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Ramsey goodness of paths versus unbalanced graphs [☆]

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ABSTRACT

Given graphs G and H, we say that G is H-*good* if the Ramsey number $R(G,H)$ equals the trivial lower bound $(|G|-1)(\chi(H)-1)+\sigma(H)$, where $\chi(H)$ denotes the usual chromatic number of H, and $\sigma(H)$ denotes the minimum size of a color class in a $\chi(H)$-coloring of H. Pokrovskiy and Sudakov (2017) [17] proved that P_n is H-good whenever $n \geq 4|H|$. In this paper, given $\varepsilon > 0$, we show that if H satisfies a special unbalance condition, then P_n is H-good whenever $n \geq (2+\varepsilon)|H|$. More specifically, we show that if $m_1, \ldots, m_k$ are such that $\varepsilon \cdot m_i \geq 2m_{i-1}^2$ for $2 \leq i \leq k$, and $n \geq (2+\varepsilon)(m_1+\cdots+m_k)$, then P_n is $K_{m_1,\ldots,m_k}$-good.

In this paper, we consider only finite and undirected graphs without loops or multiple edges. Throughout this text, given a graph G, we denote by $|G|$ the number of vertices of G. Given graphs K, G and H, we write $K \to (G,H)$ if every red–blue coloring of the edges of K contains a red copy of G or a blue copy of H; and the *Ramsey number* $R(G,H)$ is the minimum positive integer N for which $K_N \to (G,H)$. Understanding the behavior of $R(G,H)$ is one of the main problems in Extremal Combinatorics, and have been extensively explored over the last century [10] while significant results were obtained recently [8,9,12,13,20].

A natural lower bound for $R(G,H)$ is as follows. Let $\chi(H)$ be the chromatic number of H, i.e., the smallest number of colors for which there is a proper coloring of the vertices of H, and let $\sigma(H)$ be the minimum size of a smallest class in a minimum proper coloring of H, i.e., $\sigma(H) = \min\{|c^{-1}(i)| : c \text{ is a proper coloring of } H \text{ with } \chi(H) \text{ colors}, i \in [\chi(H)]\}$. Burr [6] observed that if G is a connected graph and $|G| \geq \sigma(H)$, then we have

$$R(G,H) \geq (|G|-1)(\chi(H)-1)+\sigma(H). \tag{1}$$

Indeed, put $N = (|G|-1)(\chi(H)-1)+\sigma(H)-1$, and consider the red–blue coloring of $E(K_N)$ obtained from $\chi(H)-1$ disjoint red cliques with $|G|-1$ vertices and one red clique with $\sigma(H)-1$ vertices by coloring the remaining edges blue. Such a coloring contains no red copy of G because each red component has size at most $|G|-1$; and contains no blue copy of H because the blue edges induce a $\chi(H)$-partite graph K^B, but different parts of H must fit in different parts of K^B, while no part of H fits in the part of size $\sigma(H)-1$.

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Motivated by this construction, in a seminal paper, Erdős and Burr [7] introduced the concept of *goodness*. More specifically, we say that a graph G is H*-good* if (1) holds with equality, i.e., whenever $R(G,H) = (|G|-1)\big(\chi(H)-1\big) + \sigma(H)$. Erdős and Burr [7] presented several examples of K_r-good sparse graphs, and posed a number of questions that shaped this subarea since then. The main such question is whether bounded degree are graphs K_r-good.

Although Brandt disproved the main conjecture [5], goodness of sparse graphs has been extensively studied. Nikiforov and Rousseau [16] answered all of other questions, and, in particular, their results together with the Separator Theorem of Alon, Seymour, and Thomas [2] imply that all planar graphs are K_r-good. More recently, Allen, Brightwell, and Skokan [1], among other interesting results, proved that bounded degree graphs with sublinear bandwidth are K_r-good.

