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The R_∞-property for braid groups over orientable surfaces

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Abstract
Let $\Sigma_{g,p}$ be an orientable surface of genus g and of finite type without boundary (i.e. an orientable closed surface with a finite number p of points removed). In this paper we study the R_∞-property for the surface pure braid groups $P_n(\Sigma_{g,p})$ as well as for the full surface braid groups $B_n(\Sigma_{g,p})$. We show that, with few exceptions, these groups have the R_∞-property.

Keywords Artin braid group · Surface braid group · R_∞-property

Mathematics Subject Classification Primary: 20E36; Secondary: 20F36 · 20E45

1 Introduction

Consider a group G and a fixed endomorphism φ of G. Two elements x and y of G are said to be twisted conjugate (via φ) if and only if there exists a $z \in G$ such that $x = zy\varphi(z)^{-1}$. The relation of being twisted conjugate is easily seen to be an equivalence relation and the number of equivalence classes (also referred to as twisted conjugacy classes or Reidemeister classes) is called the Reidemeister number $R(\varphi)$ of φ. This Reidemeister number is either a positive integer or ∞.

These Reidemeister numbers appear naturally in algebraic topology and to be more precise in Nielsen–Reidemeister fixed point theory. Here one is interested in the num-

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ber of fixed point classes of a selfmap f of a space X. This number is called the Reidemeister number $R(f)$ of the map f, and one can show that $R(f) = R(f_*)$, where $f_*\colon \pi_1(X) \longrightarrow \pi_1(X)$ is the induced endomorphism on the fundamental group $\pi_1(X)$ of X.

A group G is said to have the R_∞-property in case $R(\varphi) = \infty$ for all automorphisms $\varphi \in \mathrm{Aut}\,(G)$. The study of groups with that property was initiated by Fel'shtyn and Hill [20] and since the beginning of this century there has been a growing interest in the study of groups having this R_∞-property.

A non-exhaustive list of examples of groups of which we know that they have the R_∞-property, are the non-elementary Gromov hyperbolic groups [16, 30], most of the Baumslag–Solitar groups [17] and groups quasi–isometric to Baumslag–Solitar groups [37], generalized Baumslag–Solitar groups [31], many linear groups [21, 34], several families of lamplighter groups [27, 38], some spherical and affine Artin-Tits groups [8], pure virtual twin groups [33], and virtual braid (twin) groups [11].

Emil Artin introduced the braid groups of the 2-disc in 1925 and continued the study of them in 1947 [2, 3]. These groups have since then been referred to as Artin Braid groups. Zariski [40] was the first to study braids on surfaces and this was later further extended by Fox and Neuwirth to braid groups of arbitrary topological spaces by using configuration spaces as follows [22]. Let M be a topological space, and let $n \in \mathbb{N}$. The *nth ordered configuration space of* M, denoted by $F_n(M)$, is defined by:

$$F_n(M) = \left\{(x_1, \ldots, x_n) \in M^n \mid x_i \neq x_j \text{ if } i \neq j,\ i, j = 1, \ldots, n\right\}.$$

The *n-string pure braid group $P_n(M)$ of M* is defined by $P_n(M) = \pi_1(F_n(M))$. The symmetric group S_n on n letters acts freely on $F_n(M)$ by permuting coordinates, giving rise to the nth unordered configuration space $F_n(M)/S_n$. The *n-string braid group $B_n(M)$ of M* is then defined as $B_n(M) = \pi_1(F_n(M)/S_n)$. This gives rise to the following short exact sequence:

$$1 \longrightarrow P_n(M) \longrightarrow B_n(M) \xrightarrow{\sigma} S_n \longrightarrow 1. \tag{1}$$

The map $\sigma : B_n(M) \longrightarrow S_n$ is the standard homomorphism that associates to any braid in $B_n(M)$ a permutation in S_n and $\mathrm{Ker}(\sigma) = P_n(M)$.

When $M = D^2$ (the disc) then $B_n(D^2)$ (resp. $P_n(D^2)$) is the classical Artin braid group denoted by B_n (resp. the classical pure Artin braid group denoted by P_n).

The R_∞-property was studied for Artin braid groups in [18] for the whole group B_n and in [10] for the pure subgroup P_n. Let $S_{g,p}$ be a surface of finite type, i.e. $S_{g,p}$ is a closed surface of genus g (possibly non-orientable) with a finite number ($p \geq 0$) of points removed. After having obtained the results for the Artin braid groups, it is now a natural question to study the R_∞-property for the surface braid groups (resp. surface pure braid groups) $B_n(S_{g,p})$ (resp. $P_n(S_{g,p})$). For the case where $n = 1$ we have that $P_1(S_{g,p}) = B_1(S_{g,p}) = \pi_1(S_{g,p})$ and here the result is well known, and the information (in the orientable case) is given in the tables below. So, from now on, we will assume that $n \geq 2$ unless it is explicitly stated otherwise.

In this paper we will study the R_∞ property only for the case of orientable surfaces. To do this, we divide the orientable surfaces of finite type into three families.

$\mathcal{F}_1$: The punctured sphere $\mathbb{S}^2$ with p points removed for $p = 0, 1, 2$.
$\mathcal{F}_2$: a) Orientable closed surfaces different from $\mathbb{S}^2$, T^2.
b) Orientable punctured surfaces $\Sigma_{g,p}$ where g is the genus and p is the number of punctures in the closed surface Σ_g, for:
i) $g = 0$ and $p \geq 3$,
ii) $g = 1$ and $p \geq 2$,
iii) $g \geq 2$ and $p \geq 1$.
$\mathcal{F}_3$: The torus $\Sigma_{1,0} = T^2$ and $\Sigma_{1,1}$ the torus minus one point.

In Table 1 we record the information that we know until now about the R_∞-property for the surface braid groups, $P_n(\Sigma_{g,p})$, $B_n(\Sigma_{g,p})$.

Remark 1 The exceptional cases which appear in the table above come from the fact that for $n \geq 2$, the groups $P_n(\Sigma_{0,1})$, $P_n(\Sigma_{0,2})$, $B_n(\Sigma_{0,1})$, $B_n(\Sigma_{0,2})$ and $B_n(\Sigma_{0,3})$ have the R_∞-property (see [8, Theorem 1]), since there are isomorphisms among some surface braid groups and Artin-Tits groups: $P_n(\Sigma_{0,1}) \cong P(A_{n-1})$, $P_n(\Sigma_{0,2}) \cong P(B_n)$, $B_n(\Sigma_{0,1}) \cong A(A_{n-1})$, $B_n(\Sigma_{0,2}) \cong A(B_n)$ and $B_n(\Sigma_{0,3}) \cong A(\widetilde{C}_n)$. In the case of $\Sigma_{0,1}$ the result was first demonstrated in [18] for the Artin braid group and in [10] for the pure Artin braid group.

The reason for dividing the surfaces into these three families is because we need different techniques to deal with the surfaces of family $\mathcal{F}_1$ and those of family $\mathcal{F}_2$. The paper does not contain new results on the two braid groups of the two surfaces of family $\mathcal{F}_3$ as this is still work in progress.

The main results of this paper are formulated below.

