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Existence and Uniqueness of Pseudo Almost Periodic Solutions to Some Abstract Partial Neutral Functional-Differential Equations

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Existência e Unicidade de Soluções Pseudo Quase Periodicas para uma Equação Diferencial Funcional Abstrata do Tipo Neutro

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Resumo

Neste artigo estudamos a existência e unicidade de soluções fracas *pseudo quase* periodicas para as equações abstratas do tipo neutro

$$
\begin{aligned}
\frac{d}{dt}(u(t) + f(t, u_t)) &= Au(t) + g(t, u_t), \\
\frac{d}{dt}[u(t) + f(t, u(t - \gamma_1(t)))] &= Au(t) + g(t, u(t - \gamma_2(t))),
\end{aligned}
$$

onde A é gerador infinitesimal de um C_0-semigrupo de operadores lineares limitados definido sobre um espaço de Banach X e f, g, γ_i são funções apropriadas.

Existence and Uniqueness of Pseudo Almost Periodic Solutions to Some Abstract Partial Neutral Functional-Differential Equations.

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Abstract

This paper is mainly concerned with the existence and uniqueness of pseudo almost periodic (mild) solutions to some classes of first-order partial neutral functional-differential equations. Under some suitable assumptions, existence and uniqueness results are obtained. As an illustration, two relevant examples are discussed.

1 Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. In this paper we study the original problem which consists of the existence and uniqueness of *pseudo almost periodic* (mild) solutions to the following abstract partial neutral functional-differential equations

$$\frac{d}{dt}(u(t) + f(t, u_t)) = Au(t) + g(t, u_t), \quad \text{and} \tag{1}$$

$$\frac{d}{dt}[u(t) + f(t, u(t - \gamma_1(t)))] = Au(t) + g(t, u(t - \gamma_2(t))), \tag{2}$$

where A is the infinitesimal generator of an uniformly exponentially stable semigroup of linear operators on $\mathbb{X}$, the history $u_t \in C([-p, 0]; \mathbb{X})$ and f, g, γ_i $(i = 1, 2)$ are some suitable functions to be defined later on in the text.

The existence of almost periodic, asymptotically almost periodic, and pseudo almost periodic solutions is one of the most attracting topics in the qualitative theory of differential equations due to their significance and applications in physical sciences.

The concept of the pseudo almost periodicity (p.a.p. for short) which is the central questioning in this paper was first initiated by C. Y. Zhang in [34, 35, 36]– and it is a natural generalization of the well-known (Bochner) almost periodicity (a.p. for short). Thus this new concept is welcome to implement another existing generalization of the (Bochner) almost periodicity, the notion of asymptotically almost periodicity (a.a.p.) due to Fréchet, see, e.g., [5], [9], and [22]. One should mention however that the concept of the pseudo almost periodicity is a special case of the well-known Besicovitch almost periodicity, that is, the space $B_1(\mathbb{R}, \mathbb{X})$, see, e.g., [23]. For more on these and related issues, see, e.g., [1], [2], [3], [4], [6], [7], [19], [25], [34] and the references therein.

Some recent contributions within the abstract differential and partial differential equations frameworks have been made in [16, 17, 18, 7, 6, 4, 8, 19]. Existence results concerning almost periodic and asymptotically almost periodic solutions to ordinary neutral differential equations and abstract partial neutral differential equations have recently been established in [21, 27, 15]. However, the existence of pseudo almost periodic to functional-differential equations with delay, especially, abstract partial neutral differential equations, is an untreated topic and this is the main motivation of the present paper.

Neutral differential equations arise in many areas of applied mathematics and for this reason they have been of a great interest during the last few decades. The literature relative to ordinary neutral differential equations is very extensive, and hence for more on this and related issues we only refer the reader to [10]. Similarly, for more on partial neutral functional differential equations we refer to Hale [11], Wu [29, 30, 31], Adimy[1] for finite delay equations, and Hernández and Henriquez [12, 13] and Hernández [14] for unbounded delays.

In what follows we recall some definitions and notations that we need in the sequel.

