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Global dynamical aspects of a generalized Sprott E differential system

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Abstract

In this paper we consider some global dynamical aspects of the generalized Sprott E differential system

$$\dot{x} = ayz + b, \quad \dot{y} = x^2 - y, \quad \dot{z} = 1 - 4x,$$

where $a, b \in \mathbb{R}$ are parameters and $a \neq 0$. This is a very interesting chaotic differential system with one equilibria for all values of the parameters. We show that for b sufficiently small it can exhibit two limit cycles emerging from the classical Hopf bifurcation at the equilibrium point $p = (1/4, 1/16, 0)$. We use the Poincaré compactification for a polynomial vector field in $\mathbb{R}^3$ to do a global analysis of the dynamics on the sphere at infinity. To show how the solutions reach the infinity we study the existence of invariant algebraic surfaces and its Darboux integrability.

Keywords: Hopf bifurcation; Poincaré compactification; invariant algebraic surface; Sprott E system

1 Introduction and statement of the main results

Chaos, an interesting phenomenon in nonlinear dynamical systems, has been developed and intensively studied in the past decades. A chaotic system is a nonlinear deterministic system that presents a complex and unpredictable behavior. The now-classic Lorenz system has motivated a great deal of interest and investigation of 3D-autonomous chaotic systems with simple nonlinearities, such as the Lorenz system [8], the Rössler system [9] and the Chen system[2]. All these system have seven terms and have two or one quadratic nonlinearities. Sprott [10] found 19 simple chaotic systems (called Sprott systems) with no more than three equilibria, one or two quadratic nonlinearities and with less than seven terms.

In [11], Wang and Chen proposed a generalization of the Sprott E system (that we will call the Wang–Chen system) with a single stable node–focus equilibria and 1–scroll chaotic attractor. This new system is very interesting because for a 3-dimensional autonomous quadratic system with a single stable node–focus equilibrium one typically would expect the non existence of a chaotic attractor. We recall that the Lorenz system and the Rössler system are all of hyperbolic type, while the Wang–Chen system is not hyperbolic. In fact, they proposed a system which is the original Sprott E system with a new parameter that causes a delay feedback. Since the stability of the single equilibrium changes from one system to the other the Wang–Chen system is not topologically equivalent to the Sprott E system.

In the investigation of chaos theory and its applications it is very important to generate new chaotic systems or enhance complex dynamics and topological structure based on the existing of chaotic attractors. It is also very important to study the stability of the equilibria of the system. This is a very challenging task, but one of the mechanisms could be the addition of new parameters to a given system.

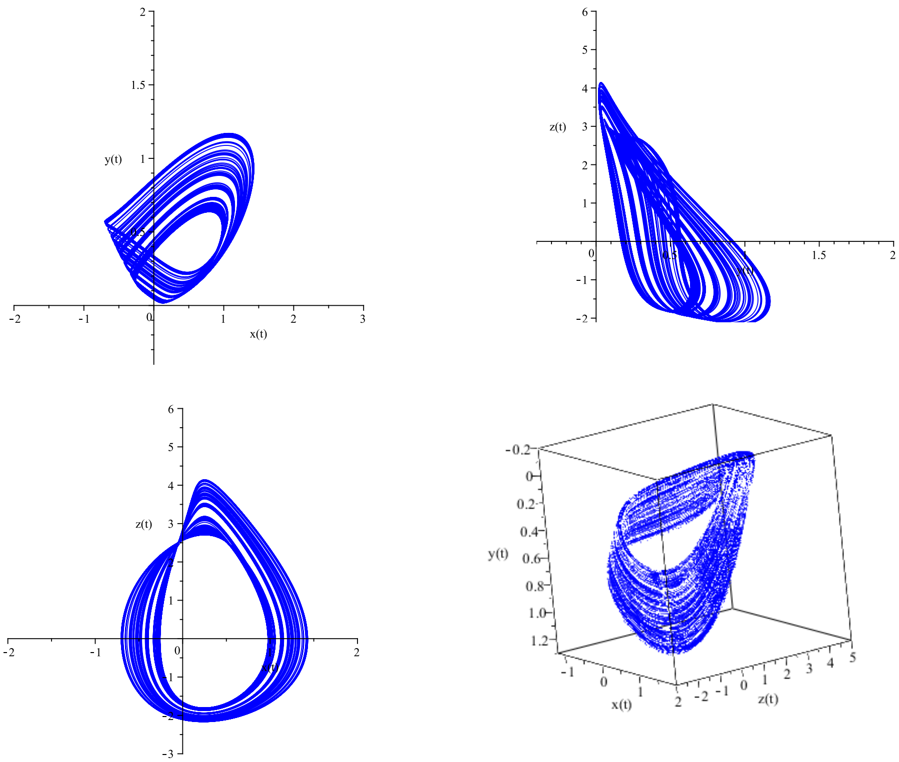

Figure 1: The chaotic atractor of system (1) when $a = 1.1$ and $b = 0.006$: 2D views on the xy-plane, yz-plane, xz-plane and the 3Dview.

In this paper we modify the Wang–Chen system by considering more parameters in the nonlinear part, expecting to cause more chaotic behavior. Although the most natural way would be to add two parameters in the quadratic part (the system has only two quadratic terms) it is easy to see that with a linear change of coordinates the system with two additional parameters can be reduced to a system with only one additional parameter. More precisely, we study the following generalization of the Sprott E system

$$\begin{aligned} \dot{x} &= ayz + b, \\ \dot{y} &= x^2 - y, \\ \dot{z} &= 1 - 4x. \end{aligned} \tag{1}$$

where $a, b \in \mathbb{R}$ are parameters and $a \neq 0$. This system for $a \neq 1$ (when $a = 1$ system (1) is the Wang–Chen system) also generates a 1-scroll chaotic attractor as shown in Figure 1.

The first goal is to study all possible bifurcations which occurs at the equilibrium point $p = (1/4, a/16, -16b/a)$ of system (1). The first bifurcations that we may study are the codimension one bifurcations. Three elementary static bifurcations are associated with a simple zero eigenvalue of the Jacobian matrix at the equilibrium point: saddle-node, transcritical and pitchfork bifurcations. Easy calculations show that for any value of $a, b \in \mathbb{R}$ with $a \neq 0$ system (1) has never a zero eigenvalue. So we will study the other codimension one bifurcation: the Hopf bifurcation. We recall that for the arguments above we will not have a zero–Hopf bifurcation. For the same reason our system (1) will not exhibit the well known codimension two bifurcation of Bogdanov–Takens.

We will study analytically all possible classical and degenerate Hopf bifurcations that occurs at the equilibrium point p of system (1). For this, we will use the classical projection method to compute the

Lyapunov coefficients associated to the Hopf bifurcations. More precisely, our first main result, concerning Hopf bifurcations is the following.

