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A study on token digraphs

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ABSTRACT

For a digraph D of order n and an integer $1 \leq k \leq n-1$, the *k-token digraph* of D is the digraph whose vertices are all k-subsets of vertices of D and, given two such k-subsets A and B, (A, B) is an arc in the k-token digraph whenever $\{a\} = A \setminus B$, $\{b\} = B \setminus A$, and there is an arc (a, b) in D. Token digraphs are a generalization of token graphs. In this paper, we study some properties of token digraphs, including strong and unilateral connectivity, kernels, girth, circumference, and Eulerianity. We also extend some known results on the clique and chromatic numbers of k-token graphs, addressing the bidirected clique number and dichromatic number of k-token digraphs. Additionally, we prove that determining whether 2-token digraphs have a kernel is NP-complete.

1. Introduction

Let G be a simple graph of order $n \geq 2$ and let k be an integer, with $1 \leq k \leq n-1$. The *k-token graph of G*, denoted by $F_k(G)$, is the graph whose vertices are all the k-subsets of $V(G)$, two of which are adjacent if their symmetric difference is a pair of adjacent vertices in G.

The name "token graph" is motivated by the following interpretation [16]. Consider a graph G and a fixed number k of indistinguishable tokens, with $1 \leq k \leq n-1$. A *k-token configuration* corresponds to the k tokens placed on distinct vertices of G, which corresponds to a subset of k vertices of G. Tokens can slide from their current vertex to an unoccupied adjacent vertex and, at each step, exactly one token can slide. Construct a graph whose vertices are the k-token configurations, and make two such configurations adjacent whenever one configuration can be reached from the other by sliding a token from one vertex to an adjacent vertex. This new graph is isomorphic to $F_k(G)$. See an example in Fig. 1.

The k-token graphs are also called the *symmetric kth power* of graphs [5]. In fact, token graphs have been defined, independently, at least four times since 1988 (see [2,16,23,27]). They have been used to study the Isomorphism Problem in graphs by means of their spectra, and, so far, some applications of token graphs to Physics and Coding Theory are known [14,15,22,27]. In addition, token graphs are related to other well-known graphs, such as Johnson graphs and doubled Johnson graphs (see, e.g., [3,15]).

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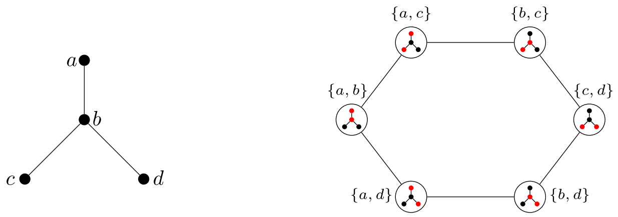

Fig. 1. A graph G and its 2-token graph $F_2(G)$.

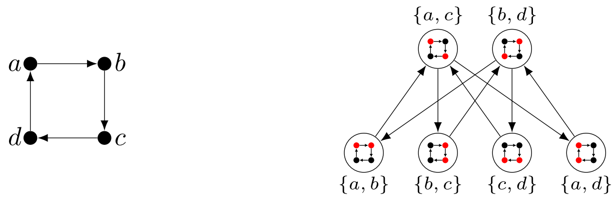

Fig. 2. A digraph D and its 2-token digraph $F_2(D)$.

In 2012, Fabila-Monroy et al. [16] reintroduced k-token graphs and proved tight lower and upper bounds on their diameter, connectivity, and chromatic number. They also characterized the cliques in token graphs in terms of the cliques in the original graph and established sufficient conditions for the existence or non-existence of a Hamiltonian path in various token graphs. They showed that if $F_k(G)$ is bipartite for some $k \geq 1$, then $F_\ell(G)$ is bipartite for all $\ell \geq 1$. Carballosa et al. [10] then characterized, for each value of k, which graphs have a regular k-token graph and which connected graphs have a planar k-token graph. Also, de Alba et al. [13] presented a tight lower bound for the matching number of $F_k(G)$ for the case in which G has either a perfect matching or an almost perfect matching, estimated the independence number for bipartite k-token graphs, and determined the exact value of the independence number for some graphs. Adame et al. [1] provided an infinite family of graphs, containing Hamiltonian and non-Hamiltonian graphs, for which their k-token graphs are Hamiltonian.

In this paper, we consider a generalization of token graphs to digraphs. A *digraph* D consists of a non-empty finite set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pairs of distinct vertices called *arcs*. Let D be a digraph on n vertices and let k be an integer with $1 \leq k \leq n-1$. The *k-token digraph of D*, denoted by $F_k(D)$, is the digraph whose vertices are all the k-subsets of $V(D)$ and, for two such k-subsets A and B, (A, B) is an arc in $F_k(D)$ whenever $\{a\} = A \setminus B$, $\{b\} = B \setminus A$, and there is an arc (a, b) in D. See Fig. 2 for an example.

As far as we know, the only generalization of token graphs to digraphs prior to our work was proposed by Gao and Shao [20] in 2009 for $k = 2$ tokens, but with the following difference: they considered 2-tuples of vertices instead of 2-subsets. This corresponds to distinct tokens.

In the final stage of this work, we became aware of a master's thesis by Fernández-Velázquez [17], which independently introduces the concept of token digraphs as proposed here. In [17], the author investigates several invariants of token digraphs, including strong connectivity, diameter, and the existence of cycles. The thesis also examines the regularity of token digraphs derived from regular tournaments, and explores the game of cops and robbers on token digraphs of specific digraph families, such as cycles, trees, and Cartesian products. Our contributions are complementary: we provide a complete characterization of strong connectivity and unilaterality through the strong component digraph of a digraph D, establish results on the directed girth, circumference, and Eulerianity of token digraphs with respect to D, and study the bidirected clique number and the dichromatic number of k-token digraphs.

A *kernel* of a digraph D is an independent set of vertices $K \subseteq V(D)$ such that, for every vertex $v \in V(D) \setminus K$, there is an arc from v to some vertex in K. The notion of a kernel is classical in digraph theory (see, e.g., Chapter 3 of [6]) and it was originally introduced by von Neumann and Morgenstern [29] in the context of game theory. Since then, it has found several applications in logic [7,8,30] and combinatorics [4,19]. Note that, in the context of token digraphs, a kernel consists of a set of configurations that are pairwise non-adjacent by a single move and which are reachable in one move from every other configuration. In this work, we also study the existence of kernels in token digraphs. Moreover, we show that deciding whether $F_2(D)$ has a kernel is NP-complete.

The paper is organized as follows. Section 2 presents some notation and states some basic properties we will use throughout the paper. In Section 3, we present results regarding strong connectivity of token digraphs while in Section 4 we present some results regarding kernels of token digraphs. In particular, Section 4 transfers the "no odd cycle" condition from D to $F_k(D)$ and shows that deciding whether $F_2(D)$ has a kernel is NP-complete (via an adaptation of Chvátal's reduction). Section 5 considers unilateral digraphs, and characterizes when their token digraphs are also unilateral. In Section 6, we analyze the relation between the directed girth and circumference of a digraph and their value for its token digraphs and, in Section 7, we study whether the fact that a digraph is Eulerian or Hamiltonian implies that its token digraphs are also Eulerian or Hamiltonian, respectively. Section 8 determines some relations between the bidirected clique number of a digraph and the bidirected clique number of its token digraphs. Finally, Section 9 studies acyclic partitions, and the dichromatic number of a digraph and its token digraphs. We conclude with some final remarks in Section 10.

2. General definitions and basic properties

The notation and terminology for digraphs used in this paper follow closely Bang-Jensen and Gutin [6].

We denote the complete graph on n vertices by K_n, the wheel graph on $n+1$ vertices by W_n, and the cycle graph on n vertices by C_n.

Let D be a digraph. For an arc $(x, y) \in A(D)$, we say x is an *in-neighbor* of y while y is an *out-neighbor* of x. The *in-degree* of a vertex v is the number of in-neighbors of v, denoted by $d^-(v)$, and the *out-degree* of v is the number of out-neighbors of v, denoted by $d^+(v)$.

A *path* of D is a sequence $(v_1, v_2, \dots, v_k)$ of vertices of D such that $v_i \neq v_j$ for all $i \neq j$ and $(v_i, v_{i+1}) \in A(D)$ for all $1 \leq i < k$. An (x, y)*-path* in D is a path from vertex x to vertex y. A *cycle* is a sequence $(v_1, v_2, \dots, v_k, v_{k+1})$ of vertices of D such that $(v_1, v_2, \dots, v_k)$ is a path, $v_{k+1} = v_1$ and $(v_k, v_1) \in A(D)$. The *length* of a path or cycle is the number of arcs it contains. A *digon* is a cycle of length 2 of D. We say that a cycle or path is odd (resp. even) if its length is odd (resp. even).

For a graph G, we denote by $\overleftrightarrow{G}$ the digraph obtained from G by replacing each edge of G by a digon. In [6], they refer to $\overleftrightarrow{G}$ as the complete biorientation of G. For a digraph D, the *reverse* of D, denoted as $\overleftarrow{D}$, is the digraph obtained from D by reversing every arc of D. The *underlying graph* of D is the undirected graph obtained by replacing each arc of D by an undirected edge between the same pair of vertices, and removing any multiple edges.

A digraph is *acyclic* if it has no cycles. An acyclic digraph is referred to as a *DAG* (acronym for directed acyclic graph). A vertex of a digraph D with out-degree zero is called a *sink*. It is known that every DAG has at least one sink.