Theorem 2 *Let $\Sigma_{g,p}$ be a finite type surface which belongs to $\mathcal{F}_1 \cup \mathcal{F}_2$. The surface pure braid group $P_n(\Sigma_{g,p})$ has the R_∞-property if and only if one of the statements below holds:*

1. *$\Sigma_{g,p}$ belongs to $\mathcal{F}_2$ and $n \geq 1$,*
2. *$\Sigma_{g,p} = \Sigma_{0,0} = \mathbb{S}^2$ and $n \geq 4$,*
3. *$\Sigma_{g,p} = \Sigma_{0,1} = \mathbb{S}^2 \backslash \{x_1\}$ and $n \geq 3$,*
4. *$\Sigma_{g,p} = \Sigma_{0,2} = \mathbb{S}^2 \backslash \{x_1, x_2\}$ and $n \geq 2$.*

Table 1 The R_∞-property for $P_n(\Sigma_{g,p})$ and $B_n(\Sigma_{g,p})$ before this paper

Family	π_1 has R_∞	$P_n(\Sigma_{g,p})$ has R_∞	$B_n(\Sigma_{g,p})$ has R_∞
$\mathcal{F}_1$	No	Unknown except, yes for $S = \Sigma_{0,2}$ and $n \geq 2$, and yes for $S = \Sigma_{0,1}$ iff $n \geq 3$	Unknown except, yes for $S = \Sigma_{0,2}$ and $n \geq 2$, and yes for $S = \Sigma_{0,1}$ iff $n \geq 3$
$\mathcal{F}_2$	Yes	Unknown	Unknown except, yes for $S = \Sigma_{0,3}$
$\mathcal{F}_3$	No for T^2 Yes for $\Sigma_{1,1}$	Unknown	Unknown

Table 2 The R_∞-property for $P_n(\Sigma_{g,p})$ and $B_n(\Sigma_{g,p})$ after this paper

Family	π_1 has R_∞	$P_n(\Sigma_{g,p})$ has R_∞	$B_n(\Sigma_{g,p})$ has R_∞
$\mathcal{F}_1$	No	Yes for most; No for few cases	Yes for most; No for few cases
$\mathcal{F}_2$	Yes	Yes	Yes
$\mathcal{F}_3$	No for T^2 Yes for $\Sigma_{1,1}$	Unknown	Unknown

In order to prove the result for the whole group $B_n(\Sigma_{g,p})$, stated in the next theorem, we shall use Theorem 2 and the following useful result: for all surfaces $S_{g,p}$ (orientable or not), $P_n(S_{g,p})$ is characteristic in $B_n(S_{g,p})$ with one exception, which is when $S_{g,p} = \Sigma_{0,2}$ and $n = 2$, see [1, Theorem 1.5].

Theorem 3 *Let $\Sigma_{g,p}$ be a surface which belongs to $\mathcal{F}_1 \cup \mathcal{F}_2$. The braid group $B_n(\Sigma_{g,p})$ has the R_∞-property, if and only if $P_n(\Sigma_{g,p})$ has the R_∞-property.*

Table 2 summarises the information obtained in this work as well the status of the question studied here for a finite type orientable surface $\Sigma_{g,p}$.

Remark 4 In the Table 2, for the family $\mathcal{F}_1$ the cases where $P_n(\Sigma_{g,p})$ $(n \geq 2)$ does not have the R_∞-property are precisely the cases $\mathbb{S}^2$ for $n = 2, 3$ and $S = D^2$ for $n = 2$. The same holds for $B_n(\Sigma_{g,p})$.

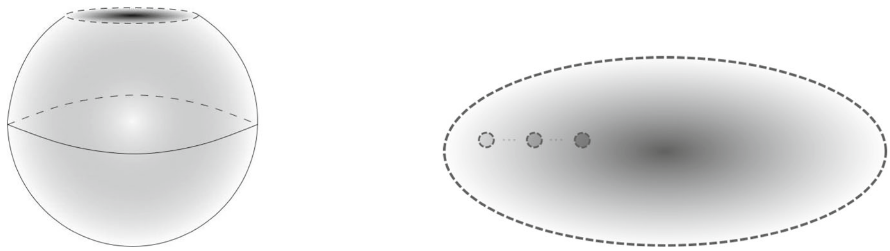
This paper is organised as follows. In Sect. 2 we show that for any finite type surface $S_{g,p}$ (orientable or not) of genus $g \geq 0$ with $p \geq 0$ points removed, there is a short exact sequence

$$1 \longrightarrow N \longrightarrow P_n(S_{g,p}) \longrightarrow \Pi_{i=1}^n(\pi_1(S_{g,p})) \longrightarrow 1 \tag{2}$$

where N is the normal closure of the Artin pure braid group in $P_n(S_{g,p})$. Then, in Sect. 3 we prove that the sequence (2) is characteristic for the surfaces of the family $\mathcal{F}_2$, being different from the sphere minus three points. In Sect. 4 we prove Theorem 2 and Theorem 3, these are the main results of the paper about the R_∞-property for orientable surface braid groups.

2 Goldberg's short exact sequence for pure braid groups over punctured surfaces (orientable or not)

Let $S_{g,p}$ be a closed surface (orientable or not) of genus $g \geq 0$ with $p \geq 0$ punctures. Let $D \subset S_{g,p}$ be a subset which is homeomorphic to the open disc of radius 1 of the plane. Denote by $D \overset{i}{\hookrightarrow} S_{g,p}$ the inclusion, and let $(z_1, \cdots, z_n)$ be a base point of $F_n(D)$ and of $F_n(S_{g,p})$. This inclusion induces a morphism $i_\#\colon P_n(D) \longrightarrow P_n(\Sigma_{g,p})$. Making use of such an embedding, in this section we prove that the pure braid groups



(a) Once-punctured. (b) With $p \geq 2$ punctures.

Fig. 1 The punctured sphere

over punctured surfaces fit into a short exact sequence of Goldberg's type [23], see Theorem 8. The proof is algebraic and in order to do that we shall use the presentations of $P_n(S_{g,p})$ given in [29, Theorem 1] for the case of the punctured sphere, in [4, Theorem 5.1] for the punctured connected sum of tori and in [12, Theorem 4.7] for the case of the punctured connected sum of projective planes. In the first three subsections we discuss in more details these presentations, where we also indicate how to identify $i_\#(P_n(D))$ in each of the surface braid groups, and in the last subsection we deal with the short exact sequence of Goldberg's type.

2.1 The punctured sphere

The once-punctured sphere is homeomorphic to the (open) disc, see Figure 1 (a), and in this case we get the classical Artin pure braid groups [2].

Let $p \geq 2$. Using the notation of [29] and considering the p-punctured sphere as being the $(p-1)$-punctured disc (see Fig. 1 (b)), one can easily see that $P_n(\Sigma_{0,p}) = P_{p-1,n}$ (so $m = p-1$). Then [29, Theorem 1] provides a presentation of $P_n(\Sigma_{0,p}) = P_{p-1,n}$, with a set of generators

$$\left\{A_{i,j} \mid 1 \leq i \leq p+n-2,\ p \leq j \leq p+n-1 \text{ and } i < j\right\}$$

subject to the following relations:

(P1) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{r,s}$ if $(i < j < r < s)$ or $(r < i < j < s)$.
(P2) $A_{i,j}^{-1} A_{j,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1}$ if $(i < j < s)$.
(P3) $A_{i,j}^{-1} A_{i,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$ if $(i < j < s)$.
(P4) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1} A_{j,s}^{-1} A_{r,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$ if $(i < r < j < s)$.

