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A series representation of a parallel system

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Abstract Willing to work in reliability theory in a general set up, under dependence conditions, we intend to represent a parallel system as a series system through its components compensator transform .

Keywords: Martingale methods in reliability theory, compensator process, series systems, parallel systems.

1.Introduction

In reliability theory a coherent system can be decomposed in a series-parallel (parallel-series) structure (Barlow and Proschan (1981)).

If the components lifetimes are denoted by $T_i, \quad i = 1, ..., n$ the system lifetime is given by

$$T = \Phi(\mathbf{T}) = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i,$$

where $\mathbf{T} = (T_1, ..., T_n)$ and where $K_j, 1 \leq j \leq k$ are minimal cut sets, that is, a minimal set of components whose joint failure causes the system to fail.

In the case in which there is only one cut set ($k = 1$), we have a parallel system and $T = \max_{1 \leq i \leq n} T_i$ and if all unitary set is a cut set ($k = n$), the system is series and $T = \min_{1 \leq i \leq n} T_i$. Even in despite of its simples structures, the series and parallel system are essential in reliability theory since that any coherent structure can be decomposed in a series-parallel (parallel-series) structure; the performance of any coherent system is always greater than the performance of a series system and lower than the performance of a parallel system; a parallel (active) redundancy at components level is always **better** than a parallel redundancy at system level are some applications of series and parallel operations.

In the case of statistically dependent components the system reliability at time t, defined by the survival function $\overline{F}(t) = P(T > t)$ is not easy to calculate, involving intricate operations with multivariate distributions. To work this dependence we consider that there exists a bijective correspondence between the space of the distributions functions and the space of the compensator processes (Jacod (1975)), that is

$$P(S_i > t|\Im_t) = \exp[-A_i^c(t)] \prod_{0\leq s\leq t} (1 - \triangle A_i(t))$$

where

$$\Im_t := \sigma(1_{\{T_i>s\}}, 1 \leq i \leq i \leq n,\ s \leq t)$$

is the filtration representing our observations, $(A_i(t))_{t\geq 0}$ is the $\Im_t$-compensator process of the counting process $(1_{\{T_i\leq t\}})_{t\geq 0}$, $A_i^c(t)$ is its continuous part and $\triangle A_i(t) = A_i(t) - A_i(-t)$.

In this context Arjas (1981) characterizes the system lifetime through its $\Im_t$-compensator process $(A_\Phi(t))_{t\geq 0}$ and the former through the components $\Im_t$-compensator processes after its corresponding critical level Y_i. The critical level Y_i is a positive and extended random variable which describes the time when component i becomes critical, i.e., the time from which on a failure of component i would lead to system failure (For a rigorous definition of critical level see Arjas (1981)). It is easy to see that the critical level of a component in a series system is the initial time 0 and that the components critical level in a parallel system of n components is the $(n-1)$-th order statistic $T_{(n-1)}$. The result from Arjas (1981) is

$$A_\Phi(t) = \sum_{i=1}^{n} [A_i(t \wedge T) - A_i(Y_i)]^+, \quad P - a.s.$$

where $a^+ = \max\{0, a\}$.

Remark: In a general set up we set $Y_i = \infty$ if either, the component i fails before it is critical or if the system fails before the component i became critical.

It follows that the system reliability is calculated by

$$\overline{F}(t) = 1 - P(T \leq t) = 1 - E\{\sum_{i=1}^{n} [A_i(t \wedge T) - A_i(Y_i)]^+\}.$$

Therefore, for a series system $\overline{F}(t) = 1 - E\{\sum_{i=1}^{n} A_i(t \wedge T)\}$ and for a parallel system $\overline{F}(t) = 1 - E\{\sum_{i=1}^{n} [A_i(t \wedge T) - A_i(T_{(n-1)})]^+\}$. However the random quantity $T_{(n-1)}$ is difficult to follow up and the calculation of $E\{[A_i(t \wedge T) - A_i(Y_i)]^+\}$ still involved in difficulties.

In this paper we intend to represent a parallel system as a series system through its components compensator transform. It have to be done in a different probability space but we can eliminate the $A_i(T_{(n-1)})$ calculation.

2. Parallel operation through compensator transform

In order to simplify the notation, we assume that relations such as $\subset, =, \leq, <, \neq$ between random variables and measurable sets, always hold with probability one.