A k*-token configuration* is a k-subset of vertices of D, and the vertices of the token digraph $F_k(D)$ are k-token configurations of D. In several of our proofs, we will use the interpretation of an arc XY of $F_k(D)$ existing when one can slide or move one token from X to Y. Also, in order to avoid confusion, we will use *nodes* to refer to vertices of token digraphs.

It is easy to see that, for every graph G on n vertices, the token graph $F_k(G)$ is isomorphic to the token graph $F_{n-k}(G)$. In particular, if n is even, this gives a non-trivial automorphism on $F_{n/2}(G)$. The following properties hold.

Property 2.1. *For a digraph D of order n, the k-token digraph $F_k(D)$ is isomorphic to the $(n-k)$-token digraph $F_{n-k}(\overleftarrow{D})$.*

Property 2.2. *For a digraph D, the k-token digraphs $F_k(D)$ and $\overleftrightarrow{F_k(\overleftarrow{D})}$ are isomorphic.*

Property 2.3. *For a graph G, the k-token digraphs $F_k(\overleftrightarrow{G})$ and $\overleftrightarrow{F_k(G)}$ are isomorphic.*

3. Strong connectivity aspects of token digraphs

Let D be a digraph. We say that D is *weakly connected*[1] if its underlying graph is connected. Note that D is weakly connected if and only if $F_k(D)$ is weakly connected. This comes from the fact that $F_k(G)$ is connected if and only if G is connected [16, Theorems 5 and 6]. In this section, we strengthen this result by establishing the analogous statement for strong connectivity. We note that this was previously proved, using a slightly different approach, in [17, Proposition 13].

A digraph D is *strongly connected* if, for every two vertices x and y of D, there is an (x, y)-path and a (y, x)-path. A digraph that is not strongly connected consists of some *strongly connected components*, which are the maximal strongly connected subdigraphs of D plus possibly some arcs connecting these components in a restricted way. Thus, to understand strong connectivity aspects of $F_k(D)$ in general, it is important to consider the case in which D is strongly connected.

Lemma 3.1. *Let D be a strongly connected digraph, and let A and B be two nodes of $F_k(D)$. There is an (A, B)-path in $F_k(D)$.*

Proof. Recall that A and B are k-subsets of $V(D)$. The proof is by induction on $|A \setminus B|$. Clearly if $|A \setminus B| = 0$, then $A = B$ and there is nothing to prove. Suppose $|A \setminus B| > 0$. Observe that this implies $|A \cap B| \leq k-1$ and, consequently, $|B \setminus A| > 0$, since $|B| = k$. Let P be a shortest path in D from a vertex in $A \setminus B$ to a vertex in $B \setminus A$. Call a the initial vertex of P and b the terminal vertex of P. The goal is to move the token from A in a to b going through the vertices in P. However there might be tokens of A and B on the way. Let $u_1, \dots, u_t$ be the vertices on P, in order, that are in $A \cap B$. Let $u_0 = a$ and $u_{t+1} = b$. For $i \in \{0, 1, \dots, t\}$ in decreasing order, we move the token from A in u_i to u_{i+1} through the vertices of P. The resulting k-token configuration is $A' = (A \setminus \{a\}) \cup \{b\}$, and the previous process describes a path from A to A' in $F_k(D)$. Now, as $|A' \setminus B| = |A \setminus B| - 1$, by induction, there is a path in $F_k(D)$ from A' to B. These two paths together contain a path from A to B in $F_k(D)$. □

[1] In [6], they use *connected* for this concept.

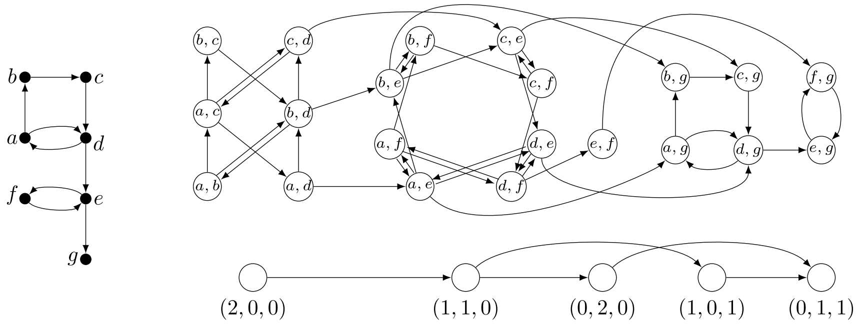

Fig. 3. A digraph D with three strongly connected components, the 2-token digraph $F_2(D)$ with five strongly connected components (for simplicity, we omit the brackets and place labels inside the vertices), and the digraph isomorphic to $\mathrm{SC}(F_2(D))$ whose vertex set is $V_2(4, 2, 1)$.

The previous result directly implies that, if D is strongly connected, then $F_k(D)$ is also strongly connected. We now address digraphs that are not necessarily strongly connected.

Let $C_1, \dots, C_t$ be the strongly connected components of D. The *strong component digraph* of D, denoted by $\mathrm{SC}(D)$, is obtained from D by collapsing all vertices in each C_i to a single vertex, which we also denote as C_i, and removing any resulting parallel arcs and loops. It is known that the strong component digraph of any digraph is a DAG. Hence we may assume without loss of generality that $C_1, \dots, C_t$ is a topological ordering of the vertices of $\mathrm{SC}(D)$, that is, all arcs in $\mathrm{SC}(D)$ go from a C_i to a C_j with $i < j$.

Let D_1 and D_2 be two digraphs. The *Cartesian product* of D_1 and D_2 is the digraph whose vertex set is the Cartesian product $V(D_1) \times V(D_2)$, and there is an arc between (u_1, u_2) and (v_1, v_2) if and only if $u_1v_1 \in A(D_1)$ and $u_2 = v_2$, or $u_2v_2 \in A(D_2)$ and $u_1 = v_1$. In what follows, we abuse notation and treat a strongly connected component of a digraph as its vertex set.

Lemma 3.2. *Let D be a digraph and $C_1, \dots, C_t$ be a topological ordering of the strongly connected components of D. Let A be a node of $F_k(D)$ and let $k_j = |A \cap C_j|$ for $j \in \{1, \dots, t\}$. Then the strongly connected component of $F_k(D)$ containing A is isomorphic to the Cartesian product of $F_{k_j}(C_j)$ for every j with $k_j > 0$.*

Proof. Let $J := \{j \in [t] \colon k_j > 0\}$. By Lemma 3.1, $F_{k_j}(C_j)$ is strongly connected for each $j \in J$. Since the Cartesian product of strongly connected digraphs is strongly connected, the strongly connected component of $F_k(D)$ containing A contains $\prod_{j \in J} F_{k_j}(C_j)$. Note that if B is a configuration obtained from A by sliding a token along an arc not contained in any strongly connected component of D, then B cannot reach A in $F_k(D)$. In particular, if $B \notin \prod_{j \in J} F_{k_j}(C_j)$, then either B is unreachable from A, or any (A, B)-path must use such an arc, implying that B cannot reach A. Therefore, the strongly connected component of $F_k(D)$ containing A is precisely $\prod_{j \in J} F_{k_j}(C_j)$. $\square$

Let D be a digraph and $C_1, \dots, C_t$ be a topological ordering of the strongly connected components of D, as in Lemma 3.2. For each integer vector $(k_1, \dots, k_t)$ such that $0 \le k_i \le |C_i|$ for every i and $\sum_{i=1}^{t} k_i = k$, there is a strongly connected component of $F_k(D)$ isomorphic to the Cartesian product of $F_{k_j}(C_j)$ for every j with $k_j > 0$. Let $V_k(c_1, \dots, c_t)$ be the set of such vectors, where $c_i = |C_i|$. For two vectors $(k_1, \dots, k_t)$ and $(k'_1, \dots, k'_t)$ in $V_k(c_1, \dots, c_t)$, we say *there is an (i, j)-move from $(k_1, \dots, k_t)$ to $(k'_1, \dots, k'_t)$* if $k_\ell = k'_\ell$ for every $\ell \neq i, j$, and $k'_i = k_i - 1$ and $k'_j = k_j + 1$. For a node A of $F_k(D)$, let $k_j = |A \cap C_j|$ for $1 \le j \le t$. We say the integer vector $(k_1, \dots, k_t)$ is the *vector associated* to A. Note that this vector is in $V_k(c_1, \dots, c_t)$.

We are ready to completely characterize the strong component digraph of $F_k(D)$ in terms of the strong component digraph of D. See Fig. 3 for an example.

Theorem 3.3. *Let D be a digraph and $C_1, \dots, C_t$ be a topological ordering of its strongly connected components. The strong component digraph $\mathrm{SC}(F_k(D))$ is isomorphic to the DAG whose vertex set is $V_k(|C_1|, \dots, |C_t|)$ and with an arc from $(k_1, \dots, k_t)$ to $(k'_1, \dots, k'_t)$ if and only if there are indices $i < j$ such that there is an (i, j)-move from $(k_1, \dots, k_t)$ to $(k'_1, \dots, k'_t)$ and there is an arc from C_i to C_j in $\mathrm{SC}(D)$.*

Proof. Lemma 3.2 characterized the strongly connected components of $F_k(D)$, and thus the vertex set of its strong component digraph $\mathrm{SC}(F_k(D))$. Let $(k_1, \dots, k_t)$ and $(k'_1, \dots, k'_t)$ be vertices of $\mathrm{SC}(F_k(D))$. Observe that there is an arc from $(k_1, \dots, k_t)$ to $(k'_1, \dots, k'_t)$ if and only if there are two nodes A and B of $F_k(D)$ with associated vectors $(k_1, \dots, k_t)$ and $(k'_1, \dots, k'_t)$ respectively, such that there is an arc in $F_k(D)$ from A to B.