In a more general setup, H-goodness is known for some specific classes of sparse graphs and any fixed graph H, not necessarily complete. Erdős, Faudree, Rousseau and Schelp [11] proved that every bounded degree tree with n vertices is H-good for every fixed H and sufficiently large $n \in \mathbb{N}$, and their result was strengthened first by Balla, Pokrovskiy and Sudakov [3], and recently by Montgomery, Pavez-Signé, and Yan [14] who proved that if $n = \Omega(|H|)$ then every bounded degree tree with n vertices is H-good. In the special case of paths, in 2017, Pokrovskiy and Sudakov [17] presented a much tighter result that P_n is H-good whenever $n \geq 4|H|$, where P_n denotes the path with n vertices. Observe that it suffices to verify the case H is a complete multipartite graph.

Theorem 1 *(Pokrovskiy–Sudakov, 2017). Given integers $m_1 \leq m_2 \leq \cdots \leq m_k$, $n \geq 3m_k + 5m_{k-1}$ and $N \geq (n-1)(k-1) + m_1$, we have $K_N \to (P_n, K_{m_1,\dots,m_k})$.*

Pokrovskiy and Sudakov [17] also observe that the multiplicative constant (4) cannot be reduced below 2, giving the following example which we present here for completeness. Fix $H = K_{m_1,m_2}$ with $m_1 \leq m_2$, and suppose $n = 2m_2 - 2 < 2(m_1+m_2) = 2|H|$. Then let $N = n - 1 + m_1$ and color K_N with disjoint (curiously) blue cliques of size $m_1 + m_2 - 1$ and $m_2 - 2$, joined by red edges. Such a coloring has no blue copy of H because each blue component has size at most $|H| - 1$; and has no red copy of P_n because any red path must alternate between the two blue cliques, and hence has size at most $2m_2 - 3$.

More recently, Pokrovskiy and Sudakov [18] obtained a result on Ramsey goodness of cycles by imposing additional conditions on the sizes the color classes of H.

Theorem 2 *(Pokrovskiy–Sudakov, 2020). If $n \geq 10^{60} m_k$, $m_1 \leq m_2 \leq \cdots \leq m_k$ satisfy $m_i \geq i^{22}$ for each i, and $N \geq (n-1)(k-1) + m_1$, then $K_N \to (C_n, K_{m_1,\dots,m_k})$.*

When we consider a simple unbalance condition in the size of the largest parts, we obtain the following straightforward consequence of Theorem 1 which brings the constant 4 closer to 3.

Corollary 3. *Let $\varepsilon \in (0,1]$. Given integers $m_1 \leq m_2 \leq \cdots \leq m_k$, such that $\varepsilon \cdot m_k \geq 2m_{k-1}$ and let $n \geq (3+\varepsilon)(m_1 + \cdots + m_k)$ and $N \geq (n-1)(k-1) + m_1$. Then $K_N \to (P_n, K_{m_1,\dots,m_k})$.*

Proof. Since $\varepsilon \cdot m_k \geq 2m_{k-1}$, we have

$$n \geq (3+\varepsilon)(m_k + m_{k-1}) \geq 3m_k + \varepsilon \cdot m_k + 3m_{k-1} \geq 3m_k + 5m_{k-1}.$$

Therefore, by Theorem 1, $K_N \to (P_n, K_{m_1,\dots,m_k})$ as desired. □

In this paper, we reduce the constant $3+\varepsilon$ to $2+\varepsilon$ by imposing a stronger unbalance condition on H. Given the example presented above, this result is somehow tight.

Theorem 4. *Let $\varepsilon \in (0,1]$ and let $m_1 \leq m_2 \leq \cdots \leq m_k$ be positive integers such that $\varepsilon \cdot m_i \geq 2m_{i-1}^2$ for every $2 \leq i \leq k$. If $n \geq (2+\varepsilon)(m_1 + \cdots + m_k)$ and $N \geq (n-1)(k-1) + m_1$, then $K_N \to (P_n, K_{m_1,\dots,m_k})$.*

Observe that both Corollary 3 and Theorem 4 imply that $K_N \to (P_n, H)$ whenever H admits a coloring with color classes satisfying their respective unbalance conditions.

Our technique borrows a few ideas from [17], but we make only a mild use of Pósa rotation-extension technique (see Lemma 6). We, alternatively, explore the structure of the graph obtained when removing a longest path, as well as relations between the neighbors of the vertices of such graph in the path (see Section 1).

