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ORIGINAL RESEARCH

Measuring and Mismeasuring Income Polarization

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Abstract
This paper addresses and clarifies key misconceptions surrounding income polarization indices commonly observed in the literature. It focuses on the measurement of the polarization of income distributions, emphasizing the crucial distinction between bipolarization and multipolarization measures. The analysis asserts that valid polarization metrics must satisfy the principle of scale invariance. Specifically, in a dichotomized income distribution (relatively poor vs. relatively rich), a valid bipolarization measure should consistently increase under regressive transfers from poorer to richer individuals or progressive transfers within either income group. The study identifies cases where bipolarization measures that fail to satisfy these conditions have been applied in previous research, leading to erroneous conclusions. Using artificial numerical examples, the paper demonstrates the behavior of different polarization measures under various scenarios, illustrating their limitations. The study concludes by highlighting errors in previously published works and offering a critical reassessment of established methodologies.

Keywords Polarization · Income distribution · Scale-invariance · Inequality

JEL Classification O15 · D31

1 Introduction

In the literature on income distribution, terms like “polarization” and “concentration” are often interchangeably used with “inequality.” Higher inequality is commonly linked to an increased concentration of income at the upper level of the distribution and a widened gap between affluent and disadvantaged segments. However, to avoid confusion and make statistical analyses clearer, it is prudent to reserve “concentration” and “polarization” for specific concepts within the broader context of the analysis of distributions. Ideally, the term “concentration” would

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signify instances where a small number of units hold a substantial proportion of total value, as exemplified by the Hirshman-Herfindahl index, frequently applied in studies of industrial concentration. Conversely, in the context of "polarization," a clear distinction between bipolarization and multipolarization is crucial. The terminology of "two poles" is common in geographic and physical contexts, such as North and South, or positive and negative. In economic development studies, the reference to "industrialization poles" could imply one, two, or multiple focal points.

The pioneering notion of bipolarization in income distribution was introduced by Foster and Wolfson (1992) working paper and then disseminated by Wolfson's journal articles in 1994 and 1997 as a means to detect the decline of the middle-income class. Esteban and Ray (1994) later offered a more inclusive measure to account for multiple peaks in a distribution, which was subsequently adapted by Duclos, Esteban, and Ray (2004) and Esteban, Gradín, and Ray (2007). Several other bipolarization measures, including Handcock and Morris (1999), Wang and Tsui (2000), Milanovic (2000), Zhang and Kanbur (2001), and Chakravarty et al. (2007), have been proposed, but some studies applying these measures often encounter challenges related to ambiguity and inaccuracies. Researchers have occasionally proposed measures that violate fundamental axioms, leading to misinterpretations, as outlined in Sects. 2, 3, 4 and 5.

The 1990s articles by Foster and Wolfson (1992), and Wolfson (1994, 1997) established that a bipolarization measure should increase in response to two types of distributional changes: "increased spread" and "increased bipolarity." This assertion has garnered support from subsequent studies by Chakravarty (2009) and Deutsch et al. (2013). Scale invariance further emerges as an important condition for bipolarization measures. While some polarization measures satisfy these three criteria, many do not.

For instance, Handcock and Morris (1999) introduced the "Median Relative Polarization" (MRP) measure, yet its practical application is misleading, as it solely captures inequality between the two halves of the distribution delimited by the median, rendering it an inadequate polarization measure.

Given the confusion surrounding the concept and application of polarization measures in income distributions, this paper aims to elucidate pertinent issues. We undertake a comprehensive discussion of various polarization measures, highlighting their strengths, limitations, and the need to abandon certain measures. Particular attention is devoted to distinguishing between bipolarization and multipolarization measures, exposing their improper application.

This study seeks to rectify misconceptions and contribute to the literature by providing clarity on polarization measures, aiding researchers in comprehending, distinguishing, and proficiently applying polarization concepts in income distributions.

The upcoming section delves into the concept of bipolarization and examines various metrics associated with it. Section 3 provides a concise overview and critical analysis of the Median Relative Polarization measure introduced by Handcock and Morris (1999). Moving to Sect. 4, we address the challenges associated with employing the multipolarization measure originally proposed by Esteban and Ray (1994) along with its subsequent modifications. We present an illustrative example using real-world data in the subsequent section. Finally, the last section serves as a comprehensive summary of our discussion.

2 Bipolarization Measures

The title of the pioneering work by Foster and Wolfson (1992) shows their concern with a critical social phenomenon, which is "the decline of the middle class". To quantify this phenomenon, they considered the division of an income distribution into two parts delimited by the median and developed a bipolarization measure that increases with both the rise in inequality between the two parts and the fall in inequality within the halves.

Consider for simplicity the analyses of an income distribution, although any cardinal variable could be considered. Let μ and m be the mean income and median income, respectively. G is the Gini index of the distribution and G_B is the Gini index of the inequality between the two halves of the distribution delimited by m. Then, the measure of bipolarization proposed by Foster and Wolfson (1992) is:

$$P_W = \left(2G_B - G\right)\frac{\mu}{m} \tag{1}$$

Defining $L(p)$ as the ordinate of the Lorenz curve at the percentile of order p, the respective discrepancy of the Lorenz curve from the line of perfect equality is $p - L(p)$ and the discrepancy at the distribution median is $D_{50} = 0.5 - L(0.5)$. Also, $G_B = D_{50}$ and letting G_W be the measure of inequality within the two halves of the distribution, the general Gini index is $G = G_B + G_W$. Then,

$$P_W = \left(2D_{50} - G\right)\frac{\mu}{m} = \left(G_B - G_W\right)\frac{\mu}{m} \tag{2}$$

Although Foster and Wolfson (1992) call it a polarization measure, it is clearly a bipolarization measure, designed to capture any reduction in the middle class in an economy. They point out that a measure of bipolarization should increase with two types of changes in distribution: "increased spread" and "increased bipolarity". The "increased spread" condition means that any regressive transfer of income across the middle (from a person below the median to one above the median) increases spread and bipolarity. Any inequality measure that obeys the Pigou-Dalton condition also increases with a regressive transfer, showing that sensitivity to "increased spread" does not distinguish measures of bipolarity from measures of inequality. Increased bipolarity occurs when progressive income transfers are made between two people belonging to the first half and/or between two people belonging to the second half. In this case the inequality measures that obey the Pigou-Dalton condition decrease, but the P_W measure increases (given the negative sign of G_W in expression (2)). The subsequent literature tended to enshrine these two conditions: a good measure of bipolarity should grow with "increased spread" and with "increased bipolarity". According to Deutsch et al. (2013, p. 5):

> The concept of bi-polarization stresses in fact two notions. The first one, "increasing spread", implies that moving from the middle position (the median) to the tails of the income distribution makes an income distribution more polarized. More precisely a rank preserving increment in incomes above the median or a rank preserving reduction in income below the median will widen the distribution, that is, extend the distance between the two groups (those above and below the median) and hence increase the degree of bi-polarization (the rich become richer and the poor poorer). The second concept, "increased bipolarity", refers to the case where the incomes below the median or those above the median get closer to each other. What is happening here is a "bunching" of the two groups in the sense that the gaps between the

incomes below the median (or those above the median) have been reduced and such a "bunching" is assumed to increase bi-polarization.