Theorem 1. *The following statements hold*

(a) Let $\mathcal{C} = \{(a,b) \in \mathbb{R}^2 : a > 0, b = 0\}$*. If* $(a,b) \in \mathcal{C}$ *then the Jacobian matrix of system* (1) *in* p *has one real eigenvalue* -1 *and a pair of purely imaginary eigenvalues* $\pm i\sqrt{a}/2$*.*

(b) The first Lyapunov coefficient at p *for the parameter values in* $\mathcal{C}$ *is given by*

$$l_1(a,0) = -\frac{a^2(16 - 40a + a^2)}{(1+a)(4+a)(256 + 69a + a^2)}.$$

If $16 - 40a + a^2 \neq 0$ *then system* (1) *has a transversal Hopf point at* p *for* $b = 0$ *and* $a > 0$*. More precisely, if* $a < 4(5 - 2\sqrt{6})$ *or* $a > 4(5 + 2\sqrt{6})$ *then the Hopf point at* p *is asymptotic stable (weak attractor focus) and for each* $b > 0$ *but sufficiently close to zero there exists a stable limit cycle near the unstable equilibrium point* p*. If* $4(5-2\sqrt{6}) < a < 4(5+2\sqrt{6})$ *then the Hopf point at* p *is unstable (weak repelling focus) and for each* $b < 0$ *but sufficiently close to zero there exists an unstable limit cycle near the asymptotically stable equilibrium point* p*.*

(c) The second Lyapunov coefficient at p *for* $a = 4(5 - 2\sqrt{6})$ *and* $b = 0$ *is given by*

$$l_2(4(5-2\sqrt{6}),0) = \frac{256(-267817529746 + 109358484143\sqrt{6})}{48101385991833} > 0.$$

Since $l_2(4(5 - 2\sqrt{6}), 0) > 0$ *system* (1) *has a transversal Hopf point of codimension 2 at* p *for the parameters* $a = 4(5 - 2\sqrt{6})$ *and* $b = 0$ *which is unstable.*

(d) The second Lyapunov coefficient at p *for* $a = 4(5 + 2\sqrt{6})$ *and* $b = 0$ *is given by*

$$l_2(4(5+2\sqrt{6}),0) = -\frac{256(267817529746 + 109358484143\sqrt{6})}{48101385991833} > 0.$$

Since $l_2(4(5 + 2\sqrt{6}), 0) < 0$ *system* (1) *has a transversal Hopf point of codimension 2 at* p *for the parameters* $a = 4(5 + 2\sqrt{6})$ *and* $b = 0$ *which is stable.*

In Theorem 1 we have analyzed the Hopf and degenerate Hopf bifurcations of system (1). We have analytically proved that there exist two points in the parameter space for which the equilibrium point p is a codimension 2 Hopf point. With the analytical data provided in the analysis of the proof of Theorem 1 we will conclude a qualitative information of the dynamical aspects of system (1). There are regions in the parameter space where system (1) has two limit cycles bifurcating from the equilibrium point p which are described as follows: for $l_1 < 0$ and $b > 0$ with $|l_1| \gg b > 0$ for the parameters where $l_2 > 0$ and for $l_1 > 0$ and $b < 0$ with $l_1 \gg |b| > 0$ for the parameters where $l_2 < 0$.

Theorem 1 will be proved in Section 3. For a review of the projection method described in [5] and the calculation of the first and second Lyapunov coefficients, see Section 2.1.

We note that system Sprott E has no parameters so it can not present any bifurcation and the Wang–Chen system has no codimension two transversal Hopf points.

Now we continue the study of the global dynamics of system (1) by studying its behavior at infinity. For that we shall use the Poincaré compactification for a polynomial vector field in $\mathbb{R}^3$ (see Section 2.2 for a brief description of this technique and for the definition of Poincaré sphere).

Theorem 2. *For all values of* $a \in \mathbb{R} \setminus \{0\}$ *and* $b \in \mathbb{R}$*, the phase portrait of system* (1) *on the Poincaré sphere* $\mathbb{S}^3$ *is topologically equivalent to the one shown in Figure 2.*

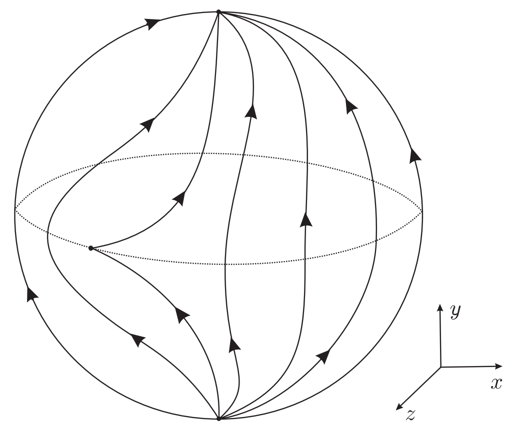

Figure 2: Global phase portrait of system (1) on the Poincaré sphere at infinity.

Theorem 2 is proved in Section 4. Note that the dynamics at infinity do not depend on the value of the parameter b because it appears in the constant terms of system (1). It depends on the parameter a but the global phase portraits at the Poincaré sphere for different values of a are topologically equivalent. From Figure 2 we have four equilibrium points at infinity, two nodes and two cusps and there are no periodic orbits on the sphere.

The Poincaré sphere at infinity is invariant by the flow of the compactified systems. A good way to understand how the solutions approach the infinity is studying the behavior of the system along of invariant algebraic surfaces, if they exist. More precisely, if system (1) has an invariant algebraic surface S, then for any orbit γ not starting on S either $\alpha(\gamma) \subset S$ and $\omega(\gamma) \subset S$, or $\alpha(\gamma) \subset \mathbb{S}^3$ and $\alpha(\gamma) \subset \mathbb{S}^3$ (for more details see Theorem 1.2 of [1]) and, $\alpha(\gamma)$ and $\omega(\gamma)$ are the $\alpha-$limit and $\omega-$limit of γ, respectively (for more details on the $\omega-$ and $\alpha-$limit sets see for instance Section 1.4 of [4]). This property is the key result which allows to describe completely the global flow of our system when it has an invariant algebraic surface and, consequently, how the dynamics approach the infinity. Guided by this we will study the existence of invariant algebraic surfaces for system (1). The knowledge of the algebraic surfaces and the so called exponential factors (see again Section 2.3 for definitions) allow us to characterize the existence of Darboux first integrals. It is worth mentioning that the existence of one or two first integrals for system (1) will much contribute to understand its dynamics and so its chaotic behavior.

Theorem 3. *The following statements holds for system* (1)

(a) It has neither invariant algebraic surfaces, nor polynomial first integrals.

(b) The unique exponential factor is $\exp(z)$ *and linear combinations of it. Moreover the cofactor of* $\exp(z)$ *is* $1 - 4x$.

(c) It has no Darboux first integrals.

The paper is organized as follows. In Section 2 we present some preliminaries. In Section 3 we prove Theorem 1. The dynamics at infinity is studied in Section 4. In Section 5 we prove Theorem 3.

2 Preliminaries

2.1 Hopf bifurcation

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In this section we present a review of the projection method used to compute the Lyapunov constants associated to Hopf bifurcations described in [5].

Consider the differential system

$$\dot{\mathrm{x}} = f(\mathrm{x}, \mu), \tag{2}$$

where $\mathrm{x} \in \mathbb{R}^3$ and $\mu \in \mathbb{R}^3$ are respectively vectors and parameters. Assume that f is quadratic and that $(\mathrm{x}, \mu) = (\mathrm{x}_0, \mu_0)$ is an equilibrium point of the system. Denoting the variable $x - x_0$ by x we write

$$F(\mathrm{x}) = f(\mathrm{x}, \mu_0) \quad \text{as} \quad F(\mathrm{x}) = A\mathrm{x} + \frac{1}{2}B(\mathrm{x}, \mathrm{x}), \tag{3}$$

where $A = f_{\mathrm{x}}(0, \mu_0)$ and, for $i = 1, 2, 3$

$$B_i(\mathrm{x}, \mathrm{y}) = \sum_{j,k=1}^{3} \frac{\partial^2 F_i(\mu)}{\partial \mu_j \partial \mu_k}\Big|_{\mu=0} x_j y_k. \tag{4}$$

Suppose that (x_0, μ_0) is an equilibrium point of (2) where the Jacobian matrix A has a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 > 0$, and admits no other eigenvalues with zero real part. Let T^c be the generalized eigenspace of A corresponding to $\lambda_{2,3}$, i.e., the largest subspace invariant by A on which the eigenvalues are $\lambda_{2,3}$.

