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RESEARCH

On the dissipation function for anisotropic porous solids with matrix behaviour based on several linear transformations

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Abstract This paper presents a theoretical procedure for obtaining the dissipation function for anisotropic rigid-plastic materials, the yielding of which is governed by the Barlat and co-workers-like plasticity criteria (Int J Plast 21:1009–1039, 2005. https://doi.org/10.1016/j.ijplas.2004.06.004) incorporating several linear transformations of the stress in generic isotropic plasticity models, extending the proposal of Karafillis and Boyce (J Mech Phys Solids 41:1859–1886, 1993. https://doi.org/10.1016/0022-5096(93)90073-o). The underlying isotropic yield criterion can be very general including possible Lode angle effects in the material behaviour. The seeked dissipation function is needed for the construction of the macroscopic behavior of voided materials when it is for instance combined to appropriate representative volume elements of these materials and associated kinematically admissible velocity fields satisfying uniform boundary conditions at their boundary in the spirit of the Gurson's approach (J Eng Mater Technol 99:2–15, 1977. https://doi.org/10.1115/1.3443401) to ductile fracture.

Keywords Porous plasticity · Matrix anisotropy · Dissipation · Ductile fracture

1 Introduction

This paper explores the modeling of voided materials with realistic matrix behavior, specifically aimed at predicting ductile fracture. It emphasizes the use of effective and widely adopted anisotropic plastic models for the matrix in the study of plastic porous materials.

Anisotropy in engineering materials has been a persistent challenge, often arising from their manufacturing processes. Rolling, extrusion, and drawing typically cause varying degrees of anisotropy in different material properties. Notably, many engineering alloys, especially aluminum alloys, exhibit considerable anisotropy in terms of strength, ductility, and plastic flow.

The microstructural features of these materials are crucial in determining their plastic anisotropy. At moderate temperatures, plastic deformation mainly occurs through dislocation glide and,

Dedicated to Professor Alan Needleman on the occasion of his 80 th Birthday.

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occasionally, twinning, both of which are limited to specific crystallographic planes and directions. The crystallographic texture (i.e., the distribution of grain orientations) plays a critical role in contributing to anisotropy, with grain shape also influencing the interactions between adjacent grains.

Among other microstructural features, but now at a another scale, the presence of second-phase particles [4], or even impurities resulting from the manufacturing processes of metals and alloys, is particularly important for understanding failure and rupture in these materials. Since these particles tend either to detach from the main phase or to crack due to their inferior mechanical properties [5], they are equivalently treated as *voids* present at the microscale [6]. During the deformation process in a portion of a ductile material, these voids at the microscale become preferential regions for stress concentration, leading to intense plastic flow in their surroundings. As these regions deform plastically, the voids can reach a configuration where adjacent voids coalesce, forming a larger void. These mechanisms are typically referred to as *nucleation*, *void growth*, and *coalescence*, respectively (Gurland [7]). Eventually, these processes can continue until unstable void growth and coalescence occur, resulting in a visible crack at the macroscale.

The anisotropy of the main phase and the presence of voids are expected to influence each other during the ductile rupture process. On one hand, an anisotropic material matrix imposes preferential directions of deformation and influences where stresses concentrate around voids. On the other hand, the presence of voids not only causes the yielding behavior to depend on both stress triaxiality and Lode angle, but may also induce preferential directions of deformation in the case of elongated or flattened voids. Therefore, a formulation of yield criteria that accounts for both the effects of matrix anisotropy and the presence of voids is required.

Regarding more realistic yield criteria for anisotropic ductile matrices, Barlat et al. [1] proposed the use of two linear transformations to operate on the arguments of an isotropic yield function, allowing more material parameters to be adjusted from experimental data. These additional material parameters arising from the second linear transformation enabled them to reproduce experimental results for aluminum alloys that were not possible to obtain using the approach of Karafillis and Boyce [2] using only one linear transformation and the concept of"Isotropy Plasticity Equivalent"(IPE). Due to its flexibility in adjusting additional material parameters, the Barlat et al.'s model [1] is widely used in modeling the plastic behavior of aluminum alloys.

In the context of yielding criteria for ductile voided materials, the Gurson model Gurson [3] is acclaimed for its pioneering nature and simplicity. It consists of a macroscopic yield function involving both hydrostatic and equivalent stresses, as well as the material porosity (i.e., the average ratio of the volume of voids to the total volume in a given material portion). Its derivation uses a limit analysis of a representative cell combined to the upper bound theorem of limit analysis by identifying a pertinent kinematically admissible velocity field $\mathbf{v}$ compatible with uniform boundary conditions $\dot{\boldsymbol{E}}$ prescribed at the boundary of this cell. By bounding the macroscopic dissipation from above, Gurson was able to obtain upper bounds to the macroscopic yield stresses of the cell. These stresses for the considered cell geometry and a for a range of macroscopic deformation rates allow to construct an upper bound yield locus for the porous material. These stresses are defined by

$$\boldsymbol{\Sigma} = \frac{\partial \Pi}{\partial \dot{\boldsymbol{E}}} \tag{1}$$

where $\Pi(\dot{\boldsymbol{E}})$ is the upper bound (to the exact macroscopic dissipation) associated to the trial kinematically admissible velocity field $\mathbf{v}$ ($\dot{\boldsymbol{\epsilon}}(\dot{\boldsymbol{E}}) = \boldsymbol{\epsilon}(\mathbf{v})$)

$$\Pi(\dot{\boldsymbol{E}}) = \frac{1}{V}\int_V \pi(\dot{\boldsymbol{\epsilon}}(\dot{\boldsymbol{E}}))dV \tag{2}$$

and $\pi(\dot{\boldsymbol{\epsilon}})$ is the dissipation function of the matrix. The main objective of the present paper is the explicit derivation of this dissipation function for anisotropic plastic behaviour based on two linear transformations such as in Barlat et al. [1] with the aim to simulate macroscopic behaviour of voided materials made of these materials according to Eqs. (1) and (2).