From now on, $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ denotes the the infinitesimal generator of an uniformly asymptotically stable semigroup of linear operators $(T(t))_{t\geq 0}$ on $\mathbb{X}$ and M, w are positive constants such that

$$\| T(t) \| \leq Me^{-wt}, \quad t \geq 0.$$

To deal with pseudo almost periodic solutions we will need to introduce some classical and new concepts. In what follows, $(\mathbb{Z}, \|\cdot\|_{\mathbb{Z}})$ and $(\mathbb{W}, \|\cdot\|_{\mathbb{W}})$ stand for Banach spaces. In addition to that $C(\mathbb{R}, \mathbb{Z})$ and $BC(\mathbb{R}, \mathbb{Z})$ denote the collection of continuous functions, and the Banach space of bounded continuous functions from $\mathbb{R}$ into $\mathbb{Z}$ equipped with the sup norm defined by $\|u\|_\infty := \sup_{t\in\mathbb{R}} \|u(t)\|$, respectively. (Similar definitions apply for both $C(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ and $BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$.) Throughout the rest of the paper the notation $B_r(x, Z)$ stands for the open ball with centered at z with radius $r > 0$ in $\mathbb{Z}$.

Definition 1 *A function $f \in C(\mathbb{R}, \mathbb{Z})$ is almost periodic (a.p. for short) if for each $\varepsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$ denoted by $\mathcal{H}(\varepsilon, f, Z)$ (i.e., there exists $\delta > 0$ such that $[a, a+\delta] \cap \mathcal{H}(\varepsilon, f, \mathbb{Z}) \neq \{\emptyset\}$ for each $a \in \mathbb{R}$) such that $\|f(t+\tau) - f(t)\|_{\mathbb{Z}} < \varepsilon$ for each $t \in \mathbb{R}$ and each $\tau \in \mathcal{H}(\varepsilon, f, \mathbb{Z})$.*

The collection of those functions will be denoted by $AP(\mathbb{Z})$.

The next Lemma is also a characterization of almost periodic functions.

Lemma 1 *[33, p. 25] A function $f \in C(\mathbb{R}, \mathbb{Z})$ is almost periodic if and only if the set of functions $\{\sigma_\tau f : \tau \in \mathbb{R}\}$, where $(\sigma_\tau f)(t) = f(t+\tau)$, is relatively compact in $C(\mathbb{R}, \mathbb{Z})$.*

Similarly, a function $F \in C(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ is almost periodic in $t \in \mathbb{R}$ uniformly in $z \in \mathbb{Z}$ if for each $\varepsilon > 0$ and for all compact $K \subset \mathbb{Z}$ there exists a relatively dense subset of $\mathbb{R}$ denoted by $\mathcal{H}(\varepsilon, F, K)$ such that $\|F(t+\tau, z) - F(t, z)\|_{\mathbb{W}} < \varepsilon$ for all $t \in \mathbb{R}$, $z \in K$, and $\tau \in \mathcal{H}(\varepsilon, F, K)$. The collection of those functions will be denoted $AP(\mathbb{Z}, \mathbb{W})$.

The notation $PAP_0(\mathbb{Z})$ stands for the space of functions

$$PAP_0(\mathbb{Z}) = \left\{ u \in BC(\mathbb{R}, \mathbb{Z}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \| u(t) \|_{\mathbb{Z}} \, dt = 0 \right\}.$$

To study issues related to delay we need to introduce the new space of functions defined for each $p \in \mathbb{R}$ by

$$PAP_0(\mathbb{Z}, p) := \left\{ u \in BC(\mathbb{R}, \mathbb{Z}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-p, t]} \| u(\theta) \|_{\mathbb{Z}} \right) dt = 0 \right\}.$$

In addition to the above-mentioned spaces, the present setting requires the introduction of the following function spaces

$$\begin{aligned} PAP_0(\mathbb{Z}, \mathbb{W}) &= \left\{ u \in BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \| u(t, z) \|_{\mathbb{W}} \, dt = 0 \right\}, \quad \text{and} \\ PAP_0(\mathbb{Z}, \mathbb{W}, p) &:= \left\{ u \in BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W}) : \lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-p, t]} \| u(\theta, z) \|_{\mathbb{W}} \right) dt = 0 \right\}, \end{aligned}$$

where in both cases the limit (as $r \mapsto \infty$) is uniform in $z \in \mathbb{Z}$.