Remark 5 In Fig. 2 we illustrate geometrically the generators $A_{i,j}$ of the group $P_n(\Sigma_{0,p})$, for $1 \leq i \leq p+n-2$, $p \leq j \leq p+n-1$ and $i < j$. We note that the set $\left\{A_{i,j} \mid p \leq i < j \leq p+n-1\right\}$ corresponds to the set of Artin generators inside $P_n(\Sigma_{0,p})$.

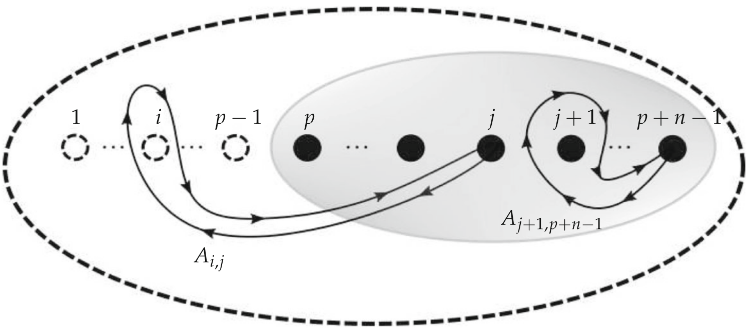


Fig. 2 Generator $A_{i,j}$ for $1 \leq i \leq p+n-2$, $p \leq j \leq p+n-1$ and $i < j$

2.2 The punctured connected sum of tori

Let $p \geq 1$ and $g \geq 1$. We shall use the presentation of $P_n(\Sigma_{g,p})$, the pure braid group of the p-punctured connected sum of g tori $\Sigma_{g,p}$, as given in [4, Theorem 5.1]. We note that this presentation had a few misprints which were corrected in [6, Theorem 12] and after private communication with P. Bellingeri ([5]) we fixed one more typo here. A set of generators of $P_n(\Sigma_{g,p})$ given in [4, Theorem 5.1] is

$$\left\{A_{i,j} \mid 1 \leq i \leq 2g+p+n-2,\ 2g+p \leq j \leq 2g+p+n-1,\ i < j\right\}$$

subject to the relations

(PR1) $A_{i,j}^{-1}A_{r,s}A_{i,j} = A_{r,s}$ if $(i < j < r < s)$ or $(r+1 < i < j < s)$, or $(i = r+1 < j < s$ for even $r < 2g$ or $r \geq 2g)$.

(PR2) $A_{i,j}^{-1}A_{j,s}A_{i,j} = A_{i,s}A_{j,s}A_{i,s}^{-1}$ if $(i < j < s)$.

(PR3) $A_{i,j}^{-1}A_{i,s}A_{i,j} = A_{i,s}A_{j,s}A_{i,s}A_{j,s}^{-1}A_{i,s}^{-1}$ if $(i < j < s)$.

(PR4) $A_{i,j}^{-1}A_{r,s}A_{i,j} = A_{i,s}A_{j,s}A_{i,s}^{-1}A_{j,s}^{-1}A_{r,s}A_{j,s}A_{i,s}A_{j,s}^{-1}A_{i,s}^{-1}$ if $(i+1 < r < j < s)$ or $(i+1 = r < j < s$ for odd $r < 2g$ or $r > 2g)$.

(ER1) $A_{r+1,j}^{-1}A_{r,s}A_{r+1,j} = A_{r,s}A_{r+1,s}A_{j,s}^{-1}A_{r+1,s}^{-1}$ if r odd, $r < 2g$ and $r+1 < j < s$.

(ER2) $A_{r-1,j}^{-1}A_{r,s}A_{r-1,j} = A_{r-1,s}A_{j,s}A_{r-1,s}^{-1}A_{r,s}A_{j,s}A_{r-1,s}A_{j,s}^{-1}A_{r-1,s}^{-1}$ if r even, $r \leq 2g$ and $r-1 < j < s$.

Remark 6 In Fig. 3 we illustrate geometrically the generators $A_{i,j}$ of the group $P_n(\Sigma_{g,p})$, for $1 \leq i \leq 2g+p+n-2$, $2g+p \leq j \leq 2g+p+n-1$ and $i < j$. We note that the set $\left\{A_{i,j} \mid 2g+p \leq i < j \leq 2g+p+n-1\right\}$ corresponds to the set of Artin generators inside $P_n(\Sigma_{g,p})$.

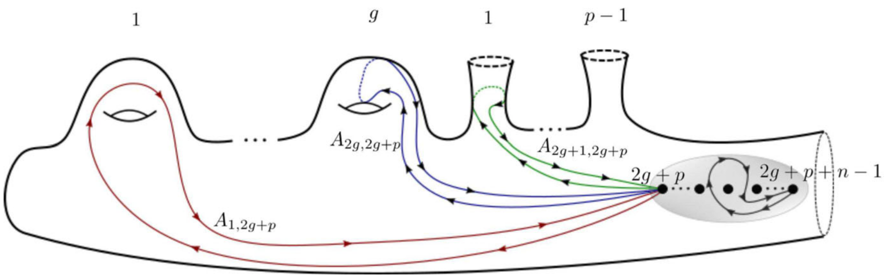


Fig. 3 Generator $A_{i,j}$ for $1 \leq i \leq 2g + p + n - 2$, $2g + p \leq j \leq 2g + p + n - 1$ and $i < j$

2.3 The punctured non orientable surfaces

For this case we use the presentation given in [12, Theorem 4.7]. Let $g,\ p \geq 1$. A set of generators of $P_n(N_{g,p})$ is given by $A_{i,j}$ and $\rho_{r,k}$ for $1 \leq i < j$, $p+1 \leq j$, $r \leq p+n$ and $1 \leq k \leq g$ subject to the relations

(a) The "Artin-type relations"

$$A_{r,s}A_{i,j}A_{r,s}^{-1} = \begin{cases} A_{i,j} & \text{if } (i < r < s < j) \text{ or } \\ & \quad (r < s < i < j) \\ A_{s,j}^{-1}A_{i,j}A_{s,j} & \text{if } i = r < s < j \\ A_{i,j}^{-1}A_{r,j}^{-1}A_{i,j}A_{r,j}A_{i,j} & \text{if } r < i = s < j \\ A_{s,j}^{-1}A_{r,j}^{-1}A_{s,j}A_{r,j}A_{i,j}A_{r,j}^{-1}A_{s,j}^{-1}A_{r,j}A_{s,j} & \text{if } r < i < s < j. \end{cases}$$

(b) For every $p+1 \leq i < j \leq p+n$ and $1 \leq k,\ l \leq g$

$$\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1} = \begin{cases} \rho_{j,l} & \text{if } k < l \\ \rho_{j,k}^{-1}A_{i,j}^{-1}\rho_{j,k}^{2} & \text{if } k = l \\ \rho_{j,k}^{-1}A_{i,j}^{-1}\rho_{j,k}A_{i,j}^{-1}\rho_{j,l}A_{i,j}\rho_{j,k}^{-1}A_{i,j}\rho_{j,k} & \text{if } k > l. \end{cases}$$

(c) The surface relation, for every $p+1 \leq j \leq p+n$

$$\prod_{l=1}^{g}\rho_{j,l}^{2} = \left(\prod_{i=1}^{j-1}A_{i,j}\right)\left(\prod_{s=1+j}^{p+n}A_{j,s}\right).$$

Note that when $j = p+n$ then in the right-hand side of the equality the second factor disappears.

