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ORIGINAL PAPER

Tiling Edge-Coloured Graphs with Few Monochromatic Bounded-Degree Graphs

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Abstract

We prove that for all integers $\Delta, r \geq 2$, there is a constant $C = C(\Delta, r) > 0$ such that the following is true for every sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ of graphs with $v(F_n) = n$ and $\Delta(F_n) \leq \Delta$, for each $n \in \mathbb{N}$. In every r-edge-coloured K_n, there is a collection of at most C monochromatic copies from $\mathcal{F}$ whose vertex-sets partition $V(K_n)$. This makes progress on a conjecture of Grinshpun and Sárközy.

Keywords Tiling · Complete edge-coloured graph · Bounded-degree graphs · Monochromatic

Mathematics Subject Classification Primary 05C55; Secondary 05C70

1 Introduction and Main Results

A conjecture of Lehel states that the vertices of any 2-edge-coloured complete graph can be partitioned into two monochromatic cycles of different colours. Here, single vertices and edges are considered cycles. This conjecture first appeared in [2], where it was also proved for some special types of colourings of K_n. Łuczak, Rödl and Szemerédi [23] proved Lehel's conjecture for sufficiently large n using the regularity

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method. Allen [1] gave an alternative proof, with a better bound on n. Finally, Bessy and Thomassé [3] proved Lehel's conjecture for all integers $n \geq 1$.

For colourings with more colours, Erdős, Gyárfás and Pyber [11] proved that the vertices of every r-edge-coloured complete graph on n vertices can be partitioned into $O(r^2 \log r)$ monochromatic cycles. They further conjectured that r cycles should be enough. The currently best-known upper bound is due to Gyárfás, Ruszinkó, Sárközy and Szemerédi [18], who showed that $O(r \log r)$ cycles suffice. However, the conjecture was refuted by Pokrovskiy [24], who showed that, for every $r \geq 3$, there exist infinitely many r-edge-coloured complete graphs which cannot be vertex-partitioned into r monochromatic cycles. Nevertheless, Pokrovskiy conjectured that in every r-edge-coloured complete graph one can find r vertex-disjoint monochromatic cycles which cover all but at most c_r vertices for some $c_r \geq 1$ only depending on r (in his counterexample $c_r = 1$ is possible).

In this paper, we study similar problems in which we are given a family of graphs $\mathcal{F}$ and an edge-coloured complete graph K_n and our goal is to partition $V(K_n)$ into monochromatic copies of graphs from $\mathcal{F}$. All families of graphs $\mathcal{F}$ we consider here are of the form $\mathcal{F} = \{F_1, F_2, \ldots\}$, where F_i is a graph on i vertices for every $i \in \mathbb{N}$. We call such a family a *sequence of graphs*. A collection $\mathcal{H}$ of vertex-disjoint subgraphs of a graph G is an $\mathcal{F}$-*tiling* of G if $\mathcal{H}$ consists of copies of graphs from $\mathcal{F}$ with $V(G) = \bigcup_{H \in \mathcal{H}} V(H)$. If G is edge-coloured, we say that $\mathcal{H}$ is *monochromatic* if every $H \in \mathcal{H}$ is monochromatic. Let $\tau_r(\mathcal{F}, n)$ be the minimum $t \in \mathbb{N}$ such that for every r-edge-coloured K_n, there is a monochromatic $\mathcal{F}$-tiling of size at most t. We define the *tiling number* of $\mathcal{F}$ as

$$\tau_r(\mathcal{F}) = \sup_{n \in \mathbb{N}} \tau_r(\mathcal{F}, n).$$

Using this notation, the results of Pokrovskiy [24] and of Gyárfás, Ruszinkó, Sárközy and Szemerédi [18] mentioned above imply that $r + 1 \leq \tau_r(\mathcal{F}_{\text{cycles}}) = O(r \log r)$, where $\mathcal{F}_{\text{cycles}}$ is the family of cycles. Note that, in general, it is not clear at all that $\tau_r(\mathcal{F})$ is finite and it is a natural question to ask for which families this is the case.

The study of such tiling problems for general families of graphs was initiated by Grinshpun and Sárközy [17]. The *maximum degree* $\Delta(\mathcal{F})$ of a sequence of graphs $\mathcal{F}$ is given by $\sup_{F \in \mathcal{F}} \Delta(F)$, where $\Delta(F)$ is the maximum degree of F. We denote by $\mathcal{F}_\Delta$ the collection of all sequences of graphs $\mathcal{F}$ with $\Delta(\mathcal{F}) \leq \Delta$. Grinshpun and Sárközy proved that $\tau_2(\mathcal{F}) \leq 2^{O(\Delta \log \Delta)}$ for all $\mathcal{F} \in \mathcal{F}_\Delta$. In particular, $\tau_2(\mathcal{F})$ is finite whenever $\Delta(\mathcal{F})$ is finite. They also proved that $\tau_2(\mathcal{F}) \leq 2^{O(\Delta)}$ for every sequence of bipartite graphs $\mathcal{F}$ of maximum degree at most Δ, and showed that this is best possible up to a constant factor in the exponent (see also Sect. 5 for a more detailed discussion on the lower bound).

Sárközy [27] further proved that $\tau_2(\mathcal{F}_{k\text{-cycles}}) = O(k^2 \log k)$, where $\mathcal{F}_{k\text{-cycles}}$ denotes the family of kth power of cycles.[1] For more than two colours less is known. Answering a question of Elekes, Soukup, Soukup and Szentmiklóssy [10], Busta-

[1] The k-th power of a graph H is the graph obtained from H by adding an edge between any two vertices at distance at most k.

mante, Frankl, Pokrovskiy, Skokan and the first author [5] proved that $\tau_r(\mathcal{F}_{k\text{-cycles}})$ is finite for all $r, k \in \mathbb{N}$. Grinshpun and Sárközy [17] conjectured that the same should be true for all families of graphs of bounded degree with an exponential bound.

Conjecture 1.1 (Grinshpun-Sárközy [17], 2016) *For every $r, \Delta \in \mathbb{N}$ and $\mathcal{F} \in \mathcal{F}_\Delta$, $\tau_r(\mathcal{F})$ is finite. Moreover, there is some $C_r > 0$ such that*

$$\tau_r(\mathcal{F}) \leq \exp(\Delta^{C_r}).$$

Our main theorem shows that $\tau_r(\mathcal{F})$ is indeed finite. For a given positive integer k, we denote by $\exp^k$ the kth-composition of the exponential function.

Theorem 1.1 *There is an absolute constant $K > 0$ such that for all integers $r, \Delta \geq 2$ and all $\mathcal{F} \in \mathcal{F}_\Delta$, we have*

$$\tau_r(\mathcal{F}) \leq \exp^2\left(r^{Kr\Delta^3}\right).$$

In particular, $\tau_r(\mathcal{F})$ is finite whenever $\Delta(\mathcal{F})$ is finite.

In order to prove Theorem 1.1, we shall prove an absorption lemma (see Lemma 4.4) whose proof relies on a *density increment argument*. This is responsible for the double exponential bound in our main theorem.

The paper is organized as follows. In Sect. 2, we present an overview of the proof of our main theorem and the proof of our absorption lemma. In Sect. 3 we collect a few lemmas regarding regular pairs and regular cylinders that we shall use repeatedly in later sections. The proof of our absorption lemma and main theorem can be found in Sect. 4.1 and Sect. 4.2, respectively. Finally, we finish the paper with some concluding remarks in Sect. 5.

2 Proof Overview

The proof of Theorem 1.1, similarly to the proof of the two colour result of Grinshpun and Sárközy [17], combines ideas from the absorption method as in the original paper of Erdős, Gyárfás and Pyber [11] with some modern approaches involving the blow-up lemma and the weak regularity lemma of Duke, Lefmann and Rödl [9]. However, in order to extend these ideas to more colours, we need to prove a significantly more complicated *absorption lemma*, requiring new ideas involving a density increment argument.

Our absorption lemma (Lemma 4.4) states that if we have $k := \Delta + 2$ disjoint sets of vertices $V_1, \ldots, V_k$ with $|V_i| \geq 2|V_1|$ for all $i = 2, \ldots, k$ such that every vertex in V_1 belongs to at least $\delta|V_2| \cdots |V_k|$ monochromatic k-cliques *transversal*[2] in $(V_1, \ldots, V_k)$, then it is possible to cover the vertices in V_1 with a constant number (depending on δ, r and Δ) of monochromatic vertex disjoint copies of graphs from $\mathcal{F}$.

[2] A k-clique is transversal in $(V_1, \ldots, V_k)$ if it contains one vertex in each one of the sets $V_1, \ldots, V_k$.

Furthermore, we can choose such a covering using no more than $|V_1|$ vertices in each $V_2, \dots, V_k$.

To deduce Theorem 1.1 from the absorption lemma, we need to partition $V(K_n)$ in a similar fashion as in [5]: first we find $k-1$ monochromatic *super-regular cylinders* $Z_1, \dots, Z_{k-1}$ covering a positive proportion of the vertices of K_n (see Sect. 3 for the definition of super-regular cylinders). Then we apply a result of Fox and Sudakov [13] to *greedily* cover with few disjoint monochromatic copies of graphs from $\mathcal{F}$ almost all of the vertices in $V(K_n)\backslash(Z_1 \cup \dots \cup Z_{k-1})$, leaving uncovered a set R_k of size much smaller than $|Z_{k-1}|$ (see Proposition 4.2).

Now we split R_k into two sets: the set R'_k of vertices belonging to at least $\delta|Z_1|\cdots|Z_{k-1}|$ monochromatic k-cliques transversal in $(R_k, Z_1, \dots, Z_{k-1})$, and the set $V_k = R_k \setminus R'_k$. Using our absorption lemma, we can cover the vertices in R'_k using no more than $|R'_k|$ vertices of each of the cylinders $Z_1, \dots, Z_{k-1}$. For each $i = 1, \dots, k-1$, let Z'_i be the set of vertices in Z_i that has not been used to cover R_1. Since we $|R'_k|$ is significantly smaller than $|Z_i|$, it follows that each Z'_i is still a super-regular cylinder. Now, if the set V_k was empty, then we would be done. Indeed, a consequence of the blow-up lemma (Theorem 3.3) guarantees that we can cover each of the cylinders $Z'_1, \dots, Z'_{k-1}$ with $k+1$ copies of vertex disjoint monochromatic graphs from $\mathcal{F}$.

So let us consider the case where V_k is non-empty. In this case, we first find a reasonably large regular cylinder Z_k in V_k, then we greedily cover most of the remaining vertices in $V_k \setminus Z_k$. Let S_k be the set of those vertices that we covered greedily and let $R_{k+1} = V_k \setminus (Z_k \cup S_k)$. Then, our assumption is that $|R_{k+1}|$ is much smaller than $|Z_k|$. If R_{k+1} is empty, then we are done with covering V_k, since the vertices in S_k are already covered and the vertices in Z_k can be covered using the blow-up lemma.

So let us assume that R_{k+1} is non-empty. Now we partition R_{k+1} similarly how we partitioned R_k: first, set apart into a set R'_{k+1} those vertices of R_{k+1} that belong to many monochromatic k-cliques transversal in R_{k+1} and $k-1$ of the cylinders $Z'_1, \dots, Z'_{k-1}, Z_k$. Remember that those vertices in R'_{k+1} can be covered using our absorption lemma and we will not use more than $|R'_{k+1}|$ vertices from $Z'_1, \dots, Z'_{k-1}, Z_k$ to cover them. Since R'_{k+1} is much smaller than any of those cylinders, we still get super-regular cylinders after removing those vertices from $Z'_1, \dots, Z'_{k-1}, Z_k$ that were used to cover R'_{k+1}. Let $V_{k+1} = R_{k+1} \setminus R'_{k+1}$ be the set of remaining vertices. If V_{k+1} is empty, then we are done, as discussed in the case where V_k is empty.