Organization of the paper. In Section 1, we present some auxiliary results that could be useful in a more general context; in Section 2 we verify Theorem 4 in the special case $k = 2$ (see Theorem 5); and in Section 3 we verify Theorem 4. For ease of notation, when dealing with bipartite graphs, we use s, t instead of m_1, m_2 for the size of its parts.

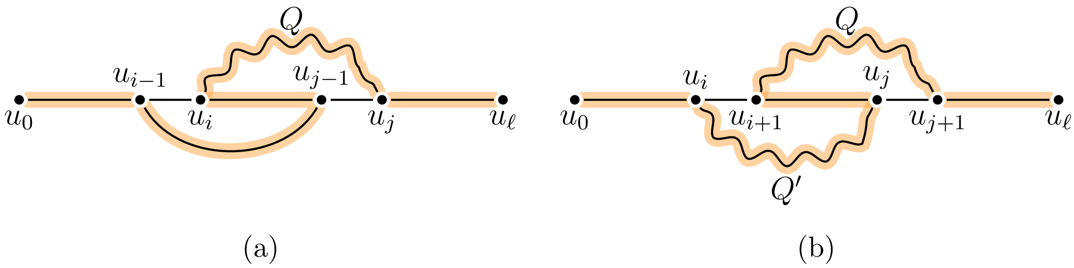

Fig. 1. Left: A longer path obtained in the case $N(F)^-$ is not an independent set; Right: A longer path obtained in the case $|N(F)^- \cap N(F')| \geq 2$.

1. Auxiliary results

Given a graph G and disjoint sets $X, Y \subseteq V(G)$, we denote by $G[X, Y]$ the bipartite graph with vertex set $X \cup Y$ and whose edges are the edges of G that join a vertex of X to a vertex of Y; and by $e(X, Y)$ the number of edges of $G[X, Y]$.

In our proof we use the following result to bound the order of a longest path in the red graph. Such a result can be proved with a simple depth-first search (see, e.g., [15, Lemma 3.3]).

Lemma 1. *Let s, t be two positive integers, and let G be a graph on N vertices. Then at least one of the following holds: (i) there is a pair S, T of disjoint sets of vertices with $|S| = s$ and $|T| = t$ for which $e(S, T) = 0$; or (ii) G contains a path of order $N - s - t + 1$.*

The following result on split graphs is also useful to find a blue copy of $K_{s,t}$.

Lemma 2. *Let s, t be two positive integers with $s \leq t$, and let G be a graph obtained from a clique X and a graph with vertex set Y with $X \cap Y = \emptyset$ by joining every vertex of X to every vertex of Y. If $|X| \geq s$ and $|X| + |Y| \geq s + t$, then G contains a $K_{s,t}$.*

Proof. Let $S \subseteq X$ be a set of size s and $T \subseteq (X \cup Y) \setminus S$ be a set of size t, then $G[S, T]$ is the desired $K_{s,t}$. □

Given a graph G and a set $X \subseteq V(G)$, we denote by $N_G(X)$ the set of vertices in $V(G) \setminus X$ that are adjacent to at least one vertex of X. We omit subscripts when it is clear from the context. A *component* of a graph G is a set F of vertices of G for which $G[F]$ is a maximal connected subgraph of G. Now, consider a path $P \subseteq G$. In what follows, we fix an ordering $P = u_0 \cdots u_\ell$ of $V(P)$. Let F be a component of $G \setminus V(P)$, and observe that $N(F) \subseteq V(P)$. We denote by $N(F)^-$ the set $\{u_{i-1} : u_i \in N(F)\}$. Also, given vertices u and u' in P, we denote by uPu' the subpath of P joining u and u'. When P is a longest path in G, we obtain the following.