(d) For every $1 \leq i < j$, $p+1 \leq j$, $k \leq p+n$, $k \neq j$ and $1 \leq l \leq g$

$$\rho_{k,l}A_{i,j}\rho_{k,l}^{-1} = \begin{cases} A_{i,j} & \text{if } k < i \text{ or } j < k \\ \rho_{j,l}^{-1}A_{i,j}^{-1}\rho_{j,l} & \text{if } k = i \\ \rho_{j,l}^{-1}A_{k,j}^{-1}\rho_{j,l}A_{k,j}^{-1}A_{i,j}A_{k,j}\rho_{j,l}^{-1}A_{k,j}\rho_{j,l} & \text{if } i < k < j. \end{cases}$$

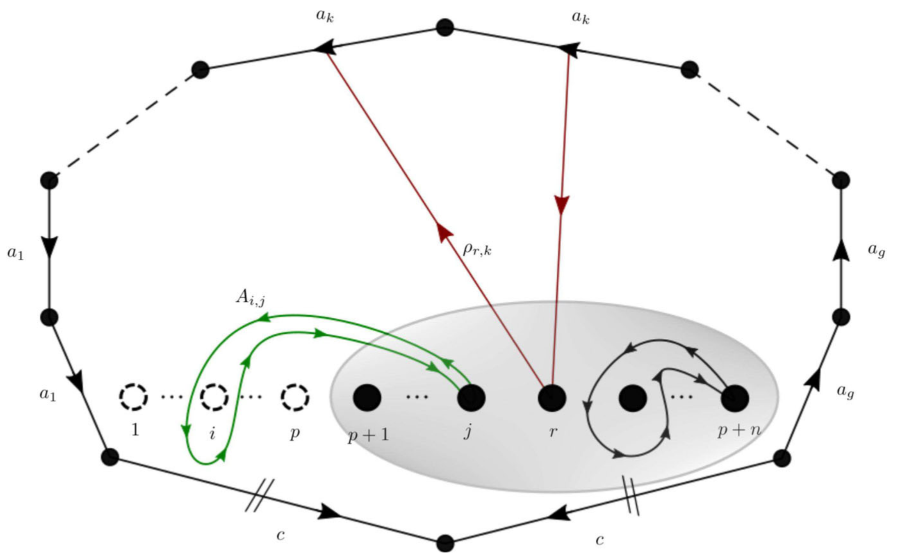


Fig. 4 Generators $A_{i,j}$ and $\rho_{r,k}$ for $1 \leq i < j$, $p+1 \leq j$, $r \leq p+n$ and $1 \leq k \leq g$

Remark 7 In Fig. 4 we illustrate geometrically the generators $A_{i,j}$ and $\rho_{r,k}$ of the group $P_n(N_{g,p})$, for $1 \leq i < j$, $p+1 \leq j$, $r \leq p+n$ and $1 \leq k \leq g$. We note that the set $\left\{A_{i,j} \mid p+1 \leq i < j \leq p+n\right\}$ corresponds to the set of Artin generators inside $P_n(N_{g,p})$.

2.4 Goldberg's short exact sequence for surface pure braid groups

The short exact sequence of the following theorem is known for closed surfaces, see [23] for $S_{g,p} \neq \mathbb{S}^2, \mathbb{R}P^2$ and [26] for $S = \mathbb{R}P^2$. We extend it here to the case of punctured surfaces.

Theorem 8 *Let $S_{g,p}$ be a closed surface (orientable or not) of genus $g \geq 0$ with $p \geq 0$ points removed. Let N denote the normal subgroup of $P_n(S_{g,p})$ generated by the image of the Artin pure braids via the inclusion of the disc D into $S_{g,p}$, $i\colon D \hookrightarrow S_{g,p}$. Then*

$$1 \longrightarrow N \longrightarrow P_n(S_{g,p}) \longrightarrow \Pi_{i=1}^{n}(\pi_1(S_{g,p})) \longrightarrow 1 \tag{3}$$

is a short exact sequence.

Proof For $\Sigma_{0,0} = \mathbb{S}^2$ this is obvious since $P_n(\mathbb{S}^2)$ is generated by the images of the pure Artin braids. For $S_{g,0} \neq \mathbb{S}^2, \mathbb{R}P^2$ this was proved by Goldberg [23] (and this was in the orientable case a conjecture of Birman [7]). For the case $N_{1,0} = \mathbb{R}P^2$ see Gonçalves and Guaschi [26].

For the remaining cases (of punctured surfaces) the proof is algebraic by considering the presentations for the given groups given in the subsections above. Let $p \geq 1$.

Case 1. First we prove the result for the punctured sphere. We consider in this case $p \geq 2$ since the pure braid group of the 1-punctured sphere $P_n(\Sigma_{0,1})$ is exactly the pure braid group of the disc $P_n(D)$ and then the result is trivial. Let $p \geq 2$. We shall use the presentation of $P_n(\Sigma_{0,p})$ given in Subsection 2.1. We note that, in this presentation of $P_n(\Sigma_{0,p})$, the generators coming from the Artin generators of $P_n(D)$, via the inclusion $i\colon D \hookrightarrow \Sigma_{0,p}$, are the elements $A_{i,j}$ for $p \leq i < j \leq p+n-1$. Now, we will describe a presentation of the quotient of $P_n(\Sigma_{0,p})$ by the normal closure subgroup generated by the Artin pure braids $A_{i,j}$, with $p \leq i < j \leq p+n-1$, that we called N.

Let us add the relations $A_{i,j} = 1$ for $p \leq i < j \leq p+n-1$ to the presentation of $P_n(\Sigma_{0,p})$ given in Subsection 2.1. This implies that for the quotient group $P_n(\Sigma_{0,p})/N$ we can take $\left\{A_{i,j} \mid 1 \leq i \leq p-1,\ p \leq j \leq p+n-1\right\}$ as the set of generators. The relations (P2) become trivial in the quotient and the relations (P1), (P3) and (P4) lead to the following relations respectively

(QR1) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{r,s}$ if $(r < i < j < s)$.
(QR2) $A_{i,j}^{-1} A_{i,s} A_{i,j} = A_{i,s}$ if $(i < j < s)$.
(QR3) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{r,s}$ if $(i < r < j < s)$.

We can actually collect the relations (QR1), (QR2) and (QR3) together to obtain the relations

(QR) $[A_{i,j}, A_{r,s}] = 1$ if $(1 \leq r, i \leq p-1)$ and $(p \leq j < s \leq p+n-1)$.

This shows that for every $\ell \in \{p, \ldots, p+n-1\}$ the group T_ℓ generated by the set $\left\{A_{k,\ell} \mid 1 \leq k \leq p-1\right\}$ is a free group of rank $p-1$ and

$$P_n(\Sigma_{0,p})/N \cong T_1 \oplus T_2 \oplus \cdots \oplus T_n \cong \left(\pi_1(\Sigma_{0,p})\right)^n .$$

This concludes the proof for the punctured sphere.

