimensional manifold, $n \geqslant 2$, either orientable or with $H_1(\overline{N};\mathbb{Z})$ finite. Given an immersion $f\colon M \longrightarrow \overline{N}$, it is possible to construct immersions from M to an n-dimensional manifold N, either non-orientable or with $H_1(N;\mathbb{Z})$ not finite, so that $N - f(M)$ has the same number of components as $\overline{N} - f(M)$. Moreover, the new immersion will have the same type as f, with respect to being orientation true, or not.*

Proof Being $f\colon M \longrightarrow \overline{N}$ a codimension one immersion, $f(M) \subset \overline{N} - D$, where D is an open n-disk contained inside one of the components of $\overline{N} - f(M)$. Making sure that the disk D does not intercept any orientation reversing loop, it is enough to guarantee the new map will be of the same type as f, regarding being orientation true, or not.

Therefore $\overline{N} - D$ imbeds into a connected sum, N, of $\overline{N}$ with either a non-orientable n-manifold or with $S^{n-1} \times S^1$.

The desired immersion is then obtained by composing $f\colon M \longrightarrow \overline{N} - D$ with the inclusion of $\overline{N} - D$ into N. □

The next proposition has a straightforward proof.

Proposition 4.2 *Let $f\colon M \longrightarrow N$ be a codimension one immersion and Y be a k-manifold. Consider $g\colon M \times Y \longrightarrow N \times Y$ given by $g = f \times \mathrm{id}_Y$. Then g is a codimension one topological immersion and $N \times Y - g(M \times Y)$ has the same number of components as $N - f(M)$. Moreover,*

(a) *If f is orientation true, then so is g.*
(b) *If $H_1(Y;\mathbb{Z})$ is finite, then the finiteness of the first homology groups of $N \times Y$ and $M \times Y$ will be the same as those of N and M, respectively.*

4.1 Non-orientation true immersions

The examples of non-orientation true immersions of $f\colon M \longrightarrow N$ presented below illustrate that any number can be realized as the number of components of $N - f(M)$.

The first three examples are such that $N - f(M)$ is connected and, therefore, they show that the minimum number of components may be realized. The last three show that, in fact, any number of components can occur.

Remark 4.3 Observe that, Proposition 4.1 can be used to modify examples of immersions $f: M \longrightarrow N$, with $H_1(N;\mathbb{Z})$ finite, to obtain other examples with the first homology group of the target space not finite.

Example 4.4 (*M orientable and N non-orientable;* $H_1(N,\mathbb{Z})$ *not finite*)

Let $f: S^1 \longrightarrow K^2$, K^2 being the Klein bottle. Consider in S^1 the semicircles a and b, so that $S^1 = a \cup b$ is oriented in counterclockwise fashion. In K^2, let α and β be the loops shown in Fig. 1. Construct f such that $f(a) = \alpha$ and $f(b) = \beta$.

It is clear that f is non-orientation true, as $f(a \cup b) = \alpha \star \beta$, and α is an orientation-reversing loop. The number of components of $K^2 - f(S^1)$ is 1, so the lower bound given by Theorem 3.5 is obtained.

Example 4.5 (*M non-orientable and N orientable;* $H_1(N,\mathbb{Z})$ *finite*)

Consider $\mathbb{Z}_4 = \{1, \eta, \eta^2, \eta^3\}$ and $\mathbb{Z}_2 = \{1, \eta^2\}$ the cyclic groups of order 4 and 2, respectively. Let $\mathbb{Z}_2$ act on S^4 via the antipodal map, and in S^5 consider the action of $\mathbb{Z}_4$ given by $\eta \cdot (z_1, z_2, z_3) = (iz_1, iz_2, iz_3)$, where $i = \sqrt{-1}$.

Consider $f: \mathbb{R}P^4 = S^4/\mathbb{Z}_2 \longrightarrow S^5/\mathbb{Z}_4$ the quotient application, induced by the natural inclusion of S^4 in S^5. Consider the sets $A = S^3/\mathbb{Z}_2 = \mathbb{R}P^3$ and $B = f(A) = S^3/\mathbb{Z}_4$.