Let $p, q \in \mathbb{C}^3$ be vectors such that

$$Aq = i\omega_0 q, \quad A^t p = i\omega_0 p, \quad < p, q >= \sum_{i=1}^{3} \overline{p}_i q_i = 1,$$

where A^T denotes the transposed matrix of A and $\overline{p}$ is the conjugate of p. Note that any $y \in T^c$ can be represented by $y = wq + \overline{wq}$, where $w =< p, q >\in \mathbb{C}$. The 2-dimensional center manifold associated to the eigenvalues $\lambda_{2,3} = \pm i\omega_0$ can be parameterized by the variables w and $\overline{w}$ by the immersion of the form $\mathrm{x} = H(w, \overline{w})$, where $H : \mathbb{C}^2 \to \mathbb{R}^3$ has a Taylor expansion of the form

$$H(w, \overline{w}) = wq + \overline{w}\,\overline{q} + \sum_{2 \le j+k \le 6} \frac{1}{j!k!} h_{jk} w^j\, \overline{w}^k + O(|w|^7),$$

with $h_{jk} \in \mathbb{C}^3$ and $h_{jk} = \overline{h}_{kj}$. So substituting this expression in 2 we get the following equation

$$H_w \dot{w} + H_{\overline{w}} \dot{\overline{w}} = F(H(w, \overline{w})) \tag{5}$$

where F is given by (3). The vectors h_{jk} are obtained solving the linear systems defined by the coefficients of (5), taking into account the coefficients of F in the expressions (3) and (4). So system (5) on the chart ω for a central manifold, is writing as

$$\dot{w} = i\omega_0 w + \frac{1}{2} w|w|^2 + \frac{1}{12} G_{32} w|w|^4 + \frac{1}{144} G_{43} w|w|^6 + O(|w|^8),$$

with $G_{jk} \in \mathbb{C}$.

More precisely, we have

$$h_{11} = -A^{-1} B(q, \overline{q}),$$
$$h_{20} = (2i\omega_0 I_3 - A)^{-1} B(q, q),$$

where I_3 is the identity matrix. For the cubic terms, the coefficients of the terms w^3 in (5), we have

$$h_{30} = 3(3i\omega_0 I_3 - A)^{-1} B(q, h_{20}).$$

From the coefficients of the terms $w^2\overline{w}$ in (5), in order to solve h_{21} must take

$$G_{21} =< p, B(\overline{q}, h_{20}) + 2B(q, h_{11}) > .$$

We define the first Lyapunov coefficient l_1 by

$$l_1 = \frac{1}{2} Re G_{21}.$$

The complex vector h_{21} can be found by solving the 4-dimensional system

$$\begin{pmatrix} i\omega_0 I_3 - A & q \\ \overline{q} & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ s \end{pmatrix} = \begin{pmatrix} B(\overline{p}, h_{20}) + 2B(q, h_{11}) - G_{21} q \\ 0 \end{pmatrix}, \tag{6}$$

with the condition $< p, h_{21} >= 0$.

From the coefficients of the terms $w^4, w^3\overline{w}$ and $w^2\overline{w}^2$ in (5) one obtain respectively

$$\begin{aligned} h_{40} &= (4i\omega_0 I_3 - A)^{-1}(3B(h_{20}, h_{20}) + 4B(q, h_{30})), \\ h_{31} &= (2i\omega_0 I_3 - A)^{-1}(3B(q, h_{21}) + B(\overline{q}, h_{30}) + 3B(h_{20}, h_{11}) - 3G_{21} h_{20}), \\ h_{22} &= -A^{-1}(2B(h_{11}, h_{11}) + 2B(q, \overline{h}_{21}) + 2B(\overline{q}, h_{21}) + B(\overline{h}_{20}, h_{20})). \end{aligned}$$

Defining

$$\mathcal{H}_{32} = 6B(h_{11}, h_{21}) + B(\overline{h}_{20}, h_{30}) + 3B(\overline{h}_{21}, h_{20}) + 3B(q, h_{22}) + 2B(\overline{q}, h_{31}) - 6G_{21} h_{21} - 3\overline{G}_{21} h_{21},$$

we have that the second Lyapunov coefficient l_2 is

$$l_2 = \frac{1}{2} Re G_{32},$$

where $G_{32} =< p, \mathcal{H}_{32} >$.

2.2 Poincaré compactification

Consider in $\mathbb{R}^3$ the polynomial differential system

$$\dot{x} = P_1(x, y, z), \quad \dot{y} = P_2(x, y, z), \quad \dot{z} = P_3(x, y, z),$$

or equivalently its associated polynomial vector field $X = (P_1, P_2, P_3)$. The degree n of X is defined as $n = \max \{\deg(P_i) : i = 1, 2, 3\}$. Let $\mathbb{S}^3 = \{y = (y_1, y_2, y_3, y_4) : ||y|| = 1\}$ be the unit sphere in $\mathbb{R}^4$ and $\mathbb{S}_+ = \{y \in \mathbb{S}^3 : y_4 > 0\}$ and $\mathbb{S}_- = \{y \in \mathbb{S}^3 : y_4 < 0\}$ be the northern and southern hemispheres of $\mathbb{S}^3$, respectively. The tangent space of $\mathbb{S}^3$ at the point y is denoted by $T_y\mathbb{S}^3$. Then the tangent plane

$$T_{(0,0,0,1)}\mathbb{S}^3 = \{(x_1, x_2, x_3, 1) \in \mathbb{R}^4 : (x_1, x_2, x_3) \in \mathbb{R}^3\}$$

can be identified with $\mathbb{R}^3$.

Consider the identification $\mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3$ and the central projection $f_\pm : T_{(0,0,0,1)}\mathbb{S}^3 \to \mathbb{S}_\pm$ defined by

$$f_\pm(x) = \pm \frac{(x_1, x_2, x_3, 1)}{\Delta(x)}, \quad \text{where} \quad \Delta(x) = \left(1 + \sum_{i=1}^{3} x_i^2\right)^{1/2}.$$

Using these central projections $\mathbb{R}^3$ is identified with the northern and southern hemispheres. The equator of $\mathbb{S}^3$ is $\mathbb{S}^2 = \{y \in \mathbb{S}^3 : y_4 = 0\}$.

The maps $f_\pm$ define two copies of X on $\mathbb{S}^3$, one $Df_+ \circ X$ in the northern hemisphere and the other $Df_- \circ X$ in the southern one. Denote by $\overline{X}$ the vector field on $\mathbb{S}^3 \setminus \mathbb{S}^2 = \mathbb{S}_+ \cup \mathbb{S}_-$, which restricted to $\mathbb{S}_+$

coincides with $Df_+ \circ X$ and restricted to $\mathbb{S}_-$ coincides with $Df_- \circ X$. Now we can extend analytically the vector field $\overline{X}(y)$ to the whole sphere $\mathbb{S}^3$ by $p(X) = y_4^{n-1}\overline{X}(y)$. This extended vector field $p(X)$ is called the Poincaré compactification of X on $\mathbb{S}^3$.

As $\mathbb{S}^3$ is a differentiable manifold in order to compute the expression for $p(X)$, we can consider the eight local charts (U_i, F_i), (V_i, G_i), where

$$U_i = \{y \in \mathbb{S}^3 : y_i > 0\} \quad \text{and} \quad V_i = \{y \in \mathbb{S}^3 : y_i < 0\}$$

for $i = 1, 2, 3, 4$. The diffeomorphisms $F_i\colon U_i \to \mathbb{R}^3$ and $G_i\colon V_i \to \mathbb{R}^3$ for $i = 1, 2, 3, 4$ are the inverse of the central projections from the origin to the tangent hyperplane at the points $(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $(0, 0, \pm 1, 0)$ and $(0, 0, 0, \pm 1)$, respectively.

