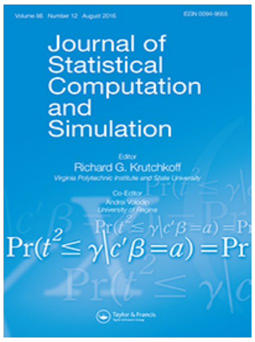

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A bimodal flexible distribution for lifetime data

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ABSTRACT
A four-parameter extended bimodal lifetime model called the exponentiated log-sinh Cauchy distribution is proposed. It extends the log-sinh Cauchy and folded Cauchy distributions. We derive some of its mathematical properties including explicit expressions for the ordinary moments and generating and quantile functions. The method of maximum likelihood is used to estimate the model parameters. We implement the fit of the model in the GAMLSS package and provide the codes. The flexibility of the model is illustrated by means of three real data sets.

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Bi-modality; exponentiated sinh Cauchy distribution; GAMLSS; lifetime distribution

1. Introduction

Generalizing lifetime distributions by introducing a few extra shape parameters is an essential method to better explore the skewness and the tails and other properties of the transformed distributions. Following the latest trend, applied statisticians are now able to construct more generalized distributions, which provide better goodness-of-fit measures when fitted to real data rather than by using the classical distributions. The Weibull, log-normal and log-logistic are very popular distributions for modelling lifetime data and phenomenon with unimodal and monotone failure rates. In these cases, they may be chosen because of their negatively and positively skewed density shapes. However, these models do not provide reasonable parametric fits for modelling phenomenon with non-monotone failure rates such as the bathtub shaped and bimodal failure rates, which are common in reliability and biological studies. In this paper, we study a four-parameter generalization of the exponentiated sinh Cauchy (ESC) distribution on the basis of the sinh Cauchy (SC) model, both proposed by Cooray,[1] for modelling bimodal and unimodal data. The advantage of this approach for constructing a parametric family of distributions lies in its flexibility to model both bathtub and bimodal failure rates even though the baseline failure rate may be monotonic. The generated model is called the *exponentiated log-sinh Cauchy* (ELSC) distribution. As we will see later, its hazard rate function (hrf) can be constant, decreasing, increasing, upside-down bathtub (unimodal), bathtub and bimodal shaped. Due to the great flexibility of the ELSC hrf, it thus provides a good alternative to many existing life distributions in modelling positive real data sets.

Cooray [1] applied the hyperbolic sine transformation to the standard Cauchy distribution by defining the SC model, whose cumulative density function (cdf) is given by

$$\Pi(y) = \frac{1}{2} + \frac{1}{\pi}\arctan\left[\nu \sinh\left(\frac{y-\mu}{\sigma}\right)\right], \quad y \in \mathbb{R}, \tag{1}$$

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where $\mu \in \mathbb{R}$ and $\sigma > 0$ are the location and scale parameters, respectively, and $\nu > 0$ is the symmetry parameter, which characterizes the bi-modality of the distribution. The SC distribution produces both bimodal and unimodal densities with a wide range of tail weights. It has a real support and therefore is not appropriate for survival data. As a better alternative, we present the *log-sinh Cauchy* (LSC) model.

Let Y be a random variable having cdf (1). The random variable $X = \mathrm{e}^Y$ defines the LSC distribution, whose cdf is given by

$$G(x) = \frac{1}{2} + \frac{1}{\pi} \arctan\left[\nu \sinh\left(\frac{\log(x) - \mu}{\sigma}\right)\right], \quad x > 0. \tag{2}$$

The SC and LSC models are not appropriate for modelling real data, even though they have some theoretical advantages due to their symmetric nature. To provide an asymmetry for the SC distribution, Cooray [1] proposed the ESC distribution using the exponentiated class of distributions.[2] The cdf of the exponentiated class is given by

$$F(x) = G(x)^{\tau}, \tag{3}$$

where $G(x)$ is the parent cdf and $\tau > 0$ denotes an extra power shape parameter. By differentiating Equation (3), the probability density function (pdf) of the exponentiated class is given by

$$f(x) = \tau G(x)^{\tau - 1} g(x), \tag{4}$$

where $g(x)$ is the baseline pdf.

The paper is outlined as follows. In Section 2, we define the ELSC model by applying the exponentiated generator to the LSC distribution. In Section 3, we derive a power series for the quantile function (qf) of this distribution. In Section 4, we obtain explicit expressions for its moments. A range of its mathematical properties is explored in Section 5 including generating function, mean deviations and order statistics. The estimation of the model parameters by maximum likelihood is addressed in Section 6. The performance of the maximum likelihood estimators (MLEs) is investigated through a simulation study in Section 7. Applications to three real data sets are addressed in Section 8 to prove empirically the flexibility of the model. In Section 9, we provide a brief discussion of the template for the ELSC distribution implemented in the 'GAMLSS' R package.[3] We also provide the computational codes used in the applications. Finally, Section 10 ends with some conclusions.

2. The ELSC model

We can add skewness for an extended LSC distribution by adopting the exponentiated class of distributions [2] given by Equation (3). Inserting Equation (2) into Equation (3), the ELSC cdf is given by

$$F(x; \mu, \sigma, \nu, \tau) = \left\{\frac{1}{2} + \frac{1}{\pi} \arctan[\nu \sinh(w)]\right\}^{\tau}, \tag{5}$$

where $w = [\log(x) - \mu]/\sigma$. For $\tau = 1$, the LSC distribution is just a special case of Equation (5). The pdf corresponding to Equation (5) is given by

$$f(x; \mu, \sigma, \nu, \tau) = \frac{\tau \nu}{x \sigma \pi} \frac{\cosh(w)}{[\nu^2 \sinh^2(w) + 1]} \left\{\frac{1}{2} + \frac{1}{\pi} \arctan[\nu \sinh(w)]\right\}^{\tau - 1}. \tag{6}$$

Henceforth, let $X \sim$ELSC(μ, σ, ν, τ) be a random variable with density function (6). We can omit sometimes the dependence on the parameters and and write simply $f(x) = f(x; \mu, \sigma, \nu, \tau)$.

The survival function and hrf of X are given by $S(x) = 1 - F(x)$ and $h(x) = f(x)/S(x)$, respectively. Plots of the ELSC density, survival and hazard functions for selected parameter values are displayed in Figures 1–3, respectively.

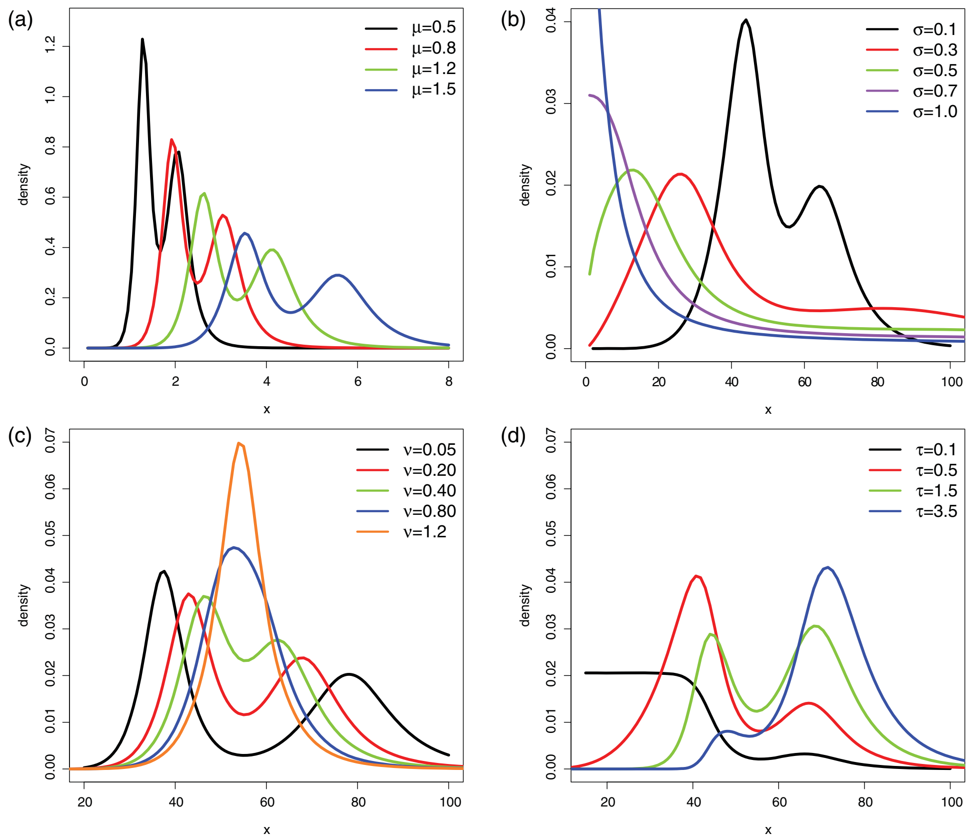

Figure 1. Plots of the ELSC density for fixed values of: (a) $\sigma = 0.1$, $\nu = 0.2$ and $\tau = 1$; (b) $\mu = 4$, $\nu = 0.3$ and $\tau = 0.7$; (c) $\mu = 4$, $\sigma = 0.1$ and $\tau = 1$; (d) $\mu = 4$, $\sigma = 0.1$ and $\nu = 0.2$.