Thus, let's assume that V_{k+1} is non-empty. Then, we find a large regular cylinder Z_{k+1} in V_{k+1} and greedily cover a set $S_{k+1} \subseteq V_{k+1} \setminus Z_{k+1}$, until the set of leftover vertices R_{k+2} is much smaller than Z_{k+1}. If R_{k+2} is empty, we are again done, since we can cover Z_{k+1} using the blow-up lemma and the vertices in S_{k+1} are already covered. Now, if R_{k+2} is non-empty, then we repeat this process to partition R_{k+2}.

Figure 1 may help to organize the placement of those sets as well as their relative size after several iterations of this process. Finally, using a lemma from [5] (see Lemma 4.6) and Ramsey's theorem, we can show that we only need to repeat this process few times. More precisely, we show that V_{N+1} must be empty, for $N = R_r(K_k)$, the r-colour Ramsey number of the graph K_k.

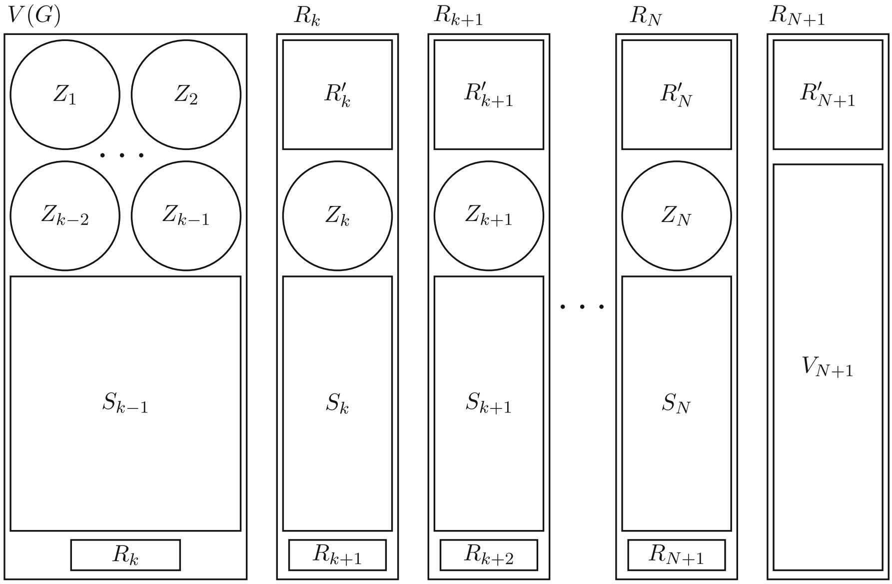

Fig. 1 A partition of $V(G)$ following the *Framework* described in the proof of Theorem 1.1. Each Z_i is an (ε, d)-super-regular k-cylinder, and for every $i \in [k, N+1]$, we have $|R_i| \leq \alpha|Z_{i-1}|$. Each S_i is a disjoint union of monochromatic copies of graphs from $\mathcal{F}$ obtained by greedily covering. Each R'_i can be covered by *previous* Z_j using the Absorption Lemma 4.4. And, finaly, we show that V_{N+1} is empty, if $N \geq R_r(k)$

In order to prove the absorption lemma, we employ a density increment argument. This is the most difficult part of the proof and the key new idea in this paper. First, we partition V_1 into r sets $V_1^{(1)}, \dots, V_1^{(r)}$ so that for every $j \in [r]$, every $v \in V_i^{(j)}$ is incident to at least $d/r \cdot |V_2| \cdots |V_k|$ monochromatic cliques of colour j which are transversal in $(V_1, \dots, V_k)$. We will cover each of these sets separately, making sure not to repeat vertices. Let us illustrate how to cover $V_1^{(1)}$.

We start by finding a large k-cylinder $Z = (U_1, \dots, U_k)$ with $U_1 \subseteq V_1^{(1)}, U_2 \subseteq V_2, \dots U_k \subseteq V_k$ which is super-regular in colour 1. We shall use Z as an *absorber* at the end of the proof to cover any small set of leftovers. Next, we greedily cover most of $V_1^{(1)} \setminus U_1$ by monochromatic copies of $\mathcal{F}$ until the set of uncovered vertices R has size much smaller then $|U_1|$. To cover the set R, we will find a partition $R = S \cup T_2 \cup \ldots \cup T_k$, where each vertex in S belongs to many monochromatic k-cliques of colour 1 which are transversal in $(S, U_2, \dots, U_k)$ (allowing S to be absorbed into the cylinder Z at the end of the proof) and each vertex in T_i, for $i \in \{2, \dots, k\}$, belongs to at least $(\delta + \eta)|V_2| \cdots |V_{i-1}||U_i| \cdots |U_k|$ monochromatic k-cliques transversal in $(T_i, V_2, \dots, V_i, U_{i+1}, \dots, U_k)$, for some $\eta \ll \delta$.

To cover the vertices in each T_i, with $i \in \{2, \dots, k\}$, we repeat the argument with $(V_1, \dots, V_k)$ replaced by $(T_i, V_2, \dots, V_i, U_{i+1}, \dots, U_k)$ and δ replaced by $\delta + \eta$. This is our density increment argument. Since every time we repeat the argument we significantly increase the density of k-cliques, we can bound the number of required repetitions in terms of the initial density of k-cliques.

While covering each of the sets $T_2, \dots, T_k$, we shall guarantee that the set of vertices $X_i \subseteq U_i$ that we use to cover them has size much smaller than $|U_i|$ for all $i = 2, \dots, k$. This way, the cylinder $Z' = (U_1 \cup S, U_2 \backslash X_2, \dots, U_k \backslash X_k)$ will be super-regular in colour 1 and thus we can cover Z' using the blow-up lemma. Repeating this for every colour $j \in [r]$, we get a covering of V_1 with $O_{\delta,r,\Delta}(1)$ many monochromatic disjoint copies of graphs from $\mathcal{F}$.

3 Regularity

In this section, we will gather all the notations and results related to the classical regularity technique which we require for the proof. We start by introducing some relevant notations. Let $G = (V_1, V_2, E)$ be a bipartite graph with parts V_1 and V_2. For any $U_i \subseteq V_i$, $i = 1, 2$, the density of the pair (U_1, U_2) in G is given by

$$d(U_1, U_2) = \frac{e(U_1, U_2)}{|U_1||U_2|}.$$

We say that G (or the pair (V_1, V_2)) is ε*-regular* if for all $U_i \subseteq V_i$ with $|U_i| \geq \varepsilon|V_i|$, $i = 1, 2$, we have

$$|d(U_1, U_2) - d(V_1, V_2)| \leq \varepsilon.$$

If additionally we have $d(V_1, V_2) \geq d$ and $\deg(v, V_i) \geq \delta|V_i|$ for all $v \in V_{3-i}$, $i = 1, 2$, then we say that G (or (V_1, V_2)) is (ε, d, δ)*-super-regular*. We often say that G is (ε, d)*-super-regular* instead of (ε, d, d)-super-regular.

We begin with some simple facts about super-regular pairs. The first one is known as the slicing lemma and roughly says that if we take a large induced subgraph in a dense regular pair we still get a dense regular pair. Its proof is straightforward from the definition of a regular pair.

Lemma 3.1 *(Slicing lemma) Let $\beta > \varepsilon > 0$, $d \in [0, 1]$ and let (V_1, V_2) be an $(\varepsilon, d, 0)$-super-regular pair. Then any pair (U_1, U_2) with $|U_i| \geq \beta|V_i|$ and $U_i \subseteq V_i$, $i = 1, 2$, is $(\varepsilon', d', 0)$-super-regular with $\varepsilon' = \max\{\varepsilon/\beta, 2\varepsilon\}$ and $d' = d - \varepsilon$.*

The following lemma essentially says that after removing few vertices from a super-regular pair and adding few new vertices with large degree, we still have a super-regular pair. The reader can find a proof of it in Appendix 1.

Lemma 3.2 *Let $0 < \varepsilon < 1/2$ and let $d, \delta \in [0, 1]$ so that $\delta \geq 4\varepsilon$. Let (V_1, V_2) be an (ε, d, δ)-super-regular pair in a graph G. Let $X_i \subseteq V_i$ for $i \in \{1, 2\}$, and let Y_1, Y_2 be disjoint subsets of $V(G) \backslash (V_1 \cup V_2)$. Suppose that for each $i \in \{1, 2\}$ we have $|X_i|, |Y_i| \leq \varepsilon^2|V_i|$ and $\deg(v, V_i) \geq \delta|V_i|$ for every $v \in Y_{3-i}$. Then the pair $((V_1 \backslash X_1) \cup Y_1, (V_2 \backslash X_2) \cup Y_2)$ is $(8\varepsilon, d - 8\varepsilon, \delta/2)$-super-regular.*

Let $k \geq 2$ be an integer and let G be a graph. Given disjoint sets of vertices $V_1, \dots, V_k \subseteq V(G)$, we call $Z = (V_1, \dots, V_k)$ a k*-cylinder* and often identify it with the induced k-partite subgraph $G[V_1, \dots, V_k]$. We write $V_i(Z) = V_i$ for every $i \in [k]$.

We say that Z is ε-*balanced* if

$$\max_{i\in[k]} |V_i(Z)| \le (1+\varepsilon) \min_{i\in[k]} |V_i(Z)|,$$

and *balanced* if it is 0-balanced. Furthermore, we say that Z is ε-regular if all the $\binom{k}{2}$ pairs (V_i, V_j) are ε-regular. If G is an r-edge-coloured graph and $i \in [r]$, we say that Z is ε-regular in colour i if it is ε-regular in G_i, the graph consisting of all edges of G with colour i. Similarly, we define (ε, d)-regular and (ε, d, δ)-super-regular cylinders.

As sketched in Sect. 2, we will use super-regular cylinders as absorbers. The following lemma, which Grinshpun and Sárközy [17] deduced from the blow-up lemma [20, 21, 26] and the Hajnal-Szemerédi theorem [19],[3] allows us to do this.

Lemma 3.3 *There is a constant K, such that for all $0 \le \delta \le d \le 1/2$, $\Delta \in \mathbb{N}$, $k = \Delta + 2$, $0 < \varepsilon \le (\delta d^{\Delta})^K$, and $\mathcal{F} \in \mathcal{F}_{\Delta}$, the following is true for every (ε, d, δ)-super-regular k-cylinder $Z = (V_1, \ldots, V_k)$.*

(i) If Z is ε-balanced, then its vertices can be partitioned into at most $\Delta + 3$ copies of graphs from $\mathcal{F}$.
(ii) If $|V_i| \ge |V_1|$ for all $i = 2, \ldots, k$, then there is a copy of a graph from $\mathcal{F}$ covering V_1 and at most $|V_1|$ vertices of each of $V_2, \ldots, V_k$.

It is important in the proof of Theorem 1.1 that we can find super-regular k-cylinders which cover linearly many vertices. The existence of such a cylinder follows from the regularity lemma. Conlon and Fox [7, Lemma 5.3] used the weak regularity lemma of Duke, Lefmann, and Rödl [9] to obtain much larger cylinders. We shall use the following coloured version of their result, the proof of which is very similar and can be found in Appendix 1. See also [17, Lemma 2] for a 2-coloured version which follows readily from the non-coloured version.

Lemma 3.4 *Let $k, r \ge 2$, $0 < \varepsilon < 1/(rk)$ and $\gamma = \varepsilon^{r^{8rk}\varepsilon^{-5}}$. Then every r-edge-coloured complete graph on $n \ge 1/\gamma$ vertices contains, in one of the colours, a balanced $(\varepsilon, 1/2r)$-super-regular k-cylinder $Z = (U_1, \ldots, U_k)$ with parts of size at least γn.*

The following lemma further guarantees that this remains possible as long as the host-graph has many k-cliques. It is also a straightforward consequence of the weak regularity lemma of Duke, Lefmann, and Rödl and we provide a proof in Appendix 1.