That happens if and only if A and B differ in exactly one token, and this token must have moved from a vertex a to a vertex b of D, through an arc of D, with a and b being in different strongly connected components of D. Let C_i and C_j be the strongly connected components of D containing a and b respectively. Then there is an arc from C_i to C_j in $\mathrm{SC}(D)$ and

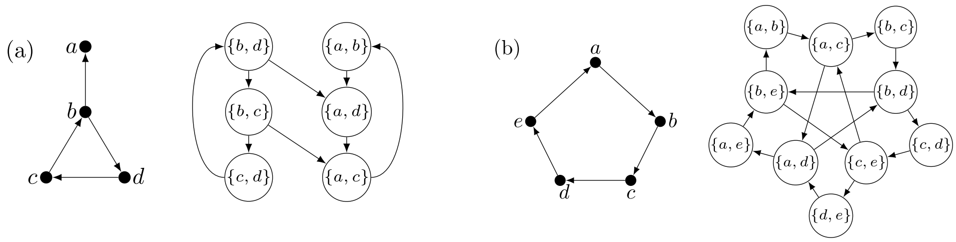

Fig. 4. Two examples of a digraph D and its 2-token digraph $F_2(D)$. (a) Digraph D has a kernel, namely the set $\{a, c\}$, while $F_2(D)$ does not. (b) The opposite occurs: the five external vertices form a kernel of $F_2(D)$.

so $i < j$. Moreover, $k_\ell = k'_\ell$ for every $\ell \neq i, j$, and $k'_i = k_i - 1$ and $k'_j = k_j + 1$. Hence there is an (i, j)-move from $(k_1, \ldots, k_t)$ to $(k'_1, \ldots, k'_t)$, and the theorem holds. □

In particular, we derive the following from Theorem 3.3.

Corollary 3.4. *The digraph $F_k(D)$ is strongly connected if and only if D is strongly connected.*

Corollary 3.5. *The digraph $F_k(D)$ is acyclic if and only if D is acyclic.*

4. Kernels

For a set S of vertices in a digraph D, let $N^-[S] = S \cup \{u \in V(D) : (u, v) \in A(D) \text{ and } v \in S\}$. Recall that a set K of vertices in D is a *kernel* if K is independent and $N^-[K] = V(D)$. Von Neumann and Morgenstern [29] showed that every DAG has a unique kernel. This result, together with Corollary 3.5, also shows that the token digraph of a DAG has a unique kernel. Indeed, the kernel of a DAG D can be obtained iteratively as follows. Start with $D' = D$ and let S_1 be the set of sinks of D'; remove $N^-[S_1]$ from D' and repeat the process until D' vanishes, that is, let S_2 be the set of sinks of the remaining D'; remove $N^-[S_2]$ from D' and so on. The set $K = S_1 \cup S_2 \cup \cdots$ is the unique kernel of D.

Richardson [26] later generalized von Neumann and Morgenstern's result to graphs with no odd cycle.

Theorem 4.1 *([26]). Every digraph with no odd cycle has a unique kernel.*

We now prove the following theorem, which, together with Theorem 4.1, implies that if D has no odd cycle, then D and $F_k(D)$ have unique kernels.

Theorem 4.2. *If D is a digraph with no odd cycles, then $F_k(D)$ has no odd cycles.*

Proof. Because D has no odd cycles, every strongly connected component C of D is bipartite. Let X and Y be a bipartition of the vertex set of C. For every i with $0 \le i \le k$, a node of $F_i(C)$ is an i-token configuration in C, and it has an even or an odd number of vertices in X. That induces a bipartition of $F_i(C)$, as a move of a token goes from X to Y or vice-versa, and always changes the parity of the configuration within X. Thus, each $F_i(C)$ is also bipartite.

Because each $F_i(C)$ is bipartite, and the Cartesian product of bipartite digraphs is bipartite (Lemma 2.6 in [28]), we conclude by Lemma 3.2 that the strongly connected components of $F_k(D)$ are bipartite, and hence $F_k(D)$ has no odd cycles. □

These results might suggest a connection between kernels in a digraph D and kernels in its token digraph $F_k(D)$, which could be valuable for our understanding of the existence of kernels in digraphs with odd cycles. However, in Fig. 4(a), we show a digraph D with a triangle that has a kernel, but for which $F_2(D)$ does not have a kernel, and, in Fig. 4(b), we show a digraph D that does not have a kernel, but for which $F_2(D)$ has a kernel.

Moreover, by adapting parts of Chvátal's proof [11] that deciding whether a digraph D has a kernel is NP-complete, we show that deciding whether $F_2(D)$ has a kernel is also NP-complete.

Chvátal's reduction is from 3-SAT. From a 3-SAT formula ϕ, he builds a digraph D such that D has a kernel if and only if ϕ is satisfied. The idea is to use the same construction, but to add a universal sink vertex u to D. The resulting digraph D' is such that ϕ has a not-all-equal satisfying assignment if and only if $F_2(D')$ has a kernel. So we would need to do the reduction from the following variant of 3-SAT, which is also NP-complete [21]. The problem NAE-3-SAT consists of, given a 3-SAT formula, to decide whether there is an assignment for the variables such that each clause has either one or two true literals. We call such an assignment a *NAE assignment*.

Theorem 4.3. *The problem of, given a digraph D, deciding whether $F_2(D)$ has a kernel is NP-complete.*

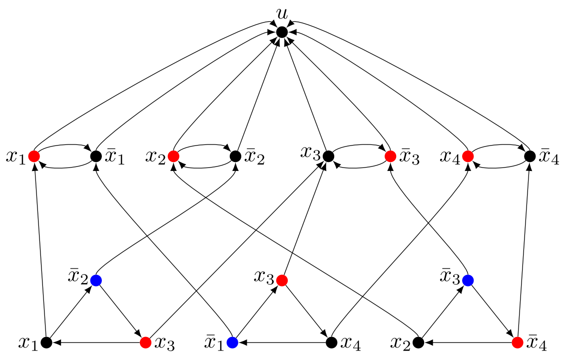

Fig. 5. The digraph D for the 3-SAT formula $\phi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \bar{x}_3 \vee \bar{x}_4)$. The vertices in red form a kernel for the digraph $D' = D - u$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Proof. Given a 3-SAT formula ϕ on variables $x_1, \ldots, x_n$, call $C_1, \ldots, C_m$ the clauses in ϕ. Consider the following digraph D built from ϕ. For each variable x_j, there is a vertex labeled x_j and a vertex labeled $\bar{x}_j$ in D. These are called *literal vertices*. For each clause C_i, there are three vertices in D, each labeled by one of the three literals in C_i. These are called *clause vertices*. The two literal vertices for variable x_j induce a digon in D, and these digons are called *variable digons*. The three vertices of a clause induce a triangle, and these are called the *clause triangles*. Additionally, there is an arc from a vertex in a clause triangle to a literal vertex whenever they have the same label. Finally, there is a vertex u, with arcs from each literal vertex to u. Let us denote by $C \subseteq V(D)$ the set of clause vertices and by $L \subseteq V(D)$ the set of literal vertices. See an example in Fig. 5.

Let D' be D without vertex u. Observe that D' is exactly the digraph from the reduction of Chvátal [11], so we know that D' has a kernel if and only if ϕ is satisfiable. Note that any kernel for D' contains a vertex in each variable digon. First we will prove that there is a NAE assignment for ϕ if and only if D' has a kernel that contains exactly one vertex in each variable digon and exactly one vertex in each clause triangle. We refer to such a kernel as *special*. Second, we will prove that D' has a special kernel if and only if $F_2(D)$ has a kernel. These two statements imply the theorem.

Suppose there is a NAE assignment for ϕ. We define a set K of vertices in D' as follows: first, include in K the true literal from each variable digon; then, for each clause triangle, include one vertex labeled by a false literal. If a clause triangle contains two vertices labeled by false literals, choose the one that is an out-neighbor of the other. The red vertices in Fig. 5 show one such set K corresponding to the assignment $x_1 = x_2 = x_4 = T$ and $x_3 = F$. Note that K is an independent set, and every vertex in D' is either in K or has an arc to a vertex in K. Thus K is a special kernel in D'.

Now, suppose K is a special kernel in D'. Consider the truth assignment that makes true exactly the literals that are labels of literal vertices in K. This is well-defined because there is exactly one vertex in K in each variable digon. We must argue that this is a NAE assignment for ϕ. In each clause triangle, there is exactly one vertex in K. Because K is independent, the literal that labels this vertex is false in the assignment, and this assignment does not satisfy all three of the literals in each clause. On the other hand, for each clause triangle, one of the two vertices not in K is not an in-neighbor of the vertex in K in this triangle. As K is a kernel, this vertex has to be an in-neighbor of the literal vertex with the same label, which means the clause is satisfied. That is, this assignment is a NAE assignment for ϕ.

Now we prove that D' has a special kernel if and only if $F_2(D)$ has a kernel. We start by arguing that if $F_2(D)$ has a kernel, then D' has a special kernel. First notice that the token graph $F_2(D)$ has two parts. One of them is isomorphic to $F_1(D')$, which in turn is isomorphic to D' itself, and corresponds to the 2-token configurations in D that have one token always in u. The second part corresponds to 2-token configurations with the two tokens being in D'. Note that all the arcs between the first and the second part go from the second part to the first one, because u is a sink. Hence, any kernel K of $F_2(D)$ induces a kernel K' in D', namely, take $K' = \{x \in V(D') \colon \{u, x\} \in K\}$. Let us argue that K' is a special kernel in D'. Recall that $C \subseteq V(D)$ is the set of clause vertices and $L \subseteq V(D)$ is the set of literal vertices. To ease the exposition, let us refer to a 2-token configuration in D as one of the following types, depending on where the two tokens are: a uL, uC, LL, LC, or a CC configuration.