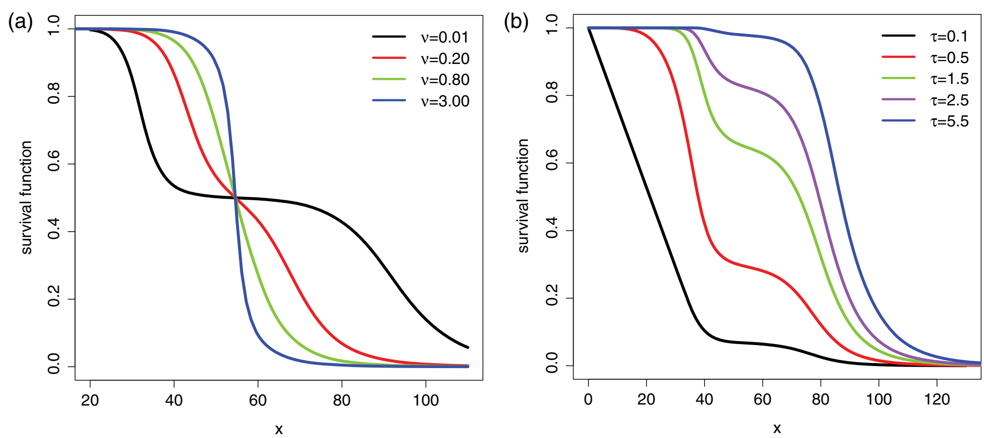

Figure 2. The ELSC survival function when $\mu = 4, \sigma = 0.1$ and: (a) for $\tau = 1$ and different values of ν; (b) for $\nu = 0.05$ and different values of τ.

In Figure 1 (a)–(b), we check the effects of the location and scale parameters μ and σ on the function $f(x)$. Figure 1(c) reveals clearly the bi-modality effect caused by the parameter ν. Further, Figure 1(d) reveals that the density of X is bimodal and symmetric, bimodal and right-skewed,

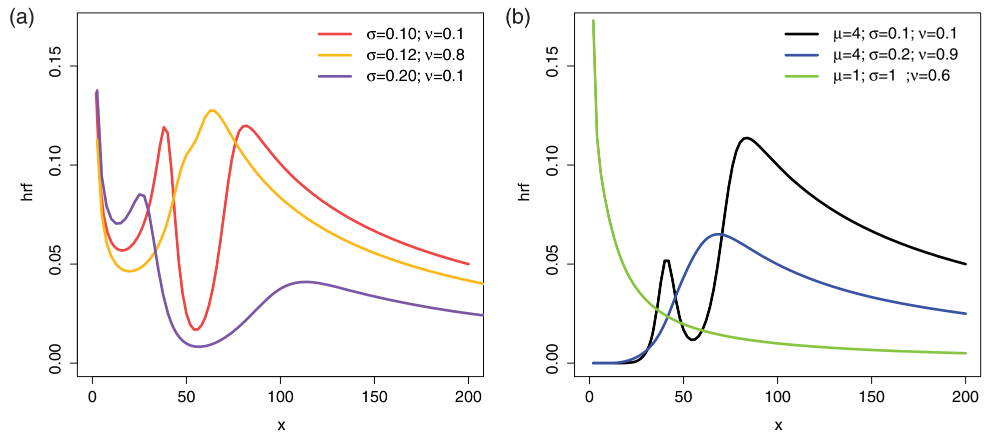

Figure 3. The ELSC hrf: (a) for $\tau = 1$ and different values of μ, σ and ν; (b) for $\mu = 4$ and $\tau = 0.01$ and different values of σ and ν.

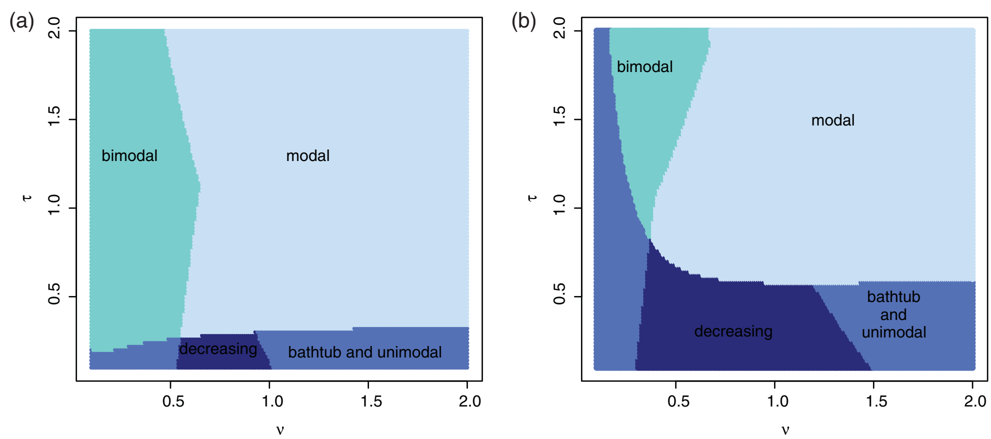

Figure 4. The ELSC hrf shapes as functions of ν and τ for $\mu = 1$ and: (a) $\sigma = 0.4$; (b) $\sigma = 0.7$.

bimodal and left-skewed depending on the parameter τ. Figure 3(a) and 3(b) indicate that the hrf of X has decreasing, unimodal and bimodal forms and double bathtub-shaped and unimodal and bathtub-shaped, respectively.

We provide in Figure 4(a)–(b) a numerical investigation to identify how the parameter values change the shapes of the hrf of X for some parameter ranges. Based on these plots, we can obtain bimodal shapes for the hrf of X for small values of the parameters ν and τ. However, large values of these parameters are necessary to obtain this characteristic when the parameter σ increases.

Because of the current computational facilities, several researchers construct new lifetime models to facilitate their use in lifetime data analysis. It is a common practical technique to fit new models to real data and develop scripts in statistical software R.[4] de Castro et al. [5] implemented some long-term survival models by taking the Weibull as the parent distribution. Rodrigues et al. [6] implemented the COM–Poisson cure rate model and illustrate its flexibility by means of a real data set. Following these ideas, the ELSC model is implemented in the R software, where a short discussion is given in Section 9.

3. Expansion of the quantile function

Inverting $F(x) = u$ (for $0 < u < 1$), we obtain the qf of X

$$x = Q(u) = \exp\left(\mu + \sigma \operatorname{arcsinh}\left\{\frac{1}{\nu}\tan[\pi(u^{1/\tau} - 0.5)]\right\}\right). \tag{7}$$

Quantiles of interest can be obtained from Equation (7) by substituting appropriate values for u. In particular, the median of X is obtained when $u = \frac{1}{2}$. We can also use Equation (7) for simulating ELSC random variables by setting u as a uniform random variable in the unit interval $(0, 1)$. The qf of the LSC distribution can be obtained by taking $\tau = 1$ in Equation (7).

Next, we derive an expansion for the qf of X to obtain some ELSC properties in the following sections. Expanding Equation (7) in power series using Mathematica, we obtain

$$Q(u) = \mathrm{e}^{\mu} \exp\left(\sum_{k=0}^{\infty} c_k z^{2k+1}\right),$$

where $z = u^{1/\tau} - 0.5$, $c_k = \sigma b_k/(2k+1)!(\pi/\nu)^{2k+1}$ and $b_0 = 1$, $b_1 = (2\nu^2 - 1)$, $b_2 = (16\nu^4 - 20\nu^2 + 9)$, $b_3 = (272\nu^6 - 616\nu^4 + 630\nu^2 - 225)$, $b_4 = (7936\nu^8 - 28160\nu^6 + 48384\nu^4 - 37800\nu^2 + 11025), \ldots$

By simple transformation of quantities, we can write

$$Q(u) = \mathrm{e}^{\mu} \exp\left(\sum_{k=1}^{\infty} \frac{d_k}{k!} z^k\right), \tag{8}$$

where

$$d_{2j} = 0 \text{ for } j = 1, 2, \ldots \text{ and } d_{2j+1} = (2j+1)!c_j \text{ for } j = 0, 1, 2, \ldots. \tag{9}$$

We can use the Bell polynomials[1] to rewrite Equation (8). The exponential partial Bell polynomials in formal double series expansion are defined by Comtet [7, p.133] as

$$\exp\left(u \sum_{m\geq 1} x_m \frac{t^m}{m!}\right) = \sum_{n,k\geq 0} \frac{B_{n,k}}{n!} t^n u^k, \tag{10}$$

where

$$B_{n,k} = B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{c_1!c_2!\ldots(1!)^{c_1}(2!)^{c_2}\ldots} x_1^{c_1} x_2^{c_2}, \ldots,$$

and the summation is over all integers $c_1, c_2, c_3, \ldots \geq 0$ such that $c_1 + 2c_2 + 3c_3 + \cdots = n$ and $c_1 + c_2 + c_3 + \cdots = k$. These exponential partial Bell polynomials can be evaluated in Mathematica and Maple using BellY[n,k,{$x_1, \ldots, x_{n-k+1}$}] and IncompleteBellB(n, k, x[1], z[2],..., x[n-k+1]).

Using the definition of the complete Bell polynomials and Equation (10), Equation (8) can be expressed as

$$Q(u) = \mathrm{e}^{\mu} \sum_{k=0}^{\infty} \frac{B_k(d_1, \ldots, d_k)}{k!} z^k,$$

where $B_k = B_k(d_1, \ldots, d_k) = \sum_{r=1}^{k} B_{k,r}(d_1, \ldots, d_{k-r+1})$ (for $k \geq 0$) is the complete Bell polynomial of order k.

The coefficients B_k can be easily obtained using Mathematica, Maple and Sage softwares. Replacing z in the last equation, the qf of X can be rewritten as

$$Q(u) = \mathrm{e}^{\mu} \sum_{k=0}^{\infty} \frac{B_k(d_1, \ldots, d_k)}{k!} (u^{1/\tau} - 0.5)^k. \tag{11}$$

By expanding the binomial term, we have

$$Q(u) = \mathrm{e}^{\mu} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k-j} u^{j/\tau}}{2^{k-j} k!} \binom{k}{j} B_k(d_1, \ldots, d_k).$$

Further, changing $\sum_{k=0}^{\infty} \sum_{j=0}^{\infty}$ by $\sum_{j=0}^{\infty} \sum_{k=j}^{\infty}$, we can write

$$Q(u) = \sum_{j=0}^{\infty} p_j u^{j/\tau}, \tag{12}$$

where the coefficients

$$p_j = \mathrm{e}^{\mu} \sum_{k=j}^{\infty} \frac{(-1)^{k-j}}{2^{k-j} k!} \binom{k}{j} B_k(d_1, \ldots, d_k) \tag{13}$$

an be evaluated using the analytical softwares cited before.