Case 2. Now we prove this result for the case of the p-punctured connected sum of g tori $\Sigma_{g,p}$. We shall use the presentation of the pure braid group $P_n(\Sigma_{g,p})$ given in Subsection 2.2. We note that the generators coming from the Artin generators of $P_n(D)$, via the inclusion $i\colon D \hookrightarrow \Sigma_{g,p}$, are the elements $A_{i,j}$ for $2g+p \leq i < j \leq 2g+p+n-1$. Now we add the relations $A_{i,j} = 1$ for $2g+p \leq i < j \leq 2g+p+n-1$ to the presentation given in Subsection 2.2 to deduce a presentation of the quotient group $P_n(\Sigma_{g,p})/N$. We claim that the set $\left\{A_{i,j} \mid 1 \leq i \leq 2g+p-1,\ 2g+p \leq j \leq 2g+p+n-1\right\}$ constitutes a set of generators for the group $P_n(\Sigma_{g,p})/N$ which are subject to the relations $[A_{i,j}, A_{r,s}] = 1$ for $1 \leq i,\ r \leq 2g+p-1$ and $2g+p \leq j,\ s \leq 2g+p+n-1$ with $j \neq s$. The claim is an immediate consequence of the following:

(QR1) $[A_{i,j}, A_{r,s}] = 1$ if $(r+1 < i < j < s)$ or $(i = r+1 < j < s$ for any $1 \leq r \leq 2g+p-2)$.

This follows directly from (PR1) when $r+1 < i < j < s$ and from (PR1) in case $i = r+1 < j < s$ for even $r < 2g$ or $r \geq 2g$ and from (ER1) when $i = r+1 < j < s$ for odd $r < 2g$.

(QR2) $[A_{i,j}, A_{i,s}] = 1$ if $i < j < s$. In fact, since $A_{j,s} = 1$ for $2g + p \leq j < s \leq 2g + p + n - 1$ then from (PR3):

$$A_{i,j}^{-1} A_{i,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$$

we obtain the relation (QR2).

(QR3) $[A_{i,j}, A_{r,s}] = 1$ if $(i + 1 < r < j < s)$ or $(i + 1 = r < j < s$ for any $2 \leq r \leq 2g + p - 2)$.

Using once more time the fact that $A_{j,s} = 1$ for $2g + p \leq j < s \leq 2g + p + n - 1$ and from (PR4):

$$A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1} A_{j,s}^{-1} A_{r,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$$

we get the relation (QR3) when $(i+1 < r < j < s)$ or $(i+1 = r < j < s$ for odd $r < 2g$ or $r > 2g)$. When $r \leq 2g$ is even the verification is similar using (ER2).

Note that all of the relations (PR2) became trivial in the quotient group $P_n(\Sigma_{g,p})/N$.

Hence the quotient group $P_n(\Sigma_{g,p})/N$ is isomorphic to $\Pi_{i=1}^{n}(\pi_1(\Sigma_{g,p}))$, the direct product of free groups of rank $2g + p - 1$.

Case 3. Finally we consider the case of the punctured connected sum of projective planes. For this case we use the presentation given in Subsection 2.3. The generators coming from the Artin generators of $P_n(D)$, via the inclusion $i : D \hookrightarrow N_{g,p}$, are the elements $A_{i,j}$ for $p + 1 \leq i < j \leq p + n$. We add the relations $A_{i,j} = 1$ for $p + 1 \leq i < j \leq p + n$ to the presentation given in Subsection 2.3 to deduce the following presentation of the quotient group $P_n(N_{g,p})/N$.

The set of generators is given by $A_{i,j}$ for $1 \leq i \leq p$ and $p+1 \leq j \leq p+n$ and $\rho_{r,k}$ for $p + 1 \leq r \leq p + n$ and $1 \leq k \leq g$ subject to the relations

(QR1) $[A_{i,j}, A_{r,s}] = 1$ for $(i < r < s < j)$ or $(i = r < s < j)$ or $(r < i < s < j)$.
This follows from item (a).

(QR2) $[\rho_{r,k}, \rho_{j,l}] = 1$ for $p + 1 \leq r < j \leq p + n$ and $1 \leq k, l \leq g$.
This follows from item (b).

(QR3) $[\rho_{k,l}, A_{i,j}] = 1$ for $j \neq k$.
This relation follows from (d).

(QSR) From item (c) we have that for every $p + 1 \leq j \leq p + n$

$$\prod_{l=1}^{g} \rho_{j,l}^2 = \prod_{i=1}^{p} A_{i,j}.$$

Recall that a presentation for the fundamental group of the punctured connected sum of projective planes, $\pi_1(N_{g,p})$, is given by

$$\left\langle A_i, \rho_l \text{ for } 1 \leq i \leq p \text{ and } 1 \leq l \leq g \mid \prod_{l=1}^{g} \rho_l^2 = \prod_{i=1}^{p} A_i \right\rangle.$$

Therefore, the quotient group $P_n(N_{g,p})/N$ is isomorphic to the direct product of n copies of $\pi_1(N_{g,p})$ (one for each $j \in \{p+1, \ldots, p+n\}$).

□

3 Automorphisms of $P_n(S)$ for S a surface of finite type

Let $\Sigma_{g,p}$ be an arbitrary finite surface of genus g with $p \geq 0$ points removed. Let $M_n^*(\Sigma_{g,p})$ denote the *extended mapping class group*, defined to be the group of isotopy classes of (possibly orientation-reversing) homeomorphisms of $(\Sigma_{g,p}, z)$, where $z = \{z_1, \ldots, z_n\}$ is a set of n distinct points in $\Sigma_{g,p}$ and $(z_1, \ldots, z_n)$ is a base point in the configuration space $F_n(\Sigma_{g,p})$. Let $h\colon \Sigma_{g,p} \longrightarrow \Sigma_{g,p}$ be a homeomorphism which leaves z invariant, i.e. $h(z) = z$, so the isotopy class of h, denoted by $[h]$ is an element of $M_n^*(\Sigma_{g,p})$. The homeomorphism h defines a permutation $\sigma_h \in S_n$ given by the following equation:

$$h(z_i) = z_{\sigma_h(i)}.$$

We will consider the morphisms $\Phi\colon M_n^*(\Sigma_{g,p}) \longrightarrow \mathrm{Aut}\left(B_n(\Sigma_{g,p})\right)$, $\Phi_0\colon M_n^*(\Sigma_{g,p}) \longrightarrow \mathrm{Aut}\left(P_n(\Sigma_{g,p})\right)$ defined as follows. For the case of Φ, given $h \in [h] \in M_n^*(\Sigma_{g,p})$ and $\{\alpha_1, \cdots, \alpha_n\}$ a set of paths between elements of z which is a representative of an element of $B_n(\Sigma_{g,p})$ define $h_{n\#}[\{\alpha_1, \cdots, \alpha_n\}]$ as the class determined by the set of paths $\{h \circ \alpha_1, \cdots, h \circ \alpha_n\}$. It is straightforward to see that this map is well defined, it is a homomorphism, and Φ is also a homomorphism. For the case of Φ_0 we have a similar definition, where we only stress the point that needs to be adapted in the description above. Given h and a representative $(\alpha_1, \cdots, \alpha_n)$ of an element of $P_n(\Sigma_{g,p}, (z_1, \cdots, z_n))$ define $h_{n\#}[(\alpha_1, \cdots, \alpha_n)]$ as the class determined by the ordered sequence of loops $(h \circ \alpha_{\sigma_h^{-1}(1)}, \cdots, h \circ \alpha_{\sigma_h^{-1}(n)})$ having base point $(z_1, \cdots, z_n)$. The rest is similar.