It is worth noting that when presenting the relative median deviation T in Foster and Wolfson (1992 and 2010, pp. 263–264) as:

$$T = \frac{1}{\mu}\left(\mu_U - \mu_L\right) \tag{3}$$

where μ_U is the mean income of those above the median, μ_L is the mean income of those below the median and μ is the overall income, they claim that

$$T = 2D_{50} = 1 - 2L(0.5) \tag{4}$$

However, indeed, T defined in (3) is equal to $4D_{50}$. As a way to obtain the correct expressions containing T in Foster and Wolfson (1992 and 2010), it is necessary to redefine T as half of expression (3).

An important aspect of Foster and Wolfson's (1992) is their proof that P_W is equal to the area under the polarization curve, whose construction is well explained in their paper. This curve is obtained using the absolute values of the differences between each quantile and the median, standardized by the median. Considering differences standardized by the median guarantees that both the polarization curve and the P_W measure are scale invariant. As the Gini index is scale-invariant, expression (1) also shows that P_W satisfies this condition.

Knowing that $G = {}^{\Delta}\!/_{2\mu}$, where Δ is the mean difference of the distribution, and letting Δ_B, Δ_L and Δ_U be the mean differences between the two halves, within the lower half and within the upper half, respectively, it follows that

$$G_B = \frac{\Delta_B}{2\mu} \text{and } G_W = \frac{\Delta_L + \Delta_U}{8\mu} \tag{5}$$

It can then be shown that

$$P_W = \left(G_B - G_W\right)\frac{\mu}{m} = \left(\Delta_B - \frac{\Delta_L + \Delta_U}{4}\right)\frac{1}{2m} \tag{6}$$

Adding a positive constant to all incomes does not affect the mean differences but causes an increase in the median. Expression (6) therefore shows that such a change in incomes causes a reduction in the Wolfson index.

A measure of polarization must be independent of the monetary unit. This means that when all incomes are multiplied by a constant, the measure of polarization will remain the same. This is a property of all widely used measures of inequality, such as the Gini index, Theil's T and L measures (and the whole family of generalized entropy indices), the percentage of income appropriated by the richest ten percent of the population, and the Palma index.[1] It makes no sense to imagine that the inequality or polarization of an income distribution changes with a change in the monetary unit or with economic growth that manifests as a proportional increase in all incomes.

[1] Scale invariance is one of the assumptions used by Shorrocks (1980) in the derivation of the generalized entropy class of inequality measures. "Mean independence" and "income homogeneity" are other expressions used to refer to this property.

There is a vast literature on absolute measures of inequality but, in part, the novelty lies only in the terminology, as these are generally measures of *dispersion*.[2] It is clear that adding a positive constant to all incomes does not change the dispersion of incomes, but it does reduce the inequality of the distribution. Also, multiplying all incomes by a constant greater than one does not change the inequality, but it does increase the dispersion.

In economic development studies, it is crucial to differentiate between the changes in income inequality and pure economic growth or the proportional increase in income for all individuals. For this purpose, measures of inequality must be scale invariant. Deaton (2013) stresses the importance of comparing relative, not absolute changes in income, showing that "equal percentage differences in income produce equal absolute shifts in life evaluation" (p. 21). To reduce poverty, public policies are either more oriented towards reducing inequality or aimed at favoring economic growth. Measures of absolute inequality are not adequate for this purpose and are usually not even considered in empirical studies, being generally confined to the more theoretical literature. Unfortunately, as far as the bipolarization of income distribution is concerned, the procedures for measuring it are not yet well established and the use of "absolute measures" has led to misinterpretations, as we will show below. It is advisable to consider as good measures of inequality or of bipolarization only those that are scale invariant.

The measure P_W has been widely used and has the three desired properties: it grows with increased spread and with increased bipolarity and it is scale invariant. An extended list of desired axioms for a measure of bipolarity is presented in the Appendix.

The second new family of measures proposed by Wang and Tsui (2000) also satisfies all three conditions. However, the first new family of measures proposed in the same paper is not scale invariant and by this criterion should be rejected. For the same reason, the proposed "Absolute indices of polarization" by Chakravarty et al. (2007) should be rejected.

Another measure proposed by Milanovic's (2000) does not satisfy the "increased bipolarity" condition. It can be verified that progressive transfers within one of the halves of the distribution do not change the value of the measure.

Also, Zhang and Kanbur (2001) proposed to measure polarization by the ratio of inequality between groups and within-group inequality. If this is done using an inequality measure that is scale invariant and obeys the Pigou-Dalton condition, the polarization measure obtained will also be scale invariant and will be sensitive to increased bipolarity. Considering the division of the distribution in two halves, the Zhang and Kanbur measure is:

$$P_{ZK} = \frac{G_B}{G_W} = 4\frac{\Delta_B}{\Delta_L + \Delta_U} \tag{7}$$

Note that this measure is not affected when adding a constant to all incomes (it is shift invariant).

Silber et al. (2007) point out that measure (7) is unbounded and propose the bipolarization measure,

[2] See, for example, Kolm (1976a, b), who states that "it is no less legitimate to attach the inequality between two incomes to their differences than to their ratio. One view must not be judged from the other's prejudice" (p. 419). In our view, the set {101, 102} shows much less inequality than the set {1, 2}, although both show the same level of dispersion.

$$P_G = \frac{G_B - G_W}{G} = \left(\Delta_B - \frac{\Delta_L + \Delta_U}{4}\right)\frac{1}{\Delta} \tag{8}$$

showing that

$$P_G = \frac{(P_{ZK} - 1)}{(P_{ZK} + 1)} \tag{9}$$

This measure is also shift invariant. However, P_{ZK} and P_G do not always grow with increased spread. For a uniform distribution with mean $a \geq \omega > 0$ and density function $f(x) = \frac{1}{2\omega}$ for $a - \omega \leq x \leq a + \omega$ and $f(x) = 0$ for $x < a - \omega$ and $x > a + \omega$, it is possible to show that $G = {}^{\omega}/_{3a}$, , $G_B = D_{50} = {}^{\omega}/_{4a}$ and $G_W = {}^{\omega}/_{12a}$. Then, for such a uniform distribution, $P_{ZK} = 3$ and $P_G = 0.5$, regardless of the values of the parameters a and ω.