Let $W(\cdot)$ be any integrable function in the positive real line. We can write from Equations (6) and (12)

$$\int_0^{\infty} W(x) f(x; \mu, \sigma, \nu, \tau)\, \mathrm{d}x = \int_0^1 W\left(\sum_{j=0}^{\infty} p_j u^{j/\tau}\right) \mathrm{d}u. \tag{14}$$

Equation (14) is an important result since it allows to obtain various mathematical properties for the ELSC distribution using integrals over $(0, 1)$. For the great majority of the applications of Equation (14), we can adopt 10 terms in the power series. Equations (12) and (14) are the main results of this section. The formulae derived throughout the paper can be easily handled in most symbolic computation software platforms such as those cited before. They have currently the ability to deal with analytic expressions of formidable size and complexity. Established explicit expressions to evaluate statistical measures can be more efficient than computing them directly by numerical integration.

4. Moments

Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis). Using Equation (4), the nth moment of X can be expressed as

$$\mu_n' = E(X^n) = \tau \int_0^{\infty} x^n G(x)^{\tau-1} g(x)\, \mathrm{d}x = \tau \int_0^1 Q_{\mathrm{LSC}}(u)^n u^{\tau-1}\, \mathrm{d}u, \tag{15}$$

where $Q_{\mathrm{LSC}}(u)$ denotes the qf of the LSC distribution.

Here, we give two explicit expressions for μ_n'. For the first one, we use the power series for $Q_{\mathrm{LSC}}(u)^n$, which follows by changing μ by $n\mu$, σ by $n\sigma$ and taking $\tau = 1$ in Equation (11). We have

$$Q_{\mathrm{LSC}}(u)^n = \mathrm{e}^{n\mu} \sum_{k=0}^{\infty} \frac{B_k(d_1^*, \ldots, d_k^*)}{k!} (u-0.5)^k, \tag{16}$$

where

$$d_{2j}^* = 0 \quad \text{for } j = 1, 2, \ldots, \quad d_{2j+1}^* = (2j+1)!c_j^* \text{ for } j = 0, 1, 2, \ldots \tag{17}$$

and $c_k^* = k\sigma b_k \pi^{2k+1}/(2k+1)!$.

Replacing Equation (16) in Equation (15), we have

$$\mu_n' = \tau \mathrm{e}^{n\mu} \sum_{k=0}^{\infty} \frac{B_k(d_1^*, \ldots, d_k^*)}{k!} \int_0^1 (u-0.5)^k u^{\tau-1}\, \mathrm{d}u.$$

Let ${}_2F_1(p, q; r; y) = \sum_{j=0}^{\infty} (p)_j (q)_j y^j / [(r)_j j!]$ be the hypergeometric function, $(p)_j$ the Pochhammer symbol defined by $(p)_j = p(p+1)\cdots(p+j-1) = \Gamma(p+j)/\Gamma(p) = (-1)^j \Gamma(1-p)/\Gamma(1-p-j)$, and $\Gamma(\cdot)$ the gamma function.

The last equation can be expressed in terms of the hypergeometric function[2] as

$$\mu_n' = \mathrm{e}^{n\mu} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} {}_2F_1(-k, \tau; \tau+1; 2) B_k(d_1^*, \ldots, d_k^*). \tag{18}$$

The hypergeometric function ${}_2F_1(p, q; r; y)$ can be evaluated from Mathematica and Maple as `HypergeometricPFQ[{p,q}, {r},y]` and `Hypergeometric([p,q],[r],y)`, respectively.

The second expression for μ_n' can be determined using Equations (7) and (12) in Equation (15) and changing μ by $n\mu$, σ by $n\sigma$ and setting $\tau = 1$. We obtain

$$\mu_n' = \tau \sum_{j=0}^{\infty} \frac{p_j^*}{j+\tau}, \tag{19}$$

where $p_j^* = \mathrm{e}^{n\mu} \sum_{k=j}^{\infty} (-1)^{k-j}/2^{k-j} k! \binom{k}{j} B_k(d_1^*, \ldots, d_k^*)$ and d_k^* is defined by Equation (17).

Equations (18) and (19) are the main results of this section. The central moments (μ_s) and cumulants (κ_s) of X are determined as $\mu_s = \sum_{k=0}^{p} (-1)^k \binom{s}{k} \mu_1'^s \mu_{s-k}'$ and $\kappa_s = \mu_s' - \sum_{k=1}^{s-1} \binom{s-1}{k-1} \kappa_k \mu_{s-k}'$, respectively, where $\kappa_1 = \mu_1'$. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ follow from the third and fourth standardized cumulants, respectively.

When these moments do not exist, for example, for the Cauchy, Lévy and Pareto distributions, alternative measures for the skewness and kurtosis, based on qfs, are sometimes more appropriate for these distributions. The measures of skewness $\mathcal{B}$ [8] and kurtosis $\mathcal{M}$ [9] are given by

$$\mathcal{B} = \frac{Q(6/8) + Q(2/8) - 2Q(4/8)}{Q(6/8) - Q(2/8)} \quad \text{and} \quad \mathcal{M} = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)},$$

respectively.

For the ELSC and LSC distributions, Galton's skewness and Moors' kurtosis can be computed using the qf (7). Figure 5 displays some plots of the measures $\mathcal{B}$ and $\mathcal{M}$ as functions of the shape and bimodality parameters. The additional shape parameter τ has substantial effect on the skewness and kurtosis of X.

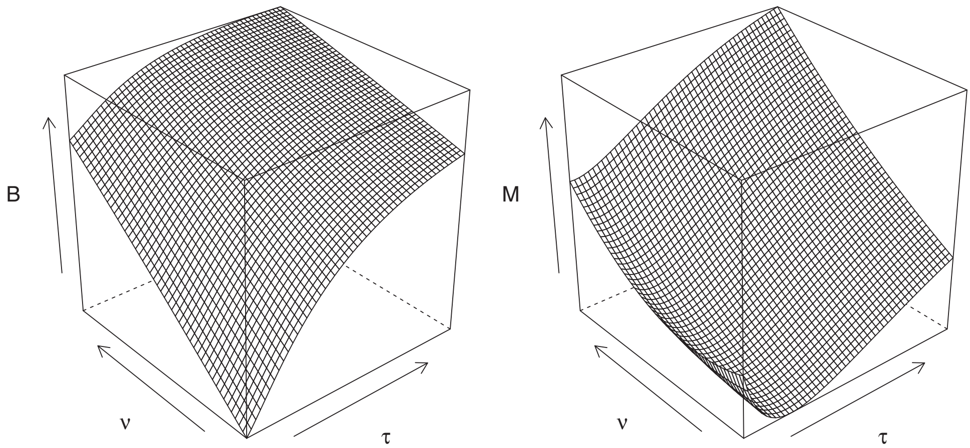

Figure 5. Plots of the measures (a) $\mathcal{B}$ and (b) $\mathcal{M}$ as functions of τ and ν for $\mu = 3$ and $\sigma = 0.2$.

5. Other measures

In this section, we derive the generating function, mean deviations and order statistics of X.

5.1. Generating function

The moment generating function (mgf) $M(t) = \mathrm{E}(\mathrm{e}^{tX})$ of X can be determined from Equation (4) in terms of its qf. We have

$$M(t) = \tau \int_0^\infty \mathrm{e}^{tx} G(x)^{\tau-1} g(x)\,\mathrm{d}x = \tau \int_0^1 u^{\tau-1} \exp[tQ_{\mathrm{LSC}}(u)]\,\mathrm{d}u.$$

Combining Equations (8) and (12) when $\tau = 1$, the mgf of X can be written as

$$M(t) = \tau\,\mathrm{e}^{tp_0} \int_0^1 u^{\tau-1} \exp\left(\sum_{j=1}^\infty \frac{p_j^{**} u^j}{j!}\right) \mathrm{d}u,$$

where $p_j^{**} = tp_j j!$ and p_j is given by Equation (13). Using again the complete Bell polynomials, we have

$$\exp\left(\sum_{j=1}^\infty \frac{p_j^{**} u^j}{j!}\right) = \sum_{j=0}^\infty \frac{B_j(p_1^{**}, \ldots, p_j^{**})}{j!} u^j,$$

and then, the mgf of X follows as

$$M(t) = \tau\,\mathrm{e}^{tp_0} \sum_{j=0}^\infty \frac{B_j(p_1^{**}, \ldots, p_j^{**})}{(\tau + j) j!}.$$

5.2. Mean deviations

For empirical purposes, the first incomplete moment $m_1(s) = \int_{-\infty}^s x f(x)\,\mathrm{d}x$ plays an important role for measuring inequality, for example, mean deviations and Lorenz and Bonferroni curves. A formula for $m_1(s)$ follows by setting $u = G(x)$ in Equation (4) as

$$m_1(s) = \tau \int_0^s Q_{\mathrm{LSC}}(u) u^{\tau-1}\, \mathrm{d}u. \tag{20}$$

Here, we provide two alternatives to compute the first incomplete moment of X. First, $m_1(s)$ can be derived from Equation (18) by taking $n = 1$ as

$$m_1(s) = \tau \mathrm{e}^{\mu} \sum_{k=0}^{\infty} \frac{(1-2s)^{-k}(s-0.5)^k s^{\tau}}{\tau k!}\, {}_2F_1(-k, \tau; \tau+1; 2s) B_k(d_1, \ldots, d_k), \tag{21}$$

where d_k is given by Equation (9). A second formula for $m_1(s)$ can be derived by inserting Equation (12) in Equation (20) and setting $\tau = 1$ as

$$m_1(s) = \tau \sum_{j=0}^{\infty} p_j \frac{s^{\tau+j}}{\tau + j}. \tag{22}$$

The main applications of Equations (21) or (22) are related to the Bonferroni and Lorenz curves defined (for a given probability π) by $B(\pi) = m_1(q)/(\pi \mu_1')$ and $L(\pi) = m_1(q)/\mu_1'$, respectively, where $\mu_1' = E(X)$ and $q = Q(\pi)$ is the qf of X at π obtained from Equation (7).

