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On Non-Commutative Multi-Rings with Involution

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Abstract: The primary motivation for this work is to develop the concept of Marshall's quotient applicable to non-commutative multi-rings endowed with involution, expanding upon the main ideas of the classical case—commutative and without involution—presented in Marshall's seminal paper. We define two multiplicative properties to address the involutive case and characterize their Marshall quotient. Moreover, this article presents various cases demonstrating that the "multi" version of rings with involution offers many examples, applications, and relatives in (multi)algebraic structures. Therefore, we established the first steps toward the development of an expansion of real algebra and real algebraic geometry to a non-commutative and involutive setting.

Keywords: multi-rings; involution; abstract real algebra

MSC: 16Y20; 11E16; 11E81

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1. Introduction

Multialgebraic structures are "algebraic-like" structures endowed with multiple valued operations: an n-ary multi-operation on set A is just a function $A^n \to P(A) \setminus \{\emptyset\}$. The definition and study of the concept of multi-group (Definition 1) began in the 1930s by Marty; in the 1950s, the commutative hyperrings were introduced by Krasner (Definition 2). Since then, research on these multi-structures and their broad range of applications has been developed. The concepts of (commutative) multi-ring and superring (Definition 2), are much more recent developments, as discussed in [1,2]. To access advances and results in the theory of multi-ring and hyperring (commutative), we recommend the following: [2–9].

Many instances of multialgebraic structures codify the nature of mathematical objects through operations. Here, we recall some basic examples and provide additional ones, focusing on the non-commutative case.

Moreover, the exploration of this subject remains substantially open compared to the classical case. The natural progression of the subject has led to the development of *polynomials* [2], *linear algebra* [10], and *orderings* [11].

The main purpose of the present work is to outline the fundamental steps necessary to expand Marshall's seminal paper [1] to the context of non-commutative multi-rings with involution. Specifically, we present and analyze the expansion of the notion of the "Marshall's quotient" (see [12]), a crucial construction in abstract concepts of real algebra and real algebraic geometry. This includes applications in the space of signs [13], abstract real spectra [14], real semigroups [15], and real reduced multi-rings [1].

Building on this foundation, future work will focus on developing a real spectrum for non-commutative rings with involution, as a preparation for establishing an abstract theory of Hermitian forms ([16]).

Within this context, we introduce the concept of the Marshall quotient for involutive (non-commutative) multi-rings and discuss some applications to quaternion algebras over formally real fields. The main technical results are presented in Theorems 3–5. To illustrate an application, in Section 5, we provide the following:

Theorem 1 (7). *Let R be a commutative ring and A be an R-algebra with involution σ. We denote*

$$Orth(A) := \{a \in A : a\sigma(a) = \sigma(a)a = 1\}.$$

If $Orth(A) \subseteq Z(A)$, then $A/_m Orth(A)$ is a (non-commutative) hyperring.

Outline

In Section 2, we provide a brief introduction to multi/super-structures relevant to this work. We offer a non-standard example that extends Krasner's hyperfield and the signal hyperfield in Example 2. In Section 3, we introduce the basic objects of the theory of (non-commutative) multi-rings with involution and invite the reader to compare this theory with the classical one. Additionally, we cover various constructions and examples, including multi-groups, products, and matrices.

In Section 4, we define Marshall's quotient on involutive multialgebras and analyze the conditions for their existence using a "coherent" approach. Theorem 3 presents two types of quotients characterized by certain multiplicative subsets. Although many relations can be considered when forming classes in the quotient, we focus on four different possibilities and show how they are similar (Lemma 4). Moreover, in developing particular examples, we verify the independence of the conditions in Theorem 3. Additionally, the available quotient provides a "concrete" framework that encodes several types. In Section 5, we explore some applications and present examples of quotients that generate well-known multi-structures. Finally, in Section 6, we present our final remarks and conclusions.

2. Multi-Structures

In this section, we provide a brief overview of multi-structures and establish the necessary notations for the reader.

Multialgebraic structures are "algebraic-like" structures endowed with multi-valued operations. An n-ary multi-operation on set A is defined as a function $f : A^n \to P(A) \setminus \{\emptyset\}$, where $P(A)$ is the power set of A. Alternatively, the same concept can be described by an $n+1$-ary relation $R_f \subseteq A^{n+1}$, which satisfies the following condition: for all $x_0, x_1, \cdots, x_{n-1} \in A$, there exists $x_n \in A$, such that $R_f(x_0, x_1, \cdots, x_{n-1}, x_n)$.

Definition 1 (Adapted from Definition 1.1 in [1]). *A **multi-group concept** is a first-order structure $(G, \cdot, r, 1)$, where G is a non-empty set, $r : G \to G$ is a function, 1 is an element of G, $\cdot \subseteq G \times G \times G$ is a ternary relation (which will play the role of a binary multi-operation, and we denote $d \in a \cdot b$ for $(a, b, d) \in \cdot$), such that for all $a, b, c, d \in G$, we have the following:*

M1 - *If $c \in a \cdot b$, then $a \in c \cdot (r(b))$ and $b \in (r(a)) \cdot c$. We write $a \cdot b^{-1}$ to simplify $a \cdot (r(b))$.*

M2 - *$b \in a \cdot 1$ iff $a = b$.*

M3 - *If there exists x, such that $x \in a \cdot b$ and $t \in x \cdot c$), then there exists y, such that $y \in b \cdot c$ and $t \in a \cdot y$. Equivalently, if $\exists\, x(x \in a \cdot b \wedge t \in x \cdot c)$, then $\exists\, y(y \in b \cdot c \wedge t \in a \cdot y)$.*

*The structure $(G, \cdot, r, 1)$ is said to be **commutative (or abelian)** if it satisfies the following condition for all $a, b, c \in G$:*

M4 - *$c \in a \cdot b$ iff $c \in b \cdot a$.*

*The structure $(G, \cdot, 1)$ is a **commutative multimonoid (with unity)** if it satisfy M3, M4, and condition $a \in 1 \cdot a$ for all $a \in G$.*

Definition 2 (Definition 5 in [2]). *A **(commutative) superring** is a tuple $(R, +, \cdot, -, 0, 1)$, satisfying the following:*

1. *$(R, +, -, 0)$ is a commutative multi-group and $(R, \cdot, 1)$ is a (commutative) multimonoid;*
2. *(Null element) $a \cdot 0 = 0$ and $0 \cdot a = 0$ for all $a \in R$;*
3. *(Weak distributive) If $x \in b + c$, then $a \cdot x \in a \cdot b + a \cdot c$ and $x \cdot a \in b \cdot a + c \cdot a$. Equivalently, $(b + c) \cdot a \subseteq b \cdot a + c \cdot a$ and $a \cdot (b + c) \subseteq a \cdot b + a \cdot c$.*

4. *The rule of signals holds:* $-(ab) = (-a)b = a(-b)$*, for all* $a, b \in R$*.*
   Note that if $a \in R$*, then* $0 = 0 \cdot a \in (1 + (-1)) \cdot a \subseteq 1 \cdot a + (-1) \cdot a$*, thus* $(-1) \cdot a = -a$*.*

R is said to be a ***multi-ring*** *if* $(R, \cdot, 1)$ *forms a monoid. A* ***hyperring*** *R is a multi-ring such that if for* $a, b, c \in R$*,* $a(b+c) = ab + ac$ *and* $(b+c)a = ba + ca$*. A multi-ring (respectively, a hyperring) R is said to be a* ***multi-domain (hyperdomain)*** *if it contains no zero divisors. A commutative multi-ring R will be a* ***multifield*** *if every non-zero element of R has a multiplicative inverse.*

If $a = 0$, then $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$. **Observe that hyperfields and multifields coincide**. Indeed, by definition, every hyperfield is a multifield, and, for a given multifield, F, if $a \neq 0$, then we have the following:

$$a^{-1}(ab + ac) \subseteq b + c \text{ implies } aa^{-1}(ab + ac) \subseteq a(b+c),$$

whenever $b, c \in F$. Therefore, $a(b+c) = ab + ac$.