Let us consider an initial situation with a substantially larger than ω. By keeping the value of a fixed and increasing ω yields increased spread, but the values of P_{ZK} and P_G do not change. Note that the values of $P_W = {}^{\omega}/_{6a}$ increases. As measures of bipolarization, P_{ZK} and P_G require a different interpretation from the one stablished by Foster and Wolfson (1992), that is, a bipolarization that does not necessarily increase with increased spread. A simple numerical example shows the possibility that the values of P_{ZK} and P_G decrease with increased spread. Consider 8 people with incomes 6, 6, 8, 8, 10, 10, 12 e 12. Then, $G = 5/36$, $G_B = D_{50} = 1/9$, $P_W = 1/12$, $P_{ZK} = 4$ and $P_G = 3/5 = 0.6$. After a regressive transfer of 4 units from a person with income 6 to a person with income 12 and another regressive transfer of 4 units from a person with income 8 to a person with income 10, the 8 incomes become 2, 4, 6, 8, 10, 12, 14 and 16. In this altered distribution, $G = 7/24$, $G_B = D_{50} = 2/9$, $P_W = 11/72$, $P_{ZK} = 3.2$ and $P_G = 11/21 = 0.5238$. Note that the two regressive transfers across the median clearly increase the spread of the distribution, making P_w increase from $1/12 = 6/72$ to $11/72$, but P_{ZK} decreases from 4 to 3.2 and P_G decreases from 0.6 to 0.5238.

Following the pioneering paper by Foster and Wolfson (1992), most of the new bipolarization measures proposed consider separating the income distribution into two parts bounded by the median. Nevertheless, one can also use other two-part divisions, with incomes in non-decreasing order. It is clear that the axioms concerning increased spread and increased bipolarity need to be reformulated, no longer limited to division into two halves: when an income distribution is divided into two parts (below and above a threshold), regressive transfers involving people who are in different parts of the distribution increase bipolarity, and progressive transfers between people who belong to one of the two parts also increase bipolarity.

Considering the delimitation by the mean (μ), it is possible to construct a measure very similar to P_W. Fig. 1 is comparable to Fig. E1 in Foster and Wolfson (1992, p. 18), or Fig. 14 in Foster and Wolfson (2010, p. 265), but instead of considering the tangent line to the Lorenz curve at the point corresponding to the median, the tangent to the Lorenz curve is drawn at the point corresponding to the mean, where the discrepancy of the Lorenz curve reaches its maximum value (D_{max}). Since this tangent line is parallel to the line of perfect equality, the parallelogram OABC has an area numerically equal to the maximum discrepancy MD = D_{max}, which, in turn, is equal to the Gini index of inequality between the two parts delimited by the mean ($G_{B\mu}$):

$$Area \text{ OABC} = D_{max} = G_{B\mu} \tag{10}$$

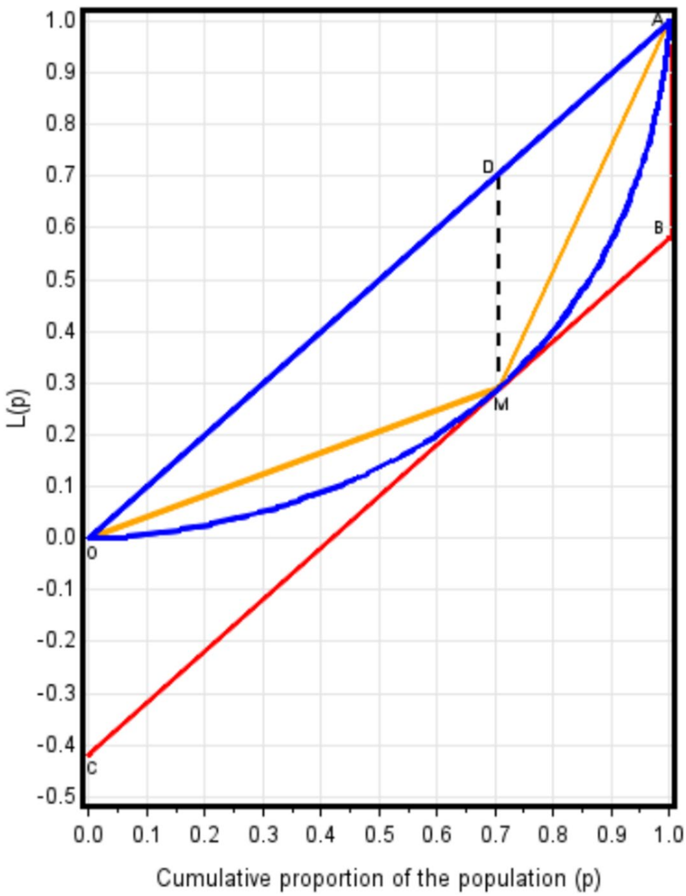

Fig. 1 The Lorenz curve and the inequality between two parts bounded by the mean of the distribution

The bipolarization measure is defined as the difference between the inequality between the two parts of the distribution delimited by μ ($G_{B\mu}$) and the inequality within these two parts ($G_{W\mu}$), analogously to expression (2):

$$P_{\mu} = G_{B\mu} - G_{W\mu} \tag{11}$$

As $G_{W\mu} = G - G_{B\mu}$, it follows that

$$P_{\mu} = 2G_{B\mu} - G = 2D_{max} - G = 2\left(D_{max} - 0.5G\right) \tag{12}$$

Since $0.5G$ is equal to the area of inequality, it turns out that the bipolarization measure P_{μ} is equal to twice the area between the Lorenz curve and the straight-line segment CB, in a manner perfectly analogous to what Foster and Wolfson (2010) show in their Fig. 14. Examining Fig. 1 it is possible to conclude that $P_{\mu} \geq 0$, with $P_{\mu} = 0$ only in the case of perfect equality in income distribution.[3]

[3] A tangent to the Lorenz curve, the line of perfect equality, and the vertical lines at $p = 0$ and at $p = 1$ form, in general, a trapezoid that includes the area of inequality. However, only for the tangents at the

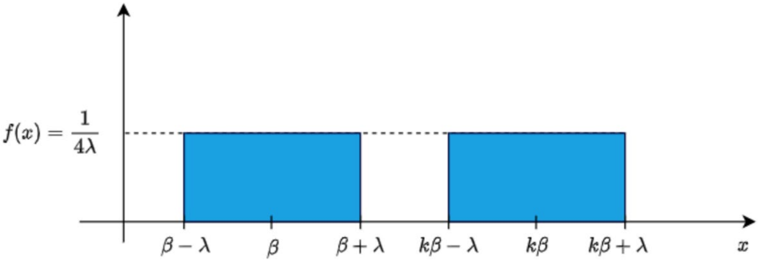

Fig. 2 A symmetric continuous distribution consisting of two uniform distributions

It turns out that P_μ has the three desired properties: it grows with increased spread and with increased bipolarity and is scale invariant. It is worth noting that in the case of P_W increased bipolarity means incomes moving away from the median towards the extremes, whereas in the case of P_μ it means incomes moving away from the mean μ.

To illustrate how the presented measures behave in a typical case of varying bipolarity, let us consider the distribution of a continuous variable x illustrated in Fig. 2.[4]

The first half is a uniform distribution from $\beta - \lambda$ to $\beta + \lambda$ and the second half is a uniform distribution from $k\beta - \lambda$ to $k\beta + \lambda$. It is assumed that $k > 1$, $\beta > 0$ and $\lambda \geq 0$. If we assume that x is never negative, we must have $\lambda \leq \beta$. It is easy to verify that both the median and the mean are $m = \mu = \frac{1}{2}(k + 1)\beta$. It can be shown (see the appendix) that the Gini index of inequality between the two halves is

$$G_B = D_{50} = D_{max} = \frac{k - 1}{2(k + 1)}, \tag{13}$$

that the Gini index for inequality within the halves is

$$G_W = \frac{\lambda}{3\beta(k + 1)} \tag{14}$$

and that the overall Gini index is

$$G = G_B + G_W = \frac{k - 1}{2(k + 1)} + \frac{\lambda}{3\beta(k + 1)} \tag{15}$$

When $k\beta - \lambda = \beta + \lambda$, or $\lambda = \frac{(k-1)\beta}{2}$, the distribution in Fig. 2 becomes a single uniform distribution.