The mean deviations about the mean ($\delta_1 = E(|X - \mu_1'|)$) and the median ($\delta_2 = E(|X - M|)$) of X are given by

$$\delta_1(X) = 2\mu_1' F(\mu_1') - 2m_1(\mu_1') \quad \text{and} \quad \delta_2(X) = \mu_1' - 2m_1(M), \tag{23}$$

respectively, where $M = \mathrm{Median}(X) = Q(0.5)$ is the median, $F(\mu_1')$ is easily evaluated from the cdf (5) and $m_1(z)$ is given by Equations (21) or (22).

5.3. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, \ldots, X_n$ is a random sample from the ELSC distribution. Let $X_{i:n}$ denote the ith order statistic. Using Equations (5) and (6), the pdf of $X_{i:n}$ can be expressed as

$$\begin{aligned} f_{i:n}(x) &= K f(x) F(x)^{i-1}\{1 - F(x)\}^{n-i} = K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x) F(x)^{j+i-1} \\ &= K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \frac{\tau \nu}{x \sigma \pi} \frac{\cosh(w)}{[\nu^2 \sinh^2(w) + 1]} \left\{ \frac{1}{2} + \frac{1}{\pi} \arctan\left[\nu \sinh(w)\right] \right\}^{(j+i)\tau - 1}, \end{aligned}$$

where $w = [\log(x) - \mu]/\sigma$ and $K = n!/[(i-1)!(n-i)!]$.

6. Inference

We consider the situation when the time-to-event is not completely observed and is subject to right censoring. Let C_i denote the censoring time. We observe $x_i = \min\{X_i, C_i\}$ and $\delta_i = I(X_i \le C_i)$, where $\delta_i = 1$ if X_i is a time-to-event and $\delta_i = 0$ if it is right censored (for $i = 1, \ldots, n$). Let c denote the parameter vector of the distribution of the time-to-event. Let X_i be a random variable following Equation (6) with the vector of parameters $\boldsymbol{\gamma} = (\mu, \sigma, \nu, \tau)^T$. From n pairs of times and censoring indicators $(x_1, \delta_1), \ldots, (x_n, \delta_n)$, the log-likelihood function under non-informative censoring is given by

$$
\begin{aligned}
l(\boldsymbol{\gamma}) = {} & r[\log(\tau\nu) - \log(\sigma\pi)] - \sum_{i\in F}\log(x_i) + \sum_{i\in F}\log\cosh(w_i) - \sum_{i\in F}\log[1+\nu^2\sinh^2(w_i)] \\
& + (\tau-1)\sum_{i\in F}\log\left\{\frac{1}{2} + \frac{1}{\pi}\arctan[\nu\sinh(w_i)]\right\} \\
& + \sum_{i\in C}\log\left(1 - \left\{\frac{1}{2} + \frac{1}{\pi}\arctan\left[\nu\sinh(w_i)\right]\right\}^{\tau}\right),
\end{aligned} \tag{24}
$$

where r is the number of failures (uncensored observations).

We can obtain the MLE $\hat{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ by maximizing the log-likelihood (24) either directly in R using the `optim` function, in SAS using the `NLMixed` procedure and in other statistical software or by solving the nonlinear likelihood equations obtained by differentiating Equation (24). The score functions for the parameters in $\boldsymbol{\gamma}$ are given by

$$
U_\mu(\boldsymbol{\gamma}) = -\sum_{i\in F}\frac{\tanh(w_i)}{\sigma} + \sum_{i\in F}\frac{\nu^2\sinh(2w_i)}{\sigma K_i} + (\tau-1)\sum_{i\in F}\frac{\nu\cosh(w_i)}{\pi\sigma J_i K_i} + \sum_{i\in C}\frac{\tau\nu\cosh(w_i)J_i^{\tau-1}}{\pi\sigma K_i(J_i^{\tau}-1)},
$$

$$
\begin{aligned}
U_\sigma(\boldsymbol{\gamma}) = {} & -\frac{r}{\sigma} - \sum_{i\in F}\frac{w_i}{\sigma}\tanh(w_i) + \sum_{i\in F}\frac{2\nu^2 w_i}{\sigma K_i}\sinh(w_i)\cosh(w_i) + (\tau-1)\sum_{i\in F}\frac{\nu w_i}{\pi\sigma J_i K_i}\cosh(w_i) \\
& + \sum_{i\in C}\frac{\tau\nu w_i J^{\tau-1}}{\pi\sigma K_i(1-J_i^{\tau})}\cosh(w_i),
\end{aligned}
$$

$$
U_\nu(\boldsymbol{\gamma}) = \frac{r}{\nu} - \sum_{i\in F}\frac{2\nu\sinh^2(w_i)}{K_i} + (\tau-1)\sum_{i\in F}\frac{\sinh(w_i)}{\pi J_i K_i} + \sum_{i\in C}\frac{\tau J_i^{\tau-1}\sinh(w_i)}{\pi K_i(J_i^{\tau}-1)}
$$

and

$$
U_\tau(\boldsymbol{\gamma}) = \frac{r}{\tau} + \sum_{i\in F}\log(J_i) + \sum_{i\in C}\frac{J_i^{\tau}}{J_i^{\tau}-1}\log(J_i),
$$

where $J_i = \frac{1}{2} + 1/\pi\arctan[\nu\sinh(w_i)]$ and $K_i = \nu^2\sinh^2(w_i) + 1$.

The numerical maximization of the log-likelihood function (24) can also be performed in the GAMLSS package in R. The advantage of this package is that we can use many maximization methods, which will depend only on the current fitted model. When there are no explanatory variables or censored observations, we can use the `gamlssML` function for fitting (24) using a nonlinear maximization algorithm. When we have censored observations, the additional package `gamlss.cens` is required to determine numerically the observed information of the likelihood function referring to the censored observations. The maximization algorithms adopted in the presence of censored data are the RS and CG procedures. All methods and algorithms are described by Rigby and Stasinopoulos [10] and Stasinopoulos and Rigby [3] and they are available in the documentation of the GAMLSS package. The RS algorithm requires the first order derivatives of the logarithm of the density function (6) given in the above equations, and the second order derivatives. The RS method, different from the CG algorithm, does not use the cross derivatives, and thus it is faster for larger data sets. The second order derivatives can be determined numerically in the script discussed in Section 8.

Under standard regularity conditions, the asymptotic distribution of $(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma})$ is $N_4(\mathbf{0}, I(\boldsymbol{\gamma})^{-1})$, where $I(\boldsymbol{\gamma})$ is the expected information matrix. This asymptotic behaviour holds if $I(\boldsymbol{\gamma})$ is replaced by $J(\hat{\boldsymbol{\gamma}})$, that is, the observed information matrix evaluated at the MLE $\hat{\boldsymbol{\gamma}}$. Thus, the multivariate normal $N_4(\mathbf{0}, J(\hat{\boldsymbol{\gamma}})^{-1})$ distribution can be used to construct approximate confidence intervals for the individual parameters.

Further, we can compute the maximum values of the log-likelihoods to obtain the likelihood ratio (LR) statistics for testing some sub-models of the ELSC distribution. For example, the test of $H_0: \tau = 1$ *versus* $H: \tau \neq 1$ is equivalent to compare the LSC and ELSC distributions. In this case, the LR statistic is given by

$$w = 2\{l(\hat{\mu}, \hat{\sigma}, \hat{\nu}, \hat{\tau}) - l(\tilde{\mu}, \tilde{\sigma}, \tilde{\nu}, 0)\},$$

where $\hat{\mu}$, $\hat{\sigma}$, $\hat{\nu}$ and $\hat{\tau}$ are the MLEs under H and $\tilde{\mu}$, $\tilde{\sigma}$ and $\tilde{\nu}$ are the estimates under H_0.

Table 1. The AEs, biases and MSEs based on 1000 simulations of the ELSC distribution for $\mu = 4$ and $\sigma = 0.1$, $\nu = 0.05, 0.6, 1.2$ and $\tau = 0.5, 1.5, 2$, and $n = 50, 150$ and 300.