Lemma 3. *Let G be a graph, let P be a longest path in G, and let F be a component of $G \setminus V(P)$. Then (i) $N(F)^- \cap N(F) = \emptyset$; and (ii) $N(F)^-$ is an independent set.*

Proof. Let $P = u_0 \cdots u_\ell$ be as above. (i) If $u_i \in N(F)^- \cap N(F)$, then $u_i, u_{i+1} \in N(F)$. Let Q be a path joining u_i to u_{i+1} whose internal vertices are in F. Then $(P \cup Q) - u_i u_{i+1}$ is a path in G with at least $|P| + 1$ vertices, a contradiction to the maximality of P. (ii) By the maximality of P we have $u_0 \notin N(F)$. Let $u_i, u_j \in N(F)$ with $1 \leq i < j$ and suppose that $u_{i-1}u_{j-1} \in E(G)$. Let Q be a path joining u_i to u_j whose internal vertices are in F. Then $(P \cup Q) - u_{i-1}u_i - u_{j-1}u_j + u_{i-1}u_{j-1}$ is a path in G with at least $|P| + 1$ vertices (see Fig. 1a), a contradiction to the maximality of P. □

The next lemma gives a bound on the number of vertices of $N(F)^-$ that can be adjacent to a vertex in F' for any two distinct components F and F' of $G \setminus V(P)$. Its proof is analogous to the proof above. We include it for completeness.

Lemma 4. *Let G be a graph, let P be a longest path in G, and let F and F' be distinct components of $G \setminus V(P)$. Then $|N(F)^- \cap N(F')| \leq 1$.*

Proof. Let $P = u_0 \cdots u_\ell$ be as above. Suppose there are two distinct vertices $u_i, u_j \in N(F)^- \cap N(F')$. By the definition of $N(F)^-$, we have $u_{i+1}, u_{j+1} \in N(F)$. Let Q (resp. Q') be a path in G connecting u_{i+1} to u_{j+1} (resp. u_i to u_j) whose internal vertices are in F (resp. in F'). Then $P^* = \big(P \cup Q \cup Q'\big) - u_i u_{i+1} - u_j u_{j+1}$ is a path in G with at least $|P| + 2$ vertices (see Fig. 1b), a contradiction to the maximality of P. □

2. Paths versus unbalanced bipartite graphs

For organizational purposes we first verify Theorem 4 in the case of bipartite graphs (see Theorem 5), i.e., when $k = 2$, to serve as the base case of our main induction argument. For that we prove the following lemma.

Lemma 5. *Let s, t, and r be positive integers with $r \leq s \leq t$, and let G be a graph with order $N \geq 2t + s^2 - 1$. Let $P \subseteq G$ be a longest path in G. If there are r components in $G \setminus V(P)$ whose union contains at least s vertices, then $K_{s,t} \subseteq \overline{G}$.*

Proof. Let $G' = G \setminus V(P)$ and let $F_1, \dots, F_r$ be components of G' with $|F_1 \cup \cdots \cup F_r| \geq s$. The proof follows by induction on r. First, suppose $r = 1$, i.e., that G' has one component F_1 of order at least s. Note that no vertex in $V(G) \setminus \big(F_1 \cup N_P(F_1)\big)$ is adjacent to a vertex in F_1. Thus, if $N - |F_1| - |N(F_1)| \geq t$, then $K_{s,t} \subseteq \overline{G}[F_1, V(G) \setminus \big(F_1 \cup N_P(F_1)\big)]$ as desired. Therefore, we may assume

$$|F_1| + |N(F_1)| \geq N - (t-1) \geq t + s^2 \geq t + s. \tag{2}$$

Now, suppose that $|N(F_1)| \geq s$. Note that, by the maximality of P, the end vertices of P are not adjacent to vertices of F_1. Thus, we have $|N_P(F_1)^-| = |N_P(F_1)| = |N(F_1)| \geq s$. By Lemma 3, no vertex of $N_P(F_1)^-$ is adjacent to a vertex of F_1, and $N_P(F_1)^-$ is an independent set. Thus, (2) implies that $\overline{G}[N_P(F_1)^- \cup F_1]$ is a graph as in the statement of Lemma 2, and hence $K_{s,t} \subseteq \overline{G}[N_P(F_1)^- \cup F_1]$, as desired.

