On Categories of o-Minimal Structures

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Our aim in this work is to look at some transfer results in model theory - mainly in the context of o-minimal structures - from the category theory viewpoint. More specifically, at first, we construct a contravariant functor $\mathcal{E}$ from a suitable category of first-order languages to the category of all locally small categories by means of which we can translate, into diagrams of categories, seminal dichotomy theorems for o-minimal structures, namely

Fact 1 (Theorem A, [5]). *Suppose that $\mathcal{R}$ is an o-minimal expansion of an ordered group $(R, <, +)$. Then exactly one of the the following holds: (a) $\mathcal{R}$ is linearly bounded (that is, for each definable function $f\colon R \to R$ there exists a definable endomorphism $\lambda\colon R \to R$ such that $|f(x)| \leq \lambda(x)$ for all sufficiently large positive arguments x); (b) $\mathcal{R}$ defines a binary operation $\cdot$ such that $(R, <, +, \cdot)$ is an ordered real closed field. If $\mathcal{R}$ is linearly bounded, then for every definable $f\colon R \to R$ there exists $c \in R$ and a definable $\lambda \in \{0\} \cup Aut(R, +)$ with $\lim_{x\to+\infty}[f(x) - \lambda(x)] = c$.*

and

Fact 2 (Theorem and Proposition, [4]): *Let $\mathcal{R}$ be an o-minimal expansion of the ordered field of real numbers $(\mathbb{R}, <, +, \cdot, 0, 1)$. If $\mathcal{R}$ is not polynomially bounded, then the exponential function is definable (without parameters) in $\mathcal{R}$. If $\mathcal{R}$ is polynomially bounded, then for every definable function $f\colon \mathbb{R} \to \mathbb{R}$, with f not identically zero for all sufficiently large positive arguments, there exist $c, r \in \mathbb{R}$ with $c \neq 0$ such that $x \mapsto x^r\colon (0, +\infty) \to \mathbb{R}$ is definable in $\mathcal{R}$ and $\lim_{x\to+\infty} f(x)/x^r = c$.*

The dichotomy on o-minimal expansions of ordered groups, asserted in Fact 1, is the analogue of the dichotomy for o-minimal expansions of the real field $\mathbb{R}$ in Fact 2. Facts 1 and 2 can be viewed as implied transfer results of o-minimality property from one structure to another, and they served as our main motivation for this work.

Also, in [1], A. Berarducci and M. Otero point out some transfer results with respect to topological properties from one o-minimal structure to another. Specifically, if $\mathcal{M}$ is an o-minimal expansion of an ordered field and φ is a first order formula in the language of the ordered rings, then the following statements concerning the definable subsets $\varphi^{\mathcal{M}}$ and $\varphi^{\mathbb{R}}$ hold: (1) $\varphi^{\mathcal{M}}$ is definably connected if and only if $\varphi^{\mathbb{R}}$ is connected; (2) $\varphi^{\mathcal{M}}$ is definably compact if and only if $\varphi^{\mathbb{R}}$ is compact; (3) there is a natural isomorphism between the homology groups $H_*^{\text{def}}(\varphi^{\mathcal{M}}) \cong H_*(\varphi^{\mathbb{R}})$; (4) there is a natural isomorphism between the fundamental groups $\pi^{\text{def}}(\varphi^{\mathcal{M}}, x_0) \cong \pi(\varphi^{\mathbb{R}}, x_0)$; and assuming that $\varphi^{\mathbb{R}}$ is compact it follows that (5) if $\varphi^{\mathcal{M}}$ is a definable manifold, then $\varphi^{\mathbb{R}}$ is a (topological) manifold; and (6) if moreover $\varphi^{\mathcal{M}}$ is definably orientable, then $\varphi^{\mathbb{R}}$ is an orientable manifold.

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We then use the Grothendieck construction in order to treat language $\mathcal{L}$ and structure $\mathcal{M}$ as just an object, namely a pair $(\mathcal{L}, \mathcal{M})$, of a larger category. The morphisms in this larger category are pairs (H, α), where H is a morphism of language and α is a morphism of structure. (The concerned dichotomy theorems can also be read in this global context, with α being the identity homomorphism.) Some suggestions of further investigation arise. For instance, we might consider even more general forms of induced functors by changing of languages as in [8] (something in this direction already ocurred in those theorems by C. Miller and S. Starchenko pointed out above). Also, it is natural to ask for examples in the setting of o-minimal structures, where the translation into the global context involves α distinct from the identity homomorphism. A phenomenon like this appears in [7], namely: if $\mathcal{M}$ is any nonstandard model of PA, with $(\mathrm{HF}^{\mathcal{M}}, \in^{\mathcal{M}})$ the corresponding nonstandard hereditary finite sets of $\mathcal{M}$ (by Ackerman coding: the natural numbers of $\mathrm{HF}^{\mathcal{M}}$ are isomorphic to $\mathcal{M}$), then for any consistent computably axiomatized theory T extending ZF in the language of set theory, there is a submodel $\mathcal{N}' \subseteq (\mathrm{HF}^{\mathcal{M}}, \in^{\mathcal{M}})$ such that $\mathcal{N}' \models T$.

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