Lemma 3.5 *Let $k \ge 2$, and let $0 < \varepsilon < 1/2$ and $2k\varepsilon \le d \le 1$. Let $\gamma = \varepsilon^{k^2\varepsilon^{-12}}$. Suppose that G is a k-partite graph with parts $V_1, \ldots, V_k$ with at least $d|V_1|\cdots|V_k|$ cliques of size k. Then there is some $\gamma' \in [\gamma, \varepsilon]$ and an $(\varepsilon, d/2)$-super-regular k-cylinder $Z = (U_1, \ldots, U_k)$ in G with $U_i \subseteq V_i$ and $|U_i| = \lfloor \gamma' |V_i| \rfloor$ for every $i \in [k]$.*

[3] The second part of the theorem is not explicitly stated in [17] but follows readily from the blow-up lemma and the Hajnal-Szemerédi theorem.

4 Proof of Theorem 1.1

In the proof, we will use the following theorem of Fox and Sudakov [13] about r-colour Ramsey numbers of bounded-degree graphs.

Theorem 4.1 *([13, Theorem 4.3]) Let $k, \Delta, r, n \in \mathbb{N}$ with $r \geq 2$ and let $H_1, \dots, H_r$ be k-partite graphs with n vertices and maximum degree at most Δ. Then*

$$R(H_1, \dots, H_r) \leq r^{2rk\Delta} n.$$

Recall that $\mathcal{F}_\Delta$ denotes the collection of all sequences of graphs $\mathcal{F}$ with $\Delta(F) \leq \Delta$, for every $F \in \mathcal{F}$, and let $\mathcal{F}_{\Delta,k}$ be the collection of sequences $\mathcal{F} \in \mathcal{F}_\Delta$ such that F is k-partite, for every $F \in \mathcal{F}$. Note that $\mathcal{F}_\Delta = \mathcal{F}_{\Delta,\Delta+1}$. The following consequence of the previous theorem states that, for each $\mathcal{F} \in \mathcal{F}_{k,\Delta}$, we can cover almost all vertices of K_n with monochromatic copies of graphs from $\mathcal{F}$.

Proposition 4.2 *Let $\Delta, k, r \in \mathbb{N}$, let $\gamma \in (0, 1]$ and let $C = 4r^{2rk\Delta} \log(1/\gamma)$. Then, for every $\mathcal{F} \in \mathcal{F}_{\Delta,k}$ and every r-edge-coloured K_n with $n \geq r^{2rk\Delta}$, it is possible to cover all but γn vertices of K_n with at most C vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$.*

Proof Let $\mathcal{F} = \{F_1, F_2, \dots\} \in \mathcal{F}_{\Delta,k}$, $t = r^{-2rk\Delta}$, $C = (4/t)\log(1/\gamma)$ and $n \geq r^{2rk\Delta}$.

Consider $n_1 = \lfloor tn \rfloor \geq tn/2$. By Theorem 4.1, since $R_r(F_{n_1}) \leq t^{-1}n_1 \leq n$, there is a monochromatic copy of F_{n_1} in K_n. Let H_1 be such copy and let $V_1 = V \setminus V(H_1)$. Note that $|V_1| = n - n_1 \leq (1 - t/2)n$.

Suppose that we have inductively found vertex-disjoint monochromatic graphs $H_1, \dots, H_i \subseteq K_n$ that are copies of graphs in $\mathcal{F}$ and such that $V_i := V(K_n) \setminus (V(H_1) \cup \dots \cup V(H_i))$ has at most $(1 - t/2)^i n$ vertices. If $|V_i| \leq 2/t$, then we cover the vertices in V_i with single vertices and stop the process. Therefore, suppose that $|V_i| \geq 2/t$. Then let $n_{i+1} = \lfloor t|V_i| \rfloor \geq t|V_i|/2$. Again by Theorem 4.1, since $R_r(F_{n_{i+1}}) \leq t^{-1}n_{i+1} \leq |V_i|$, there is a monochromatic copy of $F_{n_{i+1}}$ contained in V_i. Let H_{i+1} be such a copy. Thus the set $V_{i+1} := V(K_n) \setminus (V(H_1) \cup \dots \cup V(H_{i+1}))$ has size

$$|V_{i+1}| = |V_i| - n_{i+1} \leq (1 - t/2)|V_i| \leq (1 - t/2)^{i+1} n.$$

Now, after $C/2$ steps, we have covered all but at most

$$(1 - t/2)^{C/2} n \leq e^{-(t/4)C} n \leq \gamma n$$

vertices of K_n using at most $C/2 + 2/t \leq C$ vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$. □

In particular, by choosing $\gamma = 1/n$, we get the following corollary.

Corollary 4.3 *Let $\Delta, k, r \in \mathbb{N}$ and let $C = 4r^{2rk\Delta} \log n$. Then, for every $\mathcal{F} \in \mathcal{F}_{\Delta,k}$ and every r-edge-coloured K_n, there is a collection of at most C monochromatic vertex-disjoint copies of graphs from $\mathcal{F}$ whose vertex-sets partition $V(G)$.*

4.1 The Absorption Lemma

Given a graph G and $U \subseteq V$, recall that we denote by $G[U]$ the subgraph of G induced by U. Given disjoint sets $V_1, \dots, V_k \subseteq V(G)$, with $k \geq 2$, we denote by $G[V_1, \dots, V_k]$ the subgraph of G with vertex set $V_1 \cup \dots \cup V_k$ containing only edges that are between two of the sets $V_1, \dots, V_k$. Furthermore, for each $v \in V_1$, let

$$\deg_G(v, V_2, \dots, V_k) = |\{(v_2, \dots, v_k) \in V_2 \times \cdots \times V_k : \{v, v_2, \dots, v_k\} \text{ is a} k\text{-clique in} G\}|,$$

and

$$\mathrm{d}_G(v, V_2, \dots, V_k) := \frac{\deg_G(v, V_2, \dots, V_k)}{|V_2| \cdots |V_k|}.$$

If additionally, we have an edge colouring $\chi\colon E(G) \to [r]$ of $E(G)$, then we denote by $\deg_{G,i}(v, V_2, \dots, V_k) = \deg_{G_i}(v, V_2, \dots, V_k)$, where G_i is the graph with vertex set $V(G)$ consisting of the edges of G with colour i. We define $\mathrm{d}_{G,i}(v, V_2, \dots, V_k)$ similarly and denote $\mathrm{d}_{G,I}(v, V_2, \dots, V_k) := \sum_{i \in I} \mathrm{d}_{G,i}(v, V_2, \dots, V_k)$, for each $I \subseteq [r]$. If the graph G is clear from context, we may drop the G in the subscript.

Given a set V, we denote by $K(V)$ the complete graph with vertex set V. Given disjoint sets $V_1, \dots, V_k$, we denote by $K(V_1, \dots, V_k)$ the complete k-partite graph with parts $V_1, \dots, V_k$. Let $G = K(V_1) \cup K(V_1, \dots, V_k)$ and let $\mathcal{H}$ be a collection of subgraphs of G. We denote by $\cup\mathcal{H}$ the graph with edge set $\bigcup_{H \in \mathcal{H}} E(H)$ and vertex set $V(\mathcal{H}) := \bigcup_{H \in \mathcal{H}} V(H)$. We say that $\mathcal{H}$ *canonically covers* V_1 if $V_1 \subseteq V(\mathcal{H})$ and

$$|V(\mathcal{H}) \cap V_i| \leq |V(\mathcal{H}) \cap V_1|,$$

for all $i \in [2, k]$.[4] The following lemma is the key ingredient of the proof of our main theorem.

Lemma 4.4 *(Absorption Lemma) There is some absolute constant $K > 0$, such that the following is true for all $d > 0$, all integers $\Delta, r \geq 2$ and for every $\mathcal{F} \in \mathcal{F}_\Delta$. Let $k = \Delta + 2$ and let*

$$C = \exp^2\left(\left(\frac{r}{d}\right)^{K\Delta}\right).$$

Consider k disjoint sets $V_1, \dots, V_k$ with $|V_i| \geq 4|V_1|$, for all $i \in [2, k]$, and let $G = K(V_1) \cup K(V_1, \dots, V_k)$. Suppose that $\chi : E(G) \to [r]$ is a colouring in which for every $v \in V_1$ we have $\mathrm{d}_{[r]}(v, V_2, \dots, V_k) \geq d$. Then, there is a collection of at most C vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$ in G which canonically covers V_1.

[4] Here, we denote by $[i, j]$ the set of integers z with $i \leq z \leq j$.

The edges of G inside V_1 will only be used to find copies from $\mathcal{F}$ which lie entirely in V_1 in order to greedily cover most vertices of V_1. The difficult part is finding monochromatic copies in $K(V_1, \ldots, V_k)$ covering the remaining vertices. To do so, we will reduce the problem to only one colour within $K(V_1, \ldots, V_k)$ and then deduce Lemma 4.4 from the following lemma.

Lemma 4.5 *There is some absolute constant $K > 0$, such that the following is true for all $d > 0$, all integers $\Delta, r \geq 2$ and for every $\mathcal{F} \in \mathcal{F}_\Delta$. Let $k = \Delta + 2$ and let*

$$C = \exp^2\left(\left(\frac{r}{d}\right)^{K\Delta}\right).$$

Consider k disjoint sets $V_1, \ldots, V_k$ with $|V_i| \geq 2|V_1|$, for all $i \in [2, k]$ and let $G = K(V_1) \cup K(V_1, \ldots, V_k)$. Suppose that $\chi : E(G) \to [r]$ is a colouring in which for every $v \in V_1$ we have $\mathrm{d}_1(v, V_2, \ldots, V_k) \geq d$. Then, there is a collection of at most C vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$ in G which canonically covers V_1.

Lemma 4.4 follows routinely from Lemma 4.5.

Proof of Lemma 4.4 Let K' be the absolute constant from Lemma 4.5 and let $d' = d/(2r)$, $\gamma = d'/(kr)$, and $C' = \exp^2\left((r/d')^{K'\Delta}\right)$. Partition $V_1 = U_1 \cup \ldots \cup U_r$ such that for each $j \in [r]$ we have $\mathrm{d}_j(v, V_2, \ldots, V_k) \geq 2d'$, for all $v \in U_j$. We will inductively cover U_j, for each $j \in [k]$.

Let us first consider the base case, i.e., $j = 1$. From Proposition 4.2, there is a collection $\mathcal{H}'$ of at most[5] C' disjoint monochromatic copies of graphs from $\mathcal{F}$ covering all but $\gamma|U_1| \leq \gamma|V_1|$ vertices of $G[U_1]$. Let $V_1' = U_1 \backslash V(\mathcal{H}')$. By applying Lemma 4.5 to $G' := G[V_1' \cup V_2 \cup \cdots \cup V_k]$ (with d'), there is a collection $\mathcal{H}''$ of at most C' disjoint monochromatic copies of graphs from $\mathcal{F}$ in G' which canonically covers V_1'. Let $\mathcal{H}_1 = \mathcal{H}' \cup \mathcal{H}''$. Note that $\mathcal{H}_1$ canonically covers U_1 and covers at most $\gamma|V_1|$ vertices of V_i, for each $i \in [2, k]$.