2 Constitutive relations for the matrix

The anisotropic models based on isotropic yield criteria supplementated by linear transformations of the stress can be very efficient and are now used currently in many fields such as metal forming, impact engineering, etc... They are very flexible to incorporate a lot of experimental data whenever available and also have been seen to reproduce them at the expense of more material parameters.

Anisotropic yield functions $f(\boldsymbol{\sigma})$ considered hereafter have the form

$$f(\boldsymbol{\sigma}) = \phi(\mathbb{L}_1 : \boldsymbol{\sigma}, \mathbb{L}_2 : \boldsymbol{\sigma}) \tag{3}$$

where $\phi(\boldsymbol{s}_1, \boldsymbol{s}_2)$ is an isotropic function of the two auxiliary stresses $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. This function is also a convex function of these two arguments. The linear transformations $\mathbb{L}_1$ or $\mathbb{L}_2$ describe the anisotropy of the material. The generic fourth order tensor $\mathbb{L}$ ($\mathbb{L}_1$ or $\mathbb{L}_2$) has the usual minor and major symmetries

$$L_{ijk\ell} = L_{jik\ell} = L_{ij\ell k} = L_{ijk\ell} \tag{4}$$

Incompressibility of the plastic behaviout is included by imposing the condition

$$L_{ijkk} = 0 \qquad \text{for } i,j,k \in \{1,2,3\}, \tag{5}$$

where the summation rule on repeated indices is used. As an illustration, the so-called Yld2004-18p model proposed in Barlat et al. [1] for orthotropic symmetry is given by

$$\begin{aligned} \boldsymbol{s}_1 &= \left(\mathbb{C}^1 : \mathbb{J}\right) : \boldsymbol{\sigma} = \mathbb{L}_1 : \boldsymbol{\sigma} \\ \boldsymbol{s}_2 &= \left(\mathbb{C}^2 : \mathbb{J}\right) : \boldsymbol{\sigma} = \mathbb{L}_2 : \boldsymbol{\sigma}. \end{aligned} \tag{6}$$

where

$$\underline{\underline{\boldsymbol{C}}}^i = \begin{bmatrix} 0 & -c^i_{12} & -c^i_{13} & 0 & 0 & 0 \\ -c^i_{21} & 0 & -c^i_{23} & 0 & 0 & 0 \\ -c^i_{31} & -c^i_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c^i_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c^i_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c^i_{66} \end{bmatrix} \tag{7}$$

$$\underline{\underline{\boldsymbol{L}}}_i = \underline{\underline{\boldsymbol{C}}}^i \underline{\underline{\boldsymbol{J}}} = \frac{1}{3} \begin{bmatrix} b^i_2 + b^i_3 & -c^i_3 & -b^i_2 & 0 & 0 & 0 \\ -b^i_3 & b^i_3 + b^i_1 & -b^i_1 & 0 & 0 & 0 \\ -b^i_2 & -b^i_1 & b^i_1 + b^i_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3b^i_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3b^i_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3b^i_6 \end{bmatrix}, \tag{8}$$

$$\begin{aligned} b^i_1 &= (4c^i_{23} + c^i_{31} - 2c^i_{12})/3, \\ b^i_2 &= (4c^i_{31} + c^i_{12} - 2c^i_{23})/3, \\ b^i_3 &= (4c^i_{12} + c^i_{23} - 2c^i_{31})/3, \\ b^i_4 &= c^i_{44}, \\ b^i_5 &= c^i_{55} \text{ and} \\ b^i_6 &= c^i_{66} \end{aligned}$$

and $\underline{\underline{\boldsymbol{C}}}^i$, $\underline{\underline{\boldsymbol{L}}}_i$ and $\underline{\underline{\boldsymbol{J}}}$ correspond, in the Voigt notation, to the matrices associated to the 4 th-order tensors $\mathbb{C}^i$, $\mathbb{L}_i$ and $\mathbb{J}$, respectively. The projector into the deviatoric space $\mathbb{J}$ given by

$$J_{ijk\ell} = \frac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{k\ell}, \qquad \text{for } i,j,k \in \{1,2,3\}. \tag{9}$$

Here and also in the following, we use the Voigt notation: Second order tensors (mainly the stress and the strain rate tensors) are represented by 9×1 column vectors, while fourth-rank tensors (essentially $\mathbb{L}$) are 9×9 matrices. When the second order tensors are symmetric and the fourth order tensors have the minor symmetries, one can use 6×1 column vectors and 6×6 matrices. Fourth and second order tensors are represented in the Voigt notation by bold symbol underlined twice or once respectively (e.g. $\mathbb{L}$ and $\dot{\boldsymbol{\epsilon}}$ become $\underline{\underline{\boldsymbol{L}}}$ and $\underline{\dot{\boldsymbol{\epsilon}}}$) as also done in (7) and (8) for $\mathbb{L}$, $\mathbb{C}^1$, $\mathbb{C}^2$ and $\mathbb{J}$.