Suppose, by means of a contradiction, that K' is not special. This means that there is a triangle clause $\triangle$ with no vertex in K'. Because K' is a kernel in D', from each vertex y in $\triangle$, the arc from y to L goes to a vertex in K'. Let x be an arbitrary variable, and consider the 2-token configurations with one token in the variable digon associated with x and the other in $\triangle$. Call Z this set of LC configurations. Let us argue that no neighbor of configurations in Z outside Z are in K. There are arcs from LC configurations to uC, LL, and LC configurations. The uC configurations that receive arcs from configurations in Z are not in K, because they consist of u and a vertex in $C \setminus K'$. The LL configurations that receive arcs from configurations in Z are also not in K, because at least one of the two vertices in L is in K'. Moreover, if there is an arc from a configuration in Z to an LC configuration, the latter is also in Z. Hence, K must contain a kernel of the subdigraph of $F_2(D)$ induced by Z. See Fig. 6 for an example of such subdigraph. However, one can check that this digraph has no kernel.

It remains to show that if D' has a special kernel, then $F_2(D)$ has a kernel. Let K' be a special kernel in D'. There is exactly one vertex from K', say y, in each clause triangle $\triangle$. The vertex not in K' in $\triangle$ that is an in-neighbor of y is called

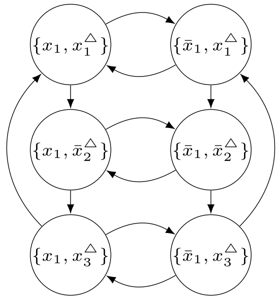

Fig. 6. The subdigraph of $F_2(D)$ for D from Fig. 5, with $\triangle$ being the triangle for clause $x_1 \vee \bar{x}_2 \vee x_3$ and $x = x_1$. We name $x_1^\triangle, \bar{x}_2^\triangle, x_3^\triangle$ the vertices in $\triangle$ whose labels are $x_1, \bar{x}_2, x_3$ respectively.

the *dominating vertex* in $\triangle$ while the other vertex in $\triangle$ not in K' is called *undominating vertex*. Note that the undominating vertex is an in-neighbor of a literal vertex in K.

Let us now describe a kernel in $F_2(D)$ from K'. For that, let $S_1 = \{\{u, v\}\colon v \in K'\}$. This is an independent set, because K' is an independent set in D'. Let $S_2 = \{\{v, v'\}\colon v, v' \in L \text{ and } v, v' \notin K'\}$. Because K' is special, there is a vertex in K' in each variable digon. Thus, as there is no arc between variable digons, the 2-token configurations in S_2 form an independent set. In fact, $S_1 \cup S_2$ is also an independent set, because the tokens in configurations of S_2 are in vertices not in K', while the tokens in literal vertices in configurations of S_1 are in vertices from K'. Let $S_3 = \{\{v, y\}\colon$ either $v \in K' \cap L$ and $y \in C$ is a dominating vertex, or $v \in L \setminus K'$ and $y \in C$ is an undominating vertex$\}$. Clearly, S_3 is an independent set. Also, because a token in C cannot move to u, and a token in an undominating vertex can only move to a vertex in L that is in K', we can see that $S_1 \cup S_2 \cup S_3$ is also an independent set. At last, let $S_4 = \{\{y, z\}\colon y, z \in C$ and either $y, z \in K'$, or y, z are both dominating vertices not neighboring a vertex in $L \cap K'$, or y, z are both undominating vertices, or both y and z are in the same clause triangle with $y \in K'$ and z being a dominating vertex$\}$. Clearly, S_4 is independent. Also, we can see that $K = S_1 \cup S_2 \cup S_3 \cup S_4$ is also an independent set, because a token in an undominating vertex can only go to a vertex in L that is in K'.

Finally, let us argue that K is a kernel of $F_2(D)$. Consider an arbitrary node S of $F_2(D)$. If $S \in K$, there is nothing to prove, so we may assume $S \notin K$. If $u \in S$, then the second vertex in S is not in K', otherwise S would be in S_1, so there is an arc from S to a configuration in S_1, because K' is a kernel in D'. If $u \notin S$, then there are three cases. In the first case, S contains only literal vertices. Then, because $S \notin S_2$, there is a vertex from K' in S, and hence there is an arc from S to a configuration in S_1. In the second case, $S = \{v, y\}$ contains a literal vertex v and a clause vertex y. If $y \in K'$, then there is an arc from S to the configuration $\{u, y\}$ in S_1. Thus either y is a dominating vertex and $v \notin K'$, or y is an undominating vertex and $v \in K'$ (otherwise S would be in S_3); either way, there is an arc from S to a configuration in S_3 (obtained by moving the token in L to the other literal vertex in the same digon). In the last case, $S = \{y, z\}$ contains only clause vertices. If $\{y, z\} \cap K' = \emptyset$, then one of them is dominating and the other (dominating or undominating) has a neighbor in $X \cap K'$, which means there is an arc from S to a configuration in S_3. Otherwise we may assume $y \in K'$, and either z is the dominating vertex in another clause triangle, or z is undominating, and there is always an arc from S to S_4. Indeed, in the former case, we can move the token in z to the vertex in K' in the same triangle and, in the latter case, if both y and z are in the same triangle, then we can move the token in z to the dominating vertex in the same triangle, and if y and z are in different clause triangles, we can move the token from y to the undominating vertex in the same triangle. $\square$

5. Unilateral digraphs and their token digraphs

A digraph D is *unilateral* if, for every pair of vertices x and y, there is an (x, y)-path or a (y, x)-path (or both). Note that a digraph can be weakly connected without being unilateral (take, for instance, the *antipath*, i.e., an oriented path that alternates the direction of its arcs). We use the next theorem to characterize when $F_k(D)$ is unilateral.

Theorem 5.1 *(Theorem 7.2 in [18]). A digraph D is unilateral if and only if the strong component digraph* $\mathrm{SC}(D)$ *has a Hamiltonian path.*

Because $\mathrm{SC}(D)$ is a DAG for every digraph D, the Hamiltonian path of $\mathrm{SC}(D)$ for a unilateral digraph D given by Theorem 5.1 is unique.

Obviously, as $F_1(D)$ is isomorphic to D, if D is unilateral, then so is $F_1(D)$. Moreover, $F_1(D)$ is isomorphic to $F_{n-1}(D)$, where n is the number of vertices in D. The next theorem addresses the remaining cases, that is, when $2 \le k \le n-2$.

Theorem 5.2. *Let D be a digraph of order n with $C_1, \ldots, C_t$ being its strongly connected components, and let k be an integer such that $2 \le k \le n-2$. Then $F_k(D)$ is unilateral if and only if D is unilateral and either $t \le 2$ or $t = 3$ with $|C_2| = 1$.*

Proof. Let us first show that if D is unilateral and either $t \le 2$ or $t = 3$ with $|C_2| = 1$, then $F_k(D)$ is unilateral. Since D is unilateral, let $C_1, \dots, C_t$ be the strongly connected components of D in the order given by the Hamiltonian path of Theorem 5.1.

We start by considering the case $t \le 2$. If $t = 1$, then D is strongly connected and so it is $F_k(D)$ by Corollary 3.4, and hence $F_k(D)$ is unilateral. For $t = 2$, according to Theorem 3.3, the vertices of $\mathrm{SC}(F_k(D))$ are pairs (k_1, k_2), for $0 \le k_i \le |C_i|$ and $k_1 + k_2 = k$. Let $k_1 = \min\{k, |C_1|\}$ and $k_2 = \min\{k, |C_2|\}$. The following is a Hamiltonian path in $\mathrm{SC}(F_k(D))$:

$$((k_1, k - k_1), (k_1 - 1, k - k_1 + 1), (k_1 - 2, k - k_1 + 2), \dots, (k - k_2, k_2)).$$

Therefore, by Theorem 5.1, $F_k(D)$ is unilateral.

Now, suppose that D is unilateral, $t = 3$ and $|C_2| = 1$. If $k \le |C_1|$, then the following is a Hamiltonian path in $\mathrm{SC}(F_k(D))$:

$$\big((k,0,0), (k-1,1,0), (k-1,0,1), (k-2,1,1), (k-2,0,2), (k-3,1,2), \dots, v\big),$$

where the last vertex v is either $(0, 0, k)$ if $k \le |C_3|$, or $(k - |C_3| - 1, 1, |C_3|)$ otherwise. If $k > |C_1|$, then the following is a Hamiltonian path in $\mathrm{SC}(F_k(D))$:

$$\big((|C_1|, 1, k - |C_1| - 1), (|C_1|, 0, k - |C_1|), (|C_1| - 1, 1, k - |C_1|),$$
$$(|C_1| - 1, 0, k - |C_1| + 1), (|C_1| - 2, 1, k - |C_1| + 1), \dots, v\big),$$

where again v is either $(0, 0, k)$ if $k \le |C_3|$, or $(k - |C_3| - 1, 1, |C_3|)$ otherwise.

For the other direction, we prove the contrapositive, that is, we show that if D is not unilateral or $t \ge 4$ or $t = 3$ with $|C_2| \ge 2$, then $F_k(D)$ is not unilateral.