		$\nu = 0.05$ and $\tau = 2$			$\nu = 0.6$ and $\tau = 2$			$\nu = 1.2$ and $\tau = 2$		
n	Parameter	AE	Bias	MSE	AE	Bias	MSE	AE	Bias	MSE
50	μ	4.001	0.001	0.001	2.913	−0.014	0.007	3.987	−0.013	0.003
	σ	0.097	−0.003	0.000	0.095	−0.005	0.001	0.099	−0.001	0.001
	ν	0.048	−0.002	0.001	0.635	0.035	1.371	1.321	0.121	0.433
	τ	2.050	0.050	0.143	2.913	0.913	42.345	2.884	0.884	7.379
150	μ	4.000	0.000	0.000	3.996	−0.004	0.003	3.989	−0.011	0.001
	σ	0.099	−0.001	0.000	0.098	−0.022	0.000	0.100	0.001	0.001
	ν	0.050	0.000	0.000	0.578	−0.022	0.026	1.209	0.009	0.093
	τ	2.014	0.014	0.045	2.181	0.181	1.051	2.368	0.368	1.044
300	μ	4.000	0.000	0.000	3.999	−0.001	0.002	3.996	−0.004	0.001
	σ	0.100	0.000	0.000	0.098	−0.002	0.000	0.100	0.001	0.001
	ν	0.050	0.000	0.000	0.580	−0.020	0.011	1.203	0.003	0.040
	τ	2.008	0.008	0.023	2.062	0.062	0.293	2.145	0.145	0.321
		$\nu = 0.05$ and $\tau = 1.5$			$\nu = 0.6$ and $\tau = 1.5$			$\nu = 1.2$ and $\tau = 1.5$		
n	Parameter	AE	Bias	MSE	AE	Bias	MSE	AE	Bias	MSE
50	μ	4.001	0.001	0.001	3.989	−0.011	0.006	3.990	−0.010	0.003
	σ	0.098	−0.002	0.001	0.097	−0.003	0.001	0.097	−0.003	0.001
	ν	0.050	0.001	0.001	0.581	−0.019	0.089	1.224	0.024	0.351
	τ	1.537	0.037	0.083	1.769	0.269	1.004	1.921	0.421	2.007
150	μ	4.001	0.001	0.001	3.995	−0.005	0.003	3.996	−0.004	0.001
	σ	0.099	−0.001	0.001	0.097	−0.003	0.001	0.101	0.001	0.001
	ν	0.050	0.001	0.001	0.578	−0.022	0.024	1.228	0.028	0.094
	τ	1.508	0.008	0.026	1.610	0.110	0.297	1.631	0.131	0.319
300	μ	4.000	0.001	0.001	3.998	−0.002	0.001	3.998	−0.002	0.001
	σ	0.100	0.001	0.001	0.099	−0.001	0.001	0.099	−0.001	0.001
	ν	0.050	0.001	0.001	0.583	−0.017	0.011	1.197	−0.003	0.040
	τ	1.508	0.008	0.013	1.550	0.050	0.129	1.562	0.062	0.107
		$\nu = 0.05$ and $\tau = 0.5$			$\nu = 0.6$ and $\tau = 0.5$			$\nu = 1.2$ and $\tau = 0.5$		
n	Parameter	AE	Bias	MSE	AE	Bias	MSE	AE	Bias	MSE
50	μ	3.998	−0.002	0.001	3.982	−0.018	0.008	4.003	0.003	0.003
	σ	0.097	−0.003	0.001	0.100	0.000	0.002	0.094	−0.006	0.002
	ν	0.049	−0.001	0.001	0.611	0.011	0.143	1.226	0.026	0.419
	τ	0.503	0.003	0.012	0.578	0.078	0.127	0.498	−0.002	0.075
150	μ	4.000	0.001	0.001	3.990	−0.010	0.003	4.006	0.006	0.001
	σ	0.099	−0.001	0.001	0.101	0.001	0.001	0.097	−0.003	0.001
	ν	0.049	−0.001	0.001	0.600	0.000	0.038	1.200	0.000	0.122
	τ	0.498	−0.002	0.004	0.538	0.038	0.040	0.485	−0.015	0.015
300	μ	4.000	0.001	0.001	3.996	−0.004	0.001	4.002	0.002	0.001
	σ	0.100	0.001	0.001	0.101	0.001	0.001	0.099	−0.001	0.001
	ν	0.050	0.001	0.001	0.602	0.002	0.018	1.205	0.005	0.054
	τ	0.500	0.001	0.002	0.516	0.016	0.015	0.493	−0.007	0.007

7. Simulation

We simulate the ELSC distribution (for $\mu = 4, \sigma = 0.1, \nu = 0.05, 0.6, 1.2$ and $\tau = 0.5, 1.5, 2$), considering bi-modality and unimodal forms, from Equation (7) by using a random variable U having a uniform distribution in $(0, 1)$. We take $n = 50$, 150 and 300 and, for each replication, we calculate the MLEs $\hat{\mu}$, $\hat{\sigma}$, $\hat{\nu}$ and $\hat{\tau}$. We repeat this process 1000 times and determine the average estimates (AEs), biases and means squared errors (MSEs). The results of the Monte Carlo study are given in Table 1. They indicate that the MSEs of the MLEs of μ, σ, ν and τ decay toward zero as the sample size increases, as expected under standard asymptotic theory.

We conclude from the figures in Table 1 that the AEs of the parameters tend to be closer to the true parameters when n increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. The normal approximation can be oftentimes improved by using bias adjustments to these estimators. Approximations to the their biases in simple models may be determined analytically. Bias correction typically does a very good job for correcting the MLEs. However, it may also increase the MSEs. Whether bias correction is useful in practice depends basically on the shape of the bias function and on the variance of the MLE. In order to improve the accuracy of these estimators using analytical bias reduction one needs to obtain several cumulants of log-likelihood derivatives, which are notoriously cumbersome for the proposed model. We illustrate the convergence in Figures 6 and 7, where the true densities are given at selected

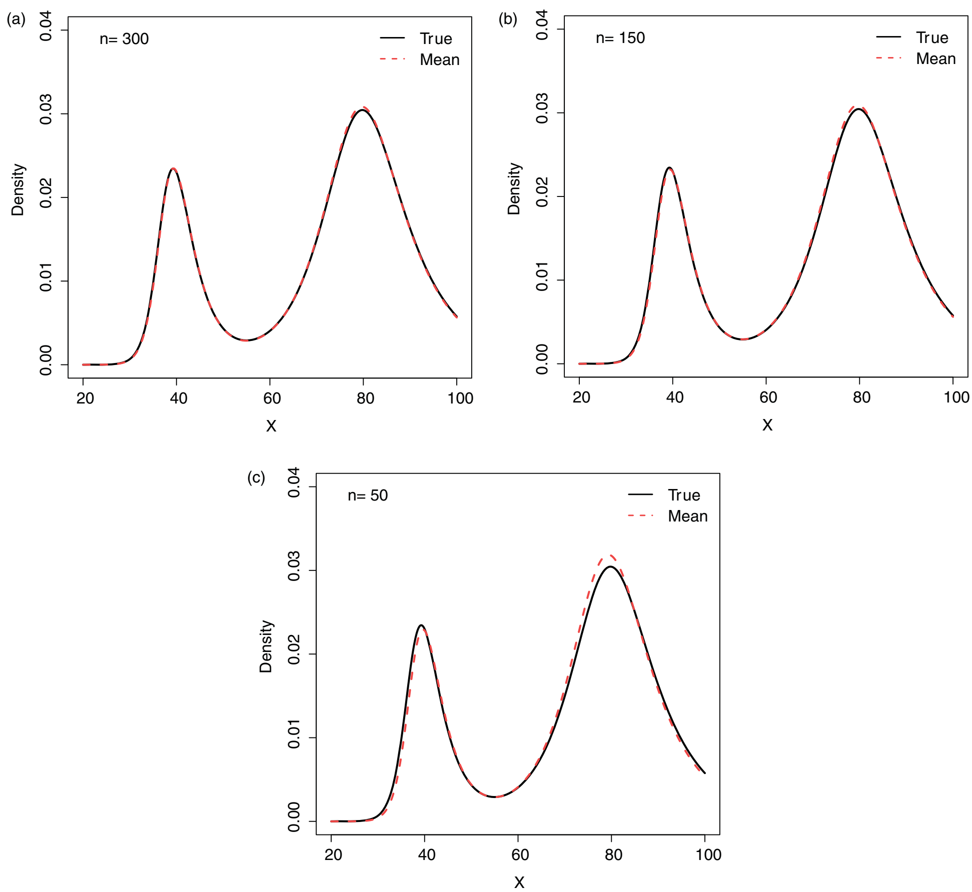

Figure 6. Some ELSC density functions at the true parameter values and at the AEs for $\mu = 4, \sigma = 0.1, \nu = 0.05$ and $\tau = 2$ when: (a) $n = 50$; (b) $n = 150$; (c) $n = 300$.

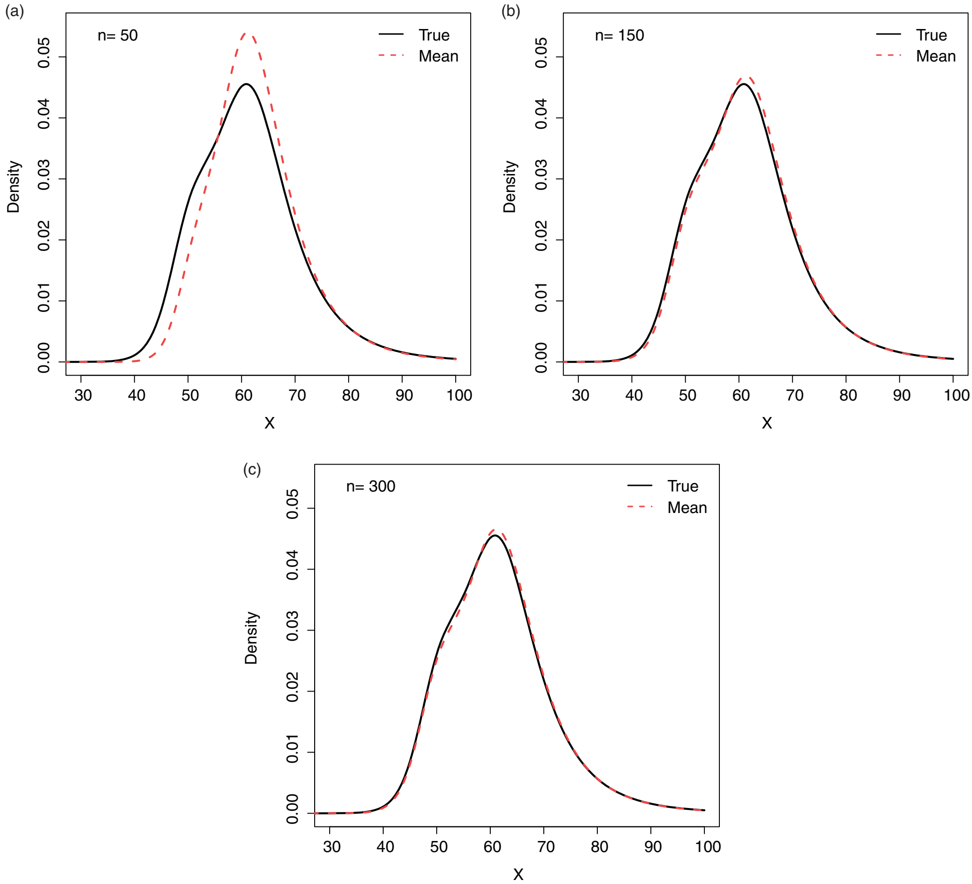

Figure 7. Some ELSC density functions at the true parameter values and at the AEs for $\mu = 4$, $\sigma = 0.1$, $\nu = 0.6$ and $\tau = 2$ when: (a) $n = 50$; (b) $n = 150$; (c) $n = 300$.

parameter values and the density functions are computed at the AEs given in Table 1 for some sample sizes and $\nu = 0.05$ and $\nu = 0.6$, respectively. In Figures 8 and 9, we present the estimated densities based on 1000 samples of the AEs of the parameters μ, σ, τ for $\nu = 0.05$ and $\nu = 0.6$, respectively, and $n = 50, 150$ and 300. These plots are in agreement with the standard asymptotic theory for the MLEs.