Now we do the computations on U_1. Suppose that the origin $(0, 0, 0, 0)$, the point $(y_1, y_2, y_3, y_4) \in \mathbb{S}^3$ and the point $(1, z_1, z_2, z_3)$ in the tangent hyperplane to $\mathbb{S}^3$ at $(1, 0, 0, 0)$ are collinear. Then we have

$$\frac{1}{y_1} = \frac{z_1}{y_2} = \frac{z_2}{y_3} = \frac{z_3}{y_4}$$

and, consequently

$$F_1(y) = (y_2/y_1, y_3/y_1, y_4/y_1) = (z_1, z_2, z_3)$$

defines the coordinates on U_1. As

$$DF_1(y) = \begin{pmatrix} -y_2/y_1^2 & 1/y_1 & 0 & 0 \\ -y_3/y_1^2 & 0 & 1/y_1 & 0 \\ -y_4/y_1^2 & 0 & 1/y_1 & 0 \end{pmatrix}$$

and $y_4^{n-1} = (z_3/\Delta(z)^{n-1})$, the analytical vector field $p(X)$ in the local chart U_1 becomes

$$\frac{z_3^n}{\Delta(z)^{n-1}}\big(-z_1P_1 + P_2, -z_2P_1 + P_3, z_3P_1\big),$$

where $P_i = P_i(1/z_3, z_1/z_3, z_2/z_3)$.

In a similar way, we can deduce the expressions of $p(X)$ in U_2 and U_3. These are

$$\frac{z_3^n}{\Delta(z)^{n-1}}\big(-z_1P_2 + P_1, -z_2P_2 + P_3, z_3P_2\big),$$

where $P_i = P_i(z_1/z_3, 1/z_3, z_2/z_3)$, in U_2 and

$$\frac{z_3^n}{\Delta(z)^{n-1}}\big(-z_1P_3 + P_1, -z_2P_3 + P_2, z_3P_3\big),$$

with $P_i = P_i(z_1/z_3, z_2/z_3, 1/z_3)$, in U_3.

The expression for $p(X)$ in U_4 is $z_3^{n+1}(P_1, P_2, P_3)$ and the expression for $p(X)$ in the local chart V_i is the same as in U_i multiplied by $(-1)^{n-1}$, where n is the degree of X, for all $i = 1, 2, 3, 4$.

Note that we can omit the common factor $1/(\Delta(z))^{n-1}$ in the expression of the compactification vector field $p(X)$ in the local charts doing a rescaling of the time variable.

From now on we will consider only the orthogonal projection of $p(X)$ from the northern hemisphere to $y_4 = 0$ which we will denote by $p(X)$ again. Observe that the projection of the closed northern hemisphere is a closed ball of radius one denoted by B, whose interior is diffeomorphic to $\mathbb{R}^3$ and whose boundary $\mathbb{S}^2$ corresponds to the infinity of $\mathbb{R}^3$. Moreover, $p(X)$ is defined in the whole closed ball B in such way that the flow on the boundary is invariant. The vector field induced by $p(X)$ on B is called the Poincaré compactification of X and B is called the Poincaré sphere.

All the points on the invariant sphere $\mathbb{S}^2$ at infinity in the coordinates of any local chart U_i and V_i have $z_3 = 0$.

2.3 Integrability theory

We start this subsection with the Darboux theory of integrability. As usual $\mathbb{C}[x, y, z]$ denotes the ring of polynomial functions in the variables x, y and z. Given $f \in \mathbb{C}[x, y, z] \setminus \mathbb{C}$ we say that the surface $f(x, y, z) = 0$ is an *invariant algebraic surface* of system (1) if there exists $k \in \mathbb{C}[x, y, z]$ such that

$$(ayz + b)\frac{\partial f}{\partial x} + (x^2 - y)\frac{\partial f}{\partial y} + (1 - 4x)\frac{\partial f}{\partial z} = kf. \tag{7}$$

The polynomial k is called the *cofactor* of the invariant algebraic surface $f = 0$ and it has degree at most 1. When $k = 0$, f is a polynomial first integral. When a real polynomial differential system has a complex invariant algebraic surface, then it has also its conjugate. It is important to consider the complex invariant algebraic surfaces of the real polynomial differential systems because sometimes these forces the real integrability of the system.

Let $f, g \in \mathbb{C}[x, y, z]$ and assume that f and g are relatively prime in the ring $\mathbb{C}[x, y, z]$, or that $g = 1$. Then the function $\exp(f/g) \notin \mathbb{C}$ is called an *exponential factor* of system (1) if for some polynomial $L \in \mathbb{C}[x, y, z]$ of degree at most 1 we have

$$(ayz + b)\frac{\partial \exp(f/g)}{\partial x} + (x^2 - y)\frac{\partial \exp(f/g)}{\partial y} + (1 - 4x)\frac{\partial \exp(f/g)}{\partial z} = L \exp(f/g). \tag{8}$$

As before we say that L is the *cofactor* of the exponential factor $\exp(f/g)$. We observe that in the definition of exponential factor if $f, g \in \mathbb{C}[x, y, z]$ then the exponential factor is a complex function. Again when a real polynomial differential system has a complex exponential factor surface, then it has also its conjugate, and both are important for the existence of real first integrals of the system. The exponential factors are related with the multiplicity of the invariant algebraic surfaces, for more details see [3], Chapter 8 of [4], and [6, 7].

Let U be an open and dense subset of $\mathbb{R}^3$, we say that a nonconstant function $H \colon U \to \mathbb{R}$ is a *first integral* of system (1) on U if $H(x(t), y(t), z(t))$ is constant for all of the values of t for which $(x(t), y(t), z(t))$ is a solution of system (1) contained in U. Obviously H is a first integral of system (1) if and only if

$$(ayz + b)\frac{\partial H}{\partial x} + (x^2 - y)\frac{\partial H}{\partial y} + (1 - 4x)\frac{\partial H}{\partial z} = 0,$$

for all $(x, y, z) \in U$.

A first integral is called a *Darboux first integral* if it is a first integral of the form

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q},$$

where $f_i = 0$ are invariant algebraic surfaces of system (1) for $i = 1, \ldots p$, and F_j are exponential factors of system (1) for $j = 1, \ldots, q$, $\lambda_i, \mu_j \in \mathbb{C}$.

The next result, proved in [4], explain how to find Darboux first integrals.

Proposition 4. *Suppose that a polynomial system* (1) *of degree m admits p invariant algebraic surfaces $f_i = 0$ with cofactors k_i for $i = 1, ..., p$ and q exponential factors* $\exp(g_j/h_j)$ *with cofactors L_j for $j = 1, ..., q$. Then, there exist λ_i and $\mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0, \tag{9}$$

if and only if the function

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \ldots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q}$$

is a Darboux first integral of system (1).

The following result whose proof is given in [6, 7] will be useful to prove statement (b) of Theorem 3.

Lemma 5. *The following statements hold.*

(a) If $\exp(f/g)$ *is an exponential factor for the polynomial differential system* (1) *and* g *is not a constant polynomial, then* $g = 0$ *is an invariant algebraic surface.*

(b) Eventually $\exp(f)$ *can be an exponential factor, coming from the multiplicity of the infinity invariant plane.*

3 Hopf bifurcation

In this section we prove Theorem 1. We will separate each of the statements in Theorem 1 in different subsections.