Now consider $j \geq 2$ and suppose that we have found a collection $\mathcal{H}_{j-1}$ of at most $2(j-1)C'$ disjoint monochromatic copies of graphs from $\mathcal{F}$ in G that canonically covers $U_1 \cup \cdots \cup U_{j-1}$ and covers at most $(j-1)\gamma|V_1|$ vertices of V_i, for each $i \in [2, k]$. From Proposition 4.2, there is a collection $\mathcal{H}'$ of at most C' disjoint monochromatic copies of graphs from $\mathcal{F}$ covering all but $\gamma|U_j| \leq \gamma|V_1|$ vertices of $G[U_j]$. Let $V_1' = U_j \backslash V(\mathcal{H}')$ and let $V_i' := V_i \backslash V(\mathcal{H}_{j-1})$, for each $i \in [2, k]$. Note that

$$|V_i'| \geq |V_i| - (j-1)\gamma|V_1| \geq 4|V_1| - r\gamma|V_i| \geq 2|V_1| \geq 2|V_1'|.$$

Also, for each $v \in V_1'$, we have

$$\deg_j(v, V_2', \ldots, V_k') \geq \deg_j(v, V_2, \ldots, V_k) - k(j-1)\gamma|V_2| \cdots |V_k|.$$

[5] Note that the constant from Proposition 4.2 is smaller than C'.

Consequently,

$$\mathrm{d}_j(v, V_2', \ldots, V_k') \geq \mathrm{d}_j(v, V_2, \ldots, V_k) - kr\gamma \geq 2d' - d' \geq d'.$$

Therefore, we can apply Lemma 4.5 to $G' := G[V_1' \cup \cdots \cup V_k']$ and get a collection $\mathcal{H}''$ of at most C' disjoint monochromatic copies of graphs from $\mathcal{F}$ in G that canonically covers V_1'. In particular, $\mathcal{H}''$ covers at most $|V_1'| \leq \gamma|V_1|$ vertices of V_i, for each $i \in [2, k]$. Let $\mathcal{H}_j = \mathcal{H}_{j-1} \cup \mathcal{H}' \cup \mathcal{H}''$. Then $\mathcal{H}_j$ is a collection of at most $2jC'$ disjoint monochromatic copies of graphs from $\mathcal{F}$ in G that canonically covers $U_1 \cup \cdots \cup U_j$ and covers at most $j\gamma|V_1|$ vertices of V_i, for each $i \in [2, k]$.

In the end, we have found a collection $\mathcal{H}_r$ of disjoint monochromatic copies of graphs from $\mathcal{F}$ that canonically covers V_1. Furthermore, $\mathcal{H}_r$ has at most $2rC' \leq \exp^2\left((r/d)^{4K'\Delta}\right)$ graphs, finishing the proof. □

The proof of Lemma 4.5 is quite long and technical (see Sect. 2 for a sketch), and we will therefore break it up into smaller claims. We use □ to denote the end of the proof of a claim and □ to denote the end of the main proof.

Proof of Lemma 4.5. Let Δ and r be given positive integers, $k = \Delta + 2$ and $\mathcal{F} \in \mathcal{F}_\Delta$. For each $d > 0$, let $C(d)$ be the smallest positive integer C such that the following holds:

($\star$) Let $V_1, \ldots, V_k$ be disjoint sets with $|V_i| \geq 2|V_1|$ for all $i \in [2, k]$, let $H \subseteq K(V_1, \ldots, V_k)$ be a graph with $\mathrm{d}_H(v, V_2, \ldots, V_k) \geq d$ for every $v \in V_1$ and $G = K(V_1) \cup H$. Let $\chi : E(G) \to [r]$ be a colouring such that every edge in $E(H)$ receives colour 1. Then, there is a collection $\mathcal{H}$ of at most C vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$ contained in G that canonically covers V_1.

Note that $C(d)$ is a decreasing function in d, and that $C(d) = 0$ for every $d > 1$. Our goal is to show that $C(d)$ is finite for every $d > 0$. We will do this by establishing a recursive upper bound (see (4.1)).

Let us first define all relevant constants used in the proof. Let K' be the universal constant given by Theorem 3.3 and fix some $0 < d \leq 1$. Define

$$\varepsilon = \left(\frac{d}{100}\right)^{2K'\Delta}, \quad \gamma = \tfrac{1}{r} \cdot \varepsilon^{k^2\varepsilon^{-12}} \quad \text{and} \quad \eta = \frac{d\gamma^k}{2}.$$

It might be of benefit for the reader to have in mind that those constants obey the following hierarchy:

$$1 \geq d \gg \varepsilon \gg \gamma \gg \eta > 0.$$

Furthermore, define

$$P(d) := 4r^{4rk^2}\log(2/\eta^2) + 1.$$

We will prove that for every $d' \geq d$ we have

$$C(d') \leq P(d) + kC\left(d' + \eta\right). \tag{4.1}$$

Since $C(d') = 0$ if $d' > 1$, it follows by iterating that $C(d) \leq (2k)^{2/\eta} P(d)$. Furthermore, we have

$$2/\eta \leq \gamma^{-2k} \leq \varepsilon^{-2rk^3\varepsilon^{-12}} \leq \exp\left(r\varepsilon^{-20}\right) \leq \exp\left((r/d)^{400K'\Delta}\right).$$

It follows that

$$C(d) \leq \exp^2\left((r/d)^{500K'\Delta}\right) P(d) \leq \exp^2\left((r/d)^{1000K'\Delta}\right),$$

concluding the proof of Lemma 4.5.

It remains to prove (4.1). Let $d' \geq d$ be fixed now and let $V_1, \ldots, V_k$, G and $\chi : E(G) \to [r]$ be as in ($\star$) (with d' playing the role of d). By assumption, there are at least $d|V_1||V_2| \cdots |V_k|$ cliques of size k in $G[V_1, V_2, \ldots, V_k]$ each of which is monochromatic in colour 1. Since $\gamma = \varepsilon^{k^2\varepsilon^{-12}}$ and $d \geq 2k\varepsilon$, we can apply Lemma 3.5 to get some $\gamma' \geq \gamma$ and a k-cylinder $Z = (U_1, \ldots, U_k)$ which is $(\varepsilon, d/2)$-super-regular with $U_i \subseteq V_i$ and $|U_i| = \lfloor \gamma' |V_i| \rfloor$ for every $i \in [k]$. Without loss of generality we may assume that $\gamma |V_i|$ is an integer for every $i \in [k]$ and that we have $\gamma' = \gamma$. By Proposition 4.2, there is a collection $\mathcal{H}_R$ of at most $4r^{4rk^2} \log(2/\eta^2)$ vertex-disjoint monochromatic copies of graphs from $\mathcal{F}$ contained in $K(V_1 \setminus U_1)$ covering all vertices in $V_1 \setminus U_1$ except for a set R with $|R| \leq \eta^2 |V_1|$. We remark here that

$$|R| \leq \eta/(4k) \cdot |U_1| \leq \varepsilon^2 |U_1|. \tag{4.2}$$

It remains now to cover the vertices in R. For each $i \in [k]$, let

$$d_i = \frac{1 - \gamma^i}{1 - \gamma^k} \cdot d' \tag{4.3}$$

and note that $(1 - \gamma)d' \leq d_1 \leq \cdots \leq d_k = d'$. For $i \in [2, k]$, let $\tilde{V}_i = V_i \setminus U_i$ and define

$$S_i = \{v \in R : \mathrm{d}(v, V_2, \ldots, V_{i-1}, V_i, U_{i+1}, \ldots, U_k) \geq d_i\},$$
$$T_i = \{v \in R : \mathrm{d}(v, V_2, \ldots, V_{i-1}, \tilde{V}_i, U_{i+1}, \ldots, U_k) > d' + 2\eta\}.$$

We will prove (4.1) using a series of claims, which we shall prove at the end.

Claim 4.1.1 *We have $R = S_1 \cup T_2 \cup \ldots \cup T_k$.*

Without loss of generality, we may assume that $S_1, T_2, \ldots, T_k$ are pairwise disjoint (more formally, we can define $T_i' := T_i \setminus (S_1 \cup T_2 \cup \ldots \cup T_{i-1})$ for all $i \in [2, k]$ and continue the proof with these sets). Our goal now is to cover each of the sets $S_1, T_2, \ldots, T_k$ one by one using the following claims.

Claim 4.1.2 *For every $i \in [2, k]$ and every set $A \subseteq V(G) \backslash T_i$ with $|A \cap V_s| \leq |R|$ for all $s \in [2, k]$, there is a collection $\mathcal{H}_i$ of at most $C(d' + \eta)$ monochromatic disjoint copies of graphs from $\mathcal{F}$ in G, such that*

(i) $V(\mathcal{H}_i) \cap V_1 = T_i$,
(ii) $V(\mathcal{H}_i) \cap A = \emptyset$, *and*
(iii) $\left|V(\mathcal{H}_i) \cap V_j\right| \leq |T_i|$ *for all* $j \in [2, k]$.

Claim 4.1.3 *For every set $A \subseteq V(G) \backslash (S_1 \cup U_1)$ with $|A \cap V_s| \leq |R|$ for all $s \in [2, k]$, there is a monochromatic copy H_1 of a graph from $\mathcal{F}$ in G, such that*

(i) $V(H_1) \cap V_1 = S_1 \cup U_1$,
(ii) $V(H_1) \cap A = \emptyset$ *and*
(iii) $\left|V(H_1) \cap V_j\right| \leq |S_1 \cup U_1|$ *for all* $j \in [2, k]$.

With these claims at hand, we can now prove (4.1). First, we apply Claim 4.1.2 repeatedly to get collections $\mathcal{H}_2, \ldots, \mathcal{H}_k$ of at most $C(d' + \eta)$ disjoint monochromatic copies of graphs from $\mathcal{F}$ that canonically covers $T_2, \ldots, T_k$, respectively, as follows. Let $i \in \{2, \ldots, k\}$ and suppose we have constructed $\mathcal{H}_2, \ldots, \mathcal{H}_{i-1}$. Let $A_i := V(\mathcal{H}_2) \cup \ldots \cup V(\mathcal{H}_{i-1})$ and note that $|A_i \cap V_s| \leq |T_2| + \cdots + |T_{i-1}| \leq |R|$ for all $s \in [2, k]$. Apply now Claim 4.1.2 for i and $A = A_i$ to get the desired collection $\mathcal{H}_i$.

Next, we apply Claim 4.1.3 with $A = V(\mathcal{H}_2) \cup \ldots \cup V(\mathcal{H}_k)$ to get a copy H_1 of a graph from $\mathcal{F}$ with the desired properties. Note that, similarly as above, we have $|A \cap V_s| \leq |R|$ for all $s \in [2, k]$. By construction $V(H_1), V(\mathcal{H}_2), \ldots, V(\mathcal{H}_k)$ and $V(\mathcal{H}_R)$ are disjoint and cover V_1. Moreover, for every $s \in [2, k]$, we have

$$\begin{aligned}|(V(H_1) \cup \ldots \cup V(\mathcal{H}_k) \cup V(\mathcal{H}_R)) \cap V_s| &\leq |S_1 \cup U_1| + |T_1| + |T_2| + \cdots + |T_k| \\ &\leq |U_1 \cup R| \leq |V_1|.\end{aligned}$$

Hence, $\{H_1\} \cup \ldots \cup \mathcal{H}_k \cup \mathcal{H}_R$ canonically covers V_1. Finally, we have $|\{H_1\} \cup \ldots|$ $|\cup \mathcal{H}_k \cup \mathcal{H}_R| \leq P(d) + kC\big(d' + \eta\big)$, proving (4.1). It remains now to prove Claims 4.1.1,4.1.2,4.1.3.