Given expressions (1), (7), (8) and (11), it is possible to verify that

$$P_W = P_\mu = (k - 1)/(2(k + 1)) - \lambda/3\beta(k + 1) \tag{16}$$

$$P_Z K = (3\beta(k - 1))/2\lambda \tag{17}$$

Footnote 3 (continued)

median and at the mean, the area of this trapezoid is equal to the Gini index of inequality between the two parts delimited by the income corresponding to the point of tangency.

[4] The same illustration was previously used by Hoffmann and Jesus (2023).

$$P_G = (3\beta(k-1) - 2\lambda)/(3\beta(k-1) + 2\lambda) \tag{18}$$

Imagine now that the values of k and β are fixed and that the value of λ increases, starting from $\lambda = 0$. Also, to avoid the two halves to overlap, consider that $k\beta - \lambda \geq \beta + \lambda$, or

$$\lambda \leq \frac{(k-1)\beta}{2} \tag{19}$$

With k and β fixed, there is maximum bipolarization if $\lambda = 0$, with half of the distribution concentrated at $x = \beta$ and the other half concentrated at $x = k\beta$. As λ increases, the bipolarization reduces, until the distribution becomes a single uniform distribution from $\beta - \frac{1}{2}(k-1)\beta$ to $k\beta + \frac{1}{2}(k-1)\beta$.

In the particular case where $k = 3$, expressions (15)–(18) become,

$$G = \frac{1}{4} + \frac{\lambda}{12\beta} \tag{20}$$

$$P_W = P_\mu = \frac{1}{4} - \frac{\lambda}{12\beta} \tag{21}$$

$$P_{ZK} = \frac{3\beta}{\lambda} \tag{22}$$

$$P_G = \frac{3\beta - \lambda}{3\beta + \lambda} \tag{23}$$

When λ increases from zero to the value of β, the Gini index of the distribution increases linearly from 1/4 to 1/3, while the bipolarization measures P_W and P_μ decrease linearly from 1/4 to 1/6, P_{ZK} decreases from very high values to 3, and P_G decreases from 1 to 0.5. Thus, in this example, inequality and bipolarization vary in opposite directions, highlighting that they are two distinct concepts. The variation in bipolarization in this case is associated with the principle of "increased bipolarity" and, therefore, the behaviour of P_{ZK} and P_G is consistent with that of P_W and P_μ, even though P_{ZK} and P_G do not always satisfy the principle of "increased spread", as demonstrated earlier.

3 The Measure of Handcock and Morris

Handcock and Morris (1999) proposed a measure called "Median Relative Polarization" (MRP) that compares two distributions. The authors explain that the measure is intended to evaluate the difference in shape between two distributions that have the same median. Let x be a variable with a distribution function $F(x)$, to be compared with the distribution of x_0, whose distribution function is $F_0(x_0)$. Let r be the value of the distribution function F_0 for each value of x, that is,

$$r = F_0(x) \tag{24}$$

The distribution of r is called the relative distribution of x with respect to x_0. Let E represents the mathematical expectation, then the index is

$$\mathrm{MRP} = 4E(|r - 0.5|) - 1 \tag{25}$$

Next, the MRP index is analyzed for the distribution presented in Fig. 2. Considering initially a distribution (x_0) with $\lambda = 0$, it is possible to observe that $r = 0$ for $x < \beta$, $r = 0.5$ for $\beta \leq x \leq k\beta$ and $r = 1$ for $x > k\beta$. This distribution, with $\lambda > 0$ and obeying the restriction (15), yields:

$$\mathrm{MRP} = 4\left(0.5\frac{1}{4} + 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 0.5\frac{1}{4}\right) - 1 = 0 \tag{26}$$

The fact that MRP equals zero regardless of the value of λ shows that this index is insensitive to the degree of polarization of the distribution. The MRP index is not an appropriate measure of the polarization of a distribution. It is clear from reading Handcock and Morris' book that the authors do not have a concept of polarization that is distinct from inequality. The "polarization" is only in the name.[5] It should be noted that MRP is not even a good measure of inequality, because it does not decrease with progressive transfers between two people who are within one of the two halves of the distribution. It turns out that this measure was misused in two published papers on "polarization" of income distribution in Brazil: Clementi and Schettino (2013, 2015). In addition to using an inappropriate measure, Clementi and Schettino (2013, 2015) applied an additive shift to the initial distribution, which reduced inequality and consequently resulted in the detection of a false increase in polarization.

In order to compare two distributions with different medians, it is necessary to first adjust one of the distributions to equalize the medians before calculating the MRP. Let us imagine that we want to compare the distribution of y with the distribution of x, with different medians. Then we should create an intermediate distribution x_0 with the same bipolarization as y and a median equal to that of x, before applying formulas (24) and (25). To do this, the values of x_0 must be obtained by multiplying the values of y by the ratio between the median of x and the median of y, the ratio $\frac{m_x}{m_y}$. This represents a translation of the logarithmic scale, which does not change the inequality or the polarization of the distribution. It also does not change the shape of the distribution of the relative incomes.

One should not equalize the medians by adding the difference $m_x - m_y$ to the values of y because this changes the inequality and the polarization of the distribution.[6] If $m_x - m_y > 0$, the sum of this difference to the y-values reduces the inequality and polarization, and the comparison between the distributions of x and x_0 will "reveal" changes artificially created by the previous translation done inappropriately. Note that the incorrect additive adjustment does not change the shape of the distribution of y, but changes the shape of the distribution of the relative incomes.

This error leads Clementi and Schettino (2013, 2015) to claim that there has been an increase in the polarization of the income distribution in Brazil from 2003 to 2012. In both articles they also calculate an index obtained by multiplying Foster and Wolfson's measure by the median.[7] This measure increases with average income, which grows substantially in

[5] According to Panek and Zwierzchowski (2020, p. 1042), "The general form of MRP index does not allow for an assessment of the degree of economic polarization in the strict sense, …".

[6] Note that adding a positive constant to the y-values changes the shape of the distribution of the relative incomes.

[7] In the authors' words: "… we construct an 'absolute' counterpart of the Foster-Wolfson index by multiplying it by the median" (Clementi and Schettino, 2015, p. 938).

Brazil from 2003 to 2012. Clementi and Schettino again find a false rise of the polarization of income distribution in Brazil. The authors fail to realize that the procedure used makes the measure sensitive to economic growth and affirm that in Brazil from 2007 to 2012 "the process of polarization in household incomes is more pronounced, with both the lower and upper tails shifting away from the median of the distribution." (Clementi and Schettino, 2015, p. 950).