Proof of Theorem 1(a). System (1) has the equilibrium point $p = (1/4, a/16, -16b/a)$ with $a \neq 0$. The proof is made computing directly the eigenvalues at the equilibrium point. The characteristic polynomial of the linear part of system (1) at the equilibrium point p is

$$p(\lambda) = -\frac{a}{4} - \big(\frac{a}{4} + 8b\big)\lambda - \lambda^2 - \lambda^3.$$

As $p(\lambda)$ is a polynomial of degree 3, it has either one, two (then one has multiplicity 2), or three real zeros. Imposing the condition

$$p(\lambda) = (\lambda - k)(\lambda^2 + \beta^2) \tag{10}$$

with $k, \beta \in \mathbb{R}$, $k \neq 0$ and $\beta > 0$ we obtain a system of three equations that correspond to the coefficients of the terms of degree $0, 1$ and 2 in λ of the polynomial in (10). This system has only the solution $k = -1, b = 0, \beta = \sqrt{a}/2$, with $a > 0$. This completes the proof. □

Proof of Theorem 1(b). We will compute the first Lyapunov coefficient at the equilibrium point p of system (1) with $(a, b) \in \mathcal{C}$. We will use the projection method described in Section 2.1 with $\omega_0 = \sqrt{a}/2$, $\mathrm{x}_0 = p$, $\mu = (a, b)$ and $\mu_0 = (a, 0)$.

The linear part of system (1) at the equilibrium point p is

$$A = \begin{pmatrix} 0 & 0 & a/16 \\ 1/2 & -1 & 0 \\ -4 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of A are $\pm i\sqrt{a}/2$ and -1. In this case, the bilinear form B evaluated at two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is given by

$$B(u, v) = \left(\frac{a}{2}(u_3 v_2 + u_2 v_3), u_1 v_1, 0\right).$$

The normalized eigenvector q of A associated to the eigenvalue $i\sqrt{a}/2$ normalized so that $\overline{q} \cdot q = 1$ is

$$q = \frac{-\sqrt{a(4+a)}}{\sqrt{256 + 69a + a^2}} \left(i, \frac{\sqrt{a(4+a)}}{\sqrt{a} - 2i}, -8\sqrt{a}\right).$$

The normalized adjoint eigenvector p such that $A^T p = -ip$, where A^T is the transpose of the matrix A, so that $\overline{p} \cdot q = 1$ is

$$p = \frac{-\sqrt{256 + 69a + a^2}}{2\sqrt{a(4+a)}} \left(i, 0, -\frac{\sqrt{2}}{8}\right).$$

The vectors h_{11} and h_{20} are

$$
\begin{aligned}
h_{11} =& \frac{a}{256+69a+a^2}\Big(0, 4+a, 128\Big),\\
h_{20} =& \frac{1}{3(256+69a+a^2)}\Bigg(32ai(\sqrt{a}+2i), \frac{a(16\sqrt{a}+(44+3a)i)}{\sqrt{a}-i}, -128\sqrt{a}(\sqrt{a}+2i)\Bigg).
\end{aligned}
$$

Moreover,

$$
G_{21} = \frac{2a^{3/2}(12\sqrt{a}+15a^{3/2}+(8-46a+6a^2)i)}{3(\sqrt{a}-i)(\sqrt{a}-2i)(256+69a+a^2)}.
$$

The first Lyapunov coefficient is given by

$$
l_1((a,0)) = -\frac{a^2(16-40a+a^2)}{(a+1)(a+4)(256+69a+a^2)}.
$$

If $16-40a+a^2 \neq 0$ then system (1) has a transversal Hopf point at p for $b=0$ and $a>0$. Note that the denominator of $l_1((a,0))$ is positive because $a>0$.

If $16-40a+a^2>0$ which corresponds to $a<4(5-2\sqrt{6})$ or $a>4(5+2\sqrt{6})$ then $l_1((a,0))$ is negative, so the Hopf point at p is a asymptotic stable (weak attractor focus).

If $16-40a+a^2<0$ which corresponds to $4(5-2\sqrt{6})<a<4(5+2\sqrt{6})$ then $l_1((a,0))$ is positive, so the Hopf point at p is unstable (weak repelling focus). This completes the proof. □

Proof of Theorem 1(c) and (d). We will only proof statement *(c)* of Theorem 1 because the proof os statement *(d)* is analogous. We consider system (1) with $b=0$ and $a=4(5-2\sqrt{6})$. Guided by Section 2.1, first we compute h_{21}, solving system (6). Doing that we get $s=0$ and

$$
h_{21} = \frac{-608\sqrt{2}}{\sqrt{1809817388739+738854273343\sqrt{6}}}\left(i, \frac{\sqrt{-20546-1815\sqrt{6}+i(29635\sqrt{2}+24189\sqrt{3})}}{114\sqrt{2}}, 4\right).
$$

The vectors h_{30}, h_{40}, h_{31} and h_{22} are, respectively

$$
\begin{aligned}
h_{30} &= \frac{8\sqrt{2(507+49\sqrt{6})}}{\sqrt{80881}(61446-25082\sqrt{6}+(55200\sqrt{3}-67692\sqrt{2})i)}(h_{30}^1, h_{30}^2, h_{30}^3),\\
h_{40} &= \frac{256}{12658259471535}(h_{40}^1, h_{40}^2, h_{40}^3),\\
h_{31} &= \frac{64}{2531651894307}(h_{31}^1, h_{31}^2, h_{31}^3),\\
h_{22} &= \frac{512}{58875625449}(0, 5(24716931-10079780\sqrt{6}), 16(-56419784+23146371\sqrt{6})),
\end{aligned}
$$

where

$$
\begin{aligned}
h_{30}^1 &= 94944-38760\sqrt{6}+(13164\sqrt{5-2\sqrt{6}}-5376\sqrt{6(5-2\sqrt{6})})i,\\
h_{30}^2 &= 19785-8077\sqrt{6}-(2373\sqrt{5-2\sqrt{6}}+967\sqrt{6(5-2\sqrt{6})})i,\\
h_{30}^3 &= -17552+7168\sqrt{6}+(12800\sqrt{5-2\sqrt{6}}-5216\sqrt{6(5-2\sqrt{6})})i,\\
h_{40}^1 &= 16(-1953106124+803446490\sqrt{6}+(70176310\sqrt{2}-80224605\sqrt{3})i),\\
h_{40}^2 &= \frac{6541736161(-7445240\sqrt{2}+6079280\sqrt{3}+(-12387403+5057280\sqrt{6})i)}{1935(5770036\sqrt{2}-4714916\sqrt{3}+(651169-263780\sqrt{6})i)},\\
h_{40}^3 &= 16(100321195+10048295\sqrt{6}+(457233346\sqrt{2}-346213144\sqrt{3})i),
\end{aligned}
$$

$$h^1_{31} = -8(-7926775675 + 3235727335\sqrt{6} + (-5501563156\sqrt{2} + 4489741563\sqrt{3})i),$$
$$h^2_{31} = \frac{-92397528797 + 37723310264\sqrt{6} + (107493145530\sqrt{2} - 87106977438\sqrt{3})i)}{19},$$
$$h^3_{31} = -16(-4076702851 + 1668298723\sqrt{6} + (2139233246\sqrt{2} - 1752971219\sqrt{3})i).$$

Finally
$$G_{32} = \frac{512(951258\sqrt{2} + 741120\sqrt{3} + (1035205 + 346030\sqrt{6})i}{387(350363749\sqrt{2} + 286072179\sqrt{3})},$$
and so
$$l_2(4(5-2\sqrt{6}), 0) = \frac{256(-267817529746 + 109358484143\sqrt{6})}{48101385991833}.$$
Since $l_2(4(5-2\sqrt{6}), 0) > 0$, system (1) has a transversal Hopf point of codimension 2 at p. This completes the proof. □

4 Compactification of Poincaré

In this section we investigate the flow of system (1) at infinity by analyzing the Poincaré compactification of the system in the local charts U_i, V_i for $i = 1, 2, 3$.