8. Applications

In this section, we provide three applications to real data to prove empirically the flexibility of the ELSC and LSC models. The computations are performed using the gamlss subroutine in the R software. In the first application, we give an application for bimodal data comparing the ELSC and LSC models with other models implemented in gamlss. In the second application, we show the flexibility of the distribution for censored data and, in the third application, we study the adequacy of the LSC model.

Recently, Cordeiro et al. [11] proposed the McDonald–Weibull (McW) model with scale parameter $\lambda > 0$, shape parameter $\gamma > 0$ and three extra shape parameters $a > 0$, $b > 0$ and $c > 0$. We focus on this model since it extends various distributions previously discussed in the lifetime literature, such as the beta Weibull (BW) [12] (for $c = 1$), Kumaraswamy Weibull (KwW) [13] (for $a = c$), exponentiated Weibull (EW) [14] (for $b = c = 1$), Weibull (for $a = b = c = 1$) and other distributions. Besides

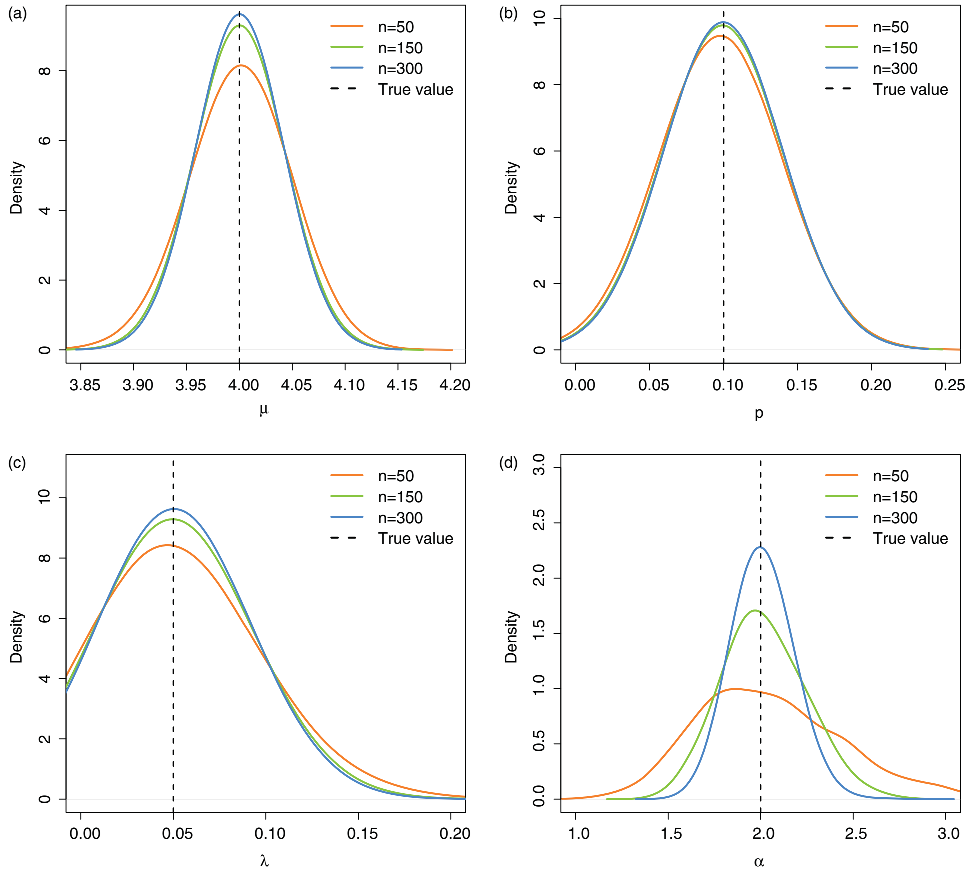

Figure 8. Estimated densities from 1000 samples for $n = 50, 150, 300$ of the parameters: (a) $\mu = 4$; (b) $\sigma = 0.1$; (c) $\nu = 0.05$; (d) $\tau = 2$ (based on selected parameter values in Table 1 for $\nu = 0.05$).

of its flexibility, the McW model can take bimodal forms and thus is a competitive model for the ELSC distribution.

All computations in this section are performed using the gamlss subroutine in R and the scripts are described in Section 9.

8.1. Eruption data

First, we provide an analysis of some data on the Old Faithful Geyser in Yellowstone National Park, Wyoming, USA. The data consist of $n = 299$ pairs of measurements referring to the times between the starts of successive eruptions. These data were collected continuously from 1 August until 15 August 1985; see [15] for more details.

We compute the Hartigans' Dip statistic D and its p-value for the test for unimodality. For i.i.d. random variables, the null hypothesis is that X_i has a unimodal distribution. Consequently, the alternative hypothesis is non-unimodal, that is, at least bimodal. The Dip test can be obtained using a function `dip.test` available in 'diptest' R package. More details about the dip test can be obtained in [16]. Applying the Dip test to verify that a unimodal distribution would be appropriate to fit the eruption data gives $D = 0.039$ with the p-value 0.002. So, we reject the null hypothesis in favour of a bimodal distribution.

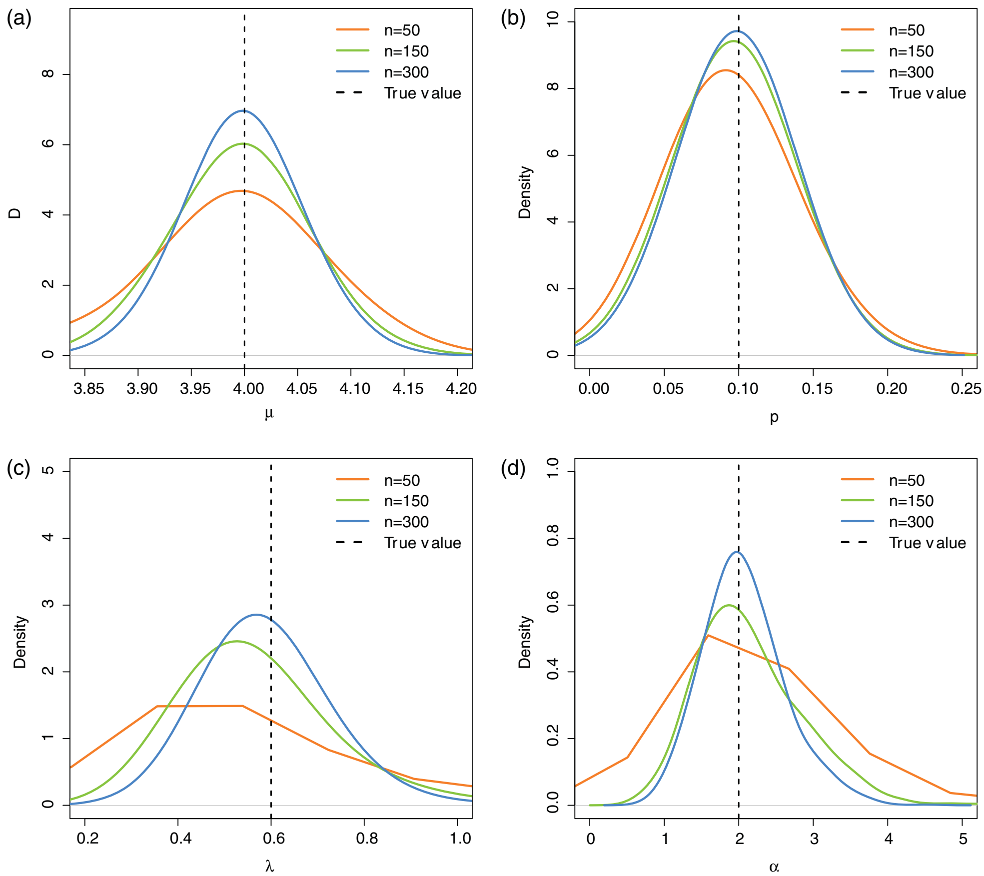

Figure 9. Estimated densities from 1000 samples for $n = 50, 150, 300$ of the parameters: (a) $\mu = 4$; (b) $\sigma = 0.1$; (c) $\nu = 0.6$; (d) $\tau = 2$ (based on selected parameter values in Table 1 for $\nu = 0.6$).

Table 2. MLEs of the model parameters for the eruption data, the corresponding SEs and the AIC and BIC statistics.