Therefore we may assume that $|N(F_1)| \leq s - 1$. Together with (2), this implies that

$$|F_1| \geq t + s(s-1) + 1 \geq t + s. \tag{3}$$

Observe that $|P| + |F_1| \leq N$, and, by (3), we have $|P| \leq N - s - t$. Therefore, by Lemma 1, we have $K_{s,t} \subseteq \overline{G}$, as desired. This proves the case $r = 1$.

Now, suppose $r \geq 2$ and let $F^* = F_1 \cup \cdots \cup F_r$. Although we can get a slightly better upper bound on $|F^*|$, here we prove that $|F^*| \leq s^2 - r(r-1)$. Indeed, if $|F_i| \geq s - (r-2)$ for some $i \in [r]$, then we can pick F_i and $r-2$ other components of $G \setminus V(P)$ and obtain $r-1$ components whose union contains s vertices. But then, by the induction hypothesis, we have $K_{s,t} \subseteq \overline{G}$. Thus, we may assume $|F_i| \leq s - (r-1)$ for every $i \in [r]$. Therefore, since $r \leq s$, we have

$$|F^*| \leq r\big(s - (r-1)\big) = r \cdot s - r(r-1) \leq s^2 - r(r-1), \tag{4}$$

as desired.

Now, analogously to the case $r = 1$, if $N - |F^*| - |N(F^*)| \geq t$, then $K_{s,t} \subseteq \overline{G}[F^*, V(G) \setminus \big(F^* \cup N(F^*)\big)]$, as desired. Thus, we may assume that

$$N - |F^*| - |N(F^*)| \leq t - 1. \tag{5}$$

Summing (4) and (5), by the hypothesis in N, we obtain

$$|N(F^*)| \geq N - t + 1 + r(r-1) - s^2 \geq t + r(r-1).$$

Now, by Lemma 4 given i and j with $i \neq j$, $N_P(F_i)^-$ has at most one vertex in $N(F_j)$. Consequently, $N_P(F_i)^-$ has at most $r-1$ vertices of $N(F^*)$. Thus, $N_P(F^*)^- = \cup_{i=1}^r N_P(F_i)^-$ has at most $r(r-1)$ vertices of $N(F^*)$. Therefore, there is a set $N^* \subseteq N_P^-(F^*) \setminus N(F^*)$ with t vertices, and hence $K_{s,t} \subseteq \overline{G}[F^*, N^*]$, as desired. □

Now, we can prove Theorem 5, which requires a slightly weaker unbalance condition than Theorem 4.

Theorem 5. *Let $\varepsilon \in (0, 1]$, and let $s \leq t$ be positive integers such that $\varepsilon \cdot t \geq s^2 - (3 + \varepsilon)s$. If $n \geq (2 + \varepsilon)(s + t)$ and $N \geq (n-1) + s$, then $K_N \to (P_n, K_{s,t})$.*

Proof. Fix a red–blue coloring of K_N and let G be the graph induced by its red edges. Let $P \subseteq G$ be a longest path in G. If $N - |P| \leq s - 1$, then $|P| \geq n$, as desired. Thus, we may assume that $N - |P| \geq s$. However, as $N = (2 + \varepsilon)(s + t) - 1 + s = 2t + (3 + \varepsilon)s + \varepsilon \cdot t - 1 \geq 2t + s^2 - 1$, applying the Lemma 5 with $r = s$ we have $K_{s,t} \subseteq \overline{G}$, as desired. □

3. Paths versus unbalanced graphs

To prove the main result of this section we need the following consequence of the well-known Pósa rotation-extension technique [5,19], which we restate for our purposes (for a proof see, e.g., [4]).

Lemma 6 *([5]). Let P be a longest path of a graph G. Then there is a set $S \subseteq V(P)$ for which $N(S) \subseteq V(P)$ and $|N(S)| \leq 2|S|$.*

Now we can prove Theorem 4.

Proof of Theorem 4. The proof follows by induction on k. If $k = 2$ the results follow from Theorem 5. Thus we may assume $k \geq 3$, and that the statement holds for $k' < k$. In this proof, we use the induction hypothesis either with $m_1, \dots, m_{k-1}$ or with $m_2, \dots, m_k$. In either case we have $\varepsilon \cdot m_i \geq 2m_{i-1}^2$ for every i in the considered interval.