From the results of Section 2.2 the expression of the Poincaré compactification $p(X)$ of system (1) in the local chart U_1 is given by

$$\begin{aligned}
\dot{z}_1 &= 1 - az_1^2 z_2 - z_1 z_3 - bz_1 z_3^2,\\
\dot{z}_2 &= -az_1 z_2^2 - 4z_3 + z_3^2 - bz_2 z_3^2,\\
\dot{z}_3 &= -z_3(az_1 z_2 + bz_3^2).
\end{aligned} \tag{11}$$

For $z_3 = 0$ (which correspond to the points on the sphere $\mathbb{S}^2$ at infinity) system (11) becomes

$$\begin{aligned}
\dot{z}_1 &= 1 - az_1^2 z_2, ,\\
\dot{z}_2 &= -az_1 z_2^2.
\end{aligned}$$

This system has no equilibria. It follows from the Flow Box Theorem that the dynamics on local chart U_1 is equivalent to the one shown in Figure **??**, whose the solutions are given by parallel straight lines.

The flow in the local chart V_1 is the same as the flow in the local chart U_1 because the compacted vector field $p(X)$ in V_1 coincides with the vector field $p(X)$ in U_1 multiplied by -1. Hence the phase portrait on the chart V_1 is the same as the one shown in the Figure 3 reserving in an appropriate way the direction of the time.

In order to obtain the expression of the Poincaré compactification $p(X)$ of system (1) in the local chart U_2 we use again the results given in Section 2. From there we get the system

$$\begin{aligned}
\dot{z}_1 &= -z_1^3 + az_2 + z_1 z_3 + bz_3^2,\\
\dot{z}_2 &= -z_1^2 z_2 - 4z_1 z_3 + z_2 z_3 + z_3^2,\\
\dot{z}_3 &= -(z_1^2 - z_3) z_3.
\end{aligned} \tag{12}$$

System (12) restricted to $z_3 = 0$ becomes

$$\begin{aligned}
\dot{z}_1 &= -z_1^3 + az_2,\\
\dot{z}_2 &= -z_1^2 z_2.
\end{aligned} \tag{13}$$

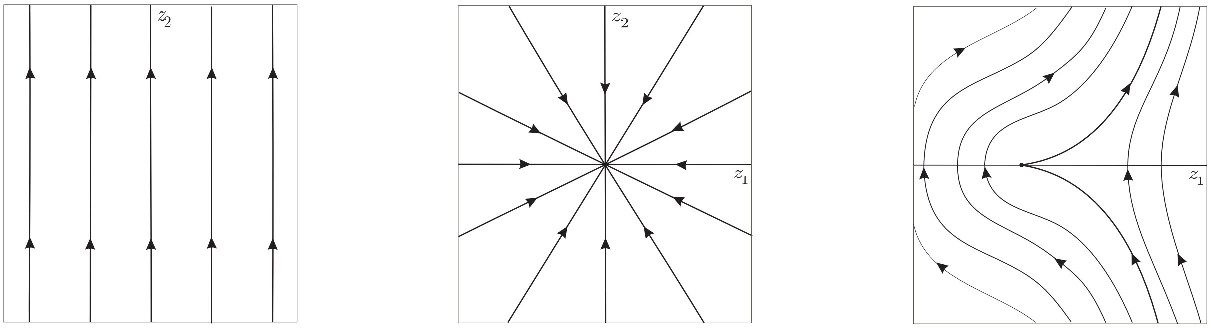

Figure 3: Phase portrait of system (1) on the Poincaré sphere at infinity in the local charts U_1 (on the left-hand side), U_2 (on the center) and U_3 (on the right-hand side).

The origin is the unique equilibria of system (13) and it is a nilpotent point. Applying Theorem 3.5 in [4] we conclude that the origin is an stable node. It local dynamics on the local chart U_2 is topologically equivalent to the one shown in Figure 3.

Again the flow in the local chart V_2 is the same as the flow in the local chart U_2 shown in Figure 3 by reversing in an appropriate way the direction of the time.

Finally, the expression of the Poincaré compactification $p(X)$ of system (1) in the local chart U_3 is

$$\begin{aligned} \dot{z}_1 &= az_2 + 4z_1^2 z_3 + bz_3^2 - z_1 z_3^2, \\ \dot{z}_2 &= z_1^2 - z_2 z_3 + 4z_1 z_2 z_3 - z_2 z_3^2, \\ \dot{z}_3 &= (4z_1 - z_3) z_3^2. \end{aligned} \tag{14}$$

Observe that system (14) restricted to the invariant z_1z_2-plane reduces to

$$\begin{aligned} \dot{z}_1 &= az_2, \\ \dot{z}_2 &= z_1^2. \end{aligned}$$

The solutions of this system behave as in Figure 3 which corresponds to the dynamics of system (1) in the local chart U_3. Indeed the origin is a nilpotent equilibrium point and from Theorem 3.5 in [4] we conclude that the origin is a cusp (in this case $f(x) \equiv 0$, $F(x) = B(x, 0) = x^2$ and $G(x) \equiv 0$, with m even). The flow at infinity in the local chart V_3 is the same as the flow in the local chart U_3 reversing appropriately the time.

Proof of Theorem 2. Considering the analysis made before and gluing the flow in the three studied charts, taking into account its orientation shown in Figure 4, we have a global picture of the dynamical behavior of system (1) at infinity. The system has four equilibrium points on the sphere, two nodes and two cusps and there are no periodic orbits. We observe that the description of the complete phase portrait of system (1) on the sphere at infinity was possible because of the invariance of these sets under the flow of the compactified system. This proves Theorem 2. We remark that the behavior of the flow at infinity does not depend on the parameter b and the global phase portrait at the sphere for different values of a are topologically equivalent. □

5 Darboux integrability

In this section we prove Theorem 3. We first prove statement (a) proceeding by contradiction. Let $f \in \mathbb{C}[x, y, z] \setminus \mathbb{C}$ be an invariant algebraic surface of system (1) with cofactor k. Then $k = k_0 + k_1 x + k_2 y + k_3 z$ for some $k_0, k_1, k_2, k_3 \in \mathbb{C}$.

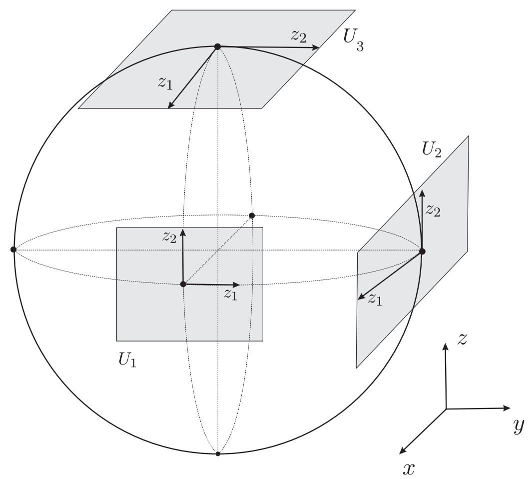

Figure 4: Orientation of the local charts U_i, $i = 1, 2, 3$ in the positive endpoints of coordinate axis x, y, z, used to drawn the phase portrait of system (1) on the Poincaré sphere at infinity (Figure 2). The charts V_i, $i = 1, 2, 3$ are diametrically opposed to U_i, in the negative endpoints of the coordinate axis.