It should be noted that the MRP measure is calculated employing the same procedure (using additive median shift) in several papers on polarization of income or consumption distribution in regions of Africa: Clementi et al. (2017), on Nigeria; Clementi et al. (2018), on Ghana; Clementi et al. (2019), on Morocco; and Clementi et al. (2021), on Sub-Saharan Africa.

Consider, for example, the work of Clementi et al. (2018), analyzing household consumption expenditure in Ghana from 1991/92 to 2012/13. As the median expenditure rose from 352.66 to 655.60, the authors implemented a large additive shift, thereby substantially reducing the inequality in the distribution that became the basis for comparison. Consequently, the calculated MRP captures a false increase in inequality (interpreted as an increase in polarization). Values of the Gini index and the Theil T measure presented in the article show an increase in inequality only in the subperiod from 1998/99 to 2005/06, but since the median grows substantially in the three subperiods (bounded by 1991/92, 1998/99, 2005/06, and 2012/13), the calculated MRP indicates a false increase in "polarization" in all three subperiods.

Nissanov and Pittau (2016) also employed an additive median shift to calculate the MRP measure for income distribution in Russia during the period 1992–2008. With a stable pattern of inequality and a median income growth exceeding 100%, this procedure resulted in a misleading increase in polarization.

Recently, the same inappropriate calculation procedure for the MRP index was applied by Fabiani (2023) to analyze income distribution in 12 European countries from 2000 to a final year ranging from 2016 to 2020, depending on the country. The MRP indicated a significant increase in polarization in 11 out of the 12 countries analyzed. The exception is Italy, which is also the only country where the median income has decreased. Fabiani (2023) also presents the values of the Gini index (G) and the Foster and Wolfson polarization measure (W) in 2000 and the final year for each of the 12 countries. By calculating the correlation between the final MRP and changes in the median value (Δm), Gini index (ΔG), and Foster and Wolfson measure (ΔW), the following results were obtained:

Correlation between Δm and MRP equals to 0.716, statistically significant at 1%.
Correlation between ΔW and MRP equals to 0.475, not significant at 10%.
Correlation between ΔW and ΔG equals to 0.925, significant at 0.01%.
Correlation between Δm and ΔG equals to −0.158, not significant.

All indications suggest that the MRP measure, calculated by inappropriately applying an additive shift to incomes, ends up reflecting a false polarization associated with median growth. The appropriate measure of bipolarization exhibits a behavior strongly linked to income inequality.

4 Measure of Multipolarization

Another pioneering paper on a measure of polarization in income distribution was published in 1994 by Esteban and Ray. This is a statistically sophisticated paper in which a measure of multipolarization is derived taking into account both identification (proximity) among people in a group and alienation (separation) between groups. This is a measure that can capture the existence of 1, 2, or more peaks in the distribution. Unlike Wolfson's measure, it was not conceived with bipolarization in mind, that is, the greater concentration of the "rich" on one side and "poor" on the other, with a concomitant decline of the "middle class".

The definition and analysis of this multipolarization measure was refined by Duclos, Esteban and Ray (2004) and Esteban, Gradín and Ray (2007). The multipolarization measures (Esteban and Ray, 1994; Duclos, Esteban and Ray, 2004; and Esteban, Gradín and Ray, 2007) were created independently of Foster and Wolfson's bipolarization measure. Certainly, there is a great analogy between "increased spread" and "alienation" on one side, and "increased bipolarity" and "identification" on the other side. However, the fact that Foster and Wolfson (1992) start by splitting the distribution into two parts leads to different effects of those concepts. A rigorous comparison of the axioms and properties of the two types of indices can be found in Esteban and Ray (2012).

Assuming that the income distribution has been divided into n strata (ordered groups), from the poorest to the richest individual, and letting π_i (with $i = 1, 2, \ldots, n$) be the fraction of the population in the i-th stratum and μ_i be the respective relative income, that is, the average income of the stratum standardized so that the overall average is equal to one ($\sum_{i=1}^{n} \pi_i \mu_i = 1$), the Esteban and Ray polarization measure is defined as:[8]

$$P_{ER}(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{\alpha+1} \pi_j \left| \mu_i - \mu_j \right| \tag{27}$$

With $\alpha = 0$ this measure is equal to twice the Gini index among the n groups. The parameter α can be associated with the degree of sensitivity to polarization. In empirical applications, Duclos, Esteban and Ray (2004) use $\alpha = 0.25$, 0.50, 0.75 or 1.00, while Esteban, Gradín and Ray (2007), for a modification of the original index, consider $\alpha = 1$, 1.3 or 1.6. There are no established criteria for choosing the value of α nor for delimiting the n groups. Esteban, Gradín and Ray (2007) recommend that, once the number (n) of groups is chosen, their delimitation should be done in a way to minimize inequality within the groups, given by $G - G_n$, where G is the general Gini index for individual data and G_n is the Gini index of inequality among the n groups. The number of groups is then left to the researcher's discretion (depending on the research objectives). They further argue that the inequality within n groups should be used to make a "correction" to the polarization measure, which becomes:

$$P_{EGR}(\alpha, \beta) = P_{ER}(\alpha) - \beta(G - G_n) \tag{28}$$

Subsequently Esteban, Gradín and Ray (2007) prove that, considering the division of the income distribution into two parts delimited by the median and adopting $\alpha = \beta = 1$, yields

[8] Note that the use of relative average income makes the measure of multipolarization scale invariant.

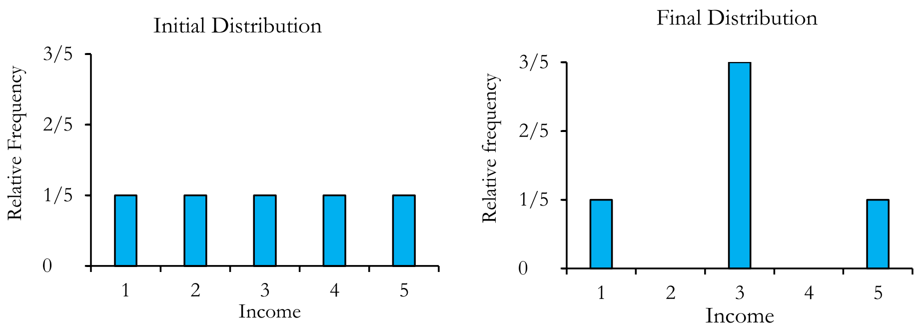

Fig. 3 Relative frequency distribution before and after progressive transfers

$$P_{EGR}(1{,}1) = 2D_{50} - G = \frac{m}{\mu}P_W \tag{29}$$

and considering the division in two parts bounded by the mean, yields the measure defined in (12):

$$P_{EGR}(1{,}1) = 2D_{max} - G = P_\mu \tag{30}$$

Exceptional results for those who appreciate mathematical generalizations: the multipolarization measure (28) practically includes, in particular cases, the bipolarization measures P_W and P_μ. Note, however, that the very comprehensive character of measure (28) is obtained at the expense of the vagueness of the delimitation of the n groups and the value of two parameters (α and β). Any practical application of measure (28) requires defining the delimitation of the groups and establishing the values of αandβ. Transforming the discrete data into a density function, as done in Duclos, Esteban, and Ray (2004), using kernel estimation procedures, circumvents the problem of group boundary, but creates a new arbitrary decision: choosing the bandwidth parameter.[9]