Fix a red–blue coloring of K_N, and let G be the graph induced by its red edges. Let P be a longest path in G. If $|P| \geq n$, then the statement follows. Thus, we may assume that $|P| \leq n-1$. We first deal with the case P is small.

Suppose that $|P| \leq n - m_2 - 2m_1 - 1$. By Lemma 6 there is a set $S \subseteq V(P)$ such that $N(S) \subseteq V(P)$ and $|N(S)| \leq 2|S|$. Thus, $N - |S \cup N(S)| \geq N - |P| \geq (n-1)(k-2) + m_2$. By induction hypothesis either $P_n \subseteq G \setminus \big(S \cup N(S)\big)$ or $K_{m_2,\dots,m_k} \subseteq \overline{G \setminus \big(S \cup N(S)\big)}$. In the former case we have $P_n \subseteq G$ as desired. Thus, we may assume that there is a copy K' of $K_{m_2,\dots,m_k}$ in $\overline{G \setminus \big(S \cup N(S)\big)}$. If $|S| \geq m_1$, then S together with $V(K')$ induces a copy of $K_{m_1,\dots,m_k}$ in $\overline{G}$, as desired. Therefore, we may assume that $|S| \leq m_1 - 1$. Now, let A be a maximum set in $V(G)$ such that $|A| \leq m_1 - 1$ and $|N(A)| \leq 2|A|$. Let $G' = G \setminus \big(A \cup N(A)\big)$ and set $N' = |V(G')| \geq N - 3m_1$. Note that for every set $A' \subseteq V(G')$ with $|N_{G'}(A')| \leq 2|A'|$ we have $|N_G(A \cup A')| = |N_G(A)| + |N_{G'}(A')| \leq 2\big(|A| + |A'|\big)$, and hence, by the maximality of A, we have $|A| + |A'| \geq m_1$.

Now, let $P' \subseteq G'$ be a longest path. Again, by Lemma 6 there is a set $S' \subseteq V(P')$ such that $N_{G'}(S') \subseteq V(P')$ and $|N_{G'}(S')| \leq 2|S'|$. Since $|N_{G'}(S')| \leq 2|S'|$, we have $|A| + |S'| \geq m_1$. Naturally, we have $|P'| \leq |P| \leq n - m_2 - 2m_1 - 1$, and since $S' \cup N_{G'}(S') \subseteq V(P')$, we have $N' - |S' \cup N_{G'}(S')| \geq N - 3m_1 - |P| \geq (n-1)(k-2) + m_2$. By induction hypothesis either $P_n \subseteq G' \setminus \big(S' \cup N_{G'}(S')\big)$ or $K_{m_2,\dots,m_k} \subseteq \overline{G' \setminus \big(S' \cup N_{G'}(S')\big)}$. In the former case we have $P_n \subseteq G$ as desired. Thus, we may assume that there is a copy K' of $K_{m_2,\dots,m_k}$ in $\overline{G' \setminus \big(S' \cup N_{G'}(S')\big)}$. Since no vertex in $A \cup S'$ is adjacent to a vertex of $G' \setminus \big(S' \cup N_{G'}(S')\big)$ and $|A \cup S'| = |A| + |S'| \geq m_1$, $A \cup S' \cup V(K')$ induces a copy of $K_{m_1,\dots,m_k}$ in $\overline{G}$, as desired.