Since $\mathbb{R}P^4$ is non-orientable and $S^5/\mathbb{Z}_4$ is orientable, f is not orientation true. Since $N = S^5/\mathbb{Z}_4$ is orientable, Γ_N is trivial. Also, since A and B are closed manifolds, Čech cohomology coincides with singular one. From Theorem 3.5 the number of components of the complement is given by $1 + \text{rank}\,(H^3(A;\mathbb{Z})) - \text{rank}\,(f|_A)^*$, where $(f|_A)^*: H^3(B;\mathbb{Z}) \longrightarrow H^3(A;\mathbb{Z})$ is multiplication by 2. Therefore the complement is connected.

Example 4.6 (*M and N non-orientable;* $H_1(N,\mathbb{Z})$ *finite*)

This example can be obtained using the immersion from Example 4.5 together with the procedure described in Propositions 4.1 and 4.2 as we take the connected sum $(S^5/\mathbb{Z}_4) \# (\mathbb{R}P^2 \times S^3)$.

Example 4.7 (*M orientable and N non-orientable;* $H_1(N,\mathbb{Z})$ *finite*)

Let $f: D^1/\!\sim\, = \mathbb{R}P^1 = S^1 \longrightarrow D^2/\!\sim\, = \mathbb{R}P^2$ be defined as follows: In S^1 consider the points labeled by 1, 2 and 3, the arcs s_1, s_2 and s_3, and in $\mathbb{R}P^2$ their image under f, as shown in Fig. 2. Then $\mathbb{R}P^2 - f(\mathbb{R}P^1)$ has two components.

The map above can be easily modified to obtain three or more components in $N - f(M)$, as follows: In order to get n components we consider $2n-1$ points in S^1, labeled by $1, 2, \ldots, 2n-1$. This gives raise to the arcs s_i, $i = 1, \ldots, 2n-1$, where

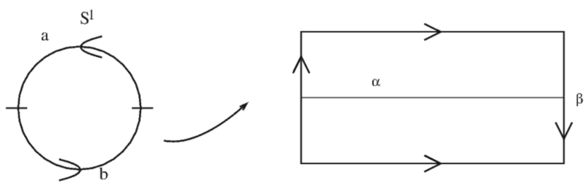

Fig. 1 *M* orientable and *N* non-orientable

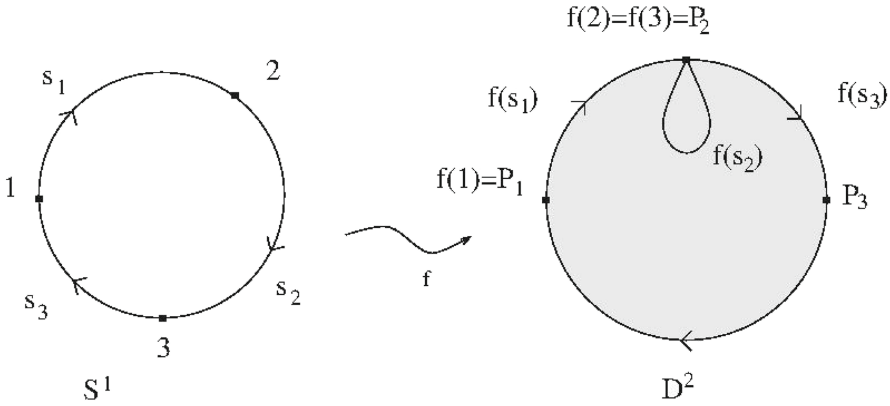

Fig. 2 M orientable and N non-orientable; $H_1(N, \mathbb{Z})$ finite

s_i is the arc between the points i and $i+1$, for $i = 1, \dots, 2n-2$, and s_{2n-1} is the arc between $2n-1$ and 1.

In the upper hemisphere of $S^1 = \partial D^2$ consider $n+1$ distinct points, with the first one and the last one being antipodal. Label them as $P_1, P_2, \dots, P_{n+1}$.

The map $f : S^1 \longrightarrow D^2/\sim = \mathbb{R}P^2$ is such that $f(1) = P_1$, $f(2) = f(3) = P_2$, $f(4) = f(5) = P_3$ and so on, until $f(2n-2) = f(2n-1) = P_n$. Also, the images of the arcs s_i under f are defined as follows: $f(s_{2i-1})$ is the arc in $S^1 = \partial D^2$ bounded by P_i and P_{i+1} for $i = 1, 2, \dots, n$ and $f(s_{2i})$ is a loop almost totally contained in the interior of D^2, having only its basepoint at $P_{i+1} \in \partial D^2$, for $i = 1, \dots, n-1$. Choosing these loops so that they do not intercept each other, we see that $N - f(M)$ has n components.