Definition 3. *Let A and B be superrings. A map* $f : A \to B$ *is a morphism if for all* $a, b, c \in A$*:*

1. $f(1) = 1$ *and* $f(0) = 0$*;*
2. $f(-a) = -f(a)$*;*
3. $f(ab) = f(a)f(b)$*;*
4. *if* $c \in a + b$ *then* $f(c) \in f(a) + f(b)$*.*

A morphism f is a ***full morphism*** *if for all* $a, b \in A$*,*

$$f(a+b) = f(a) + f(b) \text{ and } f(a \cdot b) = f(a) \cdot f(b).$$

In this text, we provide some examples and treat (non-commutative) multi-rings. For more details, we recommend the reader to check [2–9] for advances and results in multi-ring/hyperring (commutative) theory.

Example 1.

1. *Suppose that* $(G, \cdot, 1)$ *is a group. Defining* $a * b = \{a \cdot b\}$*, and* $r(g) = g^{-1}$*, we have that* $(G, *, r, 1)$ *is a multi-group. In this way, every ring, domain, and field is a multi-ring, multi-domain, and hyperfield, respectively.*
2. *Let* $K = \{0, 1\}$ *with the usual product, and the sum defined by relations* $x + 0 = 0 + x = x$*,* $x \in K$*, and* $1 + 1 = \{0, 1\}$*. This is a hyperfield referred to as* ***Krasner's hyperfield*** *[17].*
3. $Q_2 = \{-1, 0, 1\}$ *is the* ***"signal" hyperfield*** *with the usual product (in* $\mathbb{Z}$*) and the multi-valued sum defined by relations*

$$\begin{cases} 0 + x = x + 0 = x, \text{ for every } x \in Q_2 \\ 1 + 1 = 1,\ (-1) + (-1) = -1 \\ 1 + (-1) = (-1) + 1 = \{-1, 0, 1\} \end{cases}$$

4. *For every multi-ring R, we define the* ***opposite multi-ring*** R^{op}*, which has the same structure unless* $(R^{op}, \cdot^{op}, 1^{op})$ *is the opposite monoid of* $(R, \cdot, 1)$*, i.e.,* $\cdot^{op}$ *is the reverse multiplication. The null element and the weak distributive properties are satisfied on both sides in* R^{op} *because they are met on the opposite sides in R.*

The following example codifies the structure of ranks of square matrices:

Example 2 (Superrings of signed ranks)**.** *Consider* $n \in \mathbb{N}$ *and*

$$K_n^{\pm} = \{0, 1, 2, ..., n-1, n_-, n_+\}.$$

This set is endowed with a superring structure, which includes the addition $\oplus$ and multiplication $\odot$ operations, defined by the following:

(n = 0,1) $K_0^{\pm} = K_0 = \{0\}$, $K_1^{\pm} = Q_2$ *and* $K_1 = \{0,1\} = K$.

(K1) 0 *is the identity with respect to the addition* $\oplus$;

(K2) 1. $m \oplus m' = \begin{cases} [|m-m'|, m+m']; m, m' \in \{1,...,n-1\}, m+m' < n; \\ [|m-m'|, m+m'] \cup \{n_\pm\}; m, m' \in \{1,...,n-1\}, m+m' \geq n; \end{cases}$

2. $m \oplus n_\pm = [n-m, n-1] \cup \{n_\pm\}$ *whenever* $m \leq n-1$;
3. *(n is even)* $n_+ \oplus n_+ = n_- \oplus n_- = K_n^{\pm}$;
 $n_+ \oplus n_- = K_n^{\pm} \setminus \{0\}$.
4. *(n is odd)* $n_+ \oplus n_+ = n_- \oplus n_- = K_n^{\pm} \setminus \{0\}$;
 $n_+ \oplus n_- = K_n^{\pm}$.

(K3) n_+ *is the identity with respect to the multiplication* $\odot$ *and* $n_- \odot n_- = n_+$;

(K4) *For* $m, m' < n$,

$$m \odot m' = \begin{cases} [m+m'-n, min(m,m')], \text{ whenever } m+m' > n; \\ [0, min(m,m')], \text{ otherwise.} \end{cases}$$

We denote the ***superrings of ranks*** *by* $K_n = \{0,1,2,...,n-1,n\}$, *whose axioms are identical, except for* $n_+ = n_- = n$.

Example 3 (Kaleidoscope, Example 2.7 in [12]). *Let* $n \in \mathbb{N}$ *and define*

$$X_n = \{-n, ..., 0, ..., n\} \subseteq \mathbb{Z}.$$

We define the n***-kaleidoscope multi-ring*** *by* $(X_n, +, \cdot, -, 0, 1)$, *where* $- : X_n \to X_n$ *is the restriction of the opposite map in* $\mathbb{Z}$, $+ : X_n \times X_n \to \mathcal{P}(X_n) \setminus \{\emptyset\}$ *is given by the following rules:*

$$a + b = \begin{cases} \{a\}, \text{ if } b \neq -a \text{ and } |b| \leq |a| \\ \{b\}, \text{ if } b \neq -a \text{ and } |a| \leq |b| \\ \{-a, ..., 0, ..., a\} \text{ if } b = -a \end{cases},$$

and $\cdot : X_n \times X_n \to X_n$ *is given by the following rules:*

$$a \cdot b = \begin{cases} sgn(ab) \max\{|a|, |b|\} \text{ if } a, b \neq 0 \\ 0 \text{ if } a = 0 \text{ or } b = 0 \end{cases}.$$

With the above rules we have that $(X_n, +, \cdot, -, 0, 1)$ *is a multi-ring, which is not a hyperring for* $n \geq 2$ *because*

$$n(1-1) = b \cdot \{-1, 0, 1\} = \{-n, 0, n\}$$

and $n - n = X_n$. *Note that* $X_0 = \{0\}$ *and* $X_1 = \{-1, 0, 1\} = Q_2$.

Example 4 (Triangle hyperfield [18]). *Let* $\mathbb{R}_+$ *be the set of non-negative real numbers endowed with the following (multi)operations:*

$$\begin{cases} a \triangledown b = \{c \in \mathbb{R}_+ \mid |a-b| \leq c \leq |a+b|\}, \text{ for all } a, b \in \mathbb{R}_+, \\ a \cdot b = ab, \text{ the usual multiplication in } \mathbb{R}_+, \\ -a = a. \end{cases}$$

Moreover, this is a hyperfield that does not satisfy the double distributive property (see 5.1 in [18] for more details).

Example 5.

1. *The prime ideals of a commutative ring (its Zariski spectrum) are classified by equivalence classes of morphisms into algebraically closed fields; however, they can be* uniformly classified *by a multi-ring morphism into the Krasner hyperfield* $K = \{0, 1\}$.
2. *The orderings of a commutative ring (its real spectrum) are classified by classes of equivalence of ring homomorphisms into real closed fields. However, they can be* uniformly classified *by a multi-ring morphism into the signal hyperfield* $Q_2 = \{-1, 0, 1\}$.
3. *The Krull valuation on a commutative ring with a group of values* $(G, +, -, 0, \leq)$ *is just a morphism into the hyperfield* $T_G = G \cup \{\infty\}$.

3. Multialgebras with Involution

In this section, we introduce the key concept of this work: multialgebras with involution. For a multi-ring A, we denote

$$Z(A) := \{a \in A : \text{for all b} \in A,\ ab = ba\},$$

the **center** of A. Of course, if A is commutative, $Z(A) = A$. The classical theory of central algebras with involution suggests the development of this subject in a very similar way.

Definition 4.