Consider a parallel system of n components with lifetime T_i, $1 \leq i \leq n$, being finite and positive random variables defined on a complete probability space $(\Omega, \Im, P)$ with $P(T_i = T_j) = 0, \quad 1 \leq i, j \leq n$, that is, the components can be dependent but simultaneous failures are null-sets. The parallel system lifetime is $T = \Phi(\mathbf{T}) = \max_{1 \leq i \leq n} T_i$, where $\mathbf{T} := (T_1, .., T_n)$.

The system is monitored at component's level through a family of sub σ-algebras of $\Im$, denoted $(\Im_t)_{t \geq 0}$, where

$$\Im_t = \sigma(1_{\{T_i > s\}}, 1 \leq i \leq n, \ s \leq t)$$

satisfies the Dellacherie conditions of right continuity and completeness.

To work with martingales technics some concepts are necessary. A positive and extended random variable τ is an $\Im_t$-stopping time if, and only if, $\{\tau \leq t\} \in \Im_t, \forall t \geq 0$; τ is a predictable $\Im_t$-stopping time if an increasing sequence $(\tau_n)_{n \geq 1}$, of $\Im_t$-stopping time , $\tau_n < \tau$, exists such that $\tau_n \to \tau$ as $n \to \infty$; τ is totally inaccessible $\Im_t$-stopping time if $P(\tau = \sigma) < \infty) = 0$ for all predictable $\Im_t$-stopping times σ.

The counting process $(N_i(t))_{t \geq 0}$, $N_i(t) = 1_{\{T_i \leq t\}}$, is an $\Im_t$-submartingale and follows from Doob-Meyer decomposition, that there exists a unique right continuos, $\Im_t$-predictable, nondecreasing and integrable process $(A_i(t))_{t \geq 0}$, with $A_i(0) = 0$ such that $(N_i(t) - A_i(t))_{t \geq 0}$ is an zero mean $\Im_t$-martingale. $A_i(t)$ is called the $\Im_t$-compensator of $N_i(t)$. In what follows we assume that the lifetimes T_i are totally inaccessible $\Im_t$-stopping time which implies that the compensators $A_i(t)$ are continuous and differentiable.

In what follows we use Girsanov Theorem from Bremaud (1981).

Theorem 2.1 (Girsanov) Let $T_i, 1 \leq i \leq n$ be totally inaccessible $\Im_t$-stopping time representing the components lifetimes. Let $(\alpha_i(t))_{t\geq 0}, 1 \leq i \leq n$, be nonnegative, $\Im_t$-predictable processes such that for all $t \geq 0$ and all $i, 1 \leq i \leq n$,

$$A_i^*(t) = \int_0^t \alpha_i(s) dA_i(s) < \infty, P - a.s.$$

and denote $\alpha(t) = (\alpha_1(t), ..., \alpha_n(t))$ and $\alpha(\infty) = \alpha$, then

$$L_{\alpha(t)}(t) = \prod_{i=1}^{n} [\alpha_i(T_i)]^{N_i(t)} \exp[A_i(t) - A_i^*(t)]$$

is a nonnegative $\Im_t$-local martingale and a nonnegative $\Im_t$-super-martingale. Furthermore, if $E[L_\alpha(\infty)] = 1$, $A_i^*(t)$ is the unique $\Im_t$-compensator of $N_i(t)$ under the probability measure Q_α defined by the Radon Nikodyn derivative

$$\frac{dQ_\alpha}{dP} = L_\alpha(\infty).$$

If $T_i, 1 \leq i \leq n$ are independent lifetimes the $\Im_t$-compensator processes of $N_i(t) = 1_{\{T_i \leq t\}}$ are its hazard functions given by $A_i(t) = -\ln P(T_i > t|\Im_t) = -\ln(\overline{F}_i(t \wedge T_i))$ where $\overline{F_i}(t) = P(T_i > t)$ is the survival function of T_i.