In view of the previous definitions it is clear that $PAP_0(\mathbb{Z}, p)$ and $PAP_0(\mathbb{Z}, \mathbb{W}, p)$ are continuously embedded in $PAP(\mathbb{Z})$ and $PAP_0(\mathbb{Z}, \mathbb{W})$, respectively. Furthermore, it is not hard to see that $PAP_0(\mathbb{Z}, p)$ and $PAP_0(\mathbb{W}, Z, p)$ are closed in $PAP_0(\mathbb{Z})$ and $PAP_0(\mathbb{W}, \mathbb{Z})$, respectively. Consequently, using [19, Lemma 1.2], one obtains the following:

Lemma 2 *The spaces $PAP_0(\mathbb{Z}, p)$ and $PAP_0(\mathbb{W}, Z, p)$ endowed with the uniform convergence topology are Banach spaces.*

Definition 2 *A function $f \in BC(\mathbb{R}, \mathbb{Z})$ is called pseudo almost periodic (p.a.p.) if $f = g + \varphi$, where $g \in AP(\mathbb{Z})$ and $\varphi \in PAP_0(\mathbb{Z})$.*

The class of those functions will be denoted by $PAP(\mathbb{Z})$.

Definition 3 *A function $F \in BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ is called uniformly pseudo almost periodic (u.p.a.p.) if $F = G + \Phi$, where $G \in AP(\mathbb{Z}, \mathbb{W})$ and $\Phi \in PAP_0(\mathbb{Z}, \mathbb{W})$.*

The class of those functions will be denoted by $UPAP(\mathbb{Z}, \mathbb{W})$.

We need to introduce two new notions of pseudo almost periodicity that we will use in the sequel.

Definition 4 *A function $F \in BC(\mathbb{R}, \mathbb{Z})$ is called pseudo almost periodic of class p (p.a.p.p.) if $F = G + \varphi$, where $G \in AP(\mathbb{Z})$ and $\varphi \in PAP_0(\mathbb{Z}, p)$.*

The class of those functions will be denoted by $PAP(\mathbb{Z}, p)$.

Definition 5 *A function $F \in BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ is called uniformly pseudo almost periodic of class p (u.p.a.p.p.) if $F = G + \varphi$, where $G \in AP(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ and $\varphi \in PAP_0(\mathbb{Z}, \mathbb{W}, p)$.*

The class of those functions will be denoted by $UPAP(\mathbb{Z}, \mathbb{W}, p)$.

This paper is organized as follows. In Section 1 we prove some preliminaries results related to the composition of pseudo almost periodic functions of class p. The existence of pseudo almost periodic solutions for the neutral systems (1) and (2) will be investigated in Sections 3 and 4. Finally, in Section 5 we consider some applications.

2 Preliminaries

Our main results on the existence of pseudo almost periodic solutions require some preliminaries results related to the composition of pseudo almost periodic functions of class p. Basically, those results are inspired from ideas and estimates given in [19]. Thus, for the sake of clarity, proofs of those results will be given.

Throughout this section we consider Banach spaces $(\mathbb{Z}, \|\cdot\|_{\mathbb{Z}})$, $(\mathbb{W}, \|\cdot\|_{\mathbb{W}})$, previously introduced.

Theorem 1 *Let $F \in PAP(\mathbb{Z}, \mathbb{W})$ and let $h \in PAP(\mathbb{W}, p)$. Assume that there exists a function $L_F : \mathbb{R} \mapsto [0, \infty)$ satisfying*

$$\|F(t, z_1) - F(t, z_2)\|_{\mathbb{W}} \leq L_F(t) \|z_1 - z_2\|_{\mathbb{Z}}, \quad \forall t \in \mathbb{R}, \ \forall z_1, z_2 \in \mathbb{Z}. \tag{3}$$

If

$$\limsup_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-p,t]} L_F(\theta) \right) dt < \infty, \quad \text{and} \tag{4}$$

$$\lim_{r \to \infty} \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-p,t]} L_F(\theta) \right) \xi(t) dt = 0 \tag{5}$$

for each $\xi \in PAP(\mathbb{R})$, then the function $t \mapsto F(t, h(t))$ belongs to $PAP(\mathbb{W}, p)$.