Proof of Claim 4.1.1 Since $S_k = R$, it suffices to show $S_i \subseteq S_{i-1} \cup T_i$ for each $i \in [2, k]$. Let $i \in [2, k]$ and let $v \in S_i \backslash S_{i-1}$. We have

$$\begin{aligned}\deg(v, V_2, \ldots, V_{i-1}, \tilde{V}_i, U_{i+1}, \ldots, U_k) = &\deg(v, V_2, \ldots, V_{i-1}, V_i, U_{i+1}, \ldots, U_k) \\ &- \deg(v, V_2, \ldots, V_{i-1}, U_i, U_{i+1}, \ldots, U_k).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{d}(v, V_2, \ldots, V_{i-1}, \tilde{V}_i, U_{i+1}, \ldots, U_k) = &\ \mathrm{d}(v, V_2, \ldots, V_{i-1}, V_i, U_{i+1}, \ldots, U_k)\frac{|V_i|}{|\tilde{V}_i|} \\ &- \mathrm{d}(v, V_2, \ldots, V_{i-1}, U_i, U_{i+1}, \ldots, U_k)\frac{|U_i|}{|\tilde{V}_i|}\end{aligned}$$

$$
\begin{aligned}
&> d_i \frac{|V_i|}{|\tilde{V}_i|} - d_{i-1}\frac{|U_i|}{|\tilde{V}_i|}\\
&= \frac{d_i - \gamma d_{i-1}}{1-\gamma}\\
&= \frac{(1-\gamma^i)d' - \gamma(1-\gamma^{i-1})d'}{(1-\gamma)(1-\gamma^k)}\\
&= \frac{d'}{1-\gamma^k} \geq d' + 2\eta,
\end{aligned}
$$

where we use (4.3) and the definition of η to obtain the last identities. Thus $v \in T_i$ and hence $S_i \subseteq S_{i-1} \cup T_i$. □

Proof of Claim 4.1.2 Let $V_s' := V_s \backslash A$ for all $s \in [2, i-1]$, $\tilde{V}_i' := \tilde{V}_i \backslash A$ and $U_s' := U_s \backslash A$ for all $s \in [i+1, k]$. Then, by (4.2), we have

$$
\begin{aligned}
|V_s'| \geq |V_s| - |R| \geq \left(1 - \tfrac{\eta}{4k}\right)|V_s| \geq \frac{|V_s|}{2}, &\text{ for } s = 2, \ldots, i-1,\\
|\tilde{V}_i'| \geq |\tilde{V}_i| - |R| \geq \left(1 - \tfrac{\eta}{4k}\right)|\tilde{V}_i| \geq \frac{|\tilde{V}_i|}{2}, &\text{ and}\\
|U_s'| \geq |U_s| - |R| \geq \left(1 - \tfrac{\eta}{4k}\right)|U_s| \geq \frac{|U_j|}{2}, &\text{ for } s = i+1, \ldots, k.
\end{aligned}
$$

In particular, it follows that

$$
\begin{aligned}
|V_s \setminus V_s'| \leq |R| \leq \frac{\eta}{4k}|V_s| \leq \frac{\eta}{2k}|V_s'|, &\text{ for } s = 2, \ldots, i-1,\\
|V_i \setminus V_i'| \leq |R| \leq \frac{\eta}{4k}|V_i| \leq \frac{\eta}{2k}|V_i'|, &\text{ and}\\
|U_s \setminus U_s'| \leq |R| \leq \frac{\eta}{4k}|U_s| \leq \frac{\eta}{2k}|U_s'|, &\text{ for } s = i+1, \ldots, k.
\end{aligned}
$$

Therefore, for every $v \in T_i$, we have

$$
\begin{aligned}
&\mathrm{d}(v, V_2', \ldots, V_{i-1}', \tilde{V}_i', U_{i+1}', \ldots, U_k')\\
&\geq d' + 2\eta - \sum_{s=2}^{i-1} \frac{|V_s \setminus V_s'|}{|V_s'|} - \frac{|\tilde{V}_i \setminus \tilde{V}_i'|}{|\tilde{V}_i'|} - \sum_{s=i+1}^{k} \frac{|U_s \setminus U_s'|}{|U_s'|}\\
&\geq d' + 2\eta - (k-1)\frac{\eta}{2k} \geq d' + \eta.
\end{aligned}
$$

Hence, by definition of $C(d' + \eta)$ (see ($\star$)), there exists a collection $\mathcal{H}_i$ of at most $C(d' + \eta)$ monochromatic copies of graphs from $\mathcal{F}$ that canonically covers T_i in the graph

$$
K(T_i) \cup K(T_i, V_2', \ldots, V_{i-1}', \tilde{V}_i', U_{i+1}', \ldots, U_k').
$$

By construction, $\mathcal{H}_i$ satisfies the requirements of the claim (note that (iii) holds since $\mathcal{H}_i$ is a canonical covering). □

Proof of Claim 4.1.3 Let $Y_1 = S_1$ and, for each $i \in [2, k]$, let $X_i = U_i \cap A$. Observe that $|Y_1| \leq |R| \leq \varepsilon^2 |U_1|$ and $|X_i| \leq |R| \leq \varepsilon^2 |U_i|$ for all $i \in [2, k]$. Let $U_1' = U_1 \cup Y_1$ and, for each $i \in [2, k]$, let $U_i' := U_i \setminus X_i$. We now consider the cylinder $Z' := (U_1', \ldots, U_k')$. By definition of S_1, we have $\mathrm{d}(v, U_2, \ldots, U_k) \geq d_1 \geq d/2$ and in particular $\deg(v, U_i) \geq d/2 \cdot |U_i|$ for all $v \in Y_1$ and $i \in [2, k]$.

Hence, by Lemma 3.2, Z' is $(8\varepsilon, d/4)$-super-regular. Furthermore, we have $|U_1'| \leq |U_i'|$ for all $i \in [k]$. Thus, by Theorem 3.3, there is a monochromatic copy H_1 of a graph from $\mathcal{F}$ in Z that covers $U_1' = U_1 \cup S_1$ and at most $|U_1'|$ vertices from each of $U_2', \ldots, U_k'$. By construction, this copy satisfies the requirements of the claim.

This finishes the proof of Lemma 4.5. □

4.2 Proof of Theorem 1.1

In this section, we will finish the proof of Theorem 1.1. We will make use of the following lemma from [5] and follow the same proof technique. Since our proof of this lemma is short, we include it here for completeness. Given a k-uniform hypergraph $\mathcal{H}$, a vertex $v \in V(\mathcal{H})$ and sets $B_2, \ldots, B_k \subseteq V(\mathcal{H})$, we define

$$\deg_{\mathcal{H}}(v, B_2, \ldots, B_k) := |\{(v_2, \ldots, v_k) \in B_2 \times \ldots \times B_k : \{v, v_2, \ldots, v_k\} \in E(\mathcal{H})\}|.$$

Lemma 4.6 *Let k and N be positive integers and let $\mathcal{H}$ be a k-uniform hypergraph. Suppose that $B_1, \ldots, B_N \subseteq V(\mathcal{H})$ are non-empty disjoint sets such that for every $1 \leq i_1 < \cdots < i_k \leq N$, we have*

$$\deg_{\mathcal{H}}(v, B_{i_1}, \ldots, B_{i_{k-1}}) < \binom{N}{k}^{-1} |B_{i_1}| \cdots |B_{i_{k-1}}|,$$

for all $v \in B_{i_k}$. Then, there exists an independent set $\{v_1, \ldots, v_N\}$ with $v_i \in B_i$, for each $i \in [N]$.

Proof For each $i \in [N]$, let v_i be chosen uniformly at random from B_i. Let $I = \{v_1, \ldots, v_N\}$. Then we have

$$\begin{aligned}
\mathbb{P}\left[I \text{ is not an independent set}\right] &\leq \sum_{1 \leq i_1 < \cdots < i_k \leq N} \mathbb{P}\left[\{v_{i_1}, \ldots, v_{i_k}\} \in E(\mathcal{H})\right] \\
&= \sum_{1 \leq i_1 < \cdots < i_k \leq N} \frac{1}{|B_{i_k}|} \sum_{v \in B_k} \mathbb{P}\left[\{v_{i_1}, \ldots, v_{i_k}\}\right. \\
&\quad \left.\in E(\mathcal{H}) \mid v_{i_k} = v\right] \\
&= \sum_{1 \leq i_1 < \cdots < i_k \leq N} \frac{1}{|B_{i_k}|} \sum_{v \in B_k} \frac{\deg_{\mathcal{H}}(v, B_{i_1}, \ldots, B_{i_{k-1}})}{|B_{i_1}| \cdots |B_{i_{k-1}}|}
\end{aligned}$$

$$< \sum_{1 \le i_1 < \cdots < i_k \le N} \binom{N}{k}^{-1} = 1.$$

Therefore, there exists an independent set $\{v_1, \ldots, v_N\}$ with $v_i \in B_i$, for each $i \in [N]$. $\square$

We are now able to prove Theorem 1.1. The main idea is to find reasonably large cylinders that are super-regular for one of the colours, greedily cover most of the remaining vertices using Proposition 4.2 and then apply the Absorption Lemma (Lemma 4.4) to the set of remaining vertices that share many monochromatic cliques with the cylinders. We then iterate this process until no vertices remain. Using Lemma 4.6, we will show that a bounded number of iterations suffices.

Proof of Theorem 1.1 Fix $r, \Delta \ge 2$, $\mathcal{F} \in \mathcal{F}_\Delta$. Let G be an r-edge-coloured complete graph on n vertices. Let

$$k = \Delta + 2, \quad N = r^{rk}, \quad \delta = \binom{N+1}{k}^{-1} \quad \text{and} \quad d = \frac{1}{2r}.$$

In order to use Theorem 3.3 and Lemma 3.4, respectively, consider the constants

$$\varepsilon = (\delta d^\Delta)^{2K'} \quad \text{and} \quad \gamma = \varepsilon^{r^{8rk}\varepsilon^{-5}},$$

where K' is the universal constant given by Theorem 3.3. Consider also the constants

$$\alpha = \varepsilon^2 \quad \text{and} \quad C_1 = 4r^{2rk\Delta} \log\left(\frac{4}{\alpha\gamma}\right),$$

in order to use Proposition 4.2. Finally, let

$$C_2 = \exp^2((2r/\delta)^{\tilde{K}\Delta}) \le \exp^2\left(r^{16\tilde{K}r\Delta^3}\right),$$

where $\tilde{K}$ is the universal constant from Lemma 4.4, and let $K = 20\tilde{K}$.

We will build a framework consisting of many k-cylinders working as absorbers and small sets that can be absorbed by them. More precisely, our goal is to define sets with the following properties (Fig. 1 should help the reader to understand the structure of those sets as we define them):

Framework There are sets $Z_1, \ldots, Z_N, S_{k-1}, \ldots, S_N, R_k, \ldots, R_{N+1}, R'_k, \ldots, R'_{N+1}$ with the following properties.

(F.1) $V(G) = \bigcup_{i=1}^N Z_i \cup \bigcup_{i=k-1}^N S_i \cup \bigcup_{i=k}^{N+1} R'_i$ is a partition.

(F.2) $Z_1, \ldots, Z_N$[6] are k-cylinders which are (ε, d)-super-regular in one of the colours (or empty).

[6] We shall identify the cylinders with their vertex-set.

(F.3) $S_{k-1}, \ldots, S_N$ are sets of vertices which we will cover greedily by monochromatic copies of graphs from $\mathcal{F}$.
(F.4) For every $i \in [k, N]$ and every $u \in R'_{i+1}$, there exists $\{i_1, \ldots, i_{k-1}\} \subseteq [i]$ for which $\mathrm{d}_{[r]}(u, Z_{i_1}, \ldots, Z_{i_{k-1}}) \geq \delta$.
(F.5) For each $k \leq i < j \leq N+1$, we have $S_j \cup Z_j \cup R'_j \subseteq R_i$ and $|R_i| \leq \alpha|Z_{i-1}|$.