First we will show that $k_2 = k_3 = 0$. Expanding f in powers of the variable y we get $f = \sum_{j=0}^{m} f_j(x, z)y^j$ where f_j are polynomials in the variables x, z and $m \in \mathbb{N} \cup \{0\}$. Computing the terms of y^{m+1} in (7) we obtain

$$az\frac{\partial f_m}{\partial x} = k_2 f_m.$$

Solving this linear differential equation we get

$$f_m = g_m(z)\exp\left(\frac{k_2 x}{az}\right),$$

where g_m is an arbitrary smooth function in z. Since f_m must be a polynomial we must have that either $f_m = 0$ or $k_2 = 0$. If $k_2 \neq 0$ then $f = f(x, z)$ and so by (7) it must satisfy

$$(ayz + b)\frac{\partial f}{\partial x} + (1 - 4x)\frac{\partial f}{\partial z} = (k_0 + k_1 x + k_2 y + k_3 z)f. \tag{15}$$

The linear terms in y in (15) satisfy

$$az\frac{\partial f}{\partial x} = k_2 f, \quad \text{that is,} \quad f = g(z)\exp\left(\frac{k_2 x}{az}\right),$$

for some arbitrary smooth function g. Since f must be a polynomial and $k_2 \neq 0$ we must have that $f = 0$ in contradiction with the fact that f is an invariant algebraic surface. In short, $k_2 = 0$.

Expanding f in powers of the variable z we get $f = \sum_{j=0}^{m} f_j(x, y)z^j$ where f_j are polynomials in the variables x, y and $m \in \mathbb{N} \cup \{0\}$. Computing the terms of z^{n+1} in (7) we get

$$ay\frac{\partial f_m}{\partial x} = k_3 f_m \quad \text{and so} \quad f_m = g_m(y)\exp\left(\frac{k_3 x}{ay}\right),$$

where g_m is an arbitrary smooth function in the variable y. Since f_m must be a polynomial we must have that either $f_m = 0$ or $k_3 = 0$. If $k_3 \neq 0$ then $f = f(x, y)$ and so by (7)

$$(ayz + b)\frac{\partial f}{\partial x} + (x^2 - y)\frac{\partial f}{\partial y} = (k_0 + k_1 x + k_3 z)f. \tag{16}$$

The linear terms in the variable z in (16) satisfy

$$ay\frac{\partial f}{\partial x} = k_3 f, \quad \text{that is,} \quad f = g(y)\exp\Big(\frac{k_3 x}{ay}\Big)$$

for some arbitrary smooth function g. Since f must be a polynomial and $k_3 \neq 0$ we must have that $f = 0$ in contradiction with the fact that f is an invariant algebraic surface. So, $k_3 = 0$.

Let n be the degree of f. Expanding the invariant algebraic surface f in sum of its homogeneous parts we get $f = \sum_{j=0}^{n} f_j(x, y, z)$ where each $f_j(x, y, z)$ is a homogeneous polynomial in x, y, z of degree j. Without loss of generality we can assume that $f_n \neq 0$ and $n \geq 1$.

Computing the terms of degree $n+1$ in (7) we have

$$ayz\frac{\partial f_n}{\partial x} + x^2\frac{\partial f_n}{\partial y} = k_1 x f_n \tag{17}$$

or in other words, if we consider the linear partial differential operator of the form

$$M = ayz\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}, \tag{18}$$

then equation (17) can be written as

$$M f_n = k_1 x f_n. \tag{19}$$

The characteristic equations associated with the linear partial differential equation in (19) are

$$\frac{dz}{dy} = 0, \quad \frac{dx}{dy} = \frac{ayz}{x^2}.$$

This system of equations has the general solution

$$z = d_1, \quad \frac{x^3}{3} - az\frac{y^2}{2} = d_2,$$

where d_1 and d_2 are constants of integration. According with the method of characteristics, we make the change of variables

$$u = \frac{x^3}{3} - az\frac{y^2}{2}, \quad v = y, \quad w = z. \tag{20}$$

Its inverse transformation is

$$x = \Big(3u + 3aw\frac{v^2}{2}\Big)^{1/3}, \quad y = v, \quad z = w. \tag{21}$$

Under changes (20) and (21), equation (19) becomes the following ordinary differential equation (for fixed u, w):

$$\Big(3u + 3aw\frac{v^2}{2}\Big)^{1/3}\frac{d\bar{f}_n}{dv} = k_1 \bar{f}_n,$$

where $\bar{f}_n$ is f_n, written in the variables u, v and w. In what follows, we always use $\bar{\theta}$ to denote a function $\theta(x, y, z)$ written in terms of the variables u, v and w. Using that

$$\int \frac{dv}{\Big(3u + 3aw\frac{v^2}{2}\Big)^{1/3}} = \frac{2^{1/3}}{3^{1/3}} v u^{-1/3} F_1\Big(\frac{1}{3}, \frac{1}{2}, \frac{3}{2}, -\frac{av^2 w}{u}\Big), \tag{22}$$

where F_1 is the hypergeometric function defined as

$$F_1(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \quad \text{with} \quad (x)_k = x(x+1)\cdots(x+k-1) \tag{23}$$

we get that

$$\bar{f}_n = k_1 2^{1/3} \bar{g}_n(u,w) v u^{-1/3} F_1\Big(\frac{1}{3}, \frac{1}{2}, \frac{3}{2}, -\frac{av^2 w}{u}\Big),$$

being $\bar{g}_n$ an arbitrary smooth function in u and w. So,

$$f_n(x,y,z) = \bar{f}_n(u,v,w) = \bar{g}_n\Big(\frac{x^3}{3} - az\frac{y^2}{2}, z\Big) \frac{k_1 2^{1/3} y}{\big(\frac{x^3}{3} - aw\frac{y^2}{2}\big)^{1/3}} F_1\Big(\frac{1}{3}, \frac{1}{2}, \frac{3}{2}, -\frac{ay^2 z}{\frac{x^3}{3} - az\frac{y^2}{2}}\Big).$$

Note that it follows from (23) that $F_1\Big(\frac{1}{3}, \frac{1}{2}, \frac{3}{2}, -\frac{ay^2 z}{\frac{x^3}{3} - az\frac{y^2}{2}}\Big)$ is never a polynomial. So, in order that f_n is a homogeneous polynomial of degree n we must have that $k_1 = 0$ and $\bar{f}_n$ is a polynomial in the variables u and w. Consequently, the cofactor of every invariant algebraic surface of system (1) is constant, i.e, $k = k_0$ and

$$f_n = \sum_{l=0}^{[n/3]} a_l z^{n-3l} \Big(\frac{x^3}{3} - az\frac{y^2}{2}\Big)^l, \tag{24}$$

where $[\cdot]$ stands for the integer part function of a real number.