An artificial numerical example illustrates the difference in behavior between the multipolarization measure P_{ER} and the bipolarization measures P_W and P_μ. The symbol P_{ER} will be used when adopting $\alpha = 1$.[10] In other cases, the value of α will always be explicit, as in P_{ER} ($\alpha = 0.25$). Consider a population of five people with incomes 1, 2, 3, 4 and 5. Then, $m = \mu = 3$, $G = 0.267$ and $P_W = P_\mu = 0.267$. Considering the division of the distribution into 5 groups (each person is a group) and using Eq. (27) with $\mu_i = \frac{x_i}{\mu} = \frac{x_i}{3}$, where x_i is an individual income, it turns out that $P_{ER} = 0.107$.

We then make a progressive transfer of one monetary unit from the fourth to the second person. The new distribution has one person with income 1, three people with income 3,

[9] The paper by Gasparini et al. (2008) illustrates the difficulty to choose and to interpret measures of multipolarization. Analyzing data from 21 Latin American and Caribbean countries from 1989 to 2004, they computed the Wolfson index and ten variants of the multipolarization index. They concluded that "Polarization and inequality measures are highly correlated in the data. At least in the Latin American context and for the indicators used in this paper, income inequality seems a good proxy for income polarization." (p. 481).

[10] This example was previously used by Hoffmann and Jesus (2023).

and one with income 5. Figure 3 shows the relative frequency distribution before and after the progressive transfers.

Due to a progressive transfer, the Gini index decreases to $G = 0.213$. The progressive transfer decreases the inequality between the two halves of the distribution and the bipolarization measures $P_{\mu} = P_W$ decrease to 0.107. On the other hand, the increase of a peak in the middle of the distribution, together with the secondary peaks at incomes 1 and 5, raise the value of P_{ER} from 0.107 to 0.149. This example shows distinct behavior between a bipolarization measure and the P_{ER} measure. It is also evident that the increase in P_{ER} cannot be automatically associated with a reduction in the "middle class", since in this example the increase in this measure is due to the appearance of a peak in the middle of the distribution. This example also shows that the "multipolarization" measure P_{ER} can increase due to the appearance or rise of a peak. It is worth noting that there is no polarization in the absence of inequality. In the special case where all incomes are equal, with the entire distribution concentrated in a single peak, both the bipolarization and multipolarization measures are zero.

In this example, it is interesting to see what happens to Esteban and Ray's measure when $\alpha = 0.25$ and $\alpha = 0.5$. In the initial situation, with incomes equal to 1, 2, 3, 4 and 5, P_{ER} $(\alpha = 0.25) = 0.357$ and P_{ER} $(\alpha = 0.5) = 0.239$. After the progressive transfer, the values are P_{ER} $(\alpha = 0.25) = 0.319$ and P_{ER} $(\alpha = 0.5) = 0.243$. With $\alpha = 0.5$ the index grows, varying in the same direction as P_{ER} with $\alpha = 1$ but with a lower value of the parameter $(\alpha = 0.25)$ the index varies in the same direction as the Gini index.

Care must be taken to avoid misinterpreting the P_{ER} as a measure of bipolarization. In Brazil, this measure is sensitive to a peak at values equal to the minimum wage, combined with the tendency of informants to declare round numbers in surveys. In periods when the minimum wage is a round number, the peak at this point is higher, leading authors to interpret erroneously the corresponding variation in P_{ER} as an indication of a decline in middle class (e.g., Hoffmann, 2008; Hoffmann 2017; and Hoffmann and Jesus, 2023). It should be noted that the misinterpretation of the P_{ER} measure as a measure of bipolarization may lead to a wrong conclusion about the effect of real minimum wage growth in Brazil from 2001 to 2014. As will be seen in the next section, using appropriate measures, it turns out that both inequality and the bipolarization of the income distribution decreased in that period and certainly the substantial growth in the real value of the minimum wage contributed to this result.

5 Variation of the Bipolarization of Income Distribution in Brazil from 2001 to 2015

In order to illustrate the application of bipolarization measures and, more specifically, to discuss the results of Clementi and Schettino (2013, 2015), microdata from the annual Brazilian National Household Surveys—PNAD (traditional)—from 2001 to 2015, undertaken by the Brazilian Institute of Geography and Statistics (IBGE), are analyzed. Until 2003 these surveys did not cover the rural area of the former Northern Region, data from this area were excluded from 2004 to 2015. Note that in 2010 there was no PNAD data because a Demographic Census was conducted.

As used by Clementi and Schettino (2015), household income (HI) is divided by the square root of the number of people in each household to obtain the income per

Table 1 Evolution of the characteristics of the income distribution per adult equivalent (household income divided by the square root of the number of people in the household) in Brazil from 2001 to 2015: mean, median, Gini index, P_W and P_μ bipolarization measures, and MRP measure considering multiplicative or additive adjustment

Year	Mean	Median	Gini	D_{50}	D_{max}	P_W	P_μ	MRP	
								Multiplicative	Additive
2001	1959	1056	0.566	0.358	0.419	0.279	0.273	0.000	0.000
2002	1953	1064	0.559	0.354	0.415	0.272	0.270	−0.017	−0.010
2003	1821	1013	0.552	0.350	0.408	0.267	0.264	−0.023	−0.051
2004	1877	1067	0.539	0.343	0.398	0.258	0.256	−0.043	−0.036
2005	1982	1137	0.536	0.340	0.394	0.252	0.253	−0.054	−0.004
2006	2163	1258	0.531	0.337	0.391	0.244	0.250	−0.066	0.050
2007	2203	1318	0.524	0.334	0.384	0.240	0.243	−0.065	0.081
2008	2287	1390	0.514	0.328	0.376	0.234	0.237	−0.081	0.099
2009	2342	1445	0.510	0.326	0.371	0.229	0.232	−0.086	0.116
2011	2461	1575	0.500	0.320	0.362	0.218	0.224	−0.101	0.153
2012	2629	1691	0.495	0.316	0.358	0.213	0.220	−0.113	0.183
2013	2712	1746	0.496	0.317	0.358	0.214	0.220	−0.111	0.201
2014	2774	1807	0.486	0.311	0.351	0.210	0.217	−0.125	0.210
2015	2573	1669	0.486	0.311	0.351	0.211	0.216	−0.123	0.163

Source: Authors' estimation based on Microdata of PNADs-IBGE from 2001 to 2015

equivalent.[11] All monetary values are expressed in Brazilian currency (Reais) of the fourth quarter of 2021 using the National consumer price index (INPC) as deflator.

MRP values were obtained using the STATA software and the command RELDIST (relative distribution) developed by Jann (2021). MRP values are calculated using multiplicative and additive median adjustments. Although the use of additive adjustment is not correct in this case, we did it just to discuss the results obtained by Clementi and Schettino (2013, 2015).