1. *Let R be a commutative multi-ring, A be a (non-necessarily commutative) multi-ring, and $j : R \to A$ a homomorphism of multi-rings, such that $j[R] \subseteq Z(A)$, then (A, j) is an R-**multialgebra**.*
2. *A **morphism** of R-multialgebras $f : (A, j) \to (A', j')$ is a morphism of multi-rings $f : A \to A'$ such that $f \circ j = j'$.*
3. *An **involution** σ over the R-multialgebra (A, j) is an (anti)isomorphism of R-multialgebras $\sigma : (A, j) \to (A^{op}, j^{op})$ where A^{op} is the opposite multi-ring, $j^{op} : R^{op} \to A^{op}$ is a homomorphism, and $\sigma^{op} = \sigma^{-1}$. Thus, for all $a, b \in A$, $\sigma(ab) = \sigma(b)\sigma(a)$.*
4. *A **multialgebra with involution** is just a (R, τ)-multialgebra endowed with an involution, where (R, τ) is a multi-ring with involution. A **morphism of** R-**multialgebras with involution** is a morphism of R-multialgebras $f : (A, j, \sigma) \longrightarrow (A', j', \sigma')$ satisfying $f \circ \sigma = \sigma' \circ f$.*
5. *For each commutative multi-ring with involution $(\mathcal{R}, \tau)$, there exists the **category of** (R, τ)-**multialgebras with involution**, whose objects are (R, τ)-multialgebras with involution and morphisms are morphisms of R-multialgebras with involution.*

Whenever the involution τ is clear, we will omit it and write only $\mathcal{R}$. Note that item 1 implies that (R, τ) is an initial object in $\mathcal{R}$. Item 2 ensures that every morphism $f : (A, j, \sigma) \longrightarrow (A', j', \sigma')$ is represented by a commutative triangle.

$$\begin{array}{ccc} (R, \tau) & \xrightarrow{\ j\ } & (A, \sigma) \\ & \searrow^{j'} \quad \circlearrowleft & \downarrow f \\ & & (A', \sigma') \end{array} \qquad (\square)$$

We call (A, σ) a **subalgebra** of (A', σ') if the diagram $(\square)$ is satisfied by the restricted identity morphism $f = id_{A'}|_A$. An **ideal** $J \subseteq A$ is a σ-invariant ($\sigma(J) \subseteq J$) non-empty subset satisfying $J \cdot A \subseteq J$ and $x + y \subseteq J$ for all $x, y \in J$. Once J is σ-invariant and σ is an isomorphism, $A \cdot J = \sigma(\sigma(J) \cdot \sigma(A)) \subseteq \sigma(J) \subseteq J$ and, thus, J is a two-sided ideal. A **proper ideal** is an ideal $J \neq A$. We call J a **prime ideal** if J is an ideal such that $ab \in J$ implies $a \in J$ or $b \in J$ for any pair $a, b \in A$. The smallest ideal generated by $a_1, ..., a_k \in A$ is

$$J(a_1,...,a_k) = \sum_{i=1}^{k} Aa_iA + A\sigma(a_i)A.$$

We define the quotient A/J as usual (see, for instance, [2,12,19], or [20]). We have many standard and effusive constructions that raise various examples in category $\mathcal{R}$.

Let I be a non-empty set. For a given family $(A_i, j_i, \sigma_i)_{i\in I}$ of R-multialgebras with involution, the **direct product** $\prod A_i = (\prod_{i\in I}(A_i, \pi_i), \tilde{j}, \tilde{\sigma})$ is an R-multialgebra with involution such that $\pi_{i_0} : \prod A_i \longrightarrow A_{i_0}$ are projection morphisms for each $i_0 \in I$. Indeed, $\tilde{\sigma}(a_i)_{i\in I} = (\sigma_i(a_i))_{i\in I}$ is an involution over $\prod A_i$, and $\tilde{j}(r) = (j_i(r))_{i\in I} \in Z(\prod A_i)$ is a well-defined map satisfying the necessary conditions above.

Matrices over a given commutative multi-ring are natural constructions. We denote by $M_n(A)$ the set of square matrices of order n with coefficients in (A, j, σ) and set the sum and product of matrices as follows:

For all matrices $C = (c_{ij})_{n\times n}, B = (b_{ij})_{n\times n} \in M_n(A)$, we define the function $\bar{\sigma} : M_n(A) \longrightarrow M_n(A)$ by $\bar{\sigma}(B) = (\sigma(b_{ji}))_{n\times n}$ and (multi)operations, as follows:

$$C + B := \{(d_{ij}) : d_{ij} \in c_{ij} + b_{ij} \text{ for all } i,j\} \neq \emptyset$$
$$CB := \{(d_{ij}) : d_{ij} \in \sum_{k=1}^{n} c_{ik}b_{kj} = c_{i1}b_{1j} + c_{i2}b_{2j} + ... + c_{in}b_{nk} \text{ for all } i,j\} \neq \emptyset$$
$$\lambda C := (\lambda c_{ij})_{n\times n}, \text{ for all } \lambda \in R.$$

Since σ is an involution and A is a commutative multi-ring, it follows that $\bar{\sigma}$ is also an involution. Finally, let $f : (A, \sigma) \longrightarrow (M_n(A), \bar{\sigma})$ be the diagonal morphism defined by

$$f(a) = \text{diag}(a, a, ..., a) \in M_n(A),$$

which associates each $a \in A$ with a diagonal matrix in $M_n(A)$ and $\bar{j} := f \circ j$ is the injective morphism such that $\bar{j}(R) \subseteq Z(M_n(A))$. We will avoid the verification that $(M_n(A), \bar{j}, \bar{\sigma})$ is an R-multialgebra with involution, but the reader can check Section 2 of [10], Theorem 2.3, and Lemma 2.5. However, we provide an example to illustrate this construction.

Example 6. *Consider the* 2*-kaleidoscope multi-ring* $(X_2, +, \cdot, -, 0, 1)$ *as defined in* 3 *and* $(\ )^t$ *the matrix transposition. Then,* $(M_2(X_2), (\ )^t)$ *is an* X_2*-multialgebra with involution.*

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ matrices over X_2. Thus,

$$AB = \begin{bmatrix} 1\cdot 0 + 2\cdot(-1) & 1\cdot 1 + 2\cdot 1 \\ -1\cdot 0 + 0\cdot(-1) & -1\cdot 1 + 0\cdot 1 \end{bmatrix},\ A^t = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix},\ B^t = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Therefore, $(AB)^t = B^tA^t = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}$.

Example 7. *(Adapted from [21]) Let* $G^0 = G \cup \{0\}$ *be a group with* 0 *and define* $+$ *the multi-operation satisfying the following:*

$$x + 0 = 0 + x = x\ , \forall x \in G^0;$$
$$x + x = G^0 \setminus \{x\}\ , \forall x \in G^0;$$
$$x + y = \{x, y\}\ , \forall x, y \in G^0 \text{ with } x \neq y.$$

We can define an involution σ *over this structure by setting* $\sigma(x) = x^{-1}$ *for all* $x \in G$ *and* $\sigma(0) = 0$*. In fact,* σ *is additive and it is easy to verify that* $(G^0, 0, 1, +, \cdot, \sigma)$ *is a multi-ring with involution.*

4. Marshall's Quotient of Multialgebras with Involution

The notion of Marshall's quotient of a commutative (hyper)ring (resp., multi-ring) by a multiplicative subset always produces a commutative hyperring (resp., a commutative multi-ring), and is the main construction used in the abstract approaches of quadratic forms theory ([1,12]). In this section, we introduce the main technical tool, in the general setting of a (non-commutative) multi-ring with involution, developed in the present work—Marshall's quotient—which will enable us to construct a variety of interesting multialgebras with involution, derived from standard algebraic structures.

Throughout this section, we fix an R-multialgebra with involution (A, j, σ). We are interested in *Marshall-coherent subsets* satisfying at least one of the conditions in Theorem 3, i.e., *normality* or *convexity*. These conditions interact in many ways with the relations below (6) compared to the commutative case. First of all, we explore basic properties due to definitions.

Definition 5. *A subset (without zero divisors) $S \subseteq A$ is called a* ***Marshall-coherent subset*** *whenever*

- *S is a multiplicative submonoid of $(A, \cdot, 1)$*
- *$\sigma[S] \subseteq S$ (or, equivalently $\sigma[S] = S$)*

We call S ***standard*** *if $s\sigma(s) \in Z(A)^\times$, for all $s \in S$. We note that S is* ***convex*** *if $xS\sigma(x) \subseteq S$ for all $x \in A_0$ in the subset of nonzero divisors of A. If $x\sigma(x) \in S$ for all $x \in A_0$, we note that S is* 1-***convex***.

Immediately, convexity implies 1-convexity. One can check Lemma 5 and Proposition 1 for a reciprocal result. From now on, we fix a Marshall-coherent subset $S \subseteq A$.