Under the independence hypothesis we can calculate

$$P(\max_{1\leq i\leq n} \{T_i\} > t) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1\leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^{k} \overline{F}_{i_j}(t) =$$

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{1\leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^{k} \exp[-A_{i_j}(t)] =$$

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{1\leq i_1 < \ldots < i_k \leq n} \exp[-\sum_{j=1}^{k} A_{i_j}(t)] =$$

$$\exp[-\sum_{j=1}^{n} A_j(t)] \sum_{k=1}^{n} (-1)^{k-1} \sum_{1\leq i_1 < \ldots < i_k \leq n} \exp[-\sum_{j=1}^{k} A_{i_j}(t) + \sum_{j=1}^{n} A_j(t)] =$$

$$\exp[-\sum_{j=1}^{n} A_j(t)] \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_{n-k} \le n} \exp[\sum_{j=1}^{n-k} A_{i_j}(t)].$$

Therefore, in the set $\{t < T\}$ the $\Im_t$- compensator of T is $A_\Phi(t) = -\ln[P(T > t|\Im_t)] =$

$$\sum_{j=1}^{n} A_j(t) - \ln\{\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_{n-k} \le n} \exp[\sum_{j=1}^{n-k} A_{i_j}(t)]\}.$$

We intend to define a compensator transform to characterize the parallel system in the dependent case through compensator transform, preserving the above intuition. As this operation is symmetric on T_i, the idea is to combine compensator transformations on $A_i(t)$.

For $1 \le l \le n$ we define the compensator transform $A_l^*(t) = \int_0^t \alpha_l(s) dA_l(s)$ where

$$\alpha_l(s) = \frac{\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_{n-k} \le n; i_j \ne l} \exp[\sum_{j=1}^{n-k} A_{i_j}(s)]}{D(s)}$$

where

$$D(t) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_{n-k} \le n} \exp[\sum_{j=1}^{n-k} A_{i_j}(t)].$$

Lemma 2.2 Under the above hypothesis and notation we have

$$\sum_{l=1}^{n} A_l^* = \sum_{l=1}^{n} A_l - \ln D(t).$$

Proof

$$A_l^*(t) = \int_0^t \frac{\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_{n-k} \le n; i_j \ne l} \exp[\sum_{j=1}^{n-k} A_{i_j}(s)]}{D(s)} dA_l(s) =$$

$$\int_0^t \{1-\frac{\sum_{k=1;k\neq l}^{n}(-1)^{k-1}\sum_{1\leq i_1<\ldots<i_{n-k-1}\leq n}\exp[\sum_{j=1}^{n-k-1}A_{i_j}(s)]\exp[A_l(s)]}{D(s)}\}dA_l(s)=$$

$$A_l(t)-\int_0^t \frac{\sum_{k=1;k\neq l}^{n}(-1)^{k-1}\sum_{1\leq i_1<\ldots<i_{n-k-1}\leq n}\exp[\sum_{j=1}^{n-k-1}A_{i_j}(s)]\exp[A_l(s)]}{D(s)}dA_l(s).$$

However

$$dD(t)=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\leq i_1<\ldots<i_{n-k}\leq n}\exp[\sum_{j=1}^{n-k}A_{i_j}(t)]\sum_{j=1}^{n-k}dA_{i_j}(t)=$$

$$\sum_{l=1}^{n}\{\sum_{k=1;k\neq l}^{n}(-1)^{k-1}\sum_{1\leq i_1<\ldots<i_{n-k-1}\leq n;i_j\neq l}\exp[\sum_{j=1}^{n-k-1}A_{i_j}(t)]\}\exp[A_l(t)]dA_l(t)$$

and therefore

$$\sum_{l=1}^{n}A_l^*=\sum_{l=1}^{n}A_l-\int_0^t\frac{dD(s)}{D(s)}=\sum_{l=1}^{n}A_l-\ln D(t).$$

Our main result follows from an adaptation of Girsanov Theorem.

Theorem 2.3 The following process

$$L(t)=\prod_{l=1}^{n}(\alpha_l(T_l))^{1_{\{T_l\leq t\}}}\exp[\sum_{j=1}A_j(t)-\sum_{j=1}A_j^*(t)]=$$

$$\prod_{l=1}^{n}(\alpha_l(T_l))^{1_{\{T_l\leq t\}}}D(t)=$$

$$\prod_{l=1}^{n}(\alpha_l(T_l))^{1_{\{T_l\leq t\}}}\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\leq i_1<\ldots<i_{n-k}\leq n}\exp[\sum_{j=1}^{n-k}A_{i_j}(t)]$$

is a nonnegative $\Im_t$-martingale with $E[L(\infty)]=1$.