So let us construct those sets from the framework. First, if $n < 1/4\gamma$, then Corollary 4.3 gives a covering with at most C_2 monochromatic vertex-disjoint copies of graphs from $\mathcal{F}$. Therefore we may assume that $n \geq 1/4\gamma$. Hence, by applying Lemma 3.4 multiple times, we find $k-1$ vertex-disjoint k-cylinders $Z_1, \ldots, Z_{k-1}$ such that each of them is (ε, d)-super-regular in some colour (not necessarily the same) and $|Z_1| \geq \cdots \geq |Z_{k-1}| \geq \gamma n/2$. Let $V_{k-1} = V(G) \setminus (Z_1 \cup \cdots \cup Z_{k-1})$. By Proposition 4.2, there is a collection of at most C_1 monochromatic vertex-disjoint copies from $\mathcal{F}$ in V_{k-1} covering a set S_{k-1} such that the leftover vertices $R_k = V_{k-1} \backslash S_{k-1}$ satisfies $|R_k| \leq \alpha\gamma n/2 \leq \alpha|Z_{k-1}|$. Let $R'_k \subseteq R_k$ be the set of vertices $u \in R_k$ with $\mathrm{d}_{[r]}(u, Z_1, \ldots, Z_{k-1}) \geq \delta$. Let $R'_{k,[k-1]} = R'_k$ and $V_k = R_k \backslash R'_k$.

Inductively, for each $i = k, \ldots, N$, we do the following. If $|V_i| < 1/4\gamma$, we use Corollary 4.3 to cover V_i using at most C_2 monochromatic vertex-disjoint copies from $\mathcal{F}$ and let $Z_i = S_i = R_{i+1} = R'_{i+1} = V_{i+1} = \emptyset$. Otherwise, we apply Lemma 3.4 to find a monochromatic (ε, d)-super-regular k-cylinder Z_i contained in V_i with $|Z_i| \geq \gamma\,|V_i|$. By Proposition 4.2, there is a collection of at most C_1 monochromatic, vertex-disjoint copies from $\mathcal{F}$ in $V_i \setminus Z_i$ covering a set $S_i \subseteq V_i$, so that the set of leftover vertices $R_{i+1} = V_i \backslash S_i$ has size at most $\alpha\gamma|V_i| \leq \alpha\,|Z_i|$.

Let R'_{i+1} be the set of vertices u in R_{i+1} for which there is a set $I = \{i_1, \ldots, i_{k-1}\} \subseteq [i]$ such that $\mathrm{d}_{[r]}(u, Z_{i_1}, \ldots, Z_{i_{k-1}}) \geq \delta$. Let

$$R'_{i+1} = \bigcup_{I \in \binom{[i]}{k-1}} R'_{i+1,I},$$

be a partition of R'_{i+1} so that, for each $I = \{i_1, \ldots, i_{k-1}\} \subseteq [i]$, we have $\mathrm{d}_{[r]}(u, Z_{i_1}, \ldots, Z_{i_{k-1}}) \geq \delta$ for all $u \in R'_{i+1,I}$. Finally, let $V_{i+1} = R_{i+1} \setminus R'_{i+1}$.

The following claim implies that these sets partition $V(G)$ as in (F.1).

Claim 4.2.1 *The set V_{N+1} is empty.*

Proof Define a k-uniform hypergraph $\mathcal{H}$ with vertex set $U = Z_1 \cup \ldots \cup Z_N \cup V_{N+1}$ and hyperedges corresponding to monochromatic k-cliques in $G[U]$. If V_{N+1} is non-empty, then so are $Z_1, \ldots, Z_N$. In order to apply Lemma 4.6, consider the sets $B_i = Z_i$, for $i \in [N]$, and $B_{N+1} = V_{N+1}$. Since for each $i = k, \ldots, N+1$ we have $B_i \subseteq R_i \setminus R'_i$, it follows that for every $1 \leq i_1 < \cdots < i_k \leq N+1$, we have

$$\begin{aligned}\deg_{\mathcal{H}}(v, B_{i_1}, \ldots, B_{i_{k-1}}) &< \delta|B_{i_1}|\cdots|B_{i_{k-1}}| \\ &= \binom{N}{k+1}^{-1} |B_{i_1}|\cdots|B_{i_{k-1}}|.\end{aligned}$$

That is, $\mathcal{H}$ and the sets B_i satisfy the hypothesis of Lemma 4.6. Therefore, there is an independent set $\{v_1, \ldots, v_{N+1}\}$ in $\mathcal{H}$ of size $N+1$. On the other hand, since

$N \geq R_r(K_k)$, it follows that the set $\{v_1, \ldots, v_{N+1}\}$ has a monochromatic k-clique in $G[U]$, which is a contradiction.

The vertices in $S_{k-1} \cup \cdots \cup S_N$ are already covered by monochromatic copies of graphs from $\mathcal{F}$. Our goal now is to cover the sets $R'_k, \ldots, R'_{N+1}$ using Lemma 4.4 without using too many vertices from the cylinders $Z_1, \ldots, Z_N$. This way, we can cover the remaining vertices in $Z_1 \cup \cdots \cup Z_N$ using Theorem 3.3.

Claim 4.2.2 *Let $i \in \{k, \ldots, N+1\}$ and $I = \{i_2, \ldots, i_k\} \subseteq [i-1]$. Let $A \subseteq V(G) \setminus R_{i,I}$ be a set with $\left|A \cap Z_j\right| \leq \alpha \left|Z_j\right|$ for each $j \in I$. Then there is a collection of at most C_2 monochromatic vertex-disjoint copies of graphs from $\mathcal{F}$ in*

$$G' = K(R'_{i,I}) \cup K(R'_{i,I}, Z_{i_2}, \ldots, Z_{i_k}),$$

which are disjoint from A and canonically cover $R'_{i,I}$.

Proof Let $\tilde{V}_1 = R'_{i,I}$ and for $j \in [k] \setminus \{1\}$, let $\tilde{V}_j = Z_{i_j} \setminus A$. Note that $|\tilde{V}_j| \geq 4|\tilde{V}_1|$ for every $j \in [k] \setminus \{1\}$ and

$$\begin{aligned} \deg_{[r]}(v, \tilde{V}_2, \ldots, \tilde{V}_k) &\geq \deg_{[r]}(v, Z_{i_2}, \ldots, Z_{i_k}) - k\alpha |Z_{i_2}| \cdots |Z_{i_k}| \\ &\geq (\delta - k\alpha)|Z_{i_2}| \cdots |Z_{i_k}| \\ &\geq \delta/2 \cdot |Z_{i_2}| \cdots |Z_{i_k}| \end{aligned}$$

for every $v \in \tilde{V}_1$. Hence, by Lemma 4.4, there is a collection of at most C_2 vertex-disjoint copies from $\mathcal{F}$ in $\tilde{V}_1 \cup \ldots \cup \tilde{V}_k$ that canonically covers $\tilde{V}_1$, finishing the proof.

We will use Claim 4.2.2 now to cover $\bigcup_{i=k}^{N+1} R'_i$. Let $\prec$ be a linear order on $\mathcal{I} := \left\{(i, I) : i \in [k, N+1], I \in \binom{[i-1]}{k-1}\right\}$. Let $(i, I) \in \mathcal{I}$ and suppose that, for all $(i', I') \in \mathcal{I}$ with $(i', I') \prec (i, I)$, we have already constructed a family $\mathcal{H}_{i',I'}$ of monochromatic copies of graphs from $\mathcal{F}$ which canonically covers $R'_{i',I'}$ in $K(R'_{i',I'}) \cup K(R'_{i',I'}, Z_{i'_2}, \ldots, Z_{i'_k})$, where $I' = \{i'_2, \ldots, i'_k\}$, and such that the sets $V(\mathcal{H}_{i',I'})$, for $(i', I') \prec (i, I)$, are disjoint.

Let $A = \bigcup_{(i',I') \prec (i,I)} V(\mathcal{H}_{i',I'})$ be the set of already covered vertices. We claim that

$$\left|A \cap Z_j\right| \leq \alpha |Z_j| \tag{4.4}$$

for each $j \in [N]$. Indeed, given some $j \in [N]$, for all $(i', I') \in \mathcal{I}$ with $i' \leq j$, we have $V(\mathcal{H}_{i',I'}) \cap Z_j = \emptyset$, since $\mathcal{H}_{i',I'}$ canonically covers $R'_{i',I'}$ in $K(R'_{i',I'}) \cup K(R'_{i',I'}, Z_{i'_2}, \ldots, Z_{i'_k})$. Now for all $(i', I') \in \mathcal{I}$ with $i' > j$, we have $\left|V(\mathcal{H}_{i',I'}) \cap Z_j\right| \leq |R'_{i',I'}|$, again because $\mathcal{H}_{i,I}$ canonically covers $R'_{i',I'}$. Therefore,

$$\left|A \cap Z_j\right| \leq \sum_{(i',I') \prec (i,I)} \left|V(\mathcal{H}_{i',I'}) \cap Z_j\right| \leq \sum_{(i',I') \in \mathcal{I}\,:\, i' > j} \left|R'_{i',I'}\right| \leq \left|R_{j+1}\right|,$$

since the sets $\{R'_{i',I'} : (i', I') \in \mathcal{I}, i > j\}$ are disjoint subsets of R_{j+1}. Finally, since $\left|R_{j+1}\right| \leq \alpha \left|Z_j\right|$, this implies (4.4). In particular, by Claim 4.2.2, there is a collection $\mathcal{H}_{i,I}$ of monochromatic copies of graphs from $\mathcal{F}$ that canonically covers $R'_{i,I}$ in $K(R'_{i,I}) \cup K(R'_{i,I}, Z_{i_2}, \ldots, Z_{i_k})$, where $I = \{i_2, \ldots, i_k\}$, and such that $V(\mathcal{H}_{i,I})$ is disjoint from A.

It remains to cover $\bigcup_{i=1}^{N} Z_i$. Let $A := \bigcup_{(i,I)\in\mathcal{I}} V(\mathcal{H}_{i,I})$ be the set of vertices covered in the previous step. Note that, similarly as in (4.4), we have $|A \cap Z_j| \leq \alpha |Z_j|$ for all $j \in [N]$. Therefore, by Lemma 3.2, the cylinder $\tilde{Z}_j$ obtained from Z_j by removing all vertices in A is $(8\varepsilon, d/2)$-super-regular and ε-balanced for every $j \in [N]$. It follows from Theorem 3.3 that, for every $j \in [N]$, there is a collection $\mathcal{H}_j$ of at most $\Delta + 3$ monochromatic vertex-disjoint copies of graphs from $\mathcal{F}$ contained in Z_j covering $V(Z_j)$.

In total, the number of monochromatic copies we used to cover $V(G)$ is at most

$$\begin{aligned} N \cdot C_1 + N^k \cdot C_2 + N \cdot (\Delta + 3) &\leq 2N^k C_2 \\ &\leq 2r^{rk^2} \cdot \exp^2\left(r^{16\tilde{K}r\Delta^3}\right) \\ &\leq \exp^2\left(r^{Kr\Delta^3}\right). \end{aligned}$$

This concludes the proof of Theorem 1.1. □

5 Concluding Remarks

We were able to prove that sequences of graphs with maximum degree Δ have finite r-colour tiling number for every $r \geq 3$, but our bound is super-exponential in Δ. Grinshpun and Sárközy [17] conjectured that it is possible to prove an upper bound which is essentially exponential in Δ (see Conjecture 1.1). The problem becomes somewhat easier when restricted to bipartite graphs. In fact, our proof gives a double exponential upper bound in Δ for r-colour tiling numbers of sequences of bipartite graph with maximum degree Δ. Indeed, the factor k in the recursive bound (4.1) can be dropped for bipartite graphs. It would be very interesting to confirm Conjecture 1.1 for sequences of bipartite graphs.