Example 4.8 (*M non-orientable and N orientable; $H_1(N, \mathbb{Z})$ finite*)

In D^2, consider the concentric circles C_1 and C_2, and the regions R_1, bounded by C_1; R_2, bounded by C_1 and C_2; and R_3, bounded by C_2 and $S^1 = \partial D^2$.

In $S^2 = \partial D^3$, consider the parallels P_1 and P_2 indicated in Fig. 3 and the regions S_1, bounded by P_1, and S_2, bounded by P_1 and P_2.

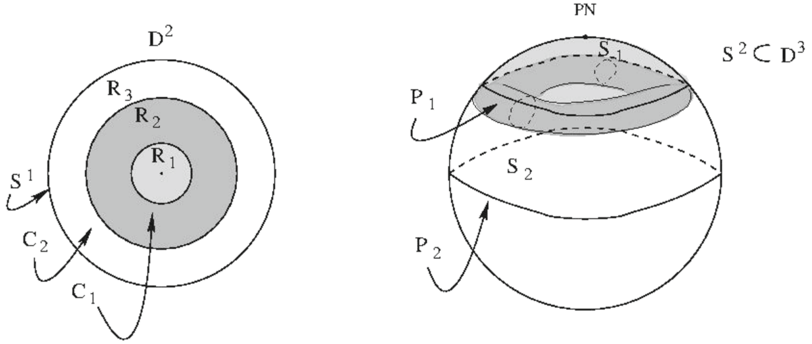

Fig. 3 M non-orientable and N orientable; $H_1(N, \mathbb{Z})$ finite

Table 1 Summary for the nonorientable case

	$H_1(N)$ finite	$H_1(N)$ not finite
M orientable and N non orientable	$\geqslant 2$ (Example 4.7)	1 (Example 4.4)
		$\geqslant 2$ (Remark 4.10)
M non orientable and N orientable	1 (Example 4.5)	$\geqslant 1$ (Remark 4.10)
	$\geqslant 2$ (Example 4.8)	
M and N non orientable	1 (Example 4.6)	$\geqslant 1$ (Remark 4.10)
	$\geqslant 2$ (Example 4.9)	

Define $g\colon D^2 \longrightarrow D^3$ by taking $g(0,0) = (1,0,0)$, $g(C_1) = g(C_2) = P_1$, $g(S^1) = P_2$; to fulfill the properties required, set $g(R_1) = S_1$, $g(R_3) = S_2$, and $g(R_2)$ is defined as a torus inside D^3 having P_1 as a parallel (P_1 is the only intersection of $g(R_2)$ with $S^2 = \partial D^3$).

Thus, define $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}P^3$ by $f \circ p_2 = p_3 \circ g$, where p_2 and p_3 are, respectively, the usual quotient map from D^2 to $\mathbb{R}P^2$, and from D^3 to $\mathbb{R}P^3$.

Then $f(\mathbb{R}P^2) = \mathbb{R}P^2 \cup T^2$ and $N - f(M)$ has two components.

Construct a map $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}P^3$ so that $N - f(M)$ has any number $n, n \geqslant 3$, of components, by modifying $g\colon D^2 \longrightarrow D^3$ as follows: In the interior of D^2, consider the concentric circles $C_1, \ldots, C_{2n-2}$, and the regions R_1, bounded by C_1; R_i, bounded by C_{i-1} and C_i, for $i = 2, \ldots, 2n-2$; and R_{2n-1}, bounded by C_{2n-2} and $S^1 = \partial D^2$.

In $S^2 = \partial D^3$, consider the parallels $P_1, \ldots, P_n$, contained in the upper hemisphere of $S^2 = \partial D^3$, with P_n being the equator of S^2. Also, consider the regions S_1, bounded by P_1 (containing the North Pole); S_i, bounded by P_{i-1} and P_i, $i = 1, \ldots, n$.

Define $g\colon D^2 \longrightarrow D^3$, by taking $g(0,0) = (1,0,0)$, $g(C_1) = g(C_2) = P_1$, $g(C_{2i-1}) = g(C_{2i}) = P_i$, $i = 1, \ldots, n-1$, and $g(S^1) = P_n$; to fulfill the properties required, set $g(R_1) = S_1$, $g(R_{2i-1}) = S_i$, $i = 2, \ldots, n$, and $g(R_{2i})$ defined as a torus inside D^3 having P_i as a parallel (P_i is the only intersection of $g(R_{2i})$ with $S^2 = \partial D^3$), $i = 1, \ldots, n-1$. We should take care that those images should not intersect, that is, the torus is taken disjoint.