The terms of degree n in (7) are

$$\begin{aligned} M f_{n-1} &= k_0 f_n + y\frac{\partial f_n}{\partial y} + 4x\frac{\partial f_n}{\partial z} \\ &= k_0 \sum_{l=0}^{[n/3]} a_l z^{n-3l} \Big(\frac{x^3}{3} - az\frac{y^2}{2}\Big)^l - azy^2 \sum_{l=0}^{[n/3]} a_l l z^{n-3l} \Big(\frac{x^3}{3} - az\frac{y^2}{2}\Big)^{l-1} \\ &+ 4x \sum_{l=0}^{[n/3]} a_l (n-3l) z^{n-3l-1} \Big(\frac{x^3}{3} - az\frac{y^2}{2}\Big)^l - 2axy^2 \sum_{l=0}^{[n/3]} a_l l z^{n-3l} \Big(\frac{x^3}{3} - az\frac{y^2}{2}\Big)^{l-1}. \end{aligned} \tag{25}$$

Using transformations (20) and (21) and working in a similar way to solve $\bar{f}_n$ we get the following ordinary differential equation (for fixed u and w):

$$\begin{aligned} \Big(3u + 3aw\frac{v^2}{2}\Big)^{2/3} \frac{d\bar{f}_{n-1}}{dv} &= k_0 \sum_{l=0}^{[n/3]} a_l w^{n-3l} u^l - awv^2 \sum_{l=0}^{[n/3]} a_l l w^{n-3l} u^{l-1} \\ &+ 4\Big(3u + 3aw\frac{v^2}{2}\Big) \sum_{l=0}^{[n/3]} a_l (n-3l) w^{n-3l-1} u^l - 2a\Big(3u + 3aw\frac{v^2}{2}\Big) v^2 \sum_{l=0}^{[n/3]} a_l l w^{n-3l} u^{l-1}. \end{aligned}$$

Integrating this equation with respect to v and using the formula in (22) together with

$$\int \frac{dv}{\Big(3u + 3aw\frac{v^2}{2}\Big)^{2/3}} = \frac{2^{2/3}}{3^{2/3}} v u^{-2/3} F_1\Big(\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, -\frac{av^2 w}{u}\Big),$$

$$\int \frac{v^2 dv}{\Big(3u + 3aw\frac{v^2}{2}\Big)^{1/3}} = \frac{2^{1/3} 3^{2/3}}{7aw} v(u + av^2 w)^{2/3} - \frac{2^{1/3} 3^{2/3}}{7aw} v u^{2/3} F_1\Big(\frac{1}{3}, \frac{1}{2}, \frac{3}{2}, -\frac{av^2 w}{u}\Big),$$

$$\int \frac{v^2 dv}{\Big(3u + 3aw\frac{v^2}{2}\Big)^{2/3}} = \frac{2^{2/3} 3^{1/3}}{5aw} v(u + av^2 w)^{1/3} - \frac{2^{2/3} 3^{1/3}}{5aw} v u^{1/3} F_1\Big(\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, -\frac{av^2 w}{u}\Big),$$

we obtain

$$\bar{f}_{n-1} = \bar{g}_{n-1}(u,w) + \frac{2^{2/3}}{3^{2/3}} vF_1\Big(\frac{1}{2},\frac{2}{3},\frac{3}{2},-\frac{av^2w}{u}\Big)\sum_{l=0}^{[n/3]} a_l\Big(k_0+\frac{3l}{5}\Big)w^{n-3l}u^{l-2/3}$$
$$-\frac{2^{2/3}3^{1/3}}{5}v(u+av^2w)^{1/3}\sum_{l=0}^{[n/3]} a_l l w^{n-3l}u^{l-1} - \frac{2^{4/3}3^{2/3}}{7}v(u+av^2w)^{2/3}\sum_{l=0}^{[n/3]} a_l l w^{n-3l-1}u^{l-1}$$
$$+2^{4/3}3^{2/3}vF_1\Big(\frac{1}{3},\frac{1}{2},\frac{3}{2},-\frac{av^2w}{u}\Big)\sum_{l=0}^{[n/3]} a_l\Big(4(n-3l)+\frac{6}{7}l\Big)w^{n-3l-1}u^{l-1/3},$$

where $\bar{g}_{n-1}$ is an arbitrary smooth function in u and w. Since f_{n-1} must be a homogeneous polynomial of degree $n-1$ and neither $F_1\Big(\frac{1}{2},\frac{2}{3},\frac{3}{2},-\frac{av^2w}{u}\Big)$ nor $F_1\Big(\frac{1}{3},\frac{1}{2},\frac{3}{2},-\frac{av^2w}{u}\Big)$ are polynomials we must have

$$\begin{aligned} &a_l\Big(k_0+\frac{3l}{5}\Big) \quad \text{for } l=0,\dots,[n/3],\\ &a_l\Big(4(n-3l)+\frac{6}{7}l\Big) \quad \text{for } l=0,\dots,[n/3]. \end{aligned} \tag{26}$$

Note that the second condition in (26) implies that

$$a_l = 0 \quad \text{for } l=0,\dots,[n/3]$$

because $l \ge 0$ and $n-3l \ge 0$ with $n \ge 1$. Therefore, from (24) we get that $f_n = 0$ which is not possible. Note that a polynomial first integral is an algebraic invariant surface with cofactor $k=0$. Proceeding as above with $k_0=k_1=k_2=k_3=0$ we also obtain that there are no polynomial first integrals. This concludes the proof of statement (a).

Now we prove statement (b). Let $E=\exp(f/g)\notin\mathbb{C}$ be an exponential factor of system (1) with cofactor L. Then $L=L_0+L_1x+L_2y+L_3z$, where $f,g\in\mathbb{C}[x,y,z]$ with $(f,g)=1$. From Theorem 3(a) and Lemma 5, we have that $E=\exp(f)$ with $f=f(x,y,z)\in\mathbb{C}[x,y,z]\notin\mathbb{C}$.

It follows from equation (8) that f satisfies

$$(ayz+b)\frac{\partial f}{\partial x}+(x^2-y)\frac{\partial f}{\partial y}+(1-4x)\frac{\partial f}{\partial z}=L_0+L_1x+L_2y+L_3z, \tag{27}$$

where we have simplified the common factor $\exp(f)$.

Let n be the degree of f. We write $f=\sum_{i=0}^n f_i(x,y,z)$, where f_i is a homogeneous polynomial of degree i. Without loss of generality we can assume that $f_n\neq 0$. Assume $n>1$. Computing the terms of degree $n+1$ in (8) we obtain

$$ayz\frac{\partial f_n}{\partial x}+x^2\frac{\partial f_n}{\partial y}=0$$

or using the operator M in (18) we have $Mf_n=0$. Proceeding as we did to solve (19) we obtain that f_n becomes as in (24). Computing the terms of degree n in (27) we obtain

$$Mf_{n-1}=y\frac{\partial f_n}{\partial y}+4x\frac{\partial f_n}{\partial z}=-azy^2\sum_{l=0}^{[n/3]}a_l l z^{n-3l}\Big(\frac{x^3}{3}-az\frac{y^2}{2}\Big)^{l-1}$$
$$+4x\sum_{l=0}^{[n/3]}a_l(n-3l)z^{n-3l-1}\Big(\frac{x^3}{3}-az\frac{y^2}{2}\Big)^{l}-2axy^2\sum_{l=0}^{[n/3]}a_l l z^{n-3l}\Big(\frac{x^3}{3}-az\frac{y^2}{2}\Big)^{l-1}.$$

which is (25) with $k_0=0$. Proceeding exactly in the same way as we did to solve (25) we get that $f_n=0$, which is not possible. So $n=1$.

We can write $f = a_1x + a_2y + a_3z$ with $a_i \in \mathbb{C}$. Imposing that f must satisfy (27) we get $f = a_3z$ with cofactor $a_3(1 - 4x)$. This concludes the proof of statement (b) of Theorem 3.

It follows from Proposition 4 and statements (a) and (b) that if system (1) has a Darboux first integral then there exist $\mu \in \mathbb{C} \setminus \{0\}$ such that (9) holds, that is, such that $\mu(1 - 4x) = 0$. But this is not possible. Hence, there are no Darboux first integrals for system (1) and the proof of statement (c) of Theorem 3 is completed.

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