Tensors $\mathbb{L}_i$ may exhibit any (or no) internal symmetries to represent all possible particular forms of anisotropy. $\mathbb{L}_i$ may contain up to 15 independent components. The linear transformation is written, using the Voigt notation in the following form $\underline{\boldsymbol{S}} = \underline{\underline{\boldsymbol{L}}} \cdot \underline{\boldsymbol{\sigma}}$, or in explicit form as

$$\begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ \sqrt{2}S_{23} \\ \sqrt{2}S_{31} \\ \sqrt{2}S_{12} \end{Bmatrix} = C \begin{bmatrix} \check{}_{11} & L_{12} & L_{13} & \sqrt{2}L_{14} & \sqrt{2}L_{15} & \sqrt{2}L_{16} \\ L_{12} & L_{22} & L_{23} & \sqrt{2}L_{24} & \sqrt{2}L_{25} & \sqrt{2}L_{26} \\ L_{13} & L_{23} & L_{33} & \sqrt{2}L_{34} & \sqrt{2}L_{35} & \sqrt{2}L_{36} \\ \sqrt{2}L_{14} & \sqrt{2}L_{24} & \sqrt{2}L_{34} & 2L_{44} & 2L_{45} & 2L_{46} \\ \sqrt{2}L_{15} & \sqrt{2}L_{25} & \sqrt{2}L_{35} & 2L_{45} & 2L_{55} & 2L_{56} \\ \sqrt{2}L_{16} & \sqrt{2}L_{26} & \sqrt{2}L_{36} & 2L_{46} & 2L_{56} & 2L_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{31} \\ \sqrt{2}\sigma_{12} \end{Bmatrix} \tag{10}$$

where C is a constant and from the pressure independence condition $\mathbb{L}_{ijkk} = 0$, $i = 1, 2, 3$ and $j = 1, 2, 3$, the following relations must hold

$$L_{12} = \frac{L_{33} - L_{11} - L_{22}}{2}, \quad L_{13} = \frac{L_{22} - L_{33} - L_{11}}{2}, \quad L_{23} = \frac{L_{11} - L_{22} - L_{33}}{2},$$
$$L_{34} = -(L_{14} + L_{24}), \quad L_{35} = -(L_{15} + L_{25}) \quad \text{and} \quad L_{36} = -(L_{16} + L_{26}).$$

Regarding the function ϕ, Barlat et al. [1] used the function

$$\begin{aligned} &\phi(\boldsymbol{s}_1, \boldsymbol{s}_2) = \phi(S_1^1, S_2^1, S_3^1, S_1^2, S_2^2, S_3^2) = \\ &\left\{ \frac{1}{4} \left(|S_1^1 - S_1^2|^m + |S_1^1 - S_2^2|^m + |S_1^1 - S_3^2|^m + \right. \right. \\ &\quad |S_2^1 - S_1^2|^m + |S_2^1 - S_2^2|^m + |S_2^1 - S_3^2|^m + \\ &\quad \left. \left. |S_3^1 - S_1^2|^m + |S_3^1 - S_2^2|^m + |S_3^1 - S_3^2|^m \right) \right\}^{1/m} \end{aligned} \tag{12}$$

which is isotropic (as a function dependent only on the principal values of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$) and reduces to the Hershey-Hosford yield function (Hershey [8], Hosford [9]) when both linear transformations are taken as $\mathbb{L}_i = \mathbb{J}$, i.e.

$$\phi(\boldsymbol{s}, \boldsymbol{s}) = \phi_1(\boldsymbol{s}) = \left\{ \frac{1}{2} \left[\left(S_1 - S_2\right)^m + \left(S_2 - S_3\right)^m + \left(S_1 - S_3\right)^m \right] \right\}^{1/m} \tag{13}$$

The associated anisotropic yield function $f(\boldsymbol{\sigma}) = \phi(\mathbb{L}_1 : \boldsymbol{\sigma}, \mathbb{L}_2 : \boldsymbol{\sigma})$ is convex because $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are linear transformations of the Cauchy stress $\boldsymbol{\sigma}$ (see for instance Rockafellar [10]).

The exponent m is here an even integer number $m = 2k$ in the range $k \geq 1$. When $m = 2$, the yield surface corresponds to the von Mises [11] yield surface, whereas when $m \to \infty$, the Tresca [12] surface is recovered.

3 Dissipation functions for plasticity models based on linear transformations

The anisotropic yield criterion, considered here, is constructed from an isotropic one. Benallal [13] considered the general case of isotropy with smooth yield domains. Ferreira et al. [14] explored the anisotropic case with one linear transformation still for the smooth case. Here, we consider the anisotropic situation with several linear transformations. We however present it for two transformations for the sake of simplicity.

Both in the case of isotropy [13] and in the anisotropic case with one linear transformation, the isotropic underlying yielding of the matrix was described by a function Φ, positive and homogeneous of degree one in the stress in the form

$$\Phi(\boldsymbol{\sigma}) - \sigma_0 = \sigma_{eq} g(\omega) - \sigma_0 \leq 0 \tag{14}$$

where σ_{eq} is the effective stress and ω is the Lode angle of the stress tensor. Function $g(\omega)$, which describes possible effects of the third stress invariant on yielding can be normalized in a number of ways (for instance by $g(0) = 1$ in which case σ_0 is the yield limit in uniaxial tension). An important requirement

for the yield surface is its convexity. For this to hold, function $g(\omega)$ must satisfy the following condition

$$g(\omega) + g''(\omega) \geq 0 \tag{15}$$

The yield surface was also considered smooth for sake of simplicity. In this case, the gradient of the yield function is given by

$$\frac{\partial \Phi}{\partial \boldsymbol{\sigma}} = \frac{3}{2} g(\omega) \frac{\mathbf{s}}{\sigma_{eq}} + \sigma_{eq} g'(\omega) \frac{\partial \omega}{\partial \boldsymbol{\sigma}} \tag{16}$$