After creating the map g, the construction of the required f proceeds as before. In this case, $f(\mathbb{R}P^2)$ will be the union of $\mathbb{R}P^2$ with $n-1$ tori so that $N - f(M)$ will have n components.

Example 4.9 (*M and N non-orientable; $H_1(N,\mathbb{Z})$ finite*)

This example can be obtained applying Proposition 4.2 to the immersion from Example 4.8, by taking the Cartesian products $M \times \mathbb{R}P^2$ and $N \times \mathbb{R}P^2$, in place of M and N, respectively.

Remark 4.10 Examples with all possible numbers of components of $N - f(M)$, with $H_1(N;\mathbb{Z})$ not finite, can be obtained from the previous examples, using Proposition 4.2.

Summary Table 1 below summarizes the collection of examples obtained in this section. The entries correspond to the number of components of $N - f(M)$, for $f\colon M \longrightarrow N$ a non-orientation true immersion.

It should be pointed out that we could not find an example of a non-orientation true immersion $f: M \longrightarrow N$, with only double points, where M is orientable, N is non-orientable and $H_1(N; \mathbb{Z})$ finite, so that $N - f(M)$ is connected.

4.2 Orientation true immersions

In this section, examples of codimension one orientation true immersions $f: M \longrightarrow N$ are listed, so that all possibilities established by Theorem 3.5 can be realized as the number of components of $N - f(M)$. We present examples in various situations, accordingly to the orientation of the manifolds involved and to the finiteness of the first homology groups of these manifolds.

Remark 4.11 Observe that, using Propositions 4.1 and 4.2, it is only needed to provide examples where both, M and N, are orientable. Moreover, Proposition 4.1 can be used to modify examples of immersions $f: M \longrightarrow N$, with $H_1(N; \mathbb{Z})$ finite, to obtain other examples with the first homology group of the target space not finite.

Example 4.12 *($H_1(M; \mathbb{Z})$ and $H_1(N; \mathbb{Z})$ both finite)*
Let $g: S^n \longrightarrow D^{n+1}$ be defined by sending the north hemisphere of S^n to the north hemisphere of $S^n = \partial D^{n+1}$, and the interior of the south hemisphere of S^n to a "cap" contained in the interior of the south hemisphere of D^{n+1}; in the equator, it coincides with the inclusion of S^{n-1} into S^n. See Fig. 4. Let $f: S^n \longrightarrow \mathbb{R}P^{n+1}$ be defined by $f = p \circ g$, where $p: D^{n+1} \longrightarrow \mathbb{R}P^{n+1}$ is the natural projection. Then f is locally injective and therefore a topological immersion. Also, $A = S^{n-1}$ and $B = f(A) = \mathbb{R}P^{n-1}$. If n is even $\mathbb{R}P^{n+1}$ is orientable. Therefore f is orientation true and as in Example 4.5 we see that the complement has two components.

Example 4.13 *($H_1(M; \mathbb{Z})$ and $H_1(N; \mathbb{Z})$ both finite)*
Consider $j: F \longrightarrow T^2$, the embedding of $F = \alpha \vee_{x_0} \beta$, where $j(\alpha)$ is a meridian circle and $j(\beta)$ is a parallel circle in the torus T^2.