Proof: Assume that $F = F_1 + \varphi$, $h = h_1 + h_2$, where $F_1 \in AP(\mathbb{Z}, \mathbb{W})$, $\varphi \in PAP(\mathbb{Z}, \mathbb{W}, p)$, $h_1 \in AP(\mathbb{Z})$ and $h_2 \in AP(\mathbb{Z}, p)$. Consider the decomposition

$$F(t, h(t)) = F_1(t, h_1(t)) + [F(t, h(t)) - F(t, h_1(t)))] + \varphi(t, h_1(t)).$$

Since $F_1(\cdot, h_1(\cdot)) \in AP(\mathbb{W})$, it remains to prove that both $[F(\cdot, h(\cdot)) - F(\cdot, h_1(\cdot)))]$ and $\varphi(\cdot, h_1(\cdot))$ belong to $PAP(\mathbb{W}, p)$. Using (3) above it follows that

$$\begin{aligned} \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-p,t]} \| F(\theta, h(\theta)) - F(\theta, h_1(\theta))) \| \right) dt & \\ \leq \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-p,t]} L_F(\theta) \| h_2(\theta) \| \right) dt & \\ \leq \frac{1}{2r} \int_{-r}^{r} (\sup_{\theta \in [t-p,t]} L_F(\theta)) \, . \, (\sup_{\theta \in [t-p,t]} \| h_2(\theta) \|) \, dt. & \end{aligned}$$

which implies that $[F(\cdot, h(\cdot)) - F(\cdot, h_1(\cdot)))] \in PAP(\mathbb{W}, p)$, by (5).

Since $h_1(\mathbb{R})$ is relatively compact in $\mathbb{Z}$ and F_1 is uniformly continuous on sets of the form $\mathbb{R} \times K$ with $K \subset \mathbb{Z}$ being compact, for $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ such that

$$\| F_1(t, z) - F_1(t, \bar{z}) \| \leq \epsilon, \qquad z, \bar{z} \in h_1(\mathbb{R})$$

with $\| z - \bar{z} \| < \delta$.

Now, fix $z_1, \ldots, z_n \in h_1(\mathbb{R})$ such that $h_1(\mathbb{R}) \subset \bigcup_{i=1}^{n} B_\delta(z_i, \mathbb{Z})$. Obviously, the sets $E_i = h_1^{-1}(B_\delta(z_i))$ form an open covering of $\mathbb{R}$, and therefore using the sets $B_1 = E_1$, $B_2 = E_2 \backslash E_1$ and $B_i = E_i \setminus \bigcup_{j=1}^{i-1} E_j$ one obtains a covering of $\mathbb{R}$ by disjoint open sets.

For $t \in B_i$, $h_1(t) \in B_\delta(z_i)$

$$\begin{aligned}
\| \varphi(t, h_1(t)) \| &\leq \| F(t, h_1(t)) - F(t, z_i) \| + \| -F_1(t, h_1(t)) + F_1(t, z_i) \| + \| \varphi(t, z_i) \| \\
&\leq L_F(t) \| h_1(t) - z_i \| + \epsilon + \| \varphi(t, z_i) \| \\
&\leq L_F(t)\epsilon + \epsilon + \| \varphi(t, x_i) \| .
\end{aligned}$$

Using the previous inequality it follows that

$$\begin{aligned}
&\frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-p,t]} \| \varphi(t, h_1(t)) \| \right) dt \\
\leq\ & \frac{1}{2r} \sum_{i=1}^{n} \int_{B_i \cap [-r,r]} \left(\sup_{\theta \in [t-p,t]} \| \varphi(\theta, h_1(\theta)) \| \right) dt \\
\leq\ & \frac{1}{2r} \sum_{i=1}^{n} \int_{B_i \cap [-r,r]} \left(\sup_{j=1,..n} \left[\sup_{\theta \in [t-p,t] \bigcap B_j} \| \varphi(\theta, h_1(\theta)) \| \right] \right) dt \\
\leq\ & \frac{1}{2r} \sum_{i=1}^{n} \int_{B_i \cap [-r,r]} \left(\sup_{j=1,..n} \left[\sup_{\theta \in [t-p,t] \bigcap B_j} \| F(\theta, h_1(\theta)) - F(\theta, z_j) \| \right] \right) dt \\
& + \frac{1}{2r} \sum_{i=1}^{n} \int_{B_i \cap [-r,r]} \left(\sup_{j=1,..n} \left[\sup_{\theta \in [t-p,t] \bigcap B_j} \| F_1(\theta, h_1(\theta)) - F_1(\theta, z_j) \| \right] \right) dt \\
& + \frac{1}{2r} \sum_{i=1}^{n} \int_{B_i \cap [-r,r]} \left(\sup_{j=1,..n} \left[\sup_{\theta \in [t-p,t] \bigcap B_j} \| \varphi(\theta, z_j) \| \right] \right) dt \\
\leq\ & \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-p,t]} L_F(\theta)\epsilon + \epsilon \right) dt + \sum_{i=1}^{n} \frac{1}{2r} \int_{-r}^{r} \left(\sup_{\theta \in [t-p,t]} \| \varphi(\theta, z_j) \| \right) dt
\end{aligned}$$