Another interesting problem is to prove a version of Theorem 1.1 for other sequences of graphs. Given a sequence of graphs $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ with $|F_i| = i$, for every $i \in \mathbb{N}$, let $\rho_r(\mathcal{F}) = \sup_{i\in\mathbb{N}} R_r(F_i)/i$. If $\rho_r(\mathcal{F})$ is finite, then we say that $\mathcal{F}$ has linear r-colour Ramsey number. If $\mathcal{F}$ is *increasing*,[7] then it follows from the pigeon-hole principle that $\tau_r(\mathcal{F}) \geq \rho_r(\mathcal{F})$. Indeed, for each $n \in \mathbb{N}$, every r-edge-coloured K_n contains a monochromatic copy from $\mathcal{F}$ of size at least $i = \lceil n/\tau_r(\mathcal{F})\rceil$. In particular, since $\mathcal{F}$ is increasing, there is a monochromatic copy of F_i in every r-edge colouring of K_n. This implies that $R_r(F_i) \leq \tau_r(\mathcal{F}) \cdot i$, and therefore $\rho_r(\mathcal{F}) \leq \tau_r(\mathcal{F})$.

Graham, Rödl and Ruciński [16] proved that there exists a sequence of bipartite graphs $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ with $\rho_2(\mathcal{F}) \geq 2^{\Omega(\Delta)}$. Grinshpun and Sárközy observed that

[7] That is, $F_i \subseteq F_{i+1}$, for every $i \in \mathbb{N}$.

one can make this sequence increasing, thereby showing that $\tau_2(\mathcal{F}) \geq 2^{\Omega(\Delta)}$ as well. Conlon, Fox and Sudakov [8] proved that for every sequence of graphs with degree at most Δ, we have $\rho_2(\mathcal{F}) \leq 2^{O(\Delta \log \Delta)}$ while Grinshpun and Sárközy [17] proved that $\tau_2(\mathcal{F}) \leq 2^{O(\Delta \log \Delta)}$. For more colours, Fox and Sudakov [13] proved that for every sequence of graphs with degree at most Δ, we have $\rho_r(\mathcal{F}) \leq 2^{O_r(\Delta^2)}$, while our main result shows that $\tau_r(\mathcal{F}) \leq \exp^3(O_r(\Delta^3))$.

With these results in mind, one can naturally ask if there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that for every sequence of graphs $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$ we have $\tau_r(\mathcal{F}) \leq f(\rho_r(\mathcal{F}))$. That is, if it is possible to bound $\tau_r(\mathcal{F})$ in terms of $\rho_r(\mathcal{F})$. In particular, this would imply that sequences of graphs with linear Ramsey number have finite tiling number. However, the following example due to Alexey Pokrovskiy (personal communication) shows that $\tau_r(\mathcal{F})$ cannot be bounded by $\rho_r(\mathcal{F})$ in general. Let S_i be a star with i vertices and let $\mathcal{S} = \{S_i : i \in \mathbb{N}\}$ be the family of stars. It follows readily from the pigeonhole principle that $R_r(S_i) \leq r(i-2)+2$, for every $i \in \mathbb{N}$, and thus $\rho_r(\mathcal{S}) \leq r$. However, the following shows that $\tau_r(\mathcal{S}) = \infty$, for every $r \geq 2$.

Example 5.1 For all $r \geq 2$ and all sufficiently large n, we have $\tau_r(\mathcal{S}, n) \geq r \cdot \log(n/8)$.

Proof Let $\tau = r \log(n/8)$ and colour $E(K_n)$ uniformly at random with r colours. Given a vertex $v \in [n]$ and a colour c, let $S_c(v)$ be the star centred at v formed by all the edges of colour c incident on v. Note that there is a monochromatic $\mathcal{S}$-tiling of size at most τ if and only if there are distinct vertices $v_1, \ldots, v_\tau$ and colours $c_1, \ldots, c_\tau \in [r]$ such that $\bigcup_{i\in[\tau]} V(S_{c_i}(v_i)) = [n]$.

Fix distinct vertices $v_1, \ldots, v_\tau \in [n]$ and colours $c_1, \ldots, c_\tau \in [r]$. Let U be the random set $U = \bigcup_{i\in[\tau]} V(S_{c_i}(v_i))$. Note that the events $\{v \in U\}$, for $v \in [n]\backslash\{v_1, \ldots, v_\tau\}$, are independent and each has probability $1-(1-1/r)^\tau$. Therefore, using $e^{-x/(1-x)} \leq 1-x \leq e^{-x}$ for all $x \leq 1$, we get

$$\begin{aligned}\mathbb{P}[U = [n]] &= \left(1-(1-1/r)^\tau\right)^{n-\tau}\\ &\leq \exp\left(-(n-\tau)(1-1/r)^\tau\right)\\ &\leq \exp\left(-n(1-1/r)^{\tau+1}\right)\\ &\leq \exp\left(-n\exp\left(-4\tau/r\right)\right)\\ &\leq \exp\left(-\sqrt{n}\right).\end{aligned}$$

Taking a union bound over all choices of $v_1, \ldots, v_\tau$ and $c_1, \ldots, c_\tau$, we conclude that the probability that there is a monochromatic $\mathcal{S}$-tiling of size τ is at most

$$(rn)^{-\tau} \cdot e^{-\sqrt{n}} < 1,$$

for all sufficiently large n. Hence, there exists an r-colouring of $E(K_n)$ without a monochromatic $\mathcal{S}$-tiling of size at most τ, finishing the proof. □

Lee [22] proved that graphs with bounded degeneracy[8] have linear Ramsey number. Example 5.1 shows however that it is not possible to extend this result to a tiling result. Nevertheless, it may be possible to allow unbounded degrees in this case.

Question 1 conj:boundedspsdegeneracy Is there a function $\omega : \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} \omega(n) = \infty$, such that the following is true for all integers $r, d \geq 2$? If $\mathcal{F} = \{F_1, F_2, \ldots\}$ is a sequence of d-degenerate graphs with $v(F_n) = n$ and $\Delta(F_n) \leq \omega(n)$ for all $n \in \mathbb{N}$, then $\tau_r(\mathcal{F}) < \infty$.

Böttcher, Kohayakawa, Taraz and Würf [4] proved an extension of the blow-up lemma to graphs H of bounded arrangeability[9] with $\Delta(H) \leq \sqrt{n}/\log(n)$. Using their result, it is possible to prove the following strengthening of Theorem 1.1.

Theorem 5.2 *For all integers $r, a \geq 2$ and all sequences of a-arrangeable graphs $\mathcal{F} = \{F_1, F_2, \ldots\}$ with $|F_n| = n$ and $\Delta(F_n) \leq \sqrt{n}/\log(n)$ for all $n \in \mathbb{N}$, we have $\tau_r(\mathcal{F}) < \infty$.*

The proof is almost identical, with the following two differences. First, instead of Theorem 3.3, we need to use the blow-up lemma mentioned above together with the following alternative to Hajnal's and Szemerédi's theorem which guarantees balanced partitions of graphs with small degree. Given a sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ of a-arrangeable graphs with $\Delta(F_n) \leq \sqrt{n}/\log(n)$ for every $n \in \mathbb{N}$, we define another sequence of graphs $\tilde{\mathcal{F}} = \{\tilde{F}_1, \tilde{F}_2, \ldots\}$ as follows. Since every a-arrangeable graph is $(a + 2)$-colourable, we can fix a partition of $V(F_n) = V_1(F_n) \cup \ldots \cup V_k(F_n)$ into independent sets, where $k = a + 2$. Then, for every $j \in \mathbb{N}$, we define $\tilde{F_{jk}}$ to be the disjoint union of k copies of F_j. Note that each $\tilde{F}_{jk}$ has a k-partition into parts of equal sizes (by rotating each copy around). Finally, for each $j \in \mathbb{N} \cup \{0\}$ and every $i \in [k-1]$, we define $\tilde{F}_{jk+i}$ to be the disjoint union of $\tilde{F}_{jk}$ and i isolated vertices (here $\tilde{F}_0$ is the empty graph). Observe that all $\tilde{F}_n$ have k-partitions into parts of almost equal sizes. Furthermore, every $\tilde{\mathcal{F}}$-tiling $\mathcal{T}$ corresponds to an $\mathcal{F}$-tiling $\tilde{\mathcal{T}}$ of size at most $(2k-1)|\mathcal{T}|$. Therefore, it suffices to prove Theorem 5.2 for graphs with balanced $(a + 2)$-partitions.

Second, we need to replace Theorem 4.1 with a similar theorem for a-arrangeable graphs G with $\Delta(G) \leq \sqrt{n}/\log(n)$, where $n = v(G)$. For two colours, such a theorem was proved by Chen and Schelp [6]. For more than two colours, this was (to the best of the author's knowledge) never explicitly stated, but is easy to obtain using modern tools (for example, by applying the above mentioned blow-up lemma for a-arrangeable graphs).

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[8] A graph G is d-degenerate if there is an ordering of its vertices so that every $v \in V(G)$ is adjacent to at most d vertices which come before v.

[9] A graph G is called *a-arrangeable* for some $a \in \mathbb{N}$ if its vertices can be ordered in such a way that for every $v \in V(G)$, there are at most a vertices to the left of v that have some common neighbour with v to the right of v.

Appendix

In this appendix, we shall prove the lemmas stated in Sect. 3 for which we could not find a proof in the literature. Their proofs however are standard and not difficult.

Proof of Lemma 3.2 Let $U_i = (V_i \setminus X_i) \cup Y_i$ for $i \in \{1, 2\}$. We will show that (U_1, U_2) is $(8\varepsilon, d - 8\varepsilon, \delta/2)$-super-regular. Let now $Z_i \subseteq U_i$ with $|Z_i| \geq 8\varepsilon|U_i|$, and let $Z_i' = Z_i \setminus Y_i$ and $Z_i'' = Z_i \cap Y_i$ for $i \in \{1, 2\}$. Note that we have

$$|Z_i| \geq 8\varepsilon|U_i| \geq \varepsilon|V_i|, \tag{A.1}$$

$$|Z_i''| \leq |Y_i| \leq \varepsilon^2|V_i| \overset{(A.1)}{\leq} \varepsilon|Z_i| \text{ and} \tag{A.2}$$

$$|Z_i'| = |Z_i| - |Z_i''| \overset{(A.2)}{\geq} (1 - \varepsilon)|Z_i| \tag{A.3}$$

for both $i \in \{1, 2\}$. We therefore have

$$e(Z_1, Z_2) \leq e(Z_1', Z_2') + e(Z_1'', Z_2) + e(Z_1, Z_2'') \overset{(A.2)}{\leq} e(Z_1', Z_2') + 2\varepsilon|Z_1||Z_2|,$$

and thus

$$d(Z_1, Z_2) \leq d(Z_1', Z_2') + 2\varepsilon.$$

On the other hand, we have

$$\begin{aligned} d(Z_1, Z_2) = \frac{e(Z_1, Z_2)}{|Z_1||Z_2|} &\geq \frac{e(Z_1', Z_2')}{|Z_1'||Z_2'|} \cdot \frac{|Z_1'||Z_2'|}{|Z_1||Z_2|} \\ &\overset{(A.3)}{\geq} d(Z_1', Z_2')(1 - \varepsilon)^2 \geq d(Z_1', Z_2') - 2\varepsilon \end{aligned}$$

and hence $d(Z_1, Z_2) = d(Z_1', Z_2') \pm 2\varepsilon$. Furthermore, by ε-regularity of (V_1, V_2), we have $d(Z_1', Z_2') = d(V_1, V_2) \pm \varepsilon$ and we conclude

$$d(Z_1, Z_2) = d(V_1, V_2) \pm 3\varepsilon.$$

This holds in particular for $Z_1 = U_1$ and $Z_2 = U_2$ and therefore the pair (U_1, U_2) is $(8\varepsilon, d - 8\varepsilon, 0)$-super-regular. Let $u_1 \in U_1$ now. By assumption, we have $\deg(u_1, V_2) \geq \delta|V_2|$ and therefore

$$\begin{aligned} \deg(u_1, U_2) \geq \deg(u_1, V_2 \setminus X_2) &\geq (\delta - \varepsilon^2)|V_2| \\ &\geq (\delta - \varepsilon^2)|U_2| \geq \delta/2 \cdot |U_2|. \end{aligned}$$

A similar statement is true for every $u_2 \in U_2$ finishing the proof. □

The following consequence of the slicing lemma will be useful when we prove Lemmas 3.4,3.5.