Model	μ	σ	ν	τ	AIC	BIC	W^*	A^*
ELSC	4.153	0.069	0.089	1.728	2328.23	2343.03	0.08	0.70
	(0.008)	(0.056)	(0.193)	(0.078)				
LSC	4.193	0.065	0.101	–	2368.26	2379.36	0.32	2.18
	(0.007)	(0.057)	(0.201)	–				
BCPEo	70.675	0.191	0.966	4.973	2387.22	2402.02	0.82	4.36
	(0.014)	(0.032)	(0.271)	(0.143)				

Further, we compare the fits of the ELSC and LSC models with the models available in the gamlss.family package. The `fitDist(..., type=c('realplus'))` function is used to fit all relevant parametric distributions. The Box–Cox power exponential (BCPEo) distribution is selected as the best model. For details on the distributions available in the package, see [17]. Table 2 lists the MLEs (and the corresponding standard errors in parentheses) of the model parameters and the values of the Akaike information criterion (AIC) and Bayesian information criterion (BIC) statistics for the fitted models.

We also evaluate the Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics described by Chen and Balakrishnan.[18] From a random sample $x_1, \ldots, x_n$ with empirical distribution function $F_n(x)$, the main objective is to test if the sample comes from a specific distribution. The W^* and A^*

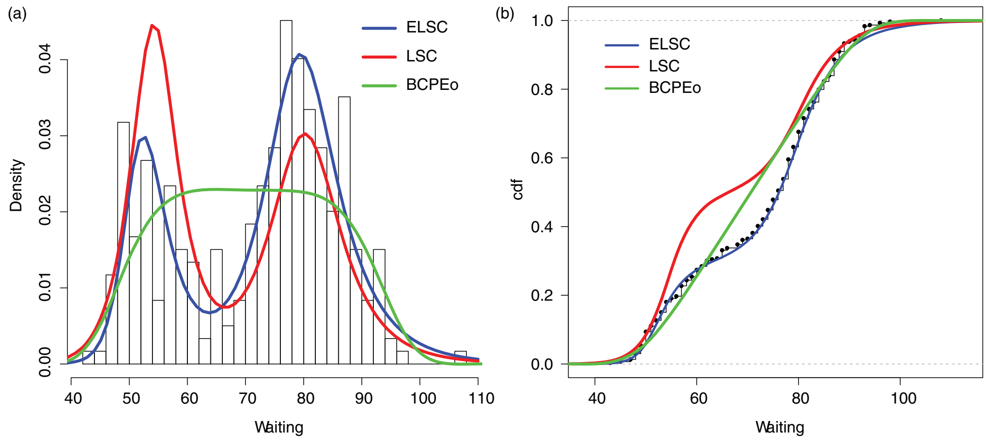

Figure 10. Estimated (a) densities and (b) cdfs for the ELSC, LSC and BCPEo models fitted to the eruption data.

statistics are given by

$$W^* = \left(n \int_{-\infty}^{+\infty} \{F_n(x) - F(x; \hat{\boldsymbol{\gamma}}_n)\}^2 \, \mathrm{d}F(x; \hat{\boldsymbol{\gamma}}_n)\right)\left(1 + \frac{0.5}{n}\right) = W^2\left(1 + \frac{0.5}{n}\right),$$

$$A^* = \left(n \int_{-\infty}^{+\infty} \frac{\{F_n(x) - F(x; \hat{\boldsymbol{\gamma}}_n)\}^2}{\{F(x; \hat{\boldsymbol{\gamma}})(1 - F(x; \hat{\boldsymbol{\gamma}}_n))\}} \, \mathrm{d}F(x; \hat{\boldsymbol{\gamma}}_n)\right)\left(1 + \frac{0.75}{n} + \frac{2.25}{n^2}\right),$$

$$= A^2\left(1 + \frac{0.75}{n} + \frac{2.25}{n^2}\right),$$

respectively, where $F_n(x)$ is the empirical distribution function and $F(x; \hat{\boldsymbol{\gamma}}_n)$ is the postulated distribution function evaluated at the MLE $\hat{\boldsymbol{\gamma}}_n$ of $\boldsymbol{\gamma}$. The W^* and A^* statistics measure the differences of $F_n(x)$ and $F(x; \hat{\boldsymbol{\gamma}}_n)$. Thus, the lower their values, the more evidence that $F(x; \hat{\boldsymbol{\gamma}}_n)$ generates the sample.

The figures in Table 2 indicate that the ELSC model has the lowest AIC and BIC values among those values of the fitted models, and therefore it could be chosen as the best model. Further, the SEs of the estimates for all fitted models are quite small.

Formal tests for the extra skewness parameters in the ELSC model can be based on the LR statistic described in Section 6. Applying the LR statistic to the eruption data, we reject the null hypothesis $H_0 : \tau = 1$ in favour of the ELSC distribution. The value of the LR statistic is $w = 42.032$ with the p-value < 0.001.

More information is provided by a visual comparison of the histogram of the data with the fitted density functions. The plots of the fitted ELSC, LSC and BCPEo densities and their cdfs are displayed in Figure 10. The plot of the ELSC hazard rate in Figure 11 reveals that this function has a bimodal shape, small at the first mode and large at the second mode.

8.2. Efron data

Second, we consider the data from a two-arm clinical trial discussed earlier by Efron.[19] Efron noted that the empirical hazard functions for both samples start near zero, suggesting an initial high-risk period at the beginning, a decline for a while, and then stabilization after about one year. He developed and illustrated a methodology for analysing the data using a combination of techniques of quantal response analysis and the spline regression methods. Specifically, Efron's data from a head and neck cancer clinical trial consist of survival times of 51 patients in arm A who were given radiation therapy

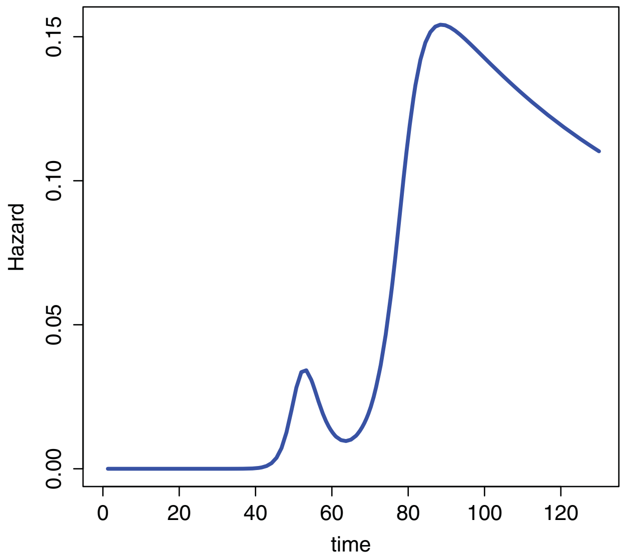

Figure 11. Estimated hrf for the ELSC distribution for eruption data.

Table 3. MLEs of the model parameters for Efron data, the corresponding SEs (given in parentheses) and the AIC and BIC statistics.

Model	μ	σ	ν	τ		AIC	BIC
ELSC	4.788	2.080	2.794	2.308		1063.9	1074.1
	(0.083)	(0.135)	(0.129)	(0.097)			
LSC	6.141	0.494	0.215	1		1074.4	1082.1
	(0.102)	(0.061)	(0.151)	–			
	λ	γ	a	b	c	AIC	BIC
McW	0.092	0.101	74.352	21.126	0.067	1088.5	1101.3
	(0.028)	(0.008)	(0.655)	(0.192)	(0.001)		
BW	0.281	0.062	167.450	60.159	1	1086.1	1096.3
	(0.106)	(0.005)	(0.406)	(0.177)	–		

and 45 patients in arm B who were given radiation plus chemotherapy. Nine patients in arm A and 14 patients in arm B were lost to follow-up and were regarded as censored.

Cordeiro et al. [11] fitted the McW regression model to these data and noted that it provides a good fit. Here, we consider only the survival times in days x_i and compare the results of the fits of the McW, ELSC and LSC models. Table 3 gives the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the AIC and BIC statistics. They indicate that the ELSC model has the lowest values of these statistics among the values of the other fitted models, and therefore it could be chosen as the best model.

By comparing the fits of the ELSC and LSC models using the LR statistic, we reject the null hypothesis $H_0 : \tau = 1$ in favour of the ELSC distribution. The LR statistic is $w = 12.552$ with the p-value < 0.001. Next, we compare the fits of the McW and BW models using the LR statistic. Applying the LR statistic for testing the null hypothesis $H_0 : c = 1$, we obtain $w = 0.00039$ with the p-value almost one. So, we could not reject the BW distribution to fit these data.

The plots of the fitted ELSC, LSC and BW densities and their estimated survival functions are displayed in Figure 12 for the current data ignoring censored observations. Clearly, the ELSC density provides a closer fit to the histogram of the data and the corresponding estimated survival function to the empirical survival function than the other models. The plot of the ELSC hrf in Figure 13 reveals that it has a modal shape.

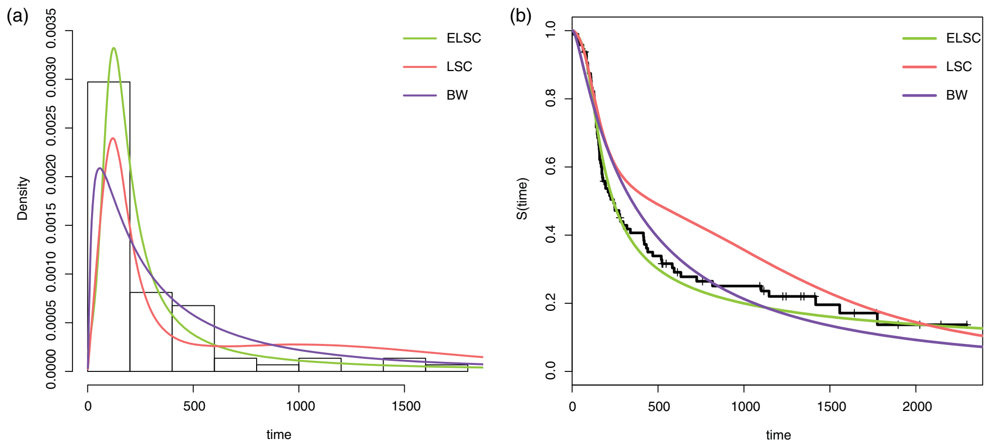

Figure 12. (a) Estimated ELSC, LSC and BW densities for Efron data. (b) Estimated ELSC and LSC survival functions and the empirical survival for Efron data.