which clearly shows that $\varphi(\cdot, h_1(\cdot)) \in PAP(\mathbb{W}, p)$ and completes the proof. ∎

Remark 1 *Note that assumptions (4) and (5) are verified by many functions. Examples include constants functions, functions in* $PAP(\mathbb{R}, p)$*, and functions of* $L^1(\mathbb{R})$ *which are decreasing on* $[0, \infty)$ *and nondecreasing on* $[-\infty, 0)$ *among others.*

To study the system (1) we need the following result:

Theorem 2 *If* $u \in PAP(\mathbb{Z}, p)$*, then* $t \to u_t$ *belongs to* $PAP(C([-p, 0], \mathbb{Z}), p)$.

Proof: Assume that $u = h + g$ where $h \in AP(\mathbb{Z})$ and $g \in PAP(\mathbb{Z}, p)$. Clearly, $u_t = h_t + g_t$ and from Lemma 1 we infer that h_t is almost periodic . On the other hand, for $r > 0$ we see that

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}\left[\sup_{\theta\in[t-p,t]}\left(\sup_{\xi\in[-p,0]}\| g(\theta+\xi)\|\right)\right]dt\\
&\quad\leq \frac{1}{2r}\int_{-r}^{r}\left(\sup_{\theta\in[t-2p,t]}\| g(\theta)\|\right)dt\\
&\quad\leq \frac{1}{2r}\int_{-r-p}^{r-p}\left(\sup_{\theta\in[t-p,t]}\| g(\theta)\|+\sup_{\theta\in[t,t+p]}\| g(\theta)\|\right)dt\\
&\quad\leq \frac{1}{2r}\int_{-r-p}^{r-p}\left(\sup_{\theta\in[t-p,t]}\| g(\theta)\|\right)dt+\frac{1}{2r}\int_{-r-p}^{r-p}\left(\sup_{\theta\in[t,t+p]}\| g(\theta)\|\right)dt\\
&\quad\leq \frac{r+p}{r}\frac{1}{2(r+p)}\int_{-r-p}^{r+p}\left(\sup_{\theta\in[t-p,t]}\| g(\theta)\|\right)dt+\frac{1}{2r}\int_{-r}^{r}\left(\sup_{\theta\in[t-p,t]}\| g(\theta)\|\right)dt
\end{aligned}$$

which enables to complete the proof. ∎

Our existence results require the next theorem.

Theorem 3 *Let $u\in PAP_0(\mathbb{Z},p)$. If v is the function defined by*

$$v(t):=\int_{-\infty}^{t}T(t-s)u(s)ds,\quad \forall t\in\mathbb{R},$$

then $v\in PAP_0(\mathbb{Z},p)$.