Lemma A.1 *Let k be a positive integer and $d, \varepsilon > 0$ with $\varepsilon \leq 1/(2k)$. If $Z = (V_1, \ldots, V_k)$ is an ε-regular k-cylinder and $d(V_i, V_j) \geq d$ for all $1 \leq i < j \leq k$, then there is some $\gamma \leq k\varepsilon$ and sets $\tilde{V}_1 \subseteq V_1, \ldots, \tilde{V}_k \subseteq V_k$ with $|\tilde{V}_i| = \lceil (1-\gamma)|V_i| \rceil$ for all $i \in [k]$ so that the k-cylinder $\tilde{Z} = (\tilde{V}_1, \ldots, \tilde{V}_k)$ is $(2\varepsilon, d - k\varepsilon)$-super-regular.*

Proof For $i \neq j \in [k]$, let $A_{i,j} := \{v \in V_i : \deg(v, V_j) < (d-\varepsilon)|V_j|\}$. By definition of ε-regularity, we have $\left|A_{i,j}\right| < \varepsilon|V_i|$ for every $i \neq j \in [k]$. For each $i \in [k]$, let $A_i = \bigcup_{j \in [k]\setminus\{i\}} A_{i,j}$. Clearly $|A_i| < (k-1)\varepsilon|V_i|$ for every $i \in [k]$, so we can add arbitrary vertices from $V_i \setminus A_i$ to A_i until $|A_i| = \lfloor (k-1)\varepsilon|V_i| \rfloor$ for every $i \in [k]$. Let now $\tilde{V}_i = V_i \setminus \tilde{A}_i$ for every $i \in [k]$ and let $\tilde{Z} = (\tilde{V}_1, \ldots, \tilde{V}_k)$. Observe that $|\tilde{V}_i| = \lceil (1-\gamma)|V_i| \rceil$ for all $i \in [k]$, where $\gamma = (k-1)\varepsilon$. It follows from Lemma 3.1 and definition of A_i that $\tilde{Z}$ is $(2\varepsilon, d-\varepsilon, d-k\varepsilon)$-super-regular. □

Given k disjoint sets $V_1, \ldots, V_k$, we call a cylinder $(U_1, \ldots, U_k)$ *relatively balanced* (w.r.t. $(V_1, \ldots, V_k)$) if there exists some $\gamma > 0$ so that $U_i \subseteq V_i$ with $|U_i| = \lfloor \gamma |V_i| \rfloor$ for every $i \in [k]$. We say that a partition $\mathcal{K}$ of $V_1 \times \cdots \times V_k$ is *cylindrical* if each partition class is of the form $W_1 \times \cdots \times W_k$ (which we associate with the k-cylinder $Z = (W_1, \ldots, W_k)$) with $W_j \subseteq V_j$ for every $j \in [k]$. Finally, we say that $\mathcal{K} = \{Z_1, \ldots, Z_N\}$ is ε-regular if

(i) $\mathcal{K}$ is a cylindrical partition of $V_1 \times \cdots \times V_k$,
(ii) each Z_i, $i \in [k]$, is a relatively balanced w.r.t. $(V_1, \ldots, V_k)$, and
(iii) all but $\varepsilon|V_1|\cdots|V_k|$ of the k-tuples $(v_1, \ldots, v_k) \in V_1 \times \cdots \times V_k$ are in ε-regular cylinders.

For technical reasons, we will allow some of the sets $V_1, \ldots, V_k$ to be empty. In this case $(A, \emptyset)$ is considered ε-regular for every set A and $\varepsilon > 0$. If G is an r-edge-coloured graph and $i \in [r]$, we say that a cylinder $\mathcal{K}$ is ε-regular in colour i if is ε-regular in G_i (the graph on $V(G)$ with all edges of colour i).

In [7], Conlon and Fox used the weak regularity lemma of Duke, Lefmann and Rödl [9] to find a reasonably large balanced k-cylinder in a k-partite graph. In order to prove a coloured version of Conlon and Fox's result, we will need the following coloured version of the weak regularity lemma of Duke, Lefmann and Rödl. Note that, like the weak regularity lemma of Frieze and Kannan [15], we get an exponential bound on the number of cylinders, in contrast to the much worse tower-type bound required by Szemerédi's regularity lemma (see [14]).

Theorem A.2 *(Duke-Lefmann-Rödl [9]) Let $0 < \varepsilon < 1/2$, $k, r \in \mathbb{N}$ and let $\beta = \varepsilon^{rk^2\varepsilon^{-5}}$. Let G be an r-edge-coloured k-partite graph with parts $V_1, \ldots, V_k$. Then there exist some $N \leq \beta^{-k}$, sets $R_1 \subseteq V_1, \ldots, R_k \subseteq V_k$ with $|R_i| \leq \beta^{-1}$ and a partition $\mathcal{K} = \{Z_1, \ldots, Z_N\}$ of $(V_1 \setminus R_1) \times \cdots \times (V_k \setminus R_k)$ so that $\mathcal{K}$ is ε-regular in every colour and $V_i(Z_j) \geq \lfloor \beta |V_i| \rfloor$ for every $i \in [k]$ and $j \in [N]$.*

Although the original statement of Duke, Lefmann and Rödl [9, Proposition 2.1] does not involve the colouring and assume that sets $V_1, \ldots, V_k$ have the same size, their proof can be easily adapted to prove Theorem A.2.

We are now ready to prove Lemmas 3.4,3.5.

Proof of Lemma 3.4 Let $k, r \geq 2$, $0 < \varepsilon < 1/(rk)$ and $\gamma = \varepsilon^{r^{8rk}\varepsilon^{-5}}$. Let $n \geq 1/\gamma$ and suppose we are given an r-edge coloured K_n. Let $\tilde{k} = r^{rk}$ and let $V_1, \ldots, V_{\tilde{k}} \subseteq [n]$ be disjoint sets of size $\lfloor n/\tilde{k} \rfloor$ and let G be the $\tilde{k}$-partite subgraph of K_n induced by $V_1, \ldots, V_{\tilde{k}}$ (inheriting the colouring). Let $\tilde{\varepsilon} = \varepsilon/2$ and $\beta = \tilde{\varepsilon}^{r^{2rk+1}\tilde{\varepsilon}^{-5}}$. We apply Theorem A.2 to get some $N \leq \beta^{-\tilde{k}}$, sets $R_1 \subseteq V_1, \ldots, R_{\tilde{k}} \subseteq V_{\tilde{k}}$ each of which of size at most β^{-1} and a partition $\mathcal{K} = \{Z_1, \ldots, Z_N\}$ of $(V_1 \backslash R_i) \times \cdots \times (V_{\tilde{k}} \backslash R_{\tilde{k}})$ which is $\tilde{\varepsilon}$-regular in every colour, and with $V_i(Z_j) \geq \lfloor \beta |V_i| \rfloor \geq 2\gamma n$ for every $i \in [\tilde{k}]$ and $j \in [N]$. Note that one of the cylinders (say Z_1) must be $\tilde{\varepsilon}$-regular in every colour and, since $(V_1, \ldots, V_k)$ is balanced, so is Z_1. We consider now the complete graph with vertex-set $\{V_1(Z_1), \ldots, V_{\tilde{k}}(Z_1)\}$ and colour every edge $V_i(Z_1)V_j(Z_1)$, $1 \leq i < j \leq \tilde{k}$, with a colour $c \in [r]$ so that the density of the pair $(V_i(Z_1), V_j(Z_1))$ in colour c is at least $1/r$. By Ramsey's theorem [12, 25], there is a colour, say 1, and k parts (say $V_1(Z_1), \ldots, V_k(Z_1)$) so that the cylinder $(V_1(Z_1), \ldots, V_k(Z_1))$ is $(\tilde{\varepsilon}, 1/r, 0)$-super-regular in colour 1. By Lemma A.1, there is an $(\varepsilon, 1/(2r))$-super-regular balanced subcylinder $\tilde{Z}_1$ with parts of size at least γn. □

Proof of Lemma 3.5 Let $k \geq 2$, and let $d, \varepsilon > 0$ with $2k\varepsilon \leq d \leq 1$. Let $\gamma = \varepsilon^{k^2\varepsilon^{-12}}$ and let G be a k-partite graph with parts $V_1, \ldots, V_k$. Let $\tilde{\varepsilon} = \varepsilon/4$ and $\beta = \tilde{\varepsilon}^{k^2\tilde{\varepsilon}^{-5}}$. We may assume that $|V_i| \geq 1/\gamma$ for every $i \in [k]$ (otherwise we set $U_i := \emptyset$ for all $i \in [k]$ with $|V_i| < 1/\gamma$). In particular, we have $|V_i| \geq k/(\tilde{\varepsilon}\beta)$ for all $i \in [k]$.

We apply Theorem A.2 (with $r = 1$) to get some $N \leq \beta^{-k}$, sets $R_1 \subseteq V_1, \ldots, R_k \subseteq V_k$, each of which of size at most β^{-1}, and an $\tilde{\varepsilon}$-regular partition $\mathcal{K} = \{Z_1, \ldots, Z_N\}$ of $(V_1 \backslash R_1) \times \cdots \times (V_k \backslash R_k)$ with $V_i(Z_j) \geq \lfloor \beta |V_i| \rfloor$ for every $i \in [k]$ and $j \in [N]$.

Note that the number of cliques of size k incident to $R = R_1 \cup \ldots \cup R_k$ is at most

$$\sum_{i=1}^{k} \beta^{-1} \prod_{j \in [k] \backslash \{i\}} |V_j| \leq \tilde{\varepsilon} |V_1| \cdots |V_k|.$$

Furthermore, since $\mathcal{K}$ is $\tilde{\varepsilon}$-regular, there are at most $\tilde{\varepsilon}|V_1| \cdots |V_k|$ cliques of size k in G that belong to a cylinder of $\mathcal{K}$ that is not ε-regular. Suppose that each cylinder $Z \in \mathcal{K}$ has at most $(d - 2\tilde{\varepsilon})|V_1(Z)| \cdots |V_k(Z)|$ cliques of size k. Then the number of k-cliques in G is at most

$$\tilde{\varepsilon}|V_1| \cdots |V_k| + \sum_{Z \in \mathcal{K}} (d - 2\tilde{\varepsilon})|V_1(Z)| \cdots |V_k(Z)| \leq (d - \tilde{\varepsilon})|V_1| \cdots |V_k|,$$

which contradicts our hypothesis over G. Therefore, there is a cylinder $\tilde{Z}$ in $\mathcal{K}$ that contains at least $(d - 2\tilde{\varepsilon})|V_1(\tilde{Z})| \cdots |V_k(\tilde{Z})|$ cliques of size k. In particular, $\tilde{Z}$ is $(\tilde{\varepsilon}, d - 2\tilde{\varepsilon}, 0)$-super-regular and relatively balanced with parts of size at least $\lfloor \beta |V_i| \rfloor$. Finally, we apply Lemma A.1 (and possibly delete a single vertex from some parts) to get a relatively balanced $(\varepsilon, d - (k+2)\tilde{\varepsilon})$-super-regular k-cylinder Z with parts of size at least $\frac{\beta}{2}|V_i| \geq \gamma |V_i|$. This completes the proof since $(k+2)\tilde{\varepsilon} \leq k\varepsilon \leq d/2$. □

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