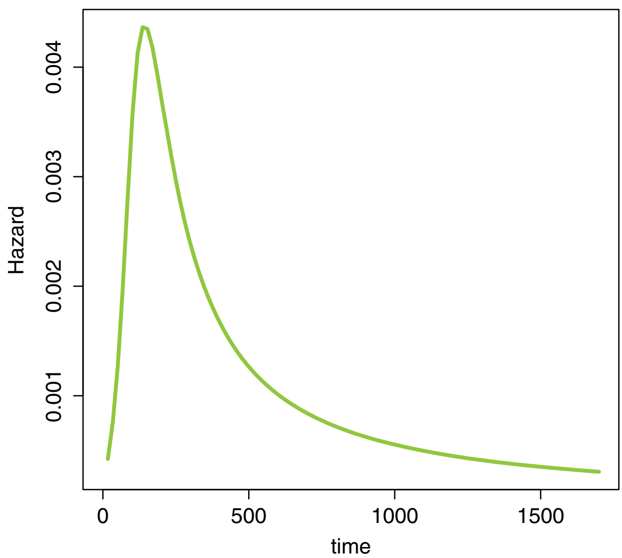

Figure 13. Estimated ELSC hazard function for Efron data.

8.3. Entomology data

Third, we consider the data from a study carried out at the Department of Entomology of the Luiz de Queiroz School of Agriculture, University of São Paulo, which aim to assess the longevity of the mediterranean fruit fly (ceratitis capitata). The need for this fly to seek food just after emerging from the larval stage has permitted the use of toxic baits for its management in Brazilian orchards for at least 50 years. This pest control technique consists of using small portions of food laced with an insecticide, generally an organophosphate, that quickly kills the flies, instead of using an insecticide alone. Recently, there have been reports of the insecticidal effect of extracts of the neem tree leading to proposals to adopt various extracts (aqueous extract of the seeds, methanol extract of the leaves and dichloromethane extract of the branches) to control pests such as the mediterranean fruit fly. For more details, see [20].

Table 4. MLEs of the model parameters for the entomology data, the corresponding SEs (given in parentheses) and the AIC and BIC statistics.

Model	μ	σ	ν	τ		AIC	BIC
ELSC	3.018	0.852	3.367	0.907		1249.0	1261.5
	(0.027)	(0.091)	(0.107)	(0.075)			
LSC	2.998	0.946	3.592	1		1247.7	1257.1
	(0.029)	(0.101)	(0.106)	–			
	s	c	k	p		AIC	BIC
BXIIGII	14.353	1.164	4.414	0.981		1270.1	1282.7
	(8.175)	(0.389)	(2.532)	(0.0211)			
BXII	34.423	2.214	2.676	1		1282.7	1292.1
	(10.386)	(0.232)	(1.284)	–			
	λ	γ	a	b	c	AIC	BIC
McW	0.079	1.718	0.883	0.329	0.049	1290.0	1305.8
	(0.007)	(0.223)	(0.313)	(0.114)	(0.013)		
BW	0.055	1.608	1.240	0.688	1	1289.7	1302.3
	(0.017)	(0.226)	(0.314)	(0.313)	–		
KwW	0.015	1.133	1	8.787	1.776	1288.9	1301.5
	(0.004)	(0.447)	–	(0.299)	(0.920)	–	
EW	0.044	1.587	1.254	1	1	1287.5	1296.9
	(0.007)	(0.275)	(0.368)	–	–		
Weibull	0.0400	1.797	1	1	1	1286.1	1292.4
	(0.002)	(0.111)	–	–	–		

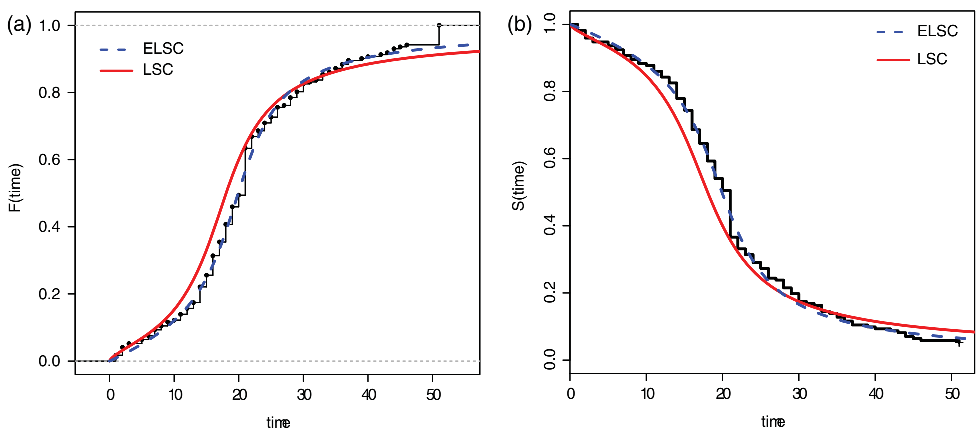

Figure 14. (a) Estimated ELSC and LSC cdfs for entomology data. (b) Estimated ELSC and LSC survival functions and the empirical survival for the entomology data.

The response variable in the experiment is the lifetime of the adult flies in days after exposure to the treatments. The experimental period was set at 51 days, so that the numbers of larvae that survived beyond this period are considered as censored observations. The total sample size is $n = 72$ because four cases are lost. Therefore, the variables used in this study are: x_i-lifetime of ceratitis capitata adults in days and δ_i-censoring indicator.

Recently, Lanjoni [21] fitted the Burr XII geometric type II (BXIIGII) distribution to these data and noted that it gives a better fit than the special Burr XII model. Now, we compare the McW and BXIIGII distributions and some of their sub-models with the ELSC and LSC models. For some fitted models, Table 4 provides the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the AIC and BIC statistics. The computations are performed using the gamlss subroutine in R. They indicate that the LSC model has the lowest AIC and BIC values among

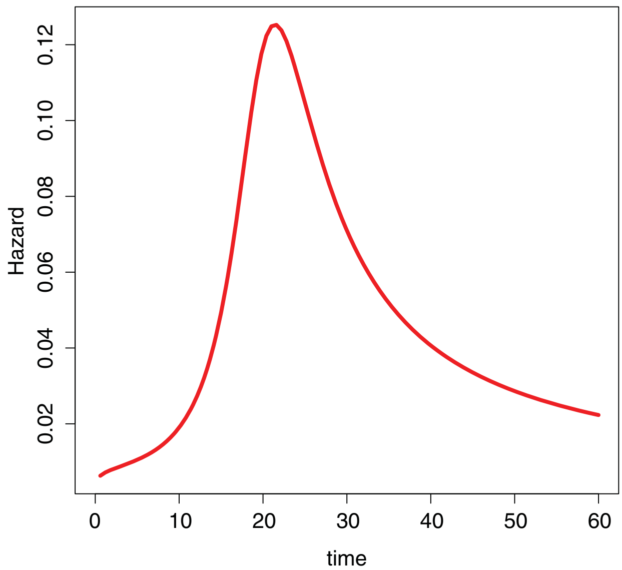

Figure 15. Estimated LSC hazard function for entomology data.

those values of the fitted models, and therefore it could be chosen as the best model. The LSC model is not able to capture asymmetry but it has the bi-modality characteristic.

In order to assess if the model is appropriate, Figure 14(a) displays the empirical and estimated cumulative distributions for the fitted ELSC and LSC models to the current data. Further, Figure 14(b) gives the plots of the empirical survival function and the estimated ELSC and LSC survival functions. They indicate the LSC model provides a good fit to these data. Further, using the LR statistic to compare the fits of these models, that is, for testing the null hypothesis $H_0 : \tau = 1$, we obtain $w = 0.748$ with the p-value$= 0.387$ and then we could accept the LSC distribution. The plot of its hrf in Figure 15 reveals a modal shape.

9. Program description

The ELSC model is implemented in the gamlss function, which is fully documented in the gamlss package.[3] Here, we will omit several functions for the gamlss package and present only the functions related to the ELSC distribution and its fit to a data set. The computational codes for the ELSC model can be downloaded from http://goo.gl/yzvoIZ. The cdf (5) and pdf (6) can be obtained using `dELSC` and `pELSC` functions, respectively. The qf given by Equation (7) can be obtained using the `qELSC` function and samples of the ELSC model can be generated using the `rELSC` function. We can use the functions listed above for the LSC sub-model by setting $\tau = 1$ with the `tau.fix=TRUE` function. To optimize the computational time, we can change the initial values of the parameters using the `parameter.fix` function. Otherwise, we can increase the number of interactions using the `n.cyc` function. The fit of the ELSC model to censored data can be performed using the additional package gamlss.cens. The structure of the gamlss function is familiar to users of the R syntax (the glm function, in particular).

10. Conclusions

The paper proposes the ELSC distribution that can be used as an alternative to mixture distributions in modelling bimodal data. Various mathematical properties of the ELSC distribution are investigated. We show that it can accommodate various shapes of the skewness, kurtosis and bi-modality.

Its model parameters are estimated by maximum likelihood. Some numerical experiments reveal that the maximum likelihood estimation procedure performs well. Three real data examples prove empirically that the ELSC distribution is very flexible, parsimonious, and a competitive model that deserves to be added to existing distributions in modelling bimodal data. The ELSC model can be fitted using the gamlss package described to facilitate its practical use by researchers from other areas.

Notes

1. http://en.wikipedia.org/wiki/Bell_polynomials
2. http://mathworld.wolfram.com/HypergeometricFunction.html

Disclosure statement

No potential conflict of interest was reported by the authors.

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