where the gradient of the Lode angle with respect to the stress is given by

$$\frac{\partial \omega}{\partial \boldsymbol{\sigma}} = -\frac{9}{2\sigma_{eq} |\sin 3\omega|} \left[\left(\frac{\mathbf{s}.\mathbf{s}}{\sigma_{eq}^2} - \frac{2}{9}\mathbf{1} \right) - \frac{1}{3} \cos 3\omega \frac{\mathbf{s}}{\sigma_{eq}} \right] \tag{17}$$

with $\mathbf{1}$ the second order unit tensor. We note that the gradient $\frac{\partial \omega}{\partial \boldsymbol{\sigma}}$, as given by Eq. (17) is always singular at $\omega = n\frac{\pi}{3}$ for $n = 0, \ldots, 5$ (axisymmetric states of stress) so that the gradient of the yield function with respect to the stress is so unless (see Eq. (16))

$$g'(n\frac{\pi}{3}) = 0, \qquad n = 0, 1, \ldots, 5 \tag{18}$$

relations that have been assumed. With this assumption, the dissipation function obtained in the isotropic case for incompressible materials is given by

$$\pi(\dot{\epsilon}) = \sigma_0 \dot{\epsilon}_{eq} G[Z(\zeta)] \tag{19}$$

when $Tr\dot{\epsilon} \neq 0$ and

$$\pi(\dot{\epsilon}) = +\infty \tag{20}$$

when when $Tr\dot{\epsilon} = 0$ with

$$G[Z(\zeta)] = \frac{1}{\sqrt{\left[g(Z(\zeta))\right]^2 + \left[g'(Z(\zeta))\right]^2}} \tag{21}$$

where $Z(\zeta)$ is the inverse function of the relation between the stess Lode angle ω and the strain rate Lode angle ζ. This relation is bijective due to the convexity of the yield surface and reads

$$\zeta = \omega + \arctan\left[\frac{g'(\omega)}{g(\omega)}\right] \Leftrightarrow \omega = Z[\zeta] \tag{22}$$

In the anisotropic case with one linear transformation, Ferreira et al. [14] have extended this formula for materials with yield behaviour defined by

$$f(\boldsymbol{\sigma}) = \Phi(\mathbb{L} : \boldsymbol{\sigma}) \tag{23}$$

In the anisotropic case with several linear transformations to be devoped afterwards, we will also introduce a function $g(\omega_1, \omega_2, \alpha)$ (see (37)) and this function is considered smooth. In the general case, its partial derivatives with respect to ω_1 and ω_2 should satisfy similar relations to (18). The g associated to the the function $\phi(\boldsymbol{s}_1, \boldsymbol{s}_2)$ adopted by Barlat et al. [1] (see(12)) satisfies these conditions.

Non-smooth g functions can be considered following two alternatives. By a limit process, when $m \to \infty$, one can consider for instance Tresca criterion as an isotropic underlying yield model but also directly but in this case, conditions (18) should be amended (see e.g. Piccolroaz and Bigoni [15]) but also instead of considering the gradient of g, one is led to work with the subdifferential of g and the cone of normals to the convex yield domain. This is not considered in this paper.

We come now to the determination of the dissipation function for materials with anisotropic behavior governed by two linear transformations. As already underlined above, extension to more linear transformations is possible but is not considered here. We denote by D the yield domain of the underlying isotropic material defined in $\mathbf{R}^{12}$ by

$$D = \{(\boldsymbol{s}_1, \boldsymbol{s}_2) \in \mathbf{R}^{12} | \, \phi(\boldsymbol{s}_1, \boldsymbol{s}_2) \leq \sigma_0\} \tag{24}$$

and consider the linear transformation A

$$\begin{aligned} A : &\mathbf{R}^6 \to \mathbf{R}^{12} \\ &\boldsymbol{\sigma} \to (\mathbb{L}_1 : \boldsymbol{\sigma}, \mathbb{L}_2 : \boldsymbol{\sigma}) \end{aligned} \tag{25}$$

The yield domain of the anisotropic material considered is the convex set C defined by

$$C = \{\boldsymbol{\sigma} \, | f(\boldsymbol{\sigma}) = \phi(\mathbb{L}_1 : \boldsymbol{\sigma}, \mathbb{L}_2 : \boldsymbol{\sigma}) \leq \sigma_0\} = A^{-1}D \tag{26}$$

where we emphasize that the notation $A^{-1}D$ stands for the inverse image of D and therefore A^{-1} is not the inverse of the linear transformation A which may not exist.

The seeked dissipation function $\pi(\dot{\boldsymbol{\epsilon}})$ of the anisotropic material is thus the support function of the convex set $A^{-1}D$, i.e.

$$\pi(\dot{\boldsymbol{\epsilon}}) = \sup_{\sigma \in A^{-1}D} \{\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}\} \tag{27}$$

The determination of this dissipation function is obtained in two steps: the first one consists in obtaining the support function of the convex set D and from this result, the second step deduces the support function of the convex set $A^{-1}D$ through standard results in convex analysis. Indeed, denoting by $\pi^{iso}(\dot{\boldsymbol{\epsilon}}^1, \dot{\boldsymbol{\epsilon}}^2)$ the support function of the set D defined by

$$\pi(\dot{\boldsymbol{\epsilon}}^1, \dot{\boldsymbol{\epsilon}}^2) = \sup_{(\boldsymbol{s}_1, \boldsymbol{s}_2) \in D} \left\{\boldsymbol{s}_1 : \dot{\boldsymbol{\epsilon}}^1 + \boldsymbol{s}_2 : \dot{\boldsymbol{\epsilon}}^2)\right\} \tag{28}$$

the support function $\pi(\dot{\boldsymbol{\epsilon}})$ of the inverse image $A^{-1}D$ of D by the linear transformation A is given by

$$\pi(\dot{\boldsymbol{\epsilon}}) = \min_{(\dot{\boldsymbol{\epsilon}}^1, \dot{\boldsymbol{\epsilon}}^2)/\dot{\boldsymbol{\epsilon}} = \mathbb{L}_1^T : \dot{\boldsymbol{\epsilon}}^1 + \mathbb{L}_2^T : \dot{\boldsymbol{\epsilon}}^2} \pi^{iso}(\dot{\boldsymbol{\epsilon}}^1, \dot{\boldsymbol{\epsilon}}^2) \tag{29}$$

(see Rockafellar [10], Corollary 16.3.1, p. 143).