Suppose that $t \ge 4$ or $t = 3$ with $|C_2| \ge 2$. Let $C' = C_2 \cup \dots \cup C_{t-1}$, so $|C'| \ge 2$. Take A to be an arbitrary k-token configuration of D such that $|A \cap C_1| \ge 1$, $|C' \setminus A| \ge 2$, and $|A \cap C_t| \ge 1$. Such a set exists because $k \ge 2$, which assures we can select A with $|A \cap C_1| \ge 1$ and $|A \cap C_t| \ge 1$, and because $|C'| \ge 2$ and $k \le n - 2 = |C_1| + |C'| + |C_t| - 2$, which assures we can also choose A that does not contain at least two vertices w and z in C'. Let x and y be vertices in $A \cap C_1$ and $A \cap C_t$, respectively. Let B be the k-token configuration $(A \setminus \{x, y\}) \cup \{w, z\}$. Then, there is no (A, B)-path in $F_k(D)$ because there is no way to move tokens from A to a configuration with fewer vertices in C_t, such as B. Also, there is no (B, A)-path because there is no way to move tokens from B to a configuration with more vertices in C_1, such as A. Therefore, $F_k(D)$ is not unilateral.

Suppose now that D is not unilateral. Then clearly $t \ge 3$. By the previous case, we may assume that $t = 3$ and $|C_2| = 1$. As we observed in Section 3, D is weakly connected if and only if $F_k(D)$ is weakly connected, so we may assume that D is weakly connected. Let y be the unique vertex in C_2. Since D is not unilateral, $\mathrm{SC}(D)$ has exactly two arcs, which are $\{(C_1, C_3), (C_2, C_3)\}$ or $\{(C_1, C_2), (C_1, C_3)\}$. Note that if $\mathrm{SC}(D)$ has the arcs $\{(C_1, C_3), (C_2, C_3)\}$, then $\mathrm{SC}(\overleftarrow{D})$ has the arcs $\{(C_1, C_2), (C_1, C_3)\}$, and vice-versa. So, given that D is unilateral if and only if $\overleftarrow{D}$ is unilateral, we may assume the former case. Now, take a configuration A with $y \notin A$ and $|A \cap C_1| \ge 1$. Let $x \in A \cap C_1$, and let $B := (A \setminus \{x\}) \cup \{y\}$. Observe that there is no (A, B)-path because no token at vertices in A can be moved to the vertex $y \in C_2$. Further, there is no (B, A)-path because no token at vertices in $B \cap (C_2 \cup C_3)$ can be moved to a vertex in C_1. Hence, $F_k(D)$ is not unilateral. □

6. Directed girth and circumference

The *directed girth* of a digraph D is the length of the shortest cycle in D, and it is denoted by $g(D)$. The *directed circumference* of D is the length of a longest cycle in D, and it is denoted by $c(D)$.

Theorem 6.1. *For every digraph D on n vertices and for each $k \in \{1, 2, \dots, n-1\}$, $g(D) = g(F_k(D))$ and $c(D) \le c(F_k(D))$.*

Proof. Let C be a cycle in D of length t. Let A be a k-token configuration in D with at least one and at most $t - 1$ tokens in C. Such a configuration exists because $1 \le k \le n - 1$. We can move around the tokens in C to obtain a cycle in $F_k(D)$ of length exactly the length of C, that is, t. This implies that $g(D) \ge g(F_k(D))$ and $c(D) \le c(F_k(D))$.

Now, let C be a cycle in $F_k(D)$ of length t. Let us argue that there is a corresponding cycle C' in D of length at most t. Let $C = (A_0, \dots, A_t)$, where $A_0 = A_t$. While traversing C, each of the k tokens traverses a path from a vertex in A_0 back to a vertex in A_0 (the same vertex or another one). Let D' be the subdigraph of D whose arc set consists of all arcs traversed in such paths of length at least one, and whose vertex set consists of the endpoints of these arcs. Hence, every vertex in D' has outdegree at least one, and thus D' contains a cycle. Clearly, this cycle has length at most t, as D' has exactly t arcs. For $t = g(F_k(D))$, this means that $g(D) \le g(D') \le t = g(F_k(D))$, which allows us to conclude that $g(D) = g(F_k(D))$. □

Now we consider the directed circumference of token digraphs. Since $F_1(D)$ is isomorphic to D and $F_{n-1}(D)$ is isomorphic to $\overleftarrow{D}$, we have that $c(F_1(D)) = c(F_{n-1}(D)) = c(D)$. On the other hand, when $2 \le k \le n-2$, there exist many digraphs D such that $c(D) < c(F_k(D))$. Indeed, $F_k(D)$ is much larger than D and, for instance, $F_k(\overleftrightarrow{K_n})$ is a Hamiltonian digraph, and so $c(F_k(\overleftrightarrow{K_n})) > c(\overleftrightarrow{K_n})$ as long as $n \ge 4$ and $2 \le k \le n - 2$.

Also, there exist non-Hamiltonian graphs whose k-token graphs are Hamiltonian (see [1]), and we can consider the digraph D obtained from such graphs by replacing each edge with the two possible arcs. It is straightforward to see that $F_k(D)$ is Hamiltonian, so the same holds for these digraphs.

Theorem 6.2. *Let D be a digraph on n vertices with $c(D) \geq 5$ and $2 \leq k \leq n-3$. Let $r = 2$ if $k = 2$ and $r = \min\{\max\{k, n-k\}, c(D) - 3\}$ otherwise. Then $c(F_k(D)) \geq r\,c(D)$.*

Proof. As $c(D) = c(\overleftarrow{D})$ and $F_k(D)$ is isomorphic to the token graph $F_{n-k}(\overleftarrow{D})$, we can assume that either $k = 2$ or $k \geq n-k$. Let $c = c(D)$ and C be a cycle of length c in D. Assume without loss of generality that $V(C) = \{0, 1, \dots, c-1\}$. In what follows, sums and subtractions are taken mod r.

First, suppose that $r = k \leq c-3$. Let $X_i = \{i, i+1, \dots, i+k\}$, for each $i \in V(C)$. Clearly $|X_i \cap X_{i+1}| = k$. Now, note that $|V(C) \setminus X_i| = c - (k+1) \geq 2$, which means that $|X_i \cap X_j| < k$ if $|j-i| \neq 1$, because i and $i+1$ are not in X_j for $j \in \{i+2, \dots, i+k-1\}$ (as the last element in X_j in this case would be at most $i+k-1+k = i-1$), $i+1$ and $i+k-1$ are not in X_{i+k} (as the last element in X_{i+k} is i), and $i+k-1$ and $i+k$ are not in X_j for $j \in \{i+k+1, \dots, i-2\}$ (as the last element in X_j in this case would be at most $i+k-2$).

Let P_i be the path in $F_k(D)$ from $X_{i-1} \cap X_i = \{i, i+1, \dots, i+k-1\}$ to $X_i \cap X_{i+1} = \{i+1, i+2, \dots, i+k\}$, obtained from moving one by one the token from j to $j+1$, for $j \in \{i, \dots, i+k-1\}$ in decreasing order, where the sum is taken mod r. It is readily seen that P_i is a path in $F_k(D)$ of length k, and that its vertices correspond to k-token configurations contained in X_i. Because of this, as $|X_i \cap X_j| < k$ if $|j-i| \neq 1$, paths P_i and P_j do not intersect if $|j-i| \neq 1$. Moreover, P_i starts at the end of P_{i-1}, hence we can concatenate $P_0, P_1, \dots, P_{c-1}$ to obtain a cycle of length kc in $F_k(D)$, which implies that $c(F_k(D)) \geq kc = rc$.

Now suppose that $r = c-3 < k$. As $k \leq n-3$, there are $k-r$ vertices outside C. Consider the subdigraph H of $F_k(D)$ generated by moving r tokens on $V(C)$, whereas the remaining $k-r$ tokens are fixed outside C. Thus, $H \simeq F_r(C)$ and, then, by the first part of the proof, we deduce that there is a cycle in H of order rc. This completes the proof. □

7. Eulerian and Hamiltonian digraphs

A digraph D is *Eulerian* if D is weakly connected and $d^+_D(v) = d^-_D(v)$ for all $v \in V(D)$.

Theorem 7.1. *For every natural k, the digraph D is Eulerian if and only if $F_k(D)$ is Eulerian.*

Proof. As observed in Section 3, a digraph D is weakly connected if and only if $F_k(D)$ is weakly connected. Therefore, to prove the statement it suffices to verify the degree condition.

Let A be a k-token configuration and denote by $e(A)$ the number of arcs of D whose both ends are in A. Note that the out-degree of A in $F_k(D)$, $d^+_{F_k(D)}(A)$, corresponds to the number of arcs in D going from a vertex in A to a vertex in $V(D) \setminus A$. Similarly, the in-degree of A in $F_k(D)$, $d^-_{F_k(D)}(A)$, corresponds to the number of arcs in D going from a vertex in $V(D) \setminus A$ to a vertex in A. We then have

$$d^+_{F_k(D)}(A) = \sum_{v \in A} d^+_D(v) - e(A) \qquad \text{and} \qquad d^-_{F_k(D)}(A) = \sum_{v \in A} d^-_D(v) - e(A).$$

Suppose first that D is Eulerian. Since $d^+_D(v) = d^-_D(v)$ for every $v \in V(D)$, we conclude directly from the observation above that $d^+_{F_k(D)}(A) = d^-_{F_k(D)}(A)$ for every A, and so $F_k(D)$ is Eulerian.