Table 1 shows for all the 14 years of data, the mean, median, Gini index, D_{50} (equal to the Gini index of inequality between the two halves of the distribution), D_{max} (equal to the Gini index of inequality between the two parts of the distribution delimited by the mean), P_W, P_μ and the MRP measures considering multiplicative or additive adjustment.

The mean and median incomes decrease from 2001 to 2003, but then systematically increase until 2014. The Gini index and the values of D_{50} and D_{max} show a clear tendency to decrease in the period. It is consensual, among researchers on the subject, that the PNAD data show a substantial fall in inequality in the period. It is known that in the PNAD data, capital incomes are especially underreported, and this leads to overestimating the fall in inequality.

The bipolarization measurements P_W and P_μ also show a clear decreasing trend. Considering the 14 values presented in Table 1, the correlation between G and year is −0.990, showing a clear decreasing tendency of income inequality from 2001 to 2015 in

[11] Excluding residents classified in the household as pensioners, domestic servants, and relatives of domestic servants.

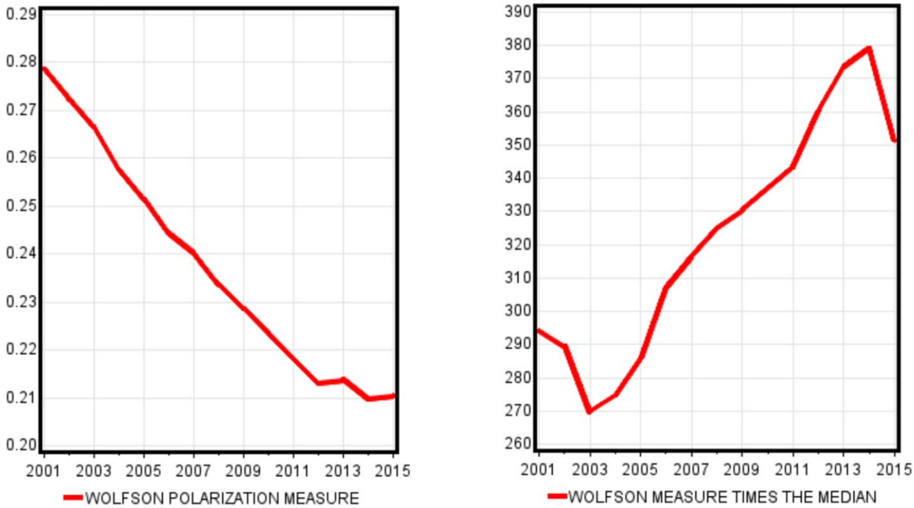

Fig. 4 Evolution of the Wolfson measure of bipolarization and the Wolfson measure multiplied by the median, considering the distribution of income per equivalent in Brazil from 2001 to 2015

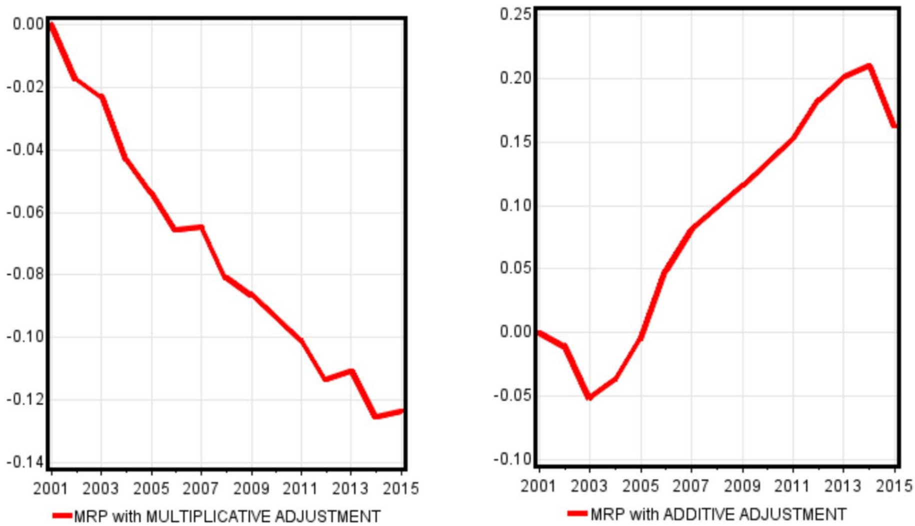

Fig. 5 Evolution of MRP (Median Relative Polarization) using multiplicative and additive adjustment, considering the distribution of income per equivalent in Brazil from 2001 to 2015

Brazil. Also, the following very high correlations are obtained: 0.996 between P_W and G; 0.998 between P_μ and G, and .997 between P_W and P_μ. Bipolarization decreased, accompanying the decrease in inequality. As seen, the MRP measure, appropriately calculated using a multiplicative median adjustment, essentially captures the evolution of inequality between the two halves of the distribution, decreasing over the years: the correlation of this measure (penultimate column of Table 1) with D_{50} is equal to 0.998.

The last column of Table 1 shows the MRP measure erroneously calculated using an additive median shift. The pattern of the change in its value from 2001 to 2012 is almost identical to that shown in Fig. 2b of Clementi and Schettino (2015, p. 937). As explained at the end of Sect. 4, the measure is sensitive to the variation of the median. The MRP index was calculated considering additive shift falls from 2001 to 2003 as a result of the decrease in the median income in this period. Also, it turns positive and increases from 2006 to 2014 because in 2006 the median is already substantially higher than in 2001 and then it increases systematically until 2014. The modification of the Wolfson index created by Clementi and Schettino (2015) (multiplying P_W by the median) behaves in a very similar way.

Figure 4 shows the Wolfson bipolarization measure (P_W) and the Wolfson measure multiplied by the median from 2001 to 2015, using the distribution of income per equivalent in Brazil. Similarly, Fig. 5 presents the MRP index with multiplicative or additive adjustment for the same data. Observe that the behavior of the correct measures presented on the left of Figs. 4 and 5 is completely different from the behavior of the the measures shown on the right of the figures and presented by Clementi and Schettino (2015).

According to PNAD data, from 2003 to 2014 there was economic growth in Brazil (52% increase on average income per equivalent and 78% increase on median), with substantial reduction in inequality and polarization of the distribution. Barros et al. (2010), Neri (2010) and Hoffmann (2018) are examples of the numerous papers that analyze the process of decreasing income inequality in Brazil after 2001. Using measures of "inequality" or "polarization" that essentially reflect economic growth does not help clarify economic or social aspects of the economy.

Clementi and Schettino (2015) use relative measures of inequality and "absolute" measures of polarization and apparently, they do not even realize they are mixing up measures with contradictory assumptions. In the Abstract of the article entitled "Declining inequality in Brazil, in the 2000s: what is hidden behind?" they stated: "Despite substantial reduction in inequality, we are able to document also increasing income polarization, …". This statement seems indefensible to us because it is based on measuring relative inequality and measuring "absolute" polarization. To avoid such confusion, our position is to require "scale invariance" of both inequality and polarization measures.