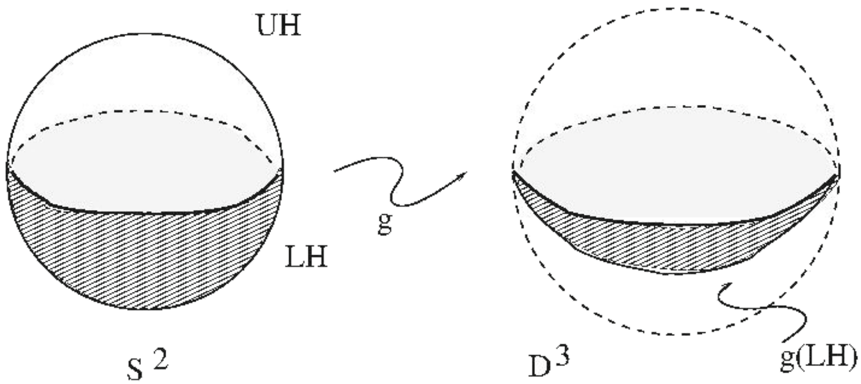

Fig. 4 $H_1(M; \mathbb{Z})$ and $H_1(N; \mathbb{Z})$ both finite

Let $S^1 = a \cup b$, a and b being, respectively, the upper half arc of S^1 and the lower half. Now, let $i : S^1 \longrightarrow F$ be the immersion of S^1 in F, given by $i(a) = \alpha$ and $i(b) = \beta$. Take $f_1 : S^1 \longrightarrow T^2$ as $f_1 = j \circ i$. Then, $T^2 - f_1(S^1)$ has only one component.

Example 4.14 *($H_1(M;\mathbb{Z})$ not finite and $H_1(N;\mathbb{Z})$ finite)*
Consider $T_1^2 = S^1 \times [-1, 1]/_{(z,-1)\sim(z,1)}$ a torus and T^2 an embedded torus in R^3 where we denote by T_S^2 the solid torus having boundary T^2. We describe the solid torus as $T_S^2 = \bigcup_{r\in[0,1]}(T_S^2)_r$, where $(T_S^2)_1$ is the 2-torus T^2 seen as a disk attached in the usual way to the bouquet $\alpha \vee \beta_1$, $(T_S^2)_0$ is the curve α and for each $0 < r < 1$, $(T_S^2)_r$ are also a 2-torus considered as a disk attached, in the usual way, to the bouquet $\alpha \vee \beta_r$, where α, β_1 and β_r are the loops indicated in Fig. 5.

In each torus $(T_S^2)_r$ for $0 < r \leqslant 1$, consider a pair of disjoint curves, γ_r^1 and γ_r^2, defined as shown in Fig. 6, so that $d(\gamma_r^1, \gamma_r^2) \to 0$ as $r \to 1$ or $r \to 0$. These curves represent the same element (2, 1) in $H_1(T^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

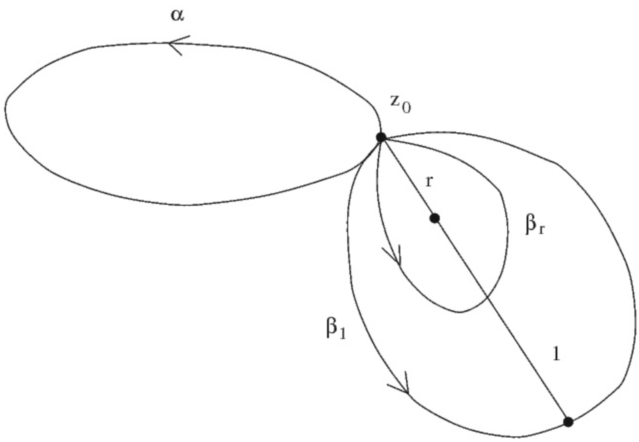

Fig. 5 $H_1(M;\mathbb{Z})$ not finite and $H_1(N;\mathbb{Z})$ finite

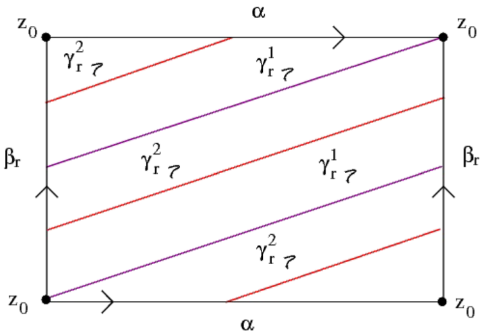

Fig. 6 Pair of disjoint curves

Define $g\colon S^1 \times [-1, 1] \longrightarrow T_S^2$ by setting

$$g(C_t)(z) = \begin{cases} \alpha(z^2), & \text{when } t = \pm 1, \\ \gamma^1_{1-t}(z), & 0 \leqslant t < 1, \\ \gamma^2_{1+t}(z), & -1 < t < 0, \end{cases}$$