Suppose now that $F_k(D)$ is Eulerian. For any two distinct vertices u and v of D, let A be a k-token configuration containing u but not v, and let $A' = A \setminus \{u\} \cup \{v\}$. Note that $A \setminus \{u\} = A' \setminus \{v\}$ and let $S^+ = \sum_{x \in A \setminus \{u\}} d^+_D(x)$ and $S^- = \sum_{x \in A \setminus \{u\}} d^-_D(x)$. Then

$$\begin{aligned} d^+_D(u) &= d^+_{F_k(D)}(A) - S^+ + e(A) \\ &= d^-_{F_k(D)}(A) - S^+ + e(A) \\ &= S^- + d^-_D(u) - e(A) - S^+ + e(A) \\ &= d^-_D(u) + S^- - S^+. \end{aligned}$$

Analogously, using A' instead of A, we can derive that $d^+_D(v) = d^-_D(v) + S^- - S^+$. Therefore $d^+_D(u) - d^-_D(u) = S^- - S^+ = d^+_D(v) - d^-_D(v)$ and, from this, we conclude that $\sum_{x \in V(D)} d^+_D(x) - \sum_{x \in V(D)} d^-_D(x) = |V(D)|(S^- - S^+)$. Because $\sum_{x \in V(D)} d^+_D(x) = \sum_{x \in V(D)} d^-_D(x)$, it must be the case that $S^- - S^+ = 0$, and hence $d^+(x) = d^-(x)$ for every $x \in V(D)$. □

For (undirected) graphs on n vertices, we know that $F_2(C_n)$ is Hamiltonian if and only if $n = 3$ or $n = 5$. The same statement holds for cycles and the token digraph. On the other hand, the digraph D shown in Fig. 7 is not Hamiltonian but its token digraph $F_2(D)$ is Hamiltonian.

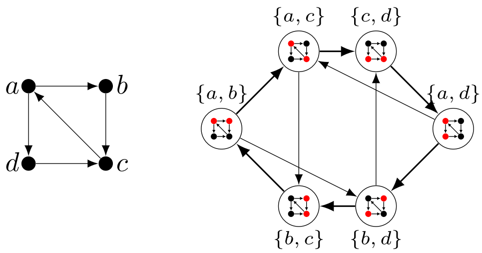

Fig. 7. A non-Hamiltonian digraph with a Hamiltonian token digraph.

8. Bidirected cliques

We call the digraph $\overleftrightarrow{K_n}$ the *complete digraph* on n vertices. A *bidirected clique* in a digraph D is a subset of $V(D)$ that induces a complete digraph in D. The *bidirected clique number* of a digraph D, denoted by $\overleftrightarrow{\omega}(D)$, is the size of the largest bidirected clique of D. We use the next known result on the clique number of undirected token graphs to characterize the bidirected clique number of $F_k(D)$.

Theorem 8.1 *(Theorem 5 in [16]). For any graph G of order n and $1 \leq k \leq n-1$, it holds that $\omega(F_k(G)) = \min\{\omega(G), \max\{n-k+1, k+1\}\}$.*

The *clean graph* of a digraph D, denoted by D^*, is the graph obtained by first taking the symmetric part of D, and then its underlying graph; that is, its vertex set is $V(D)$ and there is an edge between u and v if and only if $(u, v), (v, u) \in A(D)$. The next result states that constructing the token graph of D and then cleaning it is the same as cleaning D first and then constructing the token graph of D^*. Observe that $F_k(D)^*$ and $F_k(D^*)$ are both (undirected) graphs.

Fact 8.2. *For any digraph D of order n and $1 \leq k \leq n$, $F_k(D)^* = F_k(D^*)$.*

Proof. Let X and Y be two distinct nodes in $F_k(D)^*$. We will prove that XY is an edge in $F_k(D)^*$ if and only if XY is an edge in $F_k(D^*)$. We need to consider only nodes X and Y whose symmetric difference is a pair of vertices in D. So, assume that $(X \setminus Y) \cup (Y \setminus X) = \{x, y\}$, for some $x, y \in V(D)$. By the definition of clean graph, XY is an edge in $F_k(D)^*$ if and only if (X, Y) and (Y, X) are arcs in $F_k(D)$. Now, by the definition of token digraph, $(X, Y), (Y, X) \in A(F_k(D))$ if and only if $(x, y), (y, x) \in A(D)$. Using the definition of clean graph, $(x, y), (y, x) \in A(D)$ if and only if $xy \in E(D^*)$. Finally, $xy \in E(D^*)$ if and only if XY is an edge in $F_k(D^*)$. $\square$

Theorem 8.3. *For any digraph D of order n and $1 \leq k \leq n-1$, it holds that $\overleftrightarrow{\omega}(F_k(D)) = \min\{\overleftrightarrow{\omega}(D), \max\{n-k+1, k+1\}\}$.*

Proof. First, note that $\overleftrightarrow{\omega}(D) = \omega(D^*)$. Using this and Fact 8.2, we have that

$$\overleftrightarrow{\omega}(F_k(D)) = \omega(F_k(D)^*) = \omega(F_k(D^*))\,.$$

By Theorem 8.1, $\omega(F_k(D^*)) = \min\{\omega(D^*), \max\{n-k+1, k+1\}\}$. Therefore,

$$\begin{aligned}\overleftrightarrow{\omega}(F_k(D)) &= \min\{\omega(D^*), \max\{n-k+1, k+1\}\} \\ &= \min\{\overleftrightarrow{\omega}(D), \max\{n-k+1, k+1\}\}\,. \quad \square\end{aligned}$$

9. Acyclic partitions

An *acyclic r-partition* of a digraph D is a partition of its vertex set into r sets such that each one induces an acyclic subdigraph of D. We can see such partition as a (non-proper) coloring $c\colon V(D) \to \{1, \ldots, r\}$ of the vertices and each set as a color class. The *dichromatic number* of D, denoted by $\overleftrightarrow{\chi}(D)$, is the smallest integer r such that D has an acyclic r-partition. These notions were introduced by Neumann-Lara [25] as a generalization of proper coloring and chromatic number in undirected graphs.

By Corollary 3.5, we have $\overleftrightarrow{\chi}(D) = 1$ if and only if $\overleftrightarrow{\chi}(F_k(D)) = 1$. Neumann-Lara [25] proved that if D has no odd cycles, then $\overleftrightarrow{\chi}(D) \leq 2$. Thus, by Theorem 4.2, we can conclude that if D has no odd cycles, then $\overleftrightarrow{\chi}(F_k(D)) \leq 2$. In fact, we can show the following result.

Theorem 9.1. *For any digraph D, $\overleftrightarrow{\chi}(F_k(D)) \leq \overleftrightarrow{\chi}(D)$.*

Proof. Let $c\colon V(D) \to \{1,\dots,r\}$ be an optimal acyclic partition of D with $r = \overleftrightarrow{\chi}(D)$, and let $H_1,\dots,H_r$ be the subdigraphs of D induced by each of the color classes. As c is an acyclic partition, each H_i is acyclic. For each node $A \in F_k(D)$, let

$$c'(A) = \sum_{a\in A} c(a) \mod r\,.$$

We aim to show that c' is an acyclic partition of $F_k(D)$. For this purpose, let us define an auxiliary r-vector for the nodes of $F_k(D)$. For a node $A \in F_k(D)$, let $\tau(A) = (t_1,\dots,t_r)$ with $t_i = |A \cap V(H_i)|$ for $i \in [r]$.

Consider an arc (A, B) of $F_k(D)$, and let $(a, b) \in A(D)$ be the corresponding arc such that (A, B) is generated by sliding one token along (a, b). Observe that (a, b) belongs to H_i, for some $i \in [r]$, if and only if $\tau(A) = \tau(B)$. Thus,

$$c'(A) = c'(B) \iff \tau(A) = \tau(B)\,.$$

This observation is generalized as follows. If $J \subseteq F_k(D)$ is a weakly connected subdigraph contained in a same color class of $F_k(D)$, then $\tau(A) = \tau(B)$ for any two nodes $A, B \in J$. This fact implies that J is generated by moving $k_1,\dots,k_r$ tokens on $H_1,\dots,H_r$, respectively, where $0 \le k_i \le |V(H_i)|$ and $k_1 + \dots + k_r = k$. In particular, we have that the tokens moving on a class H_i cannot slide to any other class H_j, implying that J contains no cycle.

On the other hand, note that a color class $\mathcal{H}$ of $F_k(D)$ is a disjoint union of maximal weakly connected subgraphs of $F_k(D)$ having the same color in c', and given that each of these subgraphs contains no cycle, we conclude that $\mathcal{H}$ contains no cycle. Therefore, c' is an acyclic partition of $F_k(D)$ and, by the definition of c', we have $\overleftrightarrow{\chi}(F_k(D)) \le r = \overleftrightarrow{\chi}(D)$, as we wanted. □

Cordero-Michel and Galeana-Sánchez [12] established the following upper bound for $\overleftrightarrow{\chi}(D)$ in terms of the directed circumference $c(D)$ and the directed girth $g(D)$ of D.

Theorem 9.2 *([12]). Let D be a digraph containing at least one cycle. Then*

$$\overleftrightarrow{\chi}(D) \le \left\lceil \frac{c(D)-1}{g(D)-1} \right\rceil + 1.$$

Applying Theorem 9.1 and Theorem 9.2 yields the following immediate consequence.