Let $\Sigma_{g,p}$ be an orientable surface of genus g with $p \geq 0$ points removed. The group $\mathrm{Aut}\left(P_n(\Sigma_{g,p})\right)$ has been studied by many authors and it was proved that there exists an isomorphism

$$M_n^*(\Sigma_{g,p}) \simeq \mathrm{Aut}\left(P_n(\Sigma_{g,p})\right) \tag{4}$$

for (not necessarily closed) orientable surfaces with Euler characteristic $\chi(\Sigma_{g,p}) < -1$ by [1, Theorem 1.3]. The isomorphism (4) is the map Φ_0 as described above, for more details see [1].

So the following cases, for orientable surfaces, are not covered by the results above:

I) for $g = 1$ and $p = 0, 1$ (i.e. the surfaces of $\mathcal{F}_3$);
II) $g = 0$, $p = 0, 1, 2, 3$. (i.e. the surfaces of $\mathcal{F}_1$ and the surface $\Sigma_{0,3}$).

Now we prove that the short exact sequence of Theorem 8 is characteristic with respect to automorphisms. This will be used to study the R_∞-property of the groups $P_n(\Sigma_{g,p})$ and $B_n(\Sigma_{g,p})$ in the next section.

Lemma 9 *Let $\Sigma_{g,p}$ be a surface of the family $\mathcal{F}_2$, being different from the sphere minus three points. Let N denote the normal subgroup generated by the image of the Artin pure braids via the inclusion of the disc D into $\Sigma_{g,p}$, $i\colon D \hookrightarrow \Sigma_{g,p}$, and $n \geq 2$. If $\varphi\colon P_n(\Sigma_{g,p}) \longrightarrow P_n(\Sigma_{g,p})$ is an automorphism then $\varphi(N) \subset N$. Therefore the short exact sequence given in equation (3) is characteristic.*

Proof Let φ be an automorphism of $P_n(\Sigma_{g,p})$. To show that $\varphi(N) \subset N$ it is enough to prove that $\varphi(i_\#(P_n(D))) \subset N$, since N is the normal subgroup generated by the images of the Artin pure braids via the inclusion $i\colon D \hookrightarrow \Sigma_{g,p}$ of the disc D into $\Sigma_{g,p}$. From the hypothesis of this lemma the Euler characteristic $\chi(\Sigma_{g,p}) < -1$, and so there is a homeomorphism $h\colon \Sigma_{g,p} \longrightarrow \Sigma_{g,p}$ such that φ is induced by h, i.e. $\varphi = \Phi_0([h]) \in \mathrm{Aut}\left(P_n(\Sigma_{g,p})\right)$ (by (4) and the discussion above).

Let $\alpha \in P_n(D)$. We shall prove that $\varphi(i_\#(\alpha)) \in N$. The inclusion $i\colon D \hookrightarrow \Sigma_{g,p}$ induces an inclusion $\widehat{i}\colon F_n(D) \hookrightarrow F_n(\Sigma_{g,p})$. Geometrically, since $P_n(D) = \pi_1(F_n(D), (z_1, \dots, z_n))$, we have that α has a representative $\widehat{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ where $\alpha_j\colon [0, 1] \longrightarrow D$ is a loop in D with base point $z_j \in D$. By definition, $\Phi_0([h])(i_\#(\alpha))$ is the homotopy class of the loop $(h \circ i \circ \alpha_{\sigma_h^{-1}(1)}, \cdots, h \circ i \circ \alpha_{\sigma_h^{-1}(n)})$ based at $(z_1, \cdots, z_n)$ in $\Sigma_{g,p}$.

Now let $\widehat{\psi}\colon F_n(\Sigma_{g,p}) \longrightarrow \Pi_{i=1}^n(\Sigma_{g,p}^n)$ be the inclusion. Since $h \circ i \circ \alpha_{\sigma_h^{-1}(j)}$ is a loop based in z_j inside $h(D) \subset \Sigma_{g,p}$, for every $1 \leq j \leq n$, and since $h|_D\colon D \longrightarrow h(D)$ is a homeomorphism, we get that $\widehat{\psi}((h \circ i \circ \alpha_{\sigma_h^{-1}(1)}, \cdots, h \circ i \circ \alpha_{\sigma_h^{-1}(n)}))$ is homotopic to the n-tuple of constant maps $(c_{z_1}, \dots, c_{z_n})$.

Algebraically, the induced map $\widehat{\psi}_\#$ on the level of the fundamental groups corresponds to the surjective map $P_n(\Sigma_{g,p}) \longrightarrow \Pi_{i=1}^n(\pi_1(\Sigma_{g,p}))$ in the short exact sequence given in equation (3). Therefore, $\widehat{\psi}_\#(\Phi_0([h])(i_\#\alpha)) = 1 \in \Pi_{i=1}^n(\pi_1(\Sigma_{g,p}))$. Hence, $\Phi_0([h])(i_\#\alpha) = \varphi(i_\#\alpha) \in N$. ☐

4 The R_∞-property for $P_n(\Sigma_{g,p})$ and $B_n(\Sigma_{g,p})$

In this section we prove Theorem 2 and Theorem 3.

4.1 The Fadell-Neuwirth short exact sequence

Let S be a connected surface and let $n \in \mathbb{N}$. If $m \geq 1$, the map $p\colon F_{n+m}(S) \longrightarrow F_n(S)$, of the configuration space $F_{n+m}(S)$ onto $F_n(S)$, defined by $p(x_1, \dots, x_n, \dots, x_{n+m}) = (x_1, \dots, x_n)$ induces a homomorphism $p_*\colon P_{n+m}(S) \longrightarrow P_n(S)$. The homomorphism p_* geometrically "forgets" the last m strings. If S is without boundary, Fadell and Neuwirth showed that p is a locally-trivial fibration [14, Theorem 1], with fibre $F_m(S\backslash\{x_1,\ldots,x_n\})$ over the point $(x_1,\ldots,x_n)$, which we consider to be a subspace of the total space via the map $i\colon F_m(S\backslash\{x_1,\ldots,x_n\}) \longrightarrow F_{n+m}(S)$ defined by $i((y_1,\ldots,y_m)) = (x_1,\ldots,x_n,y_1,\ldots,y_m)$. Applying the associated long exact sequence in homotopy to this fibration, we obtain the Fadell-Neuwirth short exact sequence of pure braid groups:

$$1 \longrightarrow P_m(S\backslash\{x_1,\ldots,x_n\}) \xrightarrow{i_*} P_{n+m}(S) \xrightarrow{p_*} P_n(S) \longrightarrow 1 \tag{5}$$

where $n \geq 3$ if S is the sphere [13, 15], $n \geq 2$ if S is the real projective plane [39], and $n \geq 1$ otherwise [14], and i_* is the homomorphism induced by the map i. This sequence has been widely studied. For instance, one question studied by many authors during several years was the splitting problem for surface pure braid groups, and it was completely solved, see [25] for more details, in particular its Theorem 2. Additional information on this sequence may be seen in [28, Section 3.1].