The expansion of the theory to this non-commutative and involutive setting, inevitably, leads us to a multitude of definitions that are collapsed to a single one in the traditional commutative setting and where the involution is trivial. Therefore, we present the following:

Definition 6. *Let $a, b \in A$ and $s_1, s_2, t_1, t_2 \in S$. We define the following:*

1. *$a \sim_1 b$ iff $a = s_1bs_2$ and $b = t_1at_2$;*
2. *$a \sim_2 b$ iff $s_1as_2 = t_1bt_2$;*
3. *$a \sim_3 b$ iff $as_1 = t_1b$ and $s_2a = bt_2$;*
4. *$a \sim_4 b$ iff there is $s \in S$ such that $as\sigma(b) \in S$.*

Despite the diversity of these relations, they are interconnected and, under certain natural conditions, they may coincide. Of course, $a \sim_1 b$ implies $a \sim_2 b$. Further, $\sim_4$ is an equivalence relation when S is 1-convex. Indeed, this relation concurs with $\sim_3$ (see Lemma 4). We start our exploration of these relations and the associated properties of Marshall-coherent subsets.

Lemma 1. *For $\sim = \sim_1$, as defined above, $\sim$ is an equivalence relation and satisfies the following:*

1. *For all $a \in A$ and all $s \in S$, $\sigma(s)as \sim a$, $sa\sigma(s) \sim a$, and $abs \sim ab$, $sab \sim ab$.*
2. *For all $a, b \in A$ if $a \sim b$ then $\sigma(a) \sim \sigma(b)$.*

Proof. Of course $\sim$ is reflexive (since S has 1) and symmetric. Now, let $a \sim b$ and $b \sim c$, with $a = s_1bs_2$, $b = t_1at_2$ and $b = r_1cr_2$, $c = w_1bw_2$, $s_1, s_2, t_1, t_2, r_1, r_2, w_1, w_2 \in S$. Then

$$a = s_1bs_2 = s_1(r_1cr_2)s_2 = (s_1r_1)c(r_2s_2)$$

and

$$c = w_1bw_2 = w_1(t_1at_2)w_2 = (w_1t_1)a(t_2w_2).$$

Since S is multiplicative, we have $s_1r_1, r_2s_2, w_1t_1, t_2w_2 \in S$, which implies $a \sim c$. Hence, $\sim = \sim_1$ is an equivalence relation. Items 1 and 2 are straightforward once S is multiplicative and σ-invariant. □

Lemma 2. *If S is standard, then $\sim = \sim_2$ is an equivalence relation and satisfies the following:*

1. *For all $a \in A$ and all $s \in S$, $\sigma(s)as \sim a$, $sa\sigma(s) \sim a$, and $abs \sim ab$, $sab \sim ab$.*
2. *For all $a, b \in A$ if $a \sim b$ then $\sigma(a) \sim \sigma(b)$.*

Proof. Reflexivity and symmetry follow immediately. Note that $s\sigma(s) \in Z(A)$ enable us to rewrite the definition of $\sim = \sim_2$ as follows:

$$a \sim_2 b \text{ iff } s_1as_2 = t_1bt_2 \text{ iff } \sigma(s_1)s_1as_2\sigma(t_2) = \sigma(s_1)t_1bt_2\sigma(t_2) \text{ iff } as_1' = t_1'b,$$

for $s_1', t_1' \in S$.

Consider $a \sim b$ and $b \sim c$, which means that there exist $s_1, t_1, s_2, t_2 \in S$. such that $as_1 = t_1b$ and $bs_2 = t_2c$. Scaling the previous equation on the right by s_2, and the latter, on the left by t_1, we conclude that $a(s_1s_2) = (t_1t_2)c$. Thus, $\sim$ is transitive; that is, an equivalence relation.

For Item 1, observe that $w(\sigma(s)as)w' = (w\sigma(s))a(sw')$, and $w(abs)w' = w(ab)sw'$ for all $s, w, w' \in S$. Item 2 follows by applying σ to both sides of $as = bt$. □

Back to Example 7, we observe that normal and convex (Marshall-coherent) subgroups coincide in this type of structure. In general, this is not the case, nor is their relationship with the relations above equal. Now, we treat these two cases.

Lemma 3. *Suppose that $x \cdot S = S \cdot x$ for each $x \in A$. Let $a, a' \in A$, and the following statements are equivalent:*

1. *$\exists s, t, s', t' \in S$ such that $sat = s'a't'$*
2. *$\exists u, u' \in S$ such that $ua = u'a'$*
3. *$\exists v, v' \in S$ such that $av = a'v'$*

That is, $a \sim_2 a'$ if, and only if, $a \sim_3 a'$. Furthermore, $\sim_S = \sim_2 = \sim_3$ is an equivalence relation.

Proof. (1) $\iff$ (2) $\iff$ (3) follows immediately from the hypothesis. Thus, $\sim_i = \sim_j$ for each pair (i, j), $i, j \in \{2, 3\}$. For simplicity, denote $\sim_S = \sim_i$, for each $i \in \{2, 3\}$.

The relation $\sim_S$ is an equivalence relation: suppose that $ua = u'a'$ and $r'a' = r''a''$ for $u, u', r', r'' \in S$. Observe the following:

$$ua = u'a' \implies r'ua = r'(u'a') \therefore r'ua = r'(a'v'), \text{ for some } v' \in S.$$

Also

$$(r'u)a = (r'a')v' = (r''a'')v' \implies \exists r'u = v, v'' \in S, \text{ such that } (r'u)a = v''a''.$$

It follows that $a \sim_S a', a' \sim_S a''$ implies $a \sim_S a''$. We already prove that $\sim_S$ is transitive. Reflexivity and symmetry follow from $1 \in S$ and the equivalence of the statements 1, 2, and 3. □

Lemma 3 is a powerful tool to deal with multiplication. It improves efficiency when managing equations, but mainly, it is a sufficient condition for the Marshall's quotient (8) being a multi-ring instead of a superring (Theorem 3).

We observe that, for a given Marshall's coherent subset S, convexity is the reflexivity property of $\sim_4$ by definition. Indeed, there is a suitable relationship between the upward-selected set of relations and Marshall's coherent convex subsets.

Lemma 4. *Suppose that S is convex. Let $a, a' \in A$, and the following statements are equivalent:*

1. $a \sim_2 a'$;
2. $a \sim_3 a'$;
3. $a \sim_4 a'$.

Furthermore, $\sim_S = \sim_2 = \sim_3 = \sim_4$ is an equivalence relation. Additionally, for every 1-convex S', $\sim_4 = \sim_3 \subseteq \sim_2$.

Proof. $1 \implies 2$: There are $s_1, s_2, t_1, t_2 \in S$ such that we have the following:

$$\begin{aligned} s_1as_2 = t_1a't_2 \implies \underbrace{s_1as_2(\sigma(a)}_{\in S} a) = t_1a' \underbrace{t_2(\sigma(a)a)}_{\in S} \qquad (S \text{ is Marshall convex}) \\ \therefore \underbrace{(a'\sigma(a'))s_3}_{\in S} a = (a' \underbrace{\sigma(a'))t_1a't_2'}_{\in S} \implies sa = a't. \end{aligned} \tag{1}$$

$2 \implies 3$: Suppose that $a \sim_3 a'$. Then, there exist $s_1, t_1 \in S$, satisfying the following:

$$as_1 = t_1a' \implies as_1\sigma(a') = t_1a'\sigma(a') \in S. \tag{2}$$

$3 \implies 1$: Finally, if $a \sim_4 a'$, then $\exists s, t_1 \in S$ such that we have the following:

$$as\sigma(a') = t_1 \implies a \underbrace{s\sigma(a')a'}_{\in S} = t_1a' \therefore \quad 1 \cdot as_2 = t_1a' \cdot 1. \tag{3}$$

To prove the final assertion, consider $a \sim_S b = a \sim_4 b$, for all $a, b \in A$. Since $1 \in S$ and S is convex, $a\sigma(a) \in S$ for all $a \in A$. Moreover, as long as S is σ-invariant, $as\sigma(b) \in S$ if, and only if, $b\sigma(s)\sigma(a) \in S$. It turns out that $a \sim_S b$ if, and only if, $b \sim_S a$. Thus, $\sim_S$ is reflexive and symmetric.