where $C_t = S^1 \times \{t\}$, $t \in [-1, 1]$ and $z \in S^1$. Since $g(C_1) = g(C_{-1})$, we have that g induces a well-defined map from T_1^2 to T_S^2. Further, g is continuous because $\gamma_r^i(z)$ and $\gamma_r^i(-z)$ converge to the same point of α as r converges to 0, for $i = 1, 2$. Composing g with the inclusion of T_S^2 into S^3, we obtain an immersion $f\colon T_1^2 \longrightarrow S^3$, so that A is the circle in T^2 corresponding to the images of C_1 and C_{-1} under the identification, and B is the circle α in T_S^2. From Theorem 3.5 the number of components of $S^3 - f(T^2)$ is given by $2 + \operatorname{rank}(H^1(S^1; \mathbb{Z})) - \operatorname{rank}(\operatorname{Im}(\eta_1))$, where $\eta_1\colon H^1(S^1; \mathbb{Z}) \oplus H^1(S^1; \mathbb{Z}) \to H^1(S^1; \mathbb{Z})$ is given by $l^* - (f|_A)^*$. Since $(f|_A)^*$ is multiplication by 2, $\operatorname{rank}(\operatorname{Im}(\eta_1)) = 1$. Therefore $S^3 - f(T^2)$ has two connected components.

The following examples realize all possible number of components, above the lower bounds established by Theorem 3.5.

Example 4.15 ($H_1(M; \mathbb{Z})$ *not finite and* $H_1(N; \mathbb{Z})$ *finite*)

Let T^2 be the usual torus surface, and let T_S^2 be the solid torus. Begin by constructing a map f_1 such that $T_S^2 - f_1(T^2)$ has two components. Let C_1 and C_2 be meridians in T^2, and C be a meridian in the boundary of T_S^2. The map f_1 is defined so that, the first "half" of the torus surface is mapped onto the boundary of T_S^2, taking both meridians C_1 and C_2 of T^2 onto C. See Fig. 7.

The other "half" is then mapped onto another torus in the interior of the solid torus, except for C_1 and C_2, that are both sent to C. See Fig. 8.

Considering the inclusion i of T_S^2 into S^3, the map $f = i \circ f_1$ yields an immersion from T^2 to S^3 such that the complement $S^3 - f(T^2)$ has three components.