We are interested in (quotients by) the center of some surface braid groups. Seemingly the content of Proposition 10 related to the braid groups over the annulus is well known for the experts in braid theory, however to the best of our knowledge there is no proof in the literature of it. Hence, for the sake of completeness, we provide a proof here. The following information will be useful: From [35, Proposition 4.1], for all $n \geq 1$, the center of the group $B_n(\Sigma_{0,2})$ is isomorphic to $\mathbb{Z}$ generated by $\alpha_n \in P_n(\Sigma_{0,2})$. See [35, Figure 4.1] for a geometric description of this element.

Proposition 10 *Let $n \geq 1$. Then $Z(B_n(\Sigma_{0,2})) = Z(P_n(\Sigma_{0,2}))$ and $P_{n+1}(\Sigma_{0,2})/Z(P_{n+1}(\Sigma_{0,2})) \cong P_n(\Sigma_{0,3})$.*

Proof Let $n = 1$, then $P_1(\Sigma_{0,2}) = B_1(\Sigma_{0,2}) \cong \mathbb{Z}$ and so $Z(P_1(\Sigma_{0,2})) = Z(B_1(\Sigma_{0,2}))$. To study the situation with more than one string, we consider the short exact sequence induced from the Fadell-Neuwirth fibration

$$1 \longrightarrow P_n(\Sigma_{0,3}) \longrightarrow P_{n+1}(\Sigma_{0,2}) \xrightarrow{p_*} P_1(\Sigma_{0,2}) \longrightarrow 1$$

where p_* geometrically forgets the last n strings. We note that $P_1(\Sigma_{0,2}) = \pi_1(\Sigma_{0,2}) \cong \mathbb{Z}$ is generated by $\alpha_1 \in P_1(\Sigma_{0,2})$. Consider the section of p_* that sends $\alpha_1 \in P_1(\Sigma_{0,2})$ onto $\alpha_{n+1} \in P_{n+1}(\Sigma_{0,2})$. Since $\alpha_{n+1} \in Z(B_{n+1}(\Sigma_{0,2}))$, α_{n+1} also commutes with all elements from $P_{n+1}(\Sigma_{0,2})$ and so we find that

$$P_{n+1}(\Sigma_{0,2}) \cong P_n(\Sigma_{0,3}) \oplus \mathbb{Z},$$

where $\mathbb{Z}$ is the subgroup of $P_{n+1}(\Sigma_{0,2})$ generated by α_{n+1}. As $Z(P_n(\Sigma_{0,3})) = 1$ [35, Proposition 1.6], it now readily follows that $Z(P_{n+1}(\Sigma_{0,2})) = Z(B_{n+1}(\Sigma_{0,2}))$ and hence $P_{n+1}(\Sigma_{0,2})/Z(P_{n+1}(\Sigma_{0,2})) \cong P_n(\Sigma_{0,3})$. □

The information of some surface braid groups in the following remarks will be useful in this section.

Remark 11 Suppose $n \geq 1$.

1. Braid groups with few strings over the sphere are finite:
   $B_1(\mathbb{S}^2)$, $P_1(\mathbb{S}^2)$ and $P_2(\mathbb{S}^2)$ are trivial groups, $B_2(\mathbb{S}^2) \cong \mathbb{Z}_2$, $B_3(\mathbb{S}^2)$ is isomorphic to $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ with non-trivial action and $P_3(\mathbb{S}^2) \cong \mathbb{Z}_2$, see [15] and also [28, Section 4].
2. If $S = \mathbb{S}^2$ is the sphere, then $P_{n+3}(\mathbb{S}^2) \cong P_n(\Sigma_{0,3}) \oplus \mathbb{Z}_2$ (see [24, Theorem 4]). We remark here that $Z(P_{n+3}(\mathbb{S}^2)) = \mathbb{Z}_2$ because $Z(P_n(\Sigma_{0,3})) = 1$ [35, Proposition 1.6]. Hence $P_{n+3}(\mathbb{S}^2)/Z(P_{n+3}(\mathbb{S}^2)) \cong P_n(\Sigma_{0,3})$.
3. Suppose $S = D$ is the disc. It is an immediate consequence of the classical Artin presentation of $P_2(D)$ and $B_2(D)$ that they are isomorphic to $\mathbb{Z}$, see [3]. Let $n \geq 3$. There is a decomposition $P_n(D) \cong P_{n-2}(\Sigma_{0,3}) \oplus \mathbb{Z}$ that follows from the splitting of the Fadell-Neuwirth short exact sequence (see [24, Theorem 4]). Using [35, Proposition 1.6] again, we find that $P_{n-2}(\Sigma_{0,3}) \cong P_n(D)/Z(P_n(D))$.

4.2 The proof of Theorem 2 and Theorem 3

Recall that we split the orientable surfaces of finite type into three families, as follows:

$\mathcal{F}_1$: The punctured sphere $\mathbb{S}^2$ with p points removed for $p = 0, 1, 2$.
$\mathcal{F}_2$: a) Orientable closed surfaces different from $\mathbb{S}^2$, T^2.
b) Orientable punctured surfaces $\Sigma_{g,p}$ where g is the genus and p is the number of punctures in the closed surface Σ_g, for:
   i) $g = 0$ and $p \geq 3$,
   ii) $g = 1$ and $p \geq 2$,
   iii) $g \geq 2$ and $p \geq 1$.
$\mathcal{F}_3$: The torus and the once punctured torus.

We will show the results for the families $\mathcal{F}_1$ and $\mathcal{F}_2$, using two different arguments. The case $\mathcal{F}_3$ is work in progress.

We shall use the following technique in order to prove that the pure braid groups of surfaces $\Sigma_{g,p}$ (closed or punctured) from $\mathcal{F}_2$ have the R_∞-property whenever $\pi_1(\Sigma_{g,p})$ has the R_∞-property. If α is an automorphism of a group G and N is a normal subgroup of N with $\alpha(N) = N$ (e.g. when N is a characteristic subgroup of G) then α induces an automorphism $\bar{\alpha}$ of G/N. It is easy to see that $R(\alpha) \geq R(\bar{\alpha})$, so if $R(\bar{\alpha}) = \infty$, then also $R(\alpha) = \infty$. For all $\Sigma_{g,p}$ of family $\mathcal{F}_2$, it holds that $\pi_1(\Sigma_{g,p})$ has the R_∞-property.

This does not longer hold for the surfaces of family $\mathcal{F}_1$ and we will solve this case using a different argument.

We will state the result for the family $\mathcal{F}_2$. To prove the result we will make use of Lemma 9 for all surface $\Sigma_{g,p}$ in $\mathcal{F}_2$, except for the case in which $g = 0$ and $p = 3$, since the validity of the equation (4) is not covered by An [1], because $\chi(\Sigma_{g,p}) = -1$. So, for this special case of the family $\mathcal{F}_2$ we shall use another approach.

Proposition 12 *For any surface $\Sigma_{g,p} \in \mathcal{F}_2$ we have that $P_n(\Sigma_{g,p})$ has the R_∞-property for $n \geq 1$.*

Proof For $n = 1$ the group $P_1(\Sigma_{g,p}) = \pi_1(\Sigma_{g,p})$ is the fundamental group of the surface. For every surface in the family $\mathcal{F}_2$ its fundamental group has the R_∞-

property. Indeed, for all $\Sigma_{g,p} \in \mathcal{F}_2$, it holds that $\chi(\Sigma_{g,p}) < 0$ and so $\pi_1(\Sigma_{g,p})$ is non-elementary hyperbolic, hence the result follows from [30], see also [16].