We need therefore to obtain $\pi^{iso}(\dot{\boldsymbol{\epsilon}}^1, \dot{\boldsymbol{\epsilon}}^2)$. As the set D is convex, when it is bounded, the supremum in 29) is reached at the boundary of D at the point where the normal to D is colinear to the strain rate $(\dot{\boldsymbol{\epsilon}}^1, \dot{\boldsymbol{\epsilon}}^2)$ (see Salençon [16]). This normal is given by $(\frac{\partial \phi}{\partial \boldsymbol{s}_1}, \frac{\partial \phi}{\partial \boldsymbol{s}_2})$. Therefore we have the relation

$$\begin{aligned} \dot{\boldsymbol{\epsilon}}^1 &= \dot{\lambda} \frac{\partial \phi}{\partial \boldsymbol{s}_1} \\ \dot{\boldsymbol{\epsilon}}^2 &= \dot{\lambda} \frac{\partial \phi}{\partial \boldsymbol{s}_2}, \end{aligned} \tag{30}$$

where $\dot{\lambda}$ is a positive scalar. With the relations (30), we obtain

$$\pi^{iso}(\underline{\dot{\boldsymbol{\epsilon}}}^1, \underline{\dot{\boldsymbol{\epsilon}}}^2) = \dot{\lambda}\left(\frac{\partial \phi}{\partial \boldsymbol{s}_1} : \boldsymbol{s}_1 + \frac{\partial \phi}{\partial \boldsymbol{s}_2} : \boldsymbol{s}_2\right) = \dot{\lambda}\phi(\boldsymbol{s}_1, \boldsymbol{s}_2) = \dot{\lambda}\sigma_0 \tag{31}$$

the two last equalities on the right of (31) following from the positive homogeneity of function ϕ and its value at the boundary of the set D.

As we can see through this relation, to obtain the dissipation function, we need to obtain the plastic multiplier $\dot{\lambda}$ only in terms of the strain rates $\dot{\boldsymbol{\epsilon}}^1$ and $\dot{\boldsymbol{\epsilon}}^2$. This will be carried out in several steps now.

Using relations (30), we get

$$\dot{\lambda} = \frac{\sqrt{(\dot{\epsilon}^1_{eq})^2 + (\dot{\epsilon}^2_{eq})^2}}{\sqrt{\frac{2}{3}\left(\frac{\partial \phi}{\partial \boldsymbol{s}_1} : \frac{\partial \phi}{\partial \boldsymbol{s}_1} + \frac{\partial \phi}{\partial \boldsymbol{s}_2} : \frac{\partial \phi}{\partial \boldsymbol{s}_2}\right)}}, \tag{32}$$

However, this expression of $\dot{\lambda}$ is still dependent on the stresses as its denominator is a function of the two stresses $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. The next steps allow to write this denominator in terms of the strain rate $\dot{\boldsymbol{\epsilon}}^1$ and $\dot{\boldsymbol{\epsilon}}^2$.

The principal stress components S^1_i and S^2_j are defined in terms of the von Mises effective stresses s^i_{eq} and the Lode angles ω^i of the stress tensors $\boldsymbol{s}_1 = \mathbb{L}_1 : \boldsymbol{\sigma}$ and $\boldsymbol{s}_2 = \mathbb{L}_2 : \boldsymbol{\sigma}$ by

$$\begin{aligned} S^i_1 &= \frac{2}{3} s^i_{eq} \cos(\omega^i_1), \qquad S^i_2 = \frac{2}{3} s^i_{eq} \cos(\omega^i_2), \\ S^i_3 &= \frac{2}{3} s^i_{eq} \cos(\omega^i_3), \end{aligned} \tag{33}$$

where

$$\omega^i_1 = \omega^i, \qquad \omega^i_2 = \omega^i - \frac{2\pi}{3}, \qquad \omega^i_3 = \omega^i + \frac{2\pi}{3} \tag{34}$$

$$\omega^i = \frac{1}{3} \arccos\left(\frac{27}{2} \frac{\det \boldsymbol{s}_i}{(s^i_{eq})^3}\right), \qquad s^i_{eq} = \sqrt{\frac{3}{2} \boldsymbol{s}_i : \boldsymbol{s}_i}, \tag{35}$$

for $i = 1, 2$.