Corollary 9.3. *Let D be a digraph containing at least one cycle. Then*

$$\overleftrightarrow{\chi}(F_k(D)) \le \left\lceil \frac{c(D)-1}{g(D)-1} \right\rceil + 1.$$

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest integer k such that G has a proper k-coloring. In the undirected case, it is known that $\chi(F_k(K_n)) < \chi(K_n) = n$ for some pairs (k, n) (see, e.g., [15]); in particular, $\chi(F_2(K_n)) = n-1$ for every even n. Since $\overleftrightarrow{\chi}(\overleftrightarrow{G}) = \chi(G)$ and $F_k(\overleftrightarrow{G}) = \overleftrightarrow{F_k(G)}$ (Property 2.3), for every even n we obtain

$$\overleftrightarrow{\chi}(F_2(\overleftrightarrow{K_n})) = \overleftrightarrow{\chi}(\overleftrightarrow{F_2(K_n)}) = \chi(F_2(K_n)) = n-1 < \chi(K_n) = \overleftrightarrow{\chi}(\overleftrightarrow{K_n}).$$

It is natural to ask how $\overleftrightarrow{\chi}(F_k(D))$ behaves as k varies. Since $\overleftrightarrow{\chi}(D) = \overleftrightarrow{\chi}(\overleftarrow{D})$ and $\overleftrightarrow{\chi}(F_k(D)) = \overleftrightarrow{\chi}(F_{n-k}(\overleftarrow{D}))$ (with $n = |V(D)|$), we may restrict attention to $1 \le k \le n/2$.

Neumann-Lara [25] proved that

$$\overleftrightarrow{\chi}(D) = \max\{\overleftrightarrow{\chi}(C)\colon C \text{ is a strongly connected component of } D\}\,.$$

Let C be a strongly connected component of D such that $\overleftrightarrow{\chi}(D) = \overleftrightarrow{\chi}(C)$. If $|V(D)| \ge |V(C)| + k - 1$, then $\overleftrightarrow{\chi}(F_k(D)) \ge \overleftrightarrow{\chi}(D)$. Indeed, we can fix $k-1$ tokens in $k-1$ nodes outside C and leave one in C to derive that $F_1(C) \subseteq F_k(D)$. Consequently, $\overleftrightarrow{\chi}(F_k(D)) \ge \overleftrightarrow{\chi}(F_1(C)) = \overleftrightarrow{\chi}(C) = \overleftrightarrow{\chi}(D)$, and therefore $\overleftrightarrow{\chi}(F_k(D)) = \overleftrightarrow{\chi}(D)$ by Theorem 9.1. In particular, D must have at least two strongly connected components for $|V(D)| \ge |V(C)| + k - 1$ to hold. If D is strongly connected, then it might happen that $\overleftrightarrow{\chi}(F_k(D)) < \overleftrightarrow{\chi}(D)$ for some k, as we previously pointed out.

One might ask whether $\overleftrightarrow{\chi}(F_2(D)) < \overleftrightarrow{\chi}(D)$ for every strongly connected graph D, but this is not true. For instance, $\overleftrightarrow{\chi}(F_k(\overrightarrow{C_n})) = \overleftrightarrow{\chi}(\overrightarrow{C_n}) = 2$ for the cycle $\overrightarrow{C_n}$. (In particular, see $\overrightarrow{C_5}$ and its 2-token digraph in Fig. 4.) It would also be nice to find out the exact conditions that ensure that $\overleftrightarrow{\chi}(F_k(D)) = \overleftrightarrow{\chi}(D)$ holds, even considering only strongly connected digraphs D and $k = 2$.

While investigating this topic, we also considered the analogous question for an undirected graph G and the chromatic number χ, which coincides with the directed question for $\overleftrightarrow{G}$ and $\overleftrightarrow{\chi}$ by Property 2.3. A detailed discussion is presented in Appendix A.

Let D be a strongly connected digraph of order n. We know that $F_k(D)$ is also strongly connected and that $\overleftrightarrow{\chi}(F_1(D)) = \overleftrightarrow{\chi}(D)$. It is natural to ask whether $\overleftrightarrow{\chi}(F_k(D)) \leq \overleftrightarrow{\chi}(F_{k-1}(D))$ for $k > 1$. This is false in general: for $D = \overleftrightarrow{K_n}$ with $n \equiv 4 \pmod 6$, as Property 2.3 and [15] give

$$\overleftrightarrow{\chi}(F_2(D)) = n - 1 \qquad \text{and} \qquad \overleftrightarrow{\chi}(F_3(D)) = n.$$

This phenomenon in the undirected world (including a non-complete example exhibiting a strict decrease as k grows) is also briefly explored at the end of Appendix A.

10. Final remarks

In this paper we introduced token digraphs $F_k(D)$ and related structural and algorithmic properties of D to those of $F_k(D)$. This study is interesting on its own, and it also sheds light on the undirected case. In particular, we characterize strong connectivity via the strong component digraph of D, obtain criteria and exact conditions for when token digraphs are unilateral, prove that D is Eulerian if and only if $F_k(D)$ is Eulerian, bound the bidirected clique number through the clean graph, show that $\overleftrightarrow{\chi}(F_k(D)) \leq \overleftrightarrow{\chi}(D)$, and transfer parity obstructions while proving that deciding whether $F_2(D)$ has a kernel is NP-complete.

We found the study of acyclic partitions and the dichromatic number of token digraphs particularly challenging, leading to questions in both directed and undirected settings (see Conjecture A.1 in Appendix A).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Chromatic number of 2-token graphs

We address the chromatic number of token graphs, focusing on the case of 2-token graphs. This question has been examined for complete graphs [15], and some general lower and upper bounds are known [16]. We conjecture that $\chi(F_2(G)) < \chi(G)$ holds precisely when G is the complete graph on an even number of vertices. We further describe properties that any potential counter-example must satisfy, and determine the chromatic number for the 2-token graphs of certain families of graphs, providing evidence in support of our conjecture.

Conjecture A.1. *$\chi(F_2(G)) < \chi(G)$ if and only if $G = K_n$ for even n.*

Let G be a graph of order n. A proper coloring of $F_2(G)$ can be viewed as an edge-coloring of K_n (not necessarily proper) such that, if the edges uv and vw of K_n are assigned the same color, then $uw \notin E(G)$. (If $uw \in E(G)$, then $\{u, v\}$ and $\{v, w\}$ are adjacent 2-token configurations in $F_2(G)$.) We will use this in the figures ahead, because we find it a convenient and compact way to present a coloring for $F_2(G)$. Also, we will denote a 2-token configuration $\{u, v\}$ simply as uv, to simplify the notation, and we will call it a configuration instead of a vertex of $F_2(G)$ to avoid confusion with the vertices of G.

When $G = K_n$, such an edge-coloring of K_n is exactly a proper edge-coloring of K_n. Hence $\chi(F_2(K_n)) = \chi'(K_n)$ and thus one direction of Conjecture A.1 is already known: $\chi(F_2(K_n)) = n - 1 = \chi(G) - 1$ for n even, because there is a partition of the edges of K_n into $n - 1$ perfect matchings, that is, $\chi'(K_n) = n - 1$.

A graph G is *critical* if $\chi(G - v) = \chi(G) - 1$ for every vertex v of G. It is *k-critical* if G is critical and $\chi(G) = k$. Any potential counter-example to Conjecture A.1—that is, a graph G that is not complete but for which $\chi(F_2(G)) < \chi(G)$—must satisfy the following conditions: (a) $\chi(G) > 3$; (b) G is critical; (c) the maximum degree $\Delta(G) \geq \chi(G)$; (d) $\chi(G) > \omega(G)$.

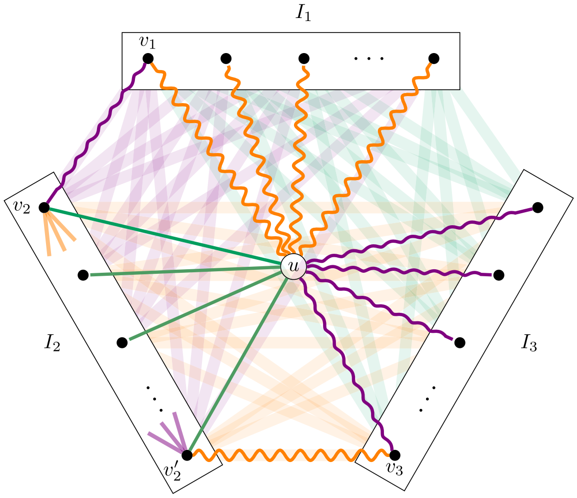

Fig. A.8. The 3-coloring of $F_2(G)$ from Lemma A.2 represented as an edge-coloring of the complete graph on $|V(G)|$ vertices.

Indeed, it is known that a graph G is bipartite if and only if $F_2(G)$ is bipartite (see [16]), which implies (a). If G is non-critical, there exists a vertex $v \in V(G)$ such that $\chi(G) = \chi(G - v)$. In this case, it suffices to consider the subgraph of $F_2(G)$ consisting of all configurations with a token at v, which is isomorphic to $F_1(G - v)$, and this in turn is isomorphic to $G - v$. We then have $\chi(F_2(G)) \geq \chi(G - v) = \chi(G)$. This proves (b). Assume now that G is critical, so G must have minimum degree at least $\chi(G) - 1$. If $\Delta(G) < \chi(G)$, then G is a $(\chi(G) - 1)$-regular graph, and, by Brook's Theorem (see [9]), G must be either an odd cycle or a complete graph, which is not the case by (a) and given that $\chi(F_2(K_n)) = \chi'(K_n) = n$ for n odd. Thus, (c) must hold. Finally, suppose $\chi(G) \leq \omega(G)$. By Theorem 8.1 we have $\omega(F_k(G)) = \min\{\omega(G), \max\{n - k + 1, k + 1\}\}$, and then, as G is not a complete graph, we have $\chi(F_2(G)) \geq \omega(F_2(G)) = \omega(G) = \chi(G)$, which proves (d).