6 Conclusion

This paper examines the theoretical foundations and empirical applications of income polarization measures, focusing specifically on multipolarization and bipolarization metrics. Building upon the seminal works of Foster and Wolfson (1992) and Esteban and Ray (1994), we analyze how different polarization measures, such as P_W, P_{ER} and P_μ respond to changes in income distribution. A key contribution of this study is the clarification of the conceptual differences between multipolarization and bipolarization metrics. While multipolarization measures, such as P_{ER}, capture the emergence of one or more peaks in an income distribution, they differ fundamentally from bipolarization measures, which focus explicitly on the growing divide between two distinct groups, such as the "rich" and the "poor." This distinction is critical for accurate interpretation and effective policy design.

Over the past few decades, measuring income polarization, particularly bipolarization, has gained increasing attention. The development of new indices has advanced the field, enabling a more nuanced understanding of income distribution. However, many of these bipolarization measures fail to satisfy essential conditions, including increased spread and bipolarity and scale invariance. Without satisfying these conditions, such measures could mislead researchers and policymakers by reflecting unintended trends.

The bipolarization index proposed by Foster and Wolfson (1992) stands out as a robust and widely accepted method for measuring bipolarization in income distribution. This index satisfies all three necessary conditions: it appropriately increases with greater spread and bipolarity, and it is scale-invariant, making it a reliable tool for understanding the division between income groups. Similarly, an alternative measure, defined as the difference between inequality across and within the two parts of the distribution divided by the mean (Eq. 11) also provides a valid framework for analyzing income polarization.

In contrast, theoretical work on "absolute" inequality measures has faced significant limitations. While these measures can describe dispersion, their lack of scale invariance makes them less suitable for studies on income distribution. The use and misinterpretation of "absolute bipolarization" measures has led to the erroneous conclusion that economic growth, which typically increases dispersion, also increases bipolarization. This conceptual misstep underscores the need for clarity in the application of these indices.

The Median Relative Polarization (MRP) index, introduced by Handcock and Morris (1999), illustrates another problematic application. While it measures inequality between the two halves of a distribution, it has been mistakenly used as a bipolarization metric, leading to misleading conclusions. The most serious error has been the calculation of an "absolute" version of the MRP, which measures dispersion (or absolute inequality) between the two halves of the distribution and its misinterpretion as a measure of polarization. This misapplication is particularly evident in studies on income distribution in countries like Brazil and some African nations. Researchers must recognize that the MRP index does not account for the broader concept of polarization, focusing instead on inter-group inequality.

The multipolarization framework proposed by Esteban and Ray (1994) offers a more comprehensive perspective. However, it is essential to distinguish this measure from bipolarization indices, as it captures the formation of one or multiple income peaks rather than a binary divide. The proper computation of the multipolarization index depends on careful parameter selection, including the number of groups (n) and the sensitivity parameter (α). Improper selection of these parameters can result in inaccurate conclusions, particularly if the outcomes are mistakenly framed as conflicts between two income groups.

To advance the field, future research should explore the interplay between specific economic variables, such as wages (especially minimum wages) and polarization. Investigating the connections between pro-poor growth and polarization reduction could also yield valuable insights. Additionally, a deeper analysis of the Esteban-Ray index, particularly regarding optimal parameter selection, would enhance our understanding of multipolarization and its broader implications.

In conclusion, the study of income polarization demands meticulous selection and application of appropriate measurement tools. Bipolarization indices, like the Foster-Wolfson measure, that satisfy the criteria of growth with spread, bipolarity, and scale invariance offer reliable insights into income distribution. However, the introduction of indices that fail to meet these criteria, along with confusion between inequality, dispersion and polarization measures, poses significant challenges. Researchers must exercise caution in applying these metrics, ensuring they align with the underlying economic dynamics they aim to capture.

Appendix

Deduction of Expression (13)

In the distribution illustrated by Fig. 2, the poorer half gets a fraction $Y_1 = \frac{1}{k+1}$ of total income and the richer half gets a fraction $Y_2 = \frac{k}{k+1}$ of total income. Calculating the Gini index between the two halves as 1 minus twice the area under the respective polygon-shaped Lorenz curve yields:

$$G_B = D_{50} = 1 - 2\left[Y_1 \cdot \frac{1}{4} + \left(1 + Y_1\right) \cdot \frac{1}{4}\right] = \frac{k-1}{2(k+1)} \tag{31}$$

Deduction of Expression (14)

Let $f(x) = \frac{1}{2\omega}$ be a uniform distribution for $a - \omega \leq x \leq a + \omega$ and $f(x) = 0$ for $x < a - \omega$ e $x > a + \omega$. Assuming that $a \geq \omega$, negative values of x are excluded. The Gini index (G), the inequality between the two halves (D_{50}) and the Wolfson bipolarization measure (P_W) of this uniform distribution are:

$$G = \omega/3a \tag{32}$$

$$D_50 = \omega/4a \tag{33}$$

$$P_W = \omega/6a \tag{34}$$

According to expression (32), the Gini index of inequality within the first half of the distribution depicted in Fig. 2 is $G_1 = \frac{\lambda}{3\beta}$ and the Gini index of inequality within the second half is $G_2 = \frac{\lambda}{3k\beta}$.

The portion of the Gini index referring to inequality within the two halves of the distribution is $G_W = \frac{1}{2}Y_1G_1 + \frac{1}{2}Y_2G_2$. Using the results obtained previously, after some algebra, we obtain:

$$G_W = \frac{\lambda}{3\beta(k+1)} \tag{35}$$

Axioms for a Measure of the Bipolarity of an Income Distribution

Assuming that the bipolarization measure considers the division of the distribution into two parts delimited by income θ (usually the mean), we adopt the following axioms, based on the contributions of Foster and Wolfson (1992 and 2010), Chakravarty (2009) and Deutsch et al. (2013):

General axioms:

1. Normalization—The measure is equal to zero if all incomes are equal.

2. Scale invariance—The measure does not change when all incomes are multiplied by a positive constant.
3. Symmetry (anonymity)—The measure does not change with any permutation of the incomes.
4. Population principle—The measure is the same for any k-tuple of the population.
5. Continuity—The measure is a continuous function of all incomes.

Specific axioms:

6. Increased spread—The measure increases with any increment of an income above θ and with any reduction in an income below θ.
7. Increased bipolarity—The measure increases with any progressive transfer among two persons above θ and with any progressive transfer between two persons below θ.

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Data Availability The data used in Table 1 are publicly available and can be downloaded from the IBGE homepage at: https://www.ibge.gov.br/estatisticas/sociais/populacao/9127-pesquisa-nacional-por-amostra-de-domicilios.html?=&t=microdados

Declarations

Conflict of interest Disclosure of potential conflicts of interest

Compliance with Ethical Standards The authors declare that they have no conflicts of interest, either financial or non-financial, related to this study. Also, this research did not involve human participants or animals.

Financial Statements The authors certify that they have no affiliations with or involvement in any organization or entity that has a financial or non-financial interest in the subject matter or materials discussed in this manuscript.

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