Finally, we prove the transitivity property. Put $a \sim_S b$ and $b \sim_S c$. Thus, by definition, it follows that

$$\exists r, s, s', s'' \in S; \begin{cases} as\sigma(b) = s' & \mathbf{1} \\ br\sigma(c) = s'' & \mathbf{2} \end{cases} \quad \overset{\mathbf{1 \cdot 2}}{\Longrightarrow} \; a \underbrace{s\sigma(b)br}_{\in S} \sigma(c) = s's'' \in S.$$

Remember that S is closed under multiplication and 1-convex. We have previously demonstrated that transitivity holds; thus, we conclude that $\sim_S$ is an equivalence relation. The final assertion follows straightforwardly. □

The following lemma summarizes and proves many results concerning the properties of Marshall-coherent subsets and the above relations.

Lemma 5. *Let S be a Marshall-coherent set in (A, σ). The following statements hold:*

1. *If $y \cdot S = S \cdot y$ for all $y \in A$ and S is 1-convex, then S is convex;*
2. *If S is convex and $x\sigma(x) \in Z(A)^\times$ for all non-zero divisors $x \in A$, then $x \cdot S = S \cdot x$ (S is normal);*
3. *If $S \subseteq A^\times$, and S is 1-convex, then $A_0 = A^\times$ denotes the set of non-zero divisors, i.e., every non-zero divisor has an inverse in A;*
4. *If S is standard, then $S \subseteq A^\times$;*
5. *If S is standard then $a \sim_1 a'$ if, and only if, $a \sim_2 a'$ if, and only if, $a \sim_3 a'$;*

Proof. 1. Let $x \in A$ be a non-zero divisor and $s \in S$. Thus, $\sigma(x)sx = z$ for some $z \in A$. Commuting s with x, it follows that $\sigma(x)xs' = y$ for a suitable $s' \in S$. Hence, 1-convexity and the closure of multiplication implies $y \in S$. Therefore, $\sigma(x)Sx \subseteq S$.

2. Let $x \in A^*$ be a non-zero divisor. For any $s_1 \in S$, $\sigma(x)s_1x = s_2$ for some s_2. Therefore $(x\sigma(x))s_1x = xs_2$, which implies $s_1x = xs_2(x\sigma(x))^{-1}$. Since $x\sigma(x) \in S^\times$ has an

inverse in S, $s_1 x = xs'_1$ for a suitable choice of s'_1. Hence, $S \cdot x \subseteq x \cdot S$. The reverse inclusive follows from symmetry.

3. By definition, $A^\times \subseteq A_0$. For the inverse inclusion, note that A_0 is a Marshall-coherent set and, let $y \in A_0$ and $1 \in S$. Thus, $\sigma(y)y = s' \neq 0$.

$$\sigma(y)y = s' \implies s'^{-1}\sigma(y)y = 1 \therefore y_l^{-1} = s'^{-1}\sigma(y) \text{ is a left inverse for } y. \tag{4}$$

The same argument shows that y has the right inverse y_r^{-1}. Note that $yy_l^{-1} = s_1 \in S$. Thus, $yy_l^{-1}y = s_1 y$ and implies $y = s_1 y$ for some $s_1 \in S$. Scaling by y_r^{-1} on both right sides of the equation, we obtain $1 = s_1$. Hence, $y^{-1} = y_l^{-1} = y_r^{-1}$.

4. By hypothesis, $s\sigma(s) \in Z(A)^\times(\cap S)$. Hence, $\exists x \in A$ such that $(x\sigma(s))s = 1$. Direct calculations confirm that this serves as a unique inverse on both sides.
5. The statement can be straightforwardly proven by scaling and division. □

Item 1 of Lemma 5 provides a sufficient condition for a normal subset to be a convex subset. On the other hand, Item 2 specifies a reciprocal condition; that is, each element, $x \in A_0$, has a norm lying in the center. In the classical theory of rings with involution (see, for instance, [16]), involution with traces $x + \sigma(x)$ and norms $x\sigma(x)$ lying in the center are called *standard*. This justifies the notation above. As we see in Section 5, standard subsets are typical examples.

For each $\sim \in \{\sim_1, \sim_2, \sim_3, \sim_4\}$, we denote an element in $A/\sim$ (whenever it exists) by $[a]$. We have well-defined rules, as follows:

$$[a] + [b] := \{[c] : c = s_1 a s_2 + t_1 b t_2 \text{ for some } s_1, s_2, t_1, t_2 \in S\} \text{ and,}$$
$$[a][b] := \{[c] : c = rasbt \text{ for some } r, s, t \in S\}.$$

Observe that the involutory structure can be defined in the very same way for superrings.

Definition 7. *A* ***superring with involution*** (A, σ) *is a superring that satisfies the (mutatis mutandis) axioms for multialgebras with involution.*

Theorem 2. *The structure* $(A/\sim_2, +, \cdot, [0], [1])$ *is a superring with involution provided by* $\sigma([a]) := [\sigma(a)]$*. If* S *is standard, then* $(A/\sim_1, +, \cdot, [0], [1])$ *is a superring with involution* $\sigma([a]) := [\sigma(a)]$*.*

Proof. We proceed with a very similar argument to the one used in Theorem 6. □

We define existing quotients for general Marshall-coherent subsets. In the sequel, we deal with normality and convexity.

Definition 8. *We define the superring* $(A/\sim, +, \cdot, [0], [1])$ *as the* ***Marshall's quotient*** *of* A *by* S*, and denote it by* $A/_m S := A/\sim$*.*

Whenever $\sim$ is chosen, we indicate the Marshall subset S by adding it to the index, i.e., writing $\sim_S$.

Theorem 3 has a central result in this section. Since the reverse image of the canonical morphism $j : R \longrightarrow A$ (see Definition 4) lifts Marshall-coherent subsets of A to R, the quotient is a multialgebra (with involution) likewise. The associated Marshall-coherent subset is $S_j = j^{-1}[S] \subseteq R$, where $S \subseteq A$ is Marshall-coherent in A and $[S] = [1]$ is the algebraic class of S under $\sim_S$.

Theorem 3. *Let* $S \subseteq A$ *be a Marshall-coherent subset of a multi-ring* A *satisfying one of the additional conditions below:*

1. *(Normal)* $xS = Sx$*, for all* $x \in A$*.*

2. *(Convex) For all $x \in A$, a nonzero divisor in A, $xS\sigma(x) \subseteq S$.*

If (A, σ) is an (R, τ)-multialgebra with involution, the set $S_j := j^{-1}[S] \subseteq R$ is a multiplicative submonoid of $(R, \cdot, 1)$. Moreover, $j_S : R/\sim_{S_j} \to A/\sim_S, [r] \mapsto [j(r)]$ defines an $R/\sim_{S_j}$-multialgebra structure over $A/\sim_S$, and $\sigma_S : A/\sim_S \to A/\sim_S, [a] \mapsto [\sigma(a)]$ is an involution over the $R/\sim_{S_j}$-multialgebra $(A/\sim_S, j_S)$. In both cases, $A/\sim_S$ is a multi-ring.

Proof. Once $j : (R, \tau) \longrightarrow (A, \sigma)$ is a homomorphism, if $s_1 r s_2 = t_1 r' t_2$ in R, then $j([r]) = [j(r)] = [j(r')] = j([r'])$. It is easy to check that S_j is a multiplicative submonoid of R and, due to S being Marshall-coherent, $\sigma(j(r)) = j(\tau(r)) \in S$ for all $r \in S_j$. Thus, $\tau(r) \in S_j$ whenever $r \in S_j$. We conclude that S_j is Marshall-coherent and, by Theorem 2, $R/\sim_{S_j}$ is a superring endowed with an involution $\tau([r]) := [\tau(r)]$.

For any two elements $[c], [d] \in [a] \cdot [b] \subseteq R/\sim_{S_j}$, $s_1 c = t_1 ab$ and $ds_2 = abt_2$ for some $s_1, t_1, s_2, t_2 \in S_j$, because R is commutative. Scaling these equations, we write $s_1 c t_2 = t_1 d s_2$, i.e., , $[a] \cdot [b] = \{[ab]\}$. Hence, $R/\sim_{S_j}$ is a multi-ring with involution.