Next we prove that Conjecture A.1 holds for two classes of 4-critical graphs. A vertex in a graph G is *universal* if it is adjacent to all other vertices of G.

Lemma A.2. *For every graph G on $n \geq 5$ vertices, with $\chi(G) = 4$ and a universal vertex u, $\chi(F_2(G)) = 4$.*

Proof. It is enough to prove that there is no proper 3-coloring for $F_2(G)$, because we already know that $\chi(F_2(G)) \leq \chi(G)$ (Theorem 6 in [16]). For a contradiction, suppose there is such a 3-coloring with colors 1, 2, 3. Let $I_j = \{v\colon \text{configuration } uv \text{ has color } j\}$ for $j \in \{1, 2, 3\}$. Note that the sets I_1, I_2, I_3 form a partition of $V(G - u)$. Fig. A.8 presents the information we are deriving on this coloring.

Because this 3-coloring of $F_2(G)$ is proper, there cannot be an edge of G with the two ends in the same I_j, that is, each I_j is an independent set in G. Moreover, since $\chi(G) = 4$, each I_j is non-empty. For every $v \in I_1$ and $w \in I_2$, the configuration vw must have color 3, because the configuration uv has color 1 and the configuration uw has color 2, and both are edges of G. Thus, all configurations vw for $v \in I_1$ and $w \in I_2$ have color 3. Similarly, all configurations vw for $v \in I_1$ and $w \in I_3$ have color 2, and all configurations vw for $v \in I_2$ and $w \in I_3$ have color 1.

Because $n \geq 5$, at least one of the sets I_j has at least two vertices. Without loss of generality, say $|I_2| \geq 2$. Because $\chi(G) = 4$, there must be an edge between a vertex $v_1 \in I_1$ and a vertex $v_2 \in I_2$. For every $v \in I_2$ distinct from v_2, as argued in the previous paragraph, the configurations v_1v_2 and v_1v have color 3. Also, as both v_2 and v are in I_2, the configurations uv_2 and uv have color 2. Thus, the configuration v_2v must have color 1. So all configurations v_2v, for every $v \in I_2$ distinct from v_2, have color 1. However, there must also be an edge between a vertex $v_3 \in I_3$ and a vertex $v_2' \in I_2$, and this implies that all configurations $v_2'v$, for every $v \in I_2$ distinct from v_2', have color 3. Because $|I_2| \geq 2$, we reach a contradiction (on the color of the configuration v_2v_2' if $v_2 \neq v_2'$, or, if $v_2 = v_2'$, on the color of the configurations v_2v for any $v \in I_2$ distinct from v_2; at least one such configuration exists because $|I_2| \geq 2$). □

A well-known class of 4-critical graphs are the odd wheels W_{2k+1}. A consequence of Lemma A.2 is that they do not contradict Conjecture A.1, because they have a universal vertex.

Let G be a graph on n vertices $v_1, \ldots, v_n$. The *Mycielski graph of* G is the graph $M(G)$ that contains a copy of G together with $n + 1$ vertices $u_0, u_1, \ldots, u_n$, where each u_i is adjacent to u_0 and to each neighbor of v_i in G, for $i \in \{1, \ldots, n\}$. Mycielski [24] proved that $\chi(M(G)) = \chi(G) + 1$. It is not hard to prove that, for an odd k, the graph $M(C_k)$ is 4-critical. Note that $M(C_k)$ does not have a universal vertex, so Lemma A.2 does not apply to $M(C_k)$. Yet we can prove the following on $F_2(M(C_k))$, which implies that $M(C_k)$ for odd k does not contradict Conjecture A.1 either.

Lemma A.3. *For every odd k, $\chi(F_2(M(C_k))) = 4$.*

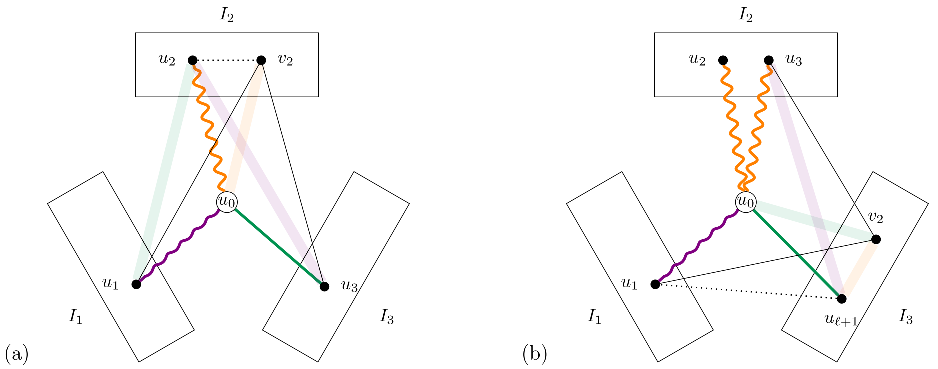

Fig. A.9. Cases for the proof of Lemma A.3.

Before presenting the proof for this lemma, we prove a seemingly unrelated result that will be used in the proof of the lemma. Let s be a string on a 3-letter alphabet $\{A, B, C\}$. We say that s is *special* if it has the form pq^jr for some $j \geq 1$, such that $\{p, q, r\} = \{A, B, C\}$.

Claim A.4. *Let $s = s_1s_2 \ldots s_n$ be a string on a 3-letter alphabet. If s contains the three letters of the alphabet, then s contains a special string as a substring.*

Proof. Let k be the smallest index such that the prefix $s_1s_2 \ldots s_k$ contains all three letters. Then $s_1s_2 \ldots s_{k-1}$ contains exactly two distinct letters. Set $r := s_k$.

Let i be the largest index with $1 \leq i < k$ such that the substring $s_is_{i+1} \ldots s_{k-1}$ contains exactly two letters, whereas $s_is_{i+1} \ldots s_k$ contains all three. Such an i exists because $s_1s_2 \ldots s_{k-1}$ already contains two letters. By the maximality of i, we must have $s_{i+1} = s_{i+2} = \cdots = s_{k-1}$. Write $q := s_{i+1}$ and $p := s_i$.

Hence the substring $s_is_{i+1} \ldots s_k$ is of the form pq^jr, with $j = k - i - 1 \geq 1$. Moreover, by the choice of i, the letters p, q, r are pairwise distinct; therefore $\{p, q, r\} = \{A, B, C\}$, and the substring is special. □

Proof of Lemma A.3. We will prove that there is no proper 3-coloring for $F_2(M(C_k))$. Let $C_k = (v_1, \ldots, v_k)$ and consider $u_0, u_1, \ldots, u_k$ as in the construction of $M(C_k)$. For a contradiction, suppose there is such a 3-coloring with colors 1, 2, 3. Let $I_j = \{v : \text{configuration } u_0v \text{ has color } j\}$ for $j \in \{1, 2, 3\}$. The sets I_1, I_2, I_3 form a partition of $V(M(C_k) - u_0)$, and each I_j is an independent set in $M(C_k) - u_0$. Because $\chi(M(C_k)) = 4$ and $u_1, \ldots, u_k$ are the neighbors of u_0, there is at least one u_i in each I_j. Furthermore, the partition $\{I_1, I_2, I_3\}$ induces a proper 3-coloring of C_k, and so there is also at least one v_i in each I_j.

Let s be the string of length k on the alphabet $\{1, 2, 3\}$ consisting of the sequence of indices j such that $u_i \in I_j$ for $i \in \{1, \ldots, k\}$. As each I_j contains some u_i, the three indices $1, 2, 3$ appear in s. We apply Claim A.4 to s, deriving, without loss of generality, that $u_1 \in I_1$, $u_2, \ldots, u_\ell \in I_2$, and $u_{\ell+1} \in I_3$, for $\ell \geq 2$. Now let us derive a contradiction by analyzing two cases.

If $\ell = 2$, then the situation is depicted in Fig. A.9(a). Because u_0 is adjacent to u_1, u_2, u_3, the configuration u_1u_2 must have color 3 and the configuration u_2u_3 must have color 1. Because v_2 is adjacent to u_1 and u_3, the configuration u_0v_2 must have color 2, which means $v_2 \in I_2$. But then the configuration u_2v_2 cannot be colored with any of the colors 1, 2, 3.

If $\ell > 2$, then the situation is depicted in Fig. A.9(b). Because v_2 is adjacent to u_1 and u_3, the configuration u_0v_2 must have color 3, which means $v_2 \in I_3$. Because u_0 is adjacent to u_3 and $u_{\ell+1}$, the configuration $u_3u_{\ell+1}$ must be of color 1, and the configuration $u_{\ell+1}v_2$ must be of color 2. But then the configuration $u_1u_{\ell+1}$ cannot be colored with any of the colors 1, 2, 3. □

We close with two brief remarks on the dependence on k. First, the monotonicity $\chi(F_k(G)) \leq \chi(F_{k-1}(G))$ for $1 < k < n - 1$ does not hold in general: for all $n \equiv 4 \pmod 6$ we have $\chi(F_2(K_n)) = n - 1$ and $\chi(F_3(K_n)) = n$ [15]. Second, beyond complete graphs, we are aware of one non-complete 6-critical example exhibiting a strict decrease: the graph obtained from K_8 by deleting a 5-cycle, for which $\chi(F_k(G)) = 6$ for all $1 \leq k \leq 3$ and $\chi(F_4(G)) = 5$. For all other graphs we examined, we observed $\chi(F_k(G)) = \chi(F_{k-1}(G))$ for every $1 < k \leq n/2$.

Data availability

No data was used for the research described in the article.

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