Proof: For $r>0$ we get

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}\left(\sup_{\theta\in[t-p,t]}\int_{-\infty}^{\theta}\| T(\theta-s)\|\| u(s)\| ds\right)dt\\
&\leq \frac{1}{2r}\int_{-r}^{r}\left(\sup_{\theta\in[t-p,t]}\int_{-\infty}^{\theta}e^{-w(\theta-s)}\| u(s)\| ds\right)dt\\
&\leq \frac{1}{2r}\int_{-r}^{r}\left(\sup_{\theta\in[t-p,t]}e^{wp}\int_{-\infty}^{\theta}e^{-w(t-s)}\| u(s)\| ds\right)dt\\
&\leq \frac{1}{2r}\int_{-r}^{r}\left(\sup_{\theta\in[t-p,t]}e^{wp}\int_{-\infty}^{t}e^{-w(t-s)}\| u(s)\| ds\right)dt\\
&\leq \frac{e^{wp}}{2r}\int_{-r}^{r}\int_{-\infty}^{t}e^{-w(t-s)}\| u(s)\| dsdt\\
&\leq \frac{e^{wp}}{2r}\int_{-\infty}^{-r}\int_{-r}^{r}e^{-w(t-s)}\| u(s)\| dtds+\frac{e^{wp}}{2r}\int_{-r}^{r}\int_{s}^{r}e^{-w(t-s)}\| u(s)\| dtds\\
&\leq \frac{e^{wp}\| u\|_\infty}{2rw}\int_{-\infty}^{-r}e^{w(s+r)}ds+\frac{e^{wp}}{2rw}\int_{-r}^{r}\| u(s)\| dt.
\end{aligned}$$

Consequently,

$$\frac{1}{2r}\int_{-r}^{r}\left(\sup_{\theta\in[t-p,t]} v(\theta)\right) dt \;\leq\; \frac{e^{wp} \parallel u \parallel_{\infty}}{2rw^2} + \frac{e^{wp}}{2r}\int_{-r}^{r} \parallel u(s) \parallel dt$$

which converges to zero as $r \longrightarrow \infty$. The proof is now complete. ■

3 Existence Results for a Neutral System with Finite Delay

This section is devoted to the existence and uniqueness of pseudo almost periodic solutions to the neutral system

$$\frac{d}{dt}(u(t) + f(t, u_t)) = Au(t) + g(t, u_t), \qquad t \in [\sigma, \sigma + a) \tag{6}$$

$$u_\sigma = \phi \in \mathcal{B} = C([-p, 0]; \mathbb{X}). \tag{7}$$

From Hernández and Henríquez [12] we adopt the concept of mild solution for the systems (6)-(7).

Definition 6 *A continuous function $u : [\sigma, \sigma + a) \longrightarrow \mathbb{X}$, $a > 0$, is a mild solution of the neutral system (6)-(7) on $[\sigma, \sigma + a)$, if the function $s \longrightarrow AT(t - s)f(s, u_s)$ is integrable on $[0, t)$ for every $\sigma < t < \sigma + a$ and*

$$\begin{aligned} u(t) &= T(t-\sigma)(\varphi(\sigma) + f(\sigma,\varphi)) - f(t,u_t) - \int_\sigma^t AT(t-s)f(s,u_s)ds \\ &\quad + \int_\sigma^t T(t-s)g(s,u_s)ds, \quad t \in [\sigma, \sigma+a). \end{aligned}$$

To discuss the existence of pseudo almost periodic solutions we need to set some assumptions on the functions f, g. In what follows, $[D(A)]$ denoted the domain of A endowed with the graph norm defined by: $\parallel u \parallel_{[D(A)]} = \parallel u \parallel + \parallel Au \parallel$ for each $u \in D(A)$.

$\mathbf{H}_1$ The functions $f, g : \mathbb{R} \times \mathcal{B} \longrightarrow \mathbb{X}$ are continuous, f is $D(A)$-valued and there exist a positive constant L_f and a continuous functions $L_g : \mathbb{R} \longrightarrow [0, \infty)$ satisfying

$$\begin{aligned} \parallel f(t,\psi_1) - f(t,\psi_2) \parallel_{[D(A)]} &\leq L_f \parallel \psi_1 - \psi_2 \parallel_{\mathcal{B}}, \\ \parallel g(t,\psi_1) - g(t,\psi_2) \parallel &\leq L_g(t) \parallel \psi_1 - \psi_2 \parallel_{\mathcal{B}}, \end{aligned}$$

for all $t \in \mathbb{R}$, $\psi_i \in \mathcal{B}$.