Let $\Sigma_{0,3}$ be the pantalon. Recall that $P_n(\Sigma_{0,3}) \cong P_{n+2}(D)/Z(P_{n+2}(D))$ (see Remarks 11 in Subsection 4.1). In [10] the group $P_{n+2}(D)/Z(P_{n+2}(D))$ was denoted by $\overline{P}_{n+2}$. Moreover, in order to prove the main result of [10] it was shown that this group has the R$_\infty$-property (see the last two lines of page 17 of [10]).

Let $n \geq 2$ and let $\Sigma_{g,p} \neq \Sigma_{0,3}$ be a surface from $\mathcal{F}_2$. From Lemma 9 the short exact sequence given in equation (3) is characteristic, then we conclude the result in these cases since $\Pi_{i=1}^n(\pi_1(S))$ has the R$_\infty$-property by [36, Corollary 4.5]. □

Now we move to the surfaces of the family $\mathcal{F}_1$. We notice that, for $n \geq 2$, the result that $P_n(\Sigma_{0,2})$ has the R$_\infty$-property was already proved in [8], using a different technique from the ones used here.

Proposition 13 *Let $\Sigma_{g,p}$ be a surface of the family $\mathcal{F}_1$. Then $P_n(\Sigma_{g,p})$ has the R_∞-property if and only if one of the following cases holds:*

1. $\Sigma_{g,p} = \mathbb{S}^2$ *and* $n \geq 4$,
2. $\Sigma_{g,p} = \mathbb{S}^2\backslash\{x_1\}$ *and* $n \geq 3$,
3. $\Sigma_{g,p} = \mathbb{S}^2\backslash\{x_1, x_2\}$ *and* $n \geq 2$.

Proof The proof is case by case, but first we deal with the exceptional cases. It is well known (see Remarks 11 in Subsection 4.1) that $P_1(\mathbb{S}^2) = P_2(\mathbb{S}^2) = \{1\}$, $P_3(\mathbb{S}^2) = \mathbb{Z}_2$, $P_1(\mathbb{S}^2\backslash\{x_1\}) = \{1\}$, $P_2(\mathbb{S}^2\backslash\{x_1\}) = \mathbb{Z}$, $P_1(\mathbb{S}^2\backslash\{x_1, x_2\}) = \mathbb{Z}$ and all these groups do not have the R$_\infty$-property.

For $\Sigma_{g,p} = \mathbb{S}^2$ and $n \geq 4$ we have that $P_n(\mathbb{S}^2)/Z(P_n(\mathbb{S}^2)) \cong P_{n-3}(\mathbb{S}^2\backslash\{x_1, x_2, x_3\})$ (see Remarks 11 in Subsection 4.1). From Proposition 12, $P_{n-3}(\mathbb{S}^2\backslash\{x_1, x_2, x_3\})$ has the R$_\infty$-property for $n-3 \geq 1$, then the result follows since $Z(P_n(\mathbb{S}^2))$ is a characteristic subgroup of $P_n(\mathbb{S}^2)$. Let $\Sigma_{g,p} = \mathbb{S}^2\backslash\{x_1\}$, then $P_n(\Sigma_{g,p})$ is the pure Artin braid group and the result follows from the main result of [10].

For $\Sigma_{g,p} = \mathbb{S}^2\backslash\{x_1, x_2\}$ and $n \geq 2$ we know $P_n(\mathbb{S}^2\backslash\{x_1, x_2\})/Z(P_n(\mathbb{S}^2\backslash\{x_1, x_2\})) \cong P_{n-1}(\mathbb{S}^2\backslash\{x_1, x_2, x_3\})$, by Proposition 10. Again by using the fact that the centre is a characteristic subgroup and using Proposition 12 for $\Sigma_{0,3}$ the result follows. □

From the discussion above we may prove the main result about the R$_\infty$-property for surface pure braid group $P_n(\Sigma_{g,p})$.

Proof of Theorem 2 The result follows immediately from Propositions 12 and 13. □

Now we consider the groups $B_n(\Sigma_{g,p})$ for $n \geq 2$, and we will make use of the short exact sequence

$$1 \longrightarrow P_n(\Sigma_{g,p}) \longrightarrow B_n(\Sigma_{g,p}) \longrightarrow S_n \longrightarrow 1.$$

We recall that by [1, Theorem 1.5] this short exact sequence is characteristic unless we are the case that $\Sigma_{g,p} = \Sigma_{0,2}$ and $n = 2$. In the cases where the short exact sequence is characteristic we will use the following result given by [32, Lemma 6]: Let $1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$ be a characteristic short exact sequence with respect to automorphisms of B such that C is finite and A has the R_∞-property. Then B has the R_∞-property. To analyse the case $B_2(\Sigma_{0,2})$ we will use a different approach.

Now we prove the main result for the groups $B_n(\Sigma_{g,p})$. We notice that, for $n \geq 2$, the result that $B_n(\Sigma_{g,p})$ has the R_∞-property was already proved, using different techniques from the ones here, in [18] for $\Sigma_{g,p} = \mathbb{S}^2$ and $\Sigma_{g,p} = D^2$, and in [8] for $\Sigma_{g,p} = \Sigma_{0,2}$ and $\Sigma_{g,p} = \Sigma_{0,3}$.

Proof of Theorem 3 Recall that for any surface $S_{g,p}$ there exists a short exact sequence

$$1 \longrightarrow P_n(S_{g,p}) \longrightarrow B_n(S_{g,p}) \longrightarrow S_n \longrightarrow 1.$$

By [1, Theorem 1.5] this short exact sequence is characteristic as long as $\Sigma_{g,p} \neq \Sigma_{0,2}$ or $n > 2$. For the if part let $P_n(\Sigma_{g,p})$ have the R_∞-property. From [32, Lemma 6] the result follows for $B_n(\Sigma_{g,p})$. The only remaining case is when $\Sigma_{g,p} = \Sigma_{0,2}$ and $n = 2$. But from [9, Proposition 2.1 (2)], $B_2(\Sigma_{0,2})$ is isomorphic to $F_2(x, y) \rtimes_\theta \mathbb{Z}$. The semi-direct product is determined by the action given by the automorphism $\theta(1)(x) = y$, $\theta(1)(y) = y^{-1}xy$. Now it follows from [19, Theorem 4.4] that $B_2(\Sigma_{0,2})$ has the R_∞-property (independent of the action).

For the only if part, observe that the only cases where $B_n(\Sigma_{g,p})$ $(n > 1)$ does not have the R_∞-property are:

a) If $\Sigma_{g,p} = \mathbb{S}^2$ for $2 \leq n \leq 3$ because the groups are finite;
b) If $\Sigma_{g,p} = \mathbb{S}^2 \backslash \{x_1\}$ for $n = 2$ because the group is isomorphic to $\mathbb{Z}$.

In both cases the groups $P_n(\Sigma_{g,p})$ also does not have the R_∞-property, see Theorem 2. So the result follows. □

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