Now, consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{j} & A & \xrightarrow{\psi} & A/\sim_S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_R} & & & \nearrow{\scriptstyle \exists! j_S} & & & \\ & & R/\sim_{S_j} & & & & & & \end{array} \tag{5}$$

(1) If $xS = Sx$, then $\sim = \sim_2$ can be read as $a \sim b$ if, and only if, $as = tb$ for some $s, t \in S$. Previous constructions (see Theorem 2) and demonstrations show that $(A/\sim_S)$ is a superring. Let $[c]$ and $[c']$ be elements in $[a] \cdot [b]$; thus, $c = abs$ and $c' = s'ab$ for some $s, s' \in S$. Scaling equations and comparing gives us $s'c = c's = s'abs$, which means that $c \sim c'$. Therefore, $[a] \cdot [b] = \{[ab]\}$ and $A/\sim$ is a multi-ring.

By the universal property of the quotient $R/\sim_{S_j}$, j_s is unique. Since all arrows are homomorphisms, $(A/\sim_S, j_S)$ is $R/\sim_{S_j}$-multialgebra. Furthermore, S is σ-invariant, which means $\sigma(aS) = \sigma(a)S$. Consequently, the induced anti-homomorphism $\sigma_S : A/\sim_S \longrightarrow A/\sim_S$ such that $\sigma([a]) = [\sigma(a)]$ is well-defined and an involution over $A/\sim_S$.

(2) Let $\sim = \sim_2$. In this case, Lemma 4 and the preceding case show that $A/\sim$ is a multi-ring. The proof is the same as before since Theorem 2 still holds. □

The above theorem provides us with two kinds of quotients lying in the class of multi-rings. One can wonder if the quotient can provide some information about the Marshall-coherent subset.

Proposition 1. *Let $A/\sim_S$ be a multi-ring, and S be a Marshall-coherent subset, such that $1 \in S$ and $\sim = \sim_2$. Then, $[1]$ is 1-convex if, and only if, $[1]$ is convex.*

Proof. (Sketch:) Note that $[S] = [1]$ is Marshall-coherent. The converse is immediate. To prove the reciprocal statement, use $[x] \cdot [s] \cdot [\sigma(x)] = [xs\sigma(x)] = [1]$ (since the quotient is a multi-ring, $\cdot$ is a usual operation) for all $s \in [1]$. We obtain $xs\sigma(x) \in [1]$ and, therefore $[1]$ is convex. □

According to the above results, some immediate examples follow below.

Example 8. *For a given (A, σ), a (R, σ')-multialgebra with involution, the following sets are Marshall-coherent:*

(a) The set of all non-zero divisors A_0;
(b) The set of all invertible elements $A^{\times}$;
(c) The set of all symmetric elements (in A_0) $Sym(A, \sigma) = \{a \in A_0 | \, a = \sigma(a)\}$;

(d) *If $x\sigma(x) \in Z(A)$ for all $x \in A$, then $A_0\sigma(A_0) = \{a\sigma(a) | a \in A_0\}$ is Marshall-coherent and convex.*

In the next section, we will provide more examples minutely. For now, we treat another kind of operation in the quotient. For $a, a' \in A$, let $a \sim_S a'$ if, and only if, there exist $s, t, s', t' \in S$ such that $sat = s'a't'$. This can be replaced in terms of the equivalent statements in 3 or 4, whether $x \cdot S = S \cdot x$ or S is convex, respectively. Hence, $\sim_S$ is an equivalence relation. Moreover, each $[a]$ is invariant under S action, $[a] = [sa]$ for all $s \in S$.

In $A/\sim_S$ define $[a] + [b] := \{[c] : \exists r_i, s_i, t_i \in S, r_0cr_1 \in s_0as_1 + t_0bt_1\}$, $-[a] := [-a]$ and $[a] \cdot [b] := [ab]$.

Theorem 4. *Suppose that $x \cdot S = S \cdot x$. Then, we have the following:*

1. *$A/\sim_S$ is a (non-commutative) multi-ring.*
2. *If A is a hyperring, then $A/\sim_S$ is a hyperring. In particular, if A is a ring, then $A/\sim_S$ is a hyperring.*
3. *It holds the universal property of Marshall's quotient for homomorphisms $f : A \to M$ and anti-homomorphisms (= homomorphism $f : A \to M^{op}$) such that $f[S] = \{1\}$.*

Proof. To demonstrate 1, we note that $+$, $\cdot$, and $-$ are well-defined as multi-group operations, and $0 = [0] = \{0\}$ is the null element because A is a multi-ring.

Suppose that $[c] \in [a] + [b]$. Thus, there exists $r, s, t \in S$, satisfying $rc \in sa + tb$ in A. Therefore, $sa \in rc + t(-b)$ (in A). Similarly, $tb \in s(-a) + rc$. Consequently, $[a] \in [c] - [b]$ and $[b] \in -[a] + [c]$.

Let $[b] \in [a] + [0]$. By definition, there exists $r \in S$ such that $rb \in sa + t0$ for some $s, t \in S$. However, it implies $[a] = [b]$. The reciprocal is obvious.

If $[x] \in [a] + [b]$ and $[t] \in [x] + [c]$, then $vt \in wx + zc$ and $r'wx \in s'a + p'b$ for $r', s', p', v, w, z \in S$. Afterward,

$$\begin{aligned} vt \in wx + zc &\implies r'vt \in r'wx + r'zc \\ \exists r'wx(r'wx \in s'a + p'b \wedge r'vt \in r'wx + r'zc) &\implies \exists y(y \in p'b + r'zc \wedge r'vt \in s'a + y) \end{aligned}$$

The last implication means $\exists[y]([y] \in [b] + [c] \wedge [t] \in [a] + [y])$. Once A is a multi-ring, $[c] \in [a] + [b]$ if, and only if, $[c] \in [b] + [a]$ follows.

We already proved that $(A/\sim_S, +, -, 0)$ is a multi-group. Note that there exists $1 = [1] = S \in A/\sim_S$ such that $[a] \cdot [1] = [a]$ for all $[a] \in A/\sim_S$. Thus, $(A/\sim_S, \cdot, 1)$ is a monoid. Moreover, $[a] \cdot 0 = 0$. Finally, let $[c] \in [a] + [b]$ and $pd \in [d] \in A/\sim_S$. By definition, exists $r, s, t \in S$ such that $rc \in sa + tb$. Since A is a multi-ring, $rcpd \in sapd + tbpd$. Using the 'normality property' of S, we rewrite it as follows:

$$r'cd \in s'ad + t'bd \therefore [c][d] \in [a][d] + [b][d].$$

Similarly, $[d][c] \in [d][a] + [d][b]$ holds. It follows that $A/\sim_S$ is a multi-ring. For the second assertion, suppose that A is a hyperring. Let $[e] \in [a][d] + [b][d]$. Thus,

$$\begin{aligned} \exists s, r, t \in S,\ se \in rad + tbd &\implies se \in (ra + tb)d && (A \text{ is hyperring}) \\ &\implies [e] \in ([a] + [b])[d] && (\text{by definition of } +). \end{aligned}$$

Therefore, $[a][d] + [b][d] = ([a] + [b])[d]$. By symmetry, $[d][a] + [d][b] = [d]([a] + [b])$ also follows.

To demonstrate the third statement, consider $f : A \longrightarrow M$ a homomorphism such that $f([S]) = 1$. Let $a \in A$ and $s \in S$. Thus, $f(sa) = f(s)f(a) = f(a)$. Define the homomorphism $\bar{f} : A/\sim_S \longrightarrow M$ with $\bar{f}([a]) = f(a)$. Hence, $\bar{f}$ is well-defined, and $f = \bar{f} \circ \psi$, with $\psi(a) = [a]$ the canonical projection. It is immediate that another homomorphism $\bar{g} : A/\sim_S \longrightarrow M$ satisfying $f = \bar{g} \circ \psi$ must coincide with $\bar{f}$. □

Remark 1. *Theorem 4 is valid if S is convex. Since both conditions normality and convexity imply $\sim_2 = \sim_3$, we are capable of proving that the distributive laws hold and the entire rest of the proof follows as above.*

The next theorem distinguishes Marshall-coherent subsets that lie in the center $Z(A)$ from an ordinary one.