Remark 2 *The assumption on f is linked to the integrability of the function $s \longrightarrow AT(t-s)f(s, u_s)$ over $[0, t)$. In general, except trivial cases, the operator function $t \longrightarrow AT(t)$ is not integrable over $[0, a]$. If f is subject to $\mathbf{H}_1$, from the Bochner's criterion for integrable functions and the estimate*

$$\begin{aligned} \parallel AT(t-s)f(s,u_s) \parallel &= \parallel T(t-s)Af(s,u_s) \parallel \\ &\leq Me^{-w(t-s)} \parallel Af(s,u_s) \parallel \\ &\leq Me^{-w(t-s)} \parallel f(s,u_s) \parallel_{[D(A)]}, \end{aligned}$$

it follows that the function $s \mapsto AT(t-s)f(s,u_s)$ is integrable over $(-\infty,t)$ for each $t>0$. For additional remarks related this type of conditions in partial neutral differential equations, see, e.g., [1, 12, 13], in particular, [14].

Definition 7 *A function $u \in BC(\mathbb{R},\mathbb{X})$ is a mild pseudo almost periodic solution to the neutral system (6)-(7) provided that the function $s \to AT(t-s)f(s,u_s)$ is integrable on $(-\infty,t)$ for each $t\in\mathbb{R}$ and*

$$u(t) \;=\; -f(t,u_t) - \int_{-\infty}^{t} AT(t-s)f(s,u_s)ds + \int_{-\infty}^{t} T(t-s)g(s,u_s)ds, \quad t\in\mathbb{R}.$$

Theorem 4 *Under assumption $\mathbf{H_1}$, there exist a unique pseudo almost periodic solution to (6)-(7) whenever*

$$\Theta = \left(L_f\left[1+\frac{M}{w}\right] + M\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-w(t-s)}L_g(s)ds\right) < 1. \tag{8}$$

Proof: In $PAP(\mathbb{X},p)$ define the operator $\Gamma: PAP(\mathbb{X},p) \to C(\mathbb{R},\mathbb{X})$ by setting

$$\Gamma u(t) := -f(t,u_t) - \int_{-\infty}^{t} AT(t-s)f(s,u_s)ds + \int_{-\infty}^{t} T(t-s)g(s,u_s)ds, \qquad t\in\mathbb{R}.$$

From previous assumptions one can easily see that Γu is well-defined and continuous. Moreover, from Theorems 1, 2 and 3 we infer that $\Gamma u \in PAP(\mathbb{X},p)$. It remains to prove that Γ is a strict contraction on $PAP(\mathbb{X},p)$. For $u,v\in PAP(\mathbb{X},p)$ we get

$$\begin{aligned}\| \Gamma u(t) - \Gamma v(t) \| \;&\leq\; L_f \| u_t - v_t \|_{\mathcal{B}} + M\int_{-\infty}^{t} L_f e^{-w(t-s)} \| u_s - v_s \|_{\mathcal{B}}\, ds \\ &\quad + M\int_{-\infty}^{t} e^{-w(t-s)}L_g(t) \| u_s - v_s \|_{\mathcal{B}}\, ds \\ &\leq \left(L_f\left[1+\frac{M}{w}\right] + M\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-w(t-s)}L_g(s)ds\right) \| u - v \|_\infty \\ &\leq \Theta \| u-v \|_\infty .\end{aligned}$$

Clearly, from (8) one obtains the existence and uniqueness of a pseudo almost periodic solution to (6)-(7). This completes the proof. ■

4 A Functional Neutral Differential System

In this section we discuss briefly the existence and uniqueness of a pseudo almost periodic solution to the abstract neutral differential equations of the form

$$\begin{aligned}\frac{d}{dt}(u(t) + f(t,u(\gamma_1(t)))) &= Au(t) + g(t,u(\gamma_2(t))), \quad t\in\mathbb{R} \qquad (9)\\ u(0) &= u_0 \in \mathbb{X}. \qquad (10)\end{aligned}$$

where $f,g:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ and $\gamma_i:\mathbb{R}\to[0,\infty)$ for $i=1,2$, are some suitable continuous functions.

The purpose here is to look for some relaxing assumptions on the function f. More precisely, one assumes the existence of a Banach space $\mathbb{Y} \hookrightarrow \mathbb{X}$ (not necessarily $[D(A)]$) such that f is a $\mathbb{Y}$-valued continuous.

From now on, $\mathbb{Y}$ denotes an arbitrary Banach space continuously embedded into $\mathbb{X}$. In this event