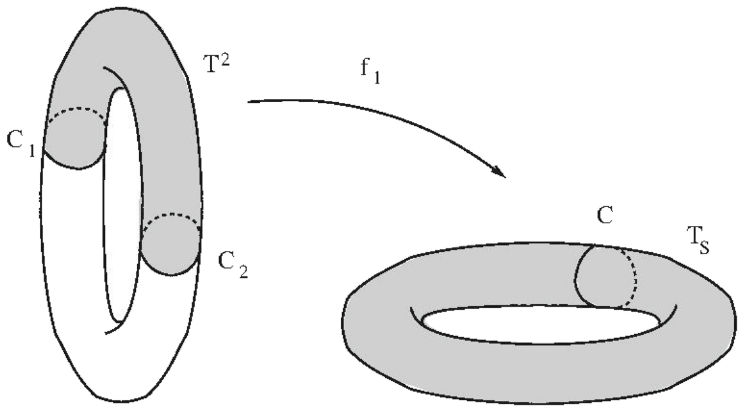

Fig. 7 $H_1(M; \mathbb{Z})$ not finite and $H_1(N; \mathbb{Z})$ finite I

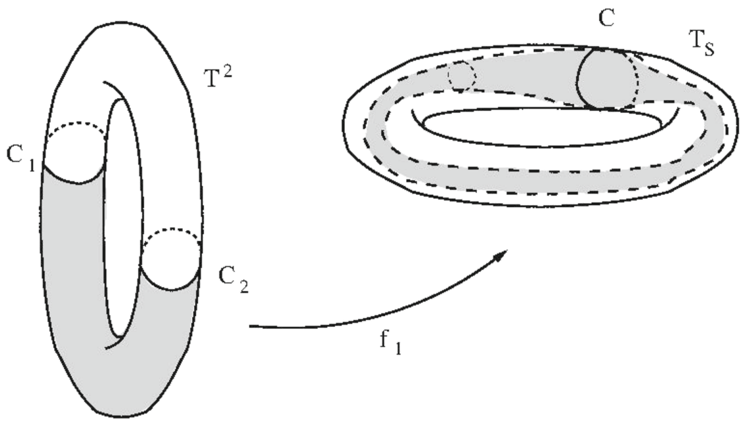

Fig. 8 $H_1(M;\mathbb{Z})$ not finite and $H_1(N;\mathbb{Z})$ finite II

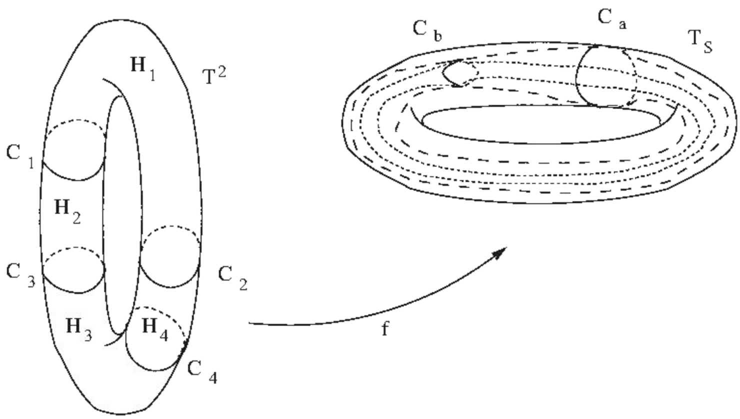

Fig. 9 $H_1(M)$ not finite and $H_1(N)$ finite

We have just described the construction of f_1 when the image of the double points, B, is just one circle. This can be easily extended to make B a set formed by more circles. The case where B is the disjoint union of two circles is showed below. For more circles, the description just gets more elaborated, but it is basically the same.

Example 4.16 ($H_1(M)$ *not finite and* $H_1(N)$ *finite*)

In Fig. 9, the map f is defined so that $f(C_1) = f(C_2) = C_a$ and $f(C_3) = f(C_4) = C_b$. Now things go a little different from the previous construction. Here, $f(H_1) = T^2 = \partial T_S^2$ and H_2 is mapped to the shorter dashed cylinder with boundary $C_a \cup C_b$ in such a way that, except for C_a, it is all contained in the interior of T_S^2. Now, H_3 is mapped to a dotted torus in the interior of T_S^2 and intercepting $f(H_2)$ only in C_b and H_4 goes to the longer dashed cylinder contained in the interior of T_S^2, except for C_a, with $C_a \cup C_b$ as its boundary. In this case $T_S^2 - f(T^2)$ will have three components.

Again, composing this map with the inclusion into S^3, yields an immersion from T^2 to S^3, such that the complement $S^3 - f(T^2)$ has four components.

The construction above can be modified, in a similar manner, to obtain any number of components.

Example 4.17 *($H_1(M;\mathbb{Z})$ and $H_1(N;\mathbb{Z})$ finite)*

In $\mathbb{R}^3$ consider two distinct parallel circles on a sphere S^2: C_1 and C_2.

Let NC be the northern cap from the North Pole to C_1; T be the region on the sphere between C_1 and C_2; and SC be the southern cap , from C_2 to the South Pole. Let S be a sphere in $\mathbb{R}^3$, and $g\colon S^2 \longrightarrow S \subset \mathbb{R}^3$ be a map such that

- $g(C_1) = g(C_2) = E$, where E is the equator of S.
- $g(NC) =$ northern hemisphere of S.
- $g(SC) =$ southern hemisphere of S.

Define the image of T, under g, as a torus around the equator of S. See Fig. 10.

Imbedding $\mathbb{R}^3$ into S^3, a map $f\colon S^2 \longrightarrow S^3$ such that $N - f(M) = S^3 - f(S^2)$ has three components is obtained.

Example 4.18 *($H_1(M;\mathbb{Z})$ and $H_1(N;\mathbb{Z})$ finite)*

In $\mathbb{R}^3$ consider $2n$ distinct parallel circles on a sphere S^2: $C_1, C_2, \ldots, C_{2n}$.

Let NC be the northern cap from the North Pole to C_1; T_i be the region on the sphere between C_i and C_{i+1} for $i = 1, 2, \ldots, (2n-1)$; and SC be the southern cap, from C_{2n} to the South Pole. Let S be a sphere in $\mathbb{R}^3$, and $g\colon S^2 \longrightarrow S \subset \mathbb{R}^3$ be a map such that

- $g(C_{2i-1}) = g(C_{2i}) = P_i$, where P_i are disjoints parallel circles on S, $i = 1, \ldots, n$.
- $g(NC)$ = northern cap of S, from the North Pole to P_1 of S.
- $g(SC) =$ southern cap of S, from the South Pole to P_n of S.
- $g(T_{2i}) =$ the region, in S, between P_i and P_{i+1}.