Theorem 5. *Let A be a multialgebra with involution and $S \subseteq A$ be a Marshall-coherent subset such that $S \subseteq Z(A)$ (thus, in particular, $xS = Sx$, for all $x \in A$). Then, $A/_m S$ is a (non-commutative) hyperring with induced involution.*

Proof. From previous considerations and Theorem 2, we prove that $A/_m S$ is a multi-ring instead of a superring, and the hyperring property still holds.

In fact, if $[c] \in [a][b]$, then $cr = asbt$ for some $r, s, t \in S \subseteq Z(A)$, which means $cr = (ab)(st)$ and $c \sim ab$. Then, $[a][b] = \{[ab]\}$, proving that $A/_m S$ is a multi-ring.

Now, let $[y] \in [c][a] + [c][b]$. Then, $[y] = [d_1] + [d_2]$ for some $[d_1] \in [c][a]$, $[d_2] \in [c][b]$, providing the following equations:

$$\begin{aligned} y &= r_1 d_1 s_1 + r_2 d_2 s_2, \\ d_1 &= t_1 c v_1 a w_1 \text{ and} \\ d_2 &= t_2 c v_2 a w_2 \end{aligned}$$

for some $r_1, r_2, s_1, s_2, t_1, t_2, v_1, v_2, w_1, w_2 \in S$. Then, we have the following:

$$\begin{aligned} y &= r_1 d_1 s_1 + r_2 d_2 s_2 \\ &= r_1[t_1 c v_1 a w_1] s_1 + r_2[t_2 c v_2 b w_2] s_2 \\ &= c(r_1 t_1 v_1) a (w_1 s_1) + c(r_2 t_2 v_2) b (w_2 s_2) \\ &= c[(r_1 t_1 v_1) a (w_1 s_1) + (r_2 t_2 v_2) b (w_2 s_2)] \end{aligned}$$

implying that $[y] \in [c]([a] + [b])$. The same reasoning provides $[ac] + [bc] \subseteq ([a] + [b])[c]$. □

5. Applications

This section focuses on results surrounding particular examples. We verify some quotients associated with typical multi-structures, a few of them presented in Section 2. Throughout the subsections below, we deal with technical results and interpret elements in the Marshall quotient as classes of isometric elements.

5.1. Orthogonal

Let R be a commutative ring and A be an R-algebra with involution σ. We denote the following:

$$\text{Orth}(A) := \{a \in A : a\sigma(a) = \sigma(a)a = 1\}.$$

Once we prove that $\text{Orth}(A)$ is a Marshall-coherent subset, then, by definition, the standard property also holds, as follows:

Lemma 6. *The set $\text{Orth}(A)$ is non-empty and if $a, b \in \text{Orth}(A)$ then $\sigma(a), ab \in \text{Orth}(A)$.*

Proof. The set $\text{Orth}(A)$ is non-empty because $1 \in \text{Orth}(A)$. For the rest, note that $\sigma(a)\sigma(\sigma(a)) = \sigma(a)a$ and $(ab)\sigma(ab) = ab[\sigma(b)\sigma(a)]$ for all $a, b \in A$. If $a, b \in \text{Orth}(A)$, these imply $\sigma(a)a = a\sigma(a) = 1$ and

$$(ab)\sigma(ab) = ab[\sigma(b)\sigma(a)] = a[b\sigma(b)]\sigma(a) = a\sigma(a) = 1.$$

□

Now let $a, b \in A$. We define

$$a \sim b \text{ if, and only if, } as = tb \text{ for some } b, t \in \mathrm{Orth}(A).$$

Note that $a \sim b$ if, and only if, $a = sbt$ for some $s, t \in \mathrm{Orth}(A)$, because $S = \mathrm{Orth}(A)$ is a Marshall-coherent standard subset.

Theorem 6. *The structure $(A/\sim, +, \cdot, [0], [1])$ is a superring with involution $\sigma([a]) := [\sigma(a)]$.*

Proof. Note that $a \sim 0$ if, and only if, $a = 0$. Moreover, from the very definitions of the sum and the product, we have for all $a, b \in A$,

$$\begin{aligned}
&[a] + [0] = [0] + [a] = \{[a]\},\ [a][1] = [1][a] = \{[a]\}, \\
&[a] + [b] = [b] + [a], \\
&\sigma([a][b]) = [\sigma(b)][\sigma(a)], \\
&[0] \in [a] + [b] \iff [b] = -[a].
\end{aligned}$$

Now, let $a, b, c \in A$ and $[e] \in ([a] + [b]) + [c]$. As a result, $[e]$ also belongs to $[x] + [c]$ for some $[x] \in [a] + [b]$. Consequently, we can express e as $s_1 x s_2 + t_1 c t_2$ and x as $v_1 a v_2 + w_1 b w_2$, where $s_1, s_2, t_1, t_2, v_1, v_2, w_1, w_2 \in \mathrm{Orth}(A)$. Then, we have the following:

$$\begin{aligned}
e &= s_1 x s_2 + t_1 c t_2 \\
&= s_1(v_1 a v_2 + w_1 b w_2) s_2 + t_1 c t_2 \\
&= (s_1 v_1) a (v_2 s_2) + (s_1 w_1) b (w_2 s_2) + t_1 c t_2 \\
&= (s_1 v_1) a (v_2 s_2) + [(s_1 w_1) b (w_2 s_2) + t_1 c t_2]
\end{aligned}$$

Let $y = (s_1 w_1) b (w_2 s_2) + t_1 c t_2$. Then, $[e] \in [a] + [y]$ with $[y] \in [b] + [c]$, implying that $[e] \in [a] + ([b] + [c])$. The same reasoning provides $[a]([b][c]) = ([a][b])[c]$.

Finally, let $[x] \in [c]([a] + [b])$. Therefore, $[x] \in [c][d]$ for some $[d] \in [a] + [b]$. These provide equations $x = rcsdt$ and $d = s_1 a s_2 + t_1 b t_2$. Thus, we have the following:

$$\begin{aligned}
x &= rcsdt \\
&= rcs[s_1 a s_2 + t_1 b t_2] t \\
&= rcss_1 a s_2 t + rcst_1 b t_2 t \\
&= rc(ss_1) a (s_2 t) + rc(st_1) b (t_2 t)
\end{aligned}$$

with $r, ss_1, s_2 t, st_1, t_2 t \in \mathrm{Orth}(A)$, concluding that $[x] \in [c][a] + [c][b]$. Similarly, we conclude that $([a] + [b])[c] \subseteq [a][c] + [b][c]$. □

Observe that S is not necessarily convex, and neither satisfies $xS = Sx$ (see Theorem 2). Thus, $A/\sim$ may not be a multi-ring.

Definition 9. *We define the superring $(A/\sim, +, \cdot, [0], [1])$ as the **orthogonal fragment** of A, and denote by $A/_m\mathrm{Orth}(A) := A/\sim$.*

Theorem 7. *If $\mathrm{Orth}(A) \subseteq Z(A)$, then $A/_m\mathrm{Orth}(A)$ is a (non-commutative) hyperring.*

Proof. This is a particular case of Theorem 5. □

Theorem 8. *Let F be a field and $A = M_2(F)$. Then $A/_m\mathrm{Orth}(A)$ consists of rotation 2×2 matrices over F.*

Proof. Note that $a \in \text{Orth}(A)$ if, and only if, $aa^t = id_2$, with $\sigma(a) = a^t$ the transpose matrix of $a = (a_{ij})_{2\times 2}$. Applying the definition of matrix product, we have to solve the following system:

$$\begin{cases} a_{11}^2 + a_{12}^2 = 1 \\ a_{21}^2 + a_{22}^2 = 1 \\ a_{11}a_{21} + a_{12}a_{22} = 0 \\ det(a)^2 = 1 \end{cases} .$$

We conclude the following:

$$\text{Orth}(A) = \left\{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \mid x^2 + y^2 = 1,\ x, y \in F \right\} \cup \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mid x^2 + y^2 = 1,\ x, y \in F \right\}. \tag{6}$$

□

If $F = \mathbb{R}$, in (6), the second subset (with the positive determinant equal to 1) is the set of orthonormal matrices or the set of linear transformations in $\mathbb{R}^2$ that are rotations by some angle $\theta \in [0, 2\pi)$ with $x = cos(\theta)$ and $y = sin(\theta)$. Moreover, consider the inner product

$$\langle a, b \rangle = \sum_{i,j=1}^{2} a_{ij} b_{ij}, \text{ for } a, b \in A.$$

One may verify that the actions of elements in $\text{Orth}(A)$ function as a set of isometries. By solving a system of equations very similar to the one discussed above, it is possible to demonstrate that these actions form a subset of isometries. The associated matrix, denoted as $T = (t_{ij})$, has a determinant different from $\pm(t_{11} - t_{12})$. Thus, this quotient describes the behavior concerning certain kinds of isometry classes considering the underlined inner product.