Therefore, we may assume that $n - 1 \geq |P| \geq n - m_2 - 2m_1$. Now, consider $G' = G \setminus V(P)$. Since $|G'| = N - |P| \geq (n-1)(k-2) + m_1$, by the induction hypothesis, either $P_n \subseteq G'$ or $K_{m_1,\dots,m_{k-1}} \subseteq \overline{G'}$. In the former case, we have $P_n \subseteq G$, as desired. Thus, we may assume that there is a copy K' of $K_{m_1,\dots,m_{k-1}}$ in $\overline{G'}$. Let K^* be the union of the components of G' that contain vertices of K'. Since $K^* \subseteq G' = G \setminus V(P)$, we have

$$|K^*| + |P| \leq N. \tag{6}$$

Moreover, if $N - |K^*| - |N(K^*)| \geq m_k$, then there is a copy of $K_{m_1,\dots,m_k}$ in $\overline{G}$ obtained from K' by adding m_k vertices of $V(G) \setminus \big(K^* \cup N(K^*)\big)$. Therefore, we may assume that

$$N - |K^*| - |N(K^*)| \leq m_k - 1. \tag{7}$$

Now, we use a simple induction to prove that for each $i \geq 2$ we have $m_i \geq 2(m_1 + \cdots + m_{i-1})$. Indeed, this holds for m_2 since $\varepsilon \cdot m_2 \geq 2m_1^2 \geq 2m_1$. Now, since $m_{i-1} \geq 2m_{i-2}^2 \geq 2m_{i-2} \geq 2$, if $m_{i-1} \geq 2(m_1 + \cdots + m_{i-2})$, we have $\varepsilon \cdot m_i \geq 2m_{i-1}^2 \geq 2m_{i-1} + m_{i-1} \geq 2m_{i-1} + 2(m_1 + \cdots + m_{i-2}) = 2(m_1 + \cdots + m_{i-1})$, as desired. Therefore, we have $m_i^2 \geq 2m_i(m_1 + \cdots + m_{i-1})$. Summing over $i \geq 2$, we have

$$\begin{aligned}
\varepsilon(m_1 + \cdots + m_k) &= \sum_{i=1}^{k} \varepsilon \cdot m_i \geq \sum_{i=0}^{k-1} 2m_i^2 \\
&= \sum_{i=0}^{k-1} m_i^2 + \sum_{i=0}^{k-1} 2m_i(m_1 + \cdots + m_{i-1}) = (m_1 + \cdots + m_{k-1})^2.
\end{aligned} \tag{8}$$

Summing (6) and (7), we obtain

$$\begin{aligned}
|N(K^*)| &\geq |P| - m_k + 1 \geq n - m_2 - 2m_1 - m_k + 1 \\
&\geq m_k + (m_1 + \cdots + m_{k-1})(m_1 + \cdots + m_{k-1} - 1),
\end{aligned}$$

where the last inequality follows by (8) because $n \geq (2 + \varepsilon)(m_1 + \cdots + m_k)$.

Finally, by Lemma 4 given two distinct vertices u and v in K^*, $N_P(u)^-$ has at most one vertex in $N(v)$. Consequently, $N_P(u)^-$ has at most $m_1 + \cdots + m_{k-1} - 1$ vertices of $\cup_{v \in V(K^*) \setminus \{u\}} N(v)$. Thus, $N_P(K^*)^- = \cup_{u \in V(K^*)} N_P(u)^-$ has at most $(m_1 + \cdots + m_{k-1})(m_1 + \cdots + m_{k-1} - 1)$ neighbors of K^*. Therefore, there is a set $N^* \subseteq N_P(K^*)^- \setminus N_P(K^*)$ with m_k vertices, and hence $N^* \cup V(K^*)$ induces a copy of $K_{m_1,\dots,m_k}$ in $\overline{G}$, as desired. □

4. Concluding remarks

In this paper we present a family of graphs H for which the family of H-good paths is almost as large as possible. We observe that the unbalance condition $\varepsilon \cdot m_i \geq 2m_{i-1}^2$ could be replaced by the slightly weaker condition $\varepsilon(m_1 + \cdots + m_i) \geq (m_1 + \cdots + m_{i-1})(m_1 + \cdots + m_{i-1} - 1)$, but this would require a longer checking on the induction hypothesis conditions, while keeping a quadratic inequality. Nevertheless, we believe that the results presented in Section 1 could be deepened in order to improve the unbalance condition, perhaps to a subquadratic inequality. For example, it's not hard to see the relation between Lemma 3 and Lemma 4. This connection suggests the existence of a more general result that considers a larger number of components of $G \setminus V(P)$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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