By combining Eq. (12) with Eq. (33) and defining an auxiliary angle

$$\alpha = \arctan\left(s^2_{eq}/s^1_{eq}\right), \qquad \alpha \in (0, \pi/2) \tag{36}$$

the yield criterion $\phi(\boldsymbol{s}_1, \boldsymbol{s}_2)$ is now expressed as a function of five arguments

$$\phi(\boldsymbol{s}_1, \boldsymbol{s}_2) = \phi(s^1_{eq}, s^2_{eq}, \omega^1, \omega^2, \alpha) = r(s^1_{eq}, s^2_{eq}) g(\omega^1, \omega^2, \alpha) \tag{37}$$

where

$$\begin{aligned} r(s^1_{eq}, s^2_{eq}) &= \sqrt{(s^1_{eq})^2 + (s^2_{eq})^2} \\ g(\omega^1, \omega^2, \alpha) &= \frac{2}{3}\left[\left(\frac{1}{4} \sum_{j=1}^{3} \sum_{i=1}^{3} |\cos\alpha \cos\omega^1_i - \sin\alpha \cos\omega^2_j|^m\right)\right]^{1/m} \end{aligned} \tag{38}$$

The derivatives of the yield function ϕ with respect to $\boldsymbol{s}_k$, $k = 1, 2$ read

$$\frac{\partial \phi}{\partial \boldsymbol{s}_k} = \left(\frac{\partial r}{\partial s^1_{eq}} \frac{\partial s^1_{eq}}{\partial \boldsymbol{s}_k} + \frac{\partial r}{\partial s^2_{eq}} \frac{\partial s^2_{eq}}{\partial \boldsymbol{s}_k} \right) g(\omega^1, \omega^2, \alpha) + r(s^1_{eq}, s^2_{eq}) \left(\frac{\partial g}{\partial \omega^1} \frac{\partial \omega^1}{\partial \boldsymbol{s}_k} + \frac{\partial g}{\partial \omega^2} \frac{\partial \omega^2}{\partial \boldsymbol{s}_k} + \frac{\partial g}{\partial \alpha} \frac{\partial \alpha}{\partial \boldsymbol{s}_k} \right). \tag{39}$$

After some algebraic manipulations, not reported here, the plastic dissipation potential π^{iso} is written concisely as

$$\pi^{iso} = \tilde{\pi}(\dot{\epsilon}^1_{eq}, \dot{\epsilon}^2_{eq}) G(\omega^1, \omega^2, \alpha), \tag{40}$$

where

$$\tilde{\pi}(\dot{\epsilon}^1_{eq}, \dot{\epsilon}^2_{eq}) = \sigma_0 \sqrt{(\dot{\epsilon}^1_{eq})^2 + (\dot{\epsilon}^2_{eq})^2} \tag{41}$$

$$G(\omega^1, \omega^2, \alpha) = \frac{1}{\sqrt{g^2 + g_\alpha^2 + (g_1 \sec \alpha)^2 + (g_2 \csc \alpha)^2}} \tag{42}$$

with

$$g_1 = \frac{\partial g}{\partial \omega^1}, \qquad g_2 = \frac{\partial g}{\partial \omega^2}, \qquad g_\alpha = \frac{\partial g}{\partial \alpha}. \tag{43}$$

Relation (42) is still dependent on three stress variables: the stress Lode angles ω^1 and ω^2 and the auxiliary stress angle α. We provide next links between these three stress variables and their counterparts associated to the strain rates $\dot{\boldsymbol{\epsilon}}^1$ and $\dot{\boldsymbol{\epsilon}}^2$. Indeed, defining the ratio β of the two effective strain rates

$$\beta = \arctan \left(\frac{\dot{\epsilon}^2_{eq}}{\dot{\epsilon}^1_{eq}} \right) \tag{44}$$

the definitions of the Lode angles and the expressions (30) lead to

$$\zeta_1 = \frac{1}{3} \arccos \left(\frac{4 \det \dot{\boldsymbol{\epsilon}}^1}{(\dot{\epsilon}^1_{eq})^3} \right) = \frac{1}{3} \arccos \left(\frac{4 \det \frac{\partial \phi}{\partial \boldsymbol{s}_1}}{\left(\frac{2}{3} \frac{\partial \phi}{\partial \boldsymbol{s}_1} : \frac{\partial \phi}{\partial \boldsymbol{s}_1} \right)^{3/2}} \right)$$

$$\zeta_2 = \frac{1}{3} \arccos \left(\frac{4 \det \dot{\boldsymbol{\epsilon}}^2}{(\dot{\epsilon}^2_{eq})^3} \right) = \frac{1}{3} \arccos \left(\frac{4 \det \frac{\partial \phi}{\partial \boldsymbol{s}_2}}{\left(\frac{2}{3} \frac{\partial \phi}{\partial \boldsymbol{s}_2} : \frac{\partial \phi}{\partial \boldsymbol{s}_2} \right)^{3/2}} \right)$$

$$\beta = \arctan \left(\frac{\sqrt{\frac{\partial \phi}{\partial \boldsymbol{s}_2} : \frac{\partial \phi}{\partial \boldsymbol{s}_2}}}{\sqrt{\frac{\partial \phi}{\partial \boldsymbol{s}_1} : \frac{\partial \phi}{\partial \boldsymbol{s}_1}}} \right) \tag{45}$$

where $\zeta_1 \in [0, 2\pi]$, $\zeta_2 \in [0, 2\pi]$ and $\beta \in (0, \pi/2)$. A full simplification of the Eq. (45) through the expressions of $\partial \phi / \partial \boldsymbol{s}_1$ and $\partial \phi / \partial \boldsymbol{s}_2$ given in Eq. (39) lead to the relation between the strain rate angles $(\zeta_1, \zeta_2, \beta)$ and stress angles $(\omega^1, \omega^2, \alpha)$ as

$$\xi^1 = \omega^1 + \arctan \left(\frac{g_1}{\cos \alpha (g \cos \alpha - g_\alpha \sin \alpha)} \right)$$

$$\xi^2 = \omega^2 + \arctan \left(\frac{g_2}{\sin \alpha (g \sin \alpha + g_\alpha \cos \alpha)} \right)$$

$$\beta = \arctan \left(\sqrt{\frac{(g_2 \csc \alpha)^2 + (g \sin \alpha + g_\alpha \cos \alpha)^2}{(g_1 \sin \alpha)^2 + (g \cos \alpha - g_\alpha \sin \alpha)^2}} \right) \tag{46}$$

Observe at this stage how the two first relations (46) compare with (22). To express the plastic multiplier (32) only in terms of the strain rates $\dot{\boldsymbol{\epsilon}}^1$ and $\dot{\boldsymbol{\epsilon}}^2$, one needs to inverse the relations (46). This inversion is not possible in the general case but can be numerically accomplished by a 4-dimensional interpolation technique (i.e. interpolation of functions whose domain is defined in a three-dimensional euclidian space). Various procedure for interpolating a structured n-dimensional set of points exist, the outcome of which are the expressions of the stress angles $(\omega^1, \omega^2, \alpha)$ in terms of the strain rate angles $(\zeta_1, \zeta_2, \beta)$ as three functions

$$\begin{aligned} \omega^1 &= Z_1(\zeta_1, \zeta_2, \beta) \\ \omega^2 &= Z_2(\zeta_1, \zeta_2, \beta) \\ \alpha &= Z_\alpha(\zeta_1, \zeta_2, \beta). \end{aligned} \tag{47}$$

With this, the plastic multiplier $\dot{\lambda}$ depends only on the strain rates $\dot{\boldsymbol{\epsilon}}^1$ and $\dot{\boldsymbol{\epsilon}}^2$. Substituting relations (47) in the expression (32) of $\dot{\lambda}$, the dissipation π^{iso} in (40) takes the final form

$$\pi^{iso}(\underline{\dot{\boldsymbol{\epsilon}}}^1, \underline{\dot{\boldsymbol{\epsilon}}}^2) = \tilde{\pi}(\dot{\epsilon}^1_{eq}, \dot{\epsilon}^2_{eq}) G(Z_1(\zeta_1, \zeta_2, \beta), Z_2(\zeta_1, \zeta_\alpha, \beta), Z_2(\zeta_1, \zeta_2, \beta)) \tag{48}$$

All what remains to do is to use Eq. (29) to obtain the dissipation function of the anisotropic material, but only after writing the strain rates $\dot{\boldsymbol{\epsilon}}^1$ and $\dot{\boldsymbol{\epsilon}}^2$ in terms of the strain rate $\dot{\boldsymbol{\epsilon}}$ as the dissipation function of the anisotropic material is only dependent on this last variable. This is done by solving the equation

$$\underline{\dot{\epsilon}} = \mathbb{L}_1^T : \dot{\epsilon}^1 + \mathbb{L}_2^T : \dot{\epsilon}^2 = \mathbb{L}_1 : \dot{\epsilon}^1 + \mathbb{L}_2 : \dot{\epsilon}^2 \tag{49}$$

appearing in the minimization contained in Eq. (29). To avoid confusions, we use the Voigt notation defined earlier and rewrite it in the equivalent form

$$\underline{\dot{\epsilon}} = \left[\underline{\underline{L}}_1 \ \underline{\underline{L}}_2 \right] \left\{ \begin{array}{c} \underline{\dot{\epsilon}}^1 \\ \underline{\dot{\epsilon}}^2 \end{array} \right\} = \underline{\underline{L}} \left\{ \begin{array}{c} \underline{\dot{\epsilon}}^1 \\ \underline{\dot{\epsilon}}^2 \end{array} \right\}. \tag{50}$$

Here, $\underline{\underline{L}}$ is a six by twelve matrix, $\underline{\dot{\epsilon}}$, $\underline{\dot{\epsilon}}^1$ and $\underline{\dot{\epsilon}}^2$ are all six order vectors. Despite the rectangular nature of $\underline{\underline{L}}$, this system can be solved using the Moore–Penrose generalized inverse $\underline{\underline{L}}^+$ of $\underline{\underline{L}}$ (Moore [17], Penrose and Todd [18]). The set of solutions $\left\{ \begin{array}{c} \underline{\dot{\epsilon}}^1 \\ \underline{\dot{\epsilon}}^2 \end{array} \right\}$ of Eq. (50) in terms of the strain rate $\underline{\dot{\epsilon}}$ is given by

$$\left\{ \begin{array}{c} \underline{\dot{\epsilon}}^1 \\ \underline{\dot{\epsilon}}^2 \end{array} \right\} = \underline{\underline{L}}^+ \cdot \underline{\dot{\epsilon}} + (\underline{\underline{I}} - \underline{\underline{L}}^+ \underline{\underline{L}}) \cdot \mathbf{w} \tag{51}$$

where $\mathbf{w} = \left\{ \begin{array}{c} \mathbf{w}_1 \\ \mathbf{w}_2 \end{array} \right\} \in \mathbb{R}^{12}$ is an arbitrary vector.