Define the image of T_{2i-1}, under g, as a torus around the circle P_i of S, for $i = 1, \ldots, n$. See Fig. 11 for the case $n = 2$.

Imbedding $\mathbb{R}^3$ into S^3, yields a map $f\colon S^2 \longrightarrow S^3$ such that $N - f(M) = S^3 - f(S^2)$ has $n + 2$ components.

Summary Table 2a and b summarize the collection of examples obtained in this section. The entries correspond to the number of components of $N - f(M)$, for $f\colon M \longrightarrow N$ an orientation true immersion.

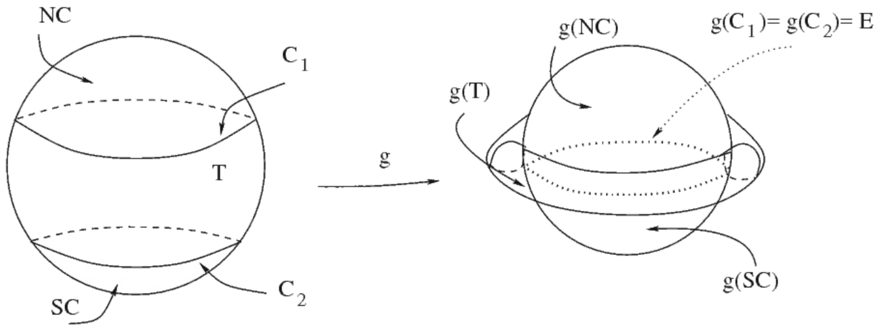

Fig. 10 $H_1(M;\mathbb{Z})$ and $H_1(N;\mathbb{Z})$ finite

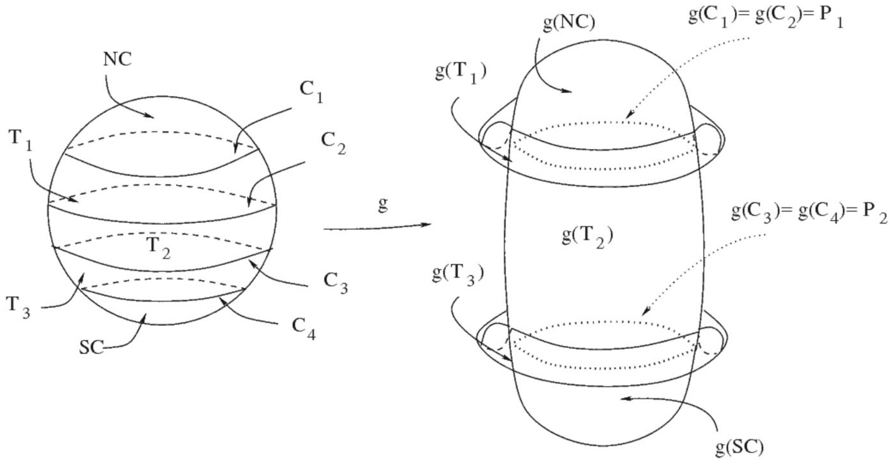

Fig. 11 $H_1(M;\mathbb{Z})$ and $H_1(N;\mathbb{Z})$ finite

Table 2 Summary for the orientable case

	$H_1(M;\mathbb{Z})$ and $H_1(N;\mathbb{Z})$ finite	$H_1(M;\mathbb{Z})$ finite $H_1(N;\mathbb{Z})$ *not* finite
(a)		
M and N orientable	2 (Example 4.12)	
	3 (Example 4.17)	$\geqslant 2$ (Remark 4.3)
	$\geqslant 4$ (Example 4.18)	
	$H_1(M;\mathbb{Z})$ *not* finite $H_1(N;\mathbb{Z})$ finite	$H_1(M;\mathbb{Z})$ and $H_1(N;\mathbb{Z})$ *not* finite
(b)		
M and N orientable	2 (Example 4.14)	1 (Example 4.13)
	$\geqslant 3$ (Example 4.15 and 4.16)	$\geqslant 2$ (Remark 4.3)

It should be pointed out that we could not find an example of an orientation true immersion $f: M \longrightarrow N$, with M and N orientable, $H_1(M;\mathbb{Z})$ finite and $H_1(N;\mathbb{Z})$ not-finite, so that $N - f(M)$ is connected.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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