Proof The process

$$\alpha_l(s) = \frac{\sum_{k=1}^{n}(-1)^{k-1} \sum_{1\leq i_1<\ldots<i_{n-k}\leq n; i_j\neq l} \exp[\sum_{j=1}^{n-k} A_{i_j}(s)]}{D(s)}$$

is an $\Im_t$-predictable process because so are the components compensator processes. As $0 < \alpha_l(s) \leq 1$ we have

$$\int_0^t \alpha_l(s) dA_l(s) \leq A_l(t) < \infty, P - a.s.$$

and we can apply Girsanov Theorem. As the lifetimes T_i are finite, $N_i(t) - A_i^*(t)$ are $\Im_t$-martingales.

Therefore, we are looking for a probability measure Q, such that, under Q, $C^*(t) = \sum_{i=1}^n A_i^*(t)$ becomes the $\Im_t$-compensator of $1_{\{\max_{1\leq i\leq n} T_i \leq t\}}$ with respect to this modified probability measure. By Girsanov Theorem the desired measure Q is given by the Radon Nikodyn derivative $\frac{dQ}{dP} = L(\infty)$.

Remarks

i) From Theorem 2.3 we have

$$L(\infty) = \prod_{l=1}^{n}(\alpha_l(T_l)) \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\leq i_1<\ldots<i_{n-k}\leq n} \exp[\sum_{j=1}^{n-k} A_{i_j}(T)].$$

ii) Based in Bueno (2005) we can define the reliability importance of a component , l, with lifetime T_l, for the system reliability, $I(l)$, with lifetime T under a parallel improvement as

$$cov(T, \alpha_l(T_l)) \int_0^{T_l} \frac{\sum_{k=1;k\neq l}^{n} (-1)^{k-1} \sum_{1\leq i_1<\ldots<i_{n-k-1}\leq n} \exp[\sum_{j=1}^{n-k-1} A_{i_j}(s)] \exp[A_l(s)]}{D(s)} dA_l(s)).$$

iii) If the components are dependent but identically distributed, we have

$$D(t) = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_{n-k}\le n} \exp[\sum_{j=1}^{n-k} A_{i_j}(t)] =$$

$$\sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_{n-k}\le n} \exp[(n-k)A(t)] =$$

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{n-k} \exp[(n-k)A(t)] =$$

$$\sum_{j=0}^{n-1}(-1)^{n-j}\binom{n}{j} \exp[jA(t)] = (1-\exp[A(t)])^n - \exp[nA(t)].$$

and

$$\sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<\ldots<i_{n-k}\le n;i_j\neq l} \exp[\sum_{j=1}^{n-k} A_{i_j}(s)] =$$

$$\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n-1}{j} \exp[jA(t)] = (1-\exp[A(t)])^{n-1}.$$

Therefore

$$\alpha_l(t) = \frac{(1-\exp[A(t)])^{n-1}}{(1-\exp[A(t)])^n - \exp[nA(t)]}.$$

iv) If the components are dependent but identically distributed with $n = 2$, the component compensator transform is

$$A^*(t) = \int_0^t \frac{1-e^{A(s)}}{1-2e^{A(s)}} dA(s)$$

and the compensator of $1_{\{T_1\vee T_2\le t\}}$ is

$$\int_0^t \frac{2-2e^{-A(s)}}{2-e^{-A(s)}} dA(s).$$

under the measure Q such that

$$\frac{dQ_\alpha}{dP} = 2 - 2\exp[-A(T)].$$

which is used in Bueno and Carmo (2007) to define active redundancy operation when the component and the spare are dependent but identically distributed.

We note that, in iv), $L(\infty) = \frac{dQ_\alpha}{dP} = 2 - 2\exp[-A(T)] < 2$.

As $\alpha_l(T_l) > 0$ on $\{T_l, \infty$ for all l, $L(\infty) > 0, P_a.s.$ and P is absolutely continuous with respect to Q and $\frac{dP}{dQ_\alpha} = \frac{1}{2-2\exp[-A(T)]} > 2$.

It follows that

$$E[T] = E_Q[\frac{1}{2 - 2\exp[-A(T)]}T] > 2E_Q[T]$$

and we get a lower bound for the mean system lifetime.

In the general case we may conclude that if $L(\infty) < (>) 1$ we get a lower (upper) bound for the mean system lifetime under P.

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