The Moore-Penrose generalized inverse exists for any real or complex matrix. For any real matrix $\underline{\underline{L}} \in \mathbb{M}(m,n)$, $\underline{\underline{L}}^+ \in \mathbb{M}(n,m)$ is a matrix that is unique and satisfies the following properties defined by Penrose and Todd [18]:

$$\underline{\underline{L}}\,\underline{\underline{L}}^+\,\underline{\underline{L}} = \underline{\underline{L}}, \tag{52}$$

$$\underline{\underline{L}}^+\,\underline{\underline{L}}\,\underline{\underline{L}}^+ = \underline{\underline{L}}_+, \tag{53}$$

$$(\underline{\underline{L}}\,\underline{\underline{L}}^+)^T = \underline{\underline{L}}\,\underline{\underline{L}}^+, \tag{54}$$

$$(\underline{\underline{L}}^+\,\underline{\underline{L}})^T = \underline{\underline{L}}^+\,\underline{\underline{L}}. \tag{55}$$

The Moore-Penrose inverse $\underline{\underline{L}}^+$ can be obtained by the Single Value Decomposition (SVD) procedure—among others (see Ben-Israel and Greville [19] and Campbell and Meyer [20])—by decomposing the original matrix $\underline{\underline{L}}$ as

$$\underline{\underline{L}} = \underline{\underline{U}}\,\underline{\underline{L}}_S\,\underline{\underline{V}}^*, \tag{56}$$

where $\underline{\underline{U}}$ is a $m \times m$ unitary matrix, $\underline{\underline{L}}$ is a $m \times n$ matrix, $\underline{\underline{V}}^*$ is a $n \times n$ unitary matrix, $\underline{\underline{L}}_S$ is a matrix whose diagonal components are nonnegative, while its other components are zero, and

$$\underline{\underline{L}}^+ = \underline{\underline{U}}\,\underline{\underline{L}}_S^+\,\underline{\underline{V}}^*, \tag{57}$$

where $\underline{\underline{L}}_S^+$ is obtained by replacing each nonzero diagonal component of $\underline{\underline{L}}_S$ by its correspondent reciprocal value.

As $\underline{\underline{L}}$ has the form $\left[\underline{\underline{L}}_1 \ \underline{\underline{L}}_2 \right]$, the above formulae imply that $\underline{\underline{L}}^+$ will have the form $\left[\begin{array}{c} \underline{\underline{K}}^1 \\ \underline{\underline{K}}^2 \end{array} \right]$ and the solutions in (51) can be written as

$$\left\{ \begin{array}{c} \underline{\dot{\epsilon}}^1 \\ \underline{\dot{\epsilon}}^2 \end{array} \right\} = \left[\begin{array}{c} \underline{\underline{K}}^1 \cdot \underline{\dot{\epsilon}} \\ \underline{\underline{K}}^2 \underline{\dot{\epsilon}} \end{array} \right] \cdot \underline{\dot{\epsilon}} + \left\{ \underline{\underline{I}} - \left[\begin{array}{c} \underline{\underline{K}}^1 \\ \underline{\underline{K}}^2 \end{array} \right] \cdot \left[\underline{\underline{L}}_1 \ \underline{\underline{L}}_2 \right] \right\} \cdot \left\{ \begin{array}{c} \mathbf{w}_1 \\ \mathbf{w}_2 \end{array} \right\} \tag{58}$$

This gives the full relations between the two strain rates $\underline{\dot{\epsilon}}^1$, $\underline{\dot{\epsilon}}^2$ and the strain rate $\underline{\dot{\epsilon}}$. Plugging these relations in (29) give the final form of the dissipation function for the anisotropic plastic model considered. As relation (58) contain the arbitrary vector $\mathbf{w}$, we obtain

$$\begin{aligned} \pi(\underline{\dot{\epsilon}}) = \min_{\mathbf{w} \in \mathbf{R}^{12}} & \tilde{\pi}(\dot{\epsilon}^1_{eq}, \dot{\epsilon}^2_{eq}) G(Z_1(\zeta_1, \zeta_2, \beta), \\ & Z_2(\zeta_1, \zeta_\alpha, \beta), Z_2(\zeta_1, \zeta_2, \beta)) \end{aligned} \tag{59}$$

whenever $\underline{\dot{\epsilon}} \neq 0$ and

$$\pi(\underline{\dot{\epsilon}}) = +\infty \tag{60}$$

where the function G has already been defined in (42), the strain rates $\underline{\dot{\epsilon}}^1$ and $\underline{\dot{\epsilon}}^2$ are given by (58), ζ_1 and ζ_2 are their Lode angles and β the ratio of their effective components. To close the determination of the dissipation function for anisotropic models based on two linear transformations, it remains now to obtain the expressions of the matrices $\underline{\underline{K}}^1$ and $\underline{\underline{K}}^2$ (or equivalently the assocaied fourth order tensors $\mathbb{K}_1$ and $\mathbb{K}_2$). The generalized inverse of matrices such that considered in this paper and more general ones are given in details in Cline [21].

4 Conclusion

We have obtained here the explicit forms of the dissipation functions associated to general anisotropic

plastic models with the main objective to use them in simulation of the macroscopic behavior of plastic porous solids. The expression of the exact dissipation contains as shown above an optimization of the vector **w** to obtain the value of the minimum contained in the above expression. This might be a difficult task. In the absence of a simple procedure to get this minimum, one could use an upper bound to the exact dissipation function given in (59). Indeed, the solution corresponding to $\mathbf{w} = \mathbf{0}$ is such an upper bound. Among all the solutions of (58), this particular solution minimizes the norm of $\left\{ \begin{array}{c} \underline{\dot{\epsilon}}^1 \\ \underline{\dot{\epsilon}}^2 \end{array} \right\}$, see Ben-Israel and Greville [19] and reads

$$\left\{ \begin{array}{c} \underline{\dot{\epsilon}}^1 \\ \underline{\dot{\epsilon}}^2 \end{array} \right\} = \underline{\underline{\boldsymbol{L}}}^{+} \cdot \underline{\dot{\epsilon}} \tag{61}$$

Moreover, for any kinematically admissible velocity field in the Gurson's approach, it will give an upper bound to the exact macroscopic dissipation and therefore maintains the consistency of the approach. This procedure in under investigation for modeling the macroscopic yielding behavior of voided materials with matrix having anisotropic plastic behavior based on several linear transformations according to relations (1) and (2).

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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