5.2. Quaternions over Real Closed Fields

Now, we explore the diversity of quotients in quaternions. Although this includes a lot of calculations, it provides quick verification of independence regarding normal and convex quotients.

Example 9. *Let R be a real closed field and*

$$H = R\{x, y\}/(x^2 + 1, y^2 + 1, xy + yx)$$

the corresponding quaternion algebra $(1, 1)_R$; it is an R-division algebra of dimension 4. Put $S = R_{>0} \cdot 1 \subseteq H$. Note that S satisfies the second condition of the Theorem 3 (it is a convex set) in the previous section and $S = \{\sigma(a) \cdot a : a \in H \setminus \{0\}\}$.

Then, $H/_m S \cong \mathbb{S}^3$ is a monoid. And, in this quotient, $\overline{x \cdot \sigma(x) \cdot x} = \overline{x}$, as $\overline{x \cdot \sigma(x)} = 1$.

We observe that, in this case, S is also standard, "normal", and 1-convex (see Lemma 5).

Example 10. *Let $\mathbb{H}$ be the quaternions real algebra endowed with the standard involution $\sigma(a) = \bar{a}$, for all $a \in \mathbb{H}$. Set $S = \mathbb{R} \setminus \{0\}$ and define $a \sim b$ iff $a = \sigma(x) b y$ for some $x, y \in S$. Thus, $[0] = \{0\}$, and for a nonzero element a, $[a]$ is the line determined by the origin and the quaternion a(without $\{0\}$), i.e., $\mathbb{H}/_m S \cong RP^3$.*

Once $S \subseteq Z(\mathbb{H})$ (S has the first "normality" property of the Theorem 3), it is easy to check that $S = [1] = [-1]$, and $[\pm a] = Sa$, and for a, a pure quaternion as well. If $a = a_0 + a_1 i + a_2 j + a_3 k$ and $b = b_0 + b_1 i + b_2 j + b_3 k$ are quaternion numbers, then we have the following:

$$[a] + [b] = \bigcup [x_0 + x_1 i + x_2 j + x_3 k], \tag{7}$$

for $x_i \in \mathbb{R}$, $x_i \in S$, or $x_i = 0$, depending on $a_i, b_i \neq 0$, or $a_i \neq 0$ and $b_i = 0$ (and vice-versa), or both $a_i = b_i = 0$, respectively, for each $i \in \{0,1,2,3\}$. Hence, $[a] + [b]$ is the plane determined by $[a]$ and $[b]$, containing (or not containing) the origin.

Example 11. *Consider the orthogonal fragment* $Orth(\mathbb{H})$*. Let* $S = \mathbb{S}^3 \subseteq \mathbb{H}$*, representing the sphere of radius with* 1 *centered at the origin.*

Clearly, $1 \in S$, and S is a multiplicative set, satisfying $x^{-1} \in S$ whenever $x \in S$. Once $|x| = x\sigma(x)$, it is immediate that S is σ-invariant. It remains to verify that the sphere qualifies as "a normal set" in $\mathbb{H}$ (item 1 of Theorem 3) and, thus, the quotient is a multi-ring. In fact, let $a \in \mathbb{H}$ and $x \in S$; given the norm is multiplicative, we have the following:

$$\begin{aligned} |ax| = |a| &\implies a\sigma(a)x\sigma(x) = \sigma(a)a \implies a\sigma(a)x = \sigma(x)^{-1}\sigma(a)a \\ &\implies \sigma(a)ax = \sigma(x)^{-1}\sigma(a)a \implies ax = (\sigma(a)^{-1}\sigma(x)^{-1}\sigma(a))a. \end{aligned} \quad (8)$$

Yet, we have the following: $|\sigma(a)^{-1}\sigma(x)^{-1}\sigma(a)| = |\sigma(a)^{-1}||\sigma(x)^{-1}||\sigma(a)| = |\sigma(a)^{-1}||\sigma(a)| = 1$; therefore,

$$y = \sigma(a)^{-1}\sigma(x)^{-1}\sigma(a) \in S.$$

We conclude that $ax = ya$ for some $y \in S$, i.e., $aS \subseteq Sa$. The reverse inclusion is followed by symmetry. Moreover, in a general division algebra with standard involution, this property holds since $S = \mathrm{Orth}(\mathbb{H})$.

Let $a \sim b$ iff $a = \sigma(x)by$, with $x, y \in S$. Hence, $a \sim b$ iff $|a| = |b|$. It is obvious that $[0] = \{0\}$ and $[1] = \mathbb{S}^3 = S$. The elements $[a]$ are spheres centered at the origin with radius $\sqrt{|a|}$. In fact, $\sqrt{|a|} = a \cdot \frac{\sigma(a)}{\sqrt{|a|}}$, with $x = \frac{\sigma(a)}{\sqrt{|a|}} \in S$. Therefore, $\sqrt{|a|} \sim a$. For $a \in [b]$, $[a] + [b]$ forms a filled sphere with radius $2\sqrt{|a|}$. If $|a| > |b|$, both triangular inequalities $|a + b| \leq |a| + |b|$ and $||a| - |b|| \leq |a - b|$ indicate that $[a] + [b]$ is the 'hollow' surface defined by two spheres with coincident centers at the origin and radii $\sqrt{|a|} + \sqrt{|b|}$ and $\sqrt{|a|} - \sqrt{|b|}$. Moreover, $\mathbb{H}/_m S \cong \mathbb{R}_+$, as a multimonoid with multi-addition, satisfies the following:

$$[a] + [b] = \begin{cases} [a - b, a + b] \text{ if } a \geq b; \\ [b - a, a + b] \text{ if } b \geq a. \end{cases}$$

Thus, this is the triangle hyperfield Example 4. In the last example, S does not satisfy the convexity property. At the same time, Example 9 shows Marshall-coherent sets satisfying many properties simultaneously. These examples illustrate that the definitions provided in the previous section encapsulate elements of different types of structures and demonstrate the independence between the statements outlined in Theorem 3.

6. Conclusions

We have extended the concept of the (commutative) multi-ring, as presented in Marshall's seminal paper [1], to the setting of (non-commutative) and involutive multi-rings (Definition 4). Additionally, we have expanded the concept of Marshall's coherent subset to this new setting (Definition 5) and introduced and studied several equivalence relations related to this notion (Definition 6; Lemmas 4 and 5). Furthermore, we have broadened the concept of Marshall's quotient (Definition 8; Theorems 2 and 3) to accommodate this framework, which serves as a key technical tool for constructing many interesting examples of multialgebras with involution. These examples are derived from standard algebraic structures such as orthogonal groups and quaternion algebras, as thoroughly developed in Section 5.

Thus, we have established the groundwork for extending real algebra and real algebraic geometry into the non-commutative and involutive settings, broadening the abstract methodologies utilized in the space of signs [13], abstract real spectra [14], real semigroups [15], and real reduced multi-rings [1]. Notably, the theory presented here lends itself to model-theoretic methods since every n-multi-operation corresponds to a $n+1$-relation, satisfying an $\forall\exists$ axiom. This is an area we intend to explore in future work. Moreover, the continued development of the theory on non-commutative multialgebras with involution should lay a robust foundation for establishing an abstract theory of Hermitian forms ([16]), similar to how the theory of special groups ([22]) serves as an abstract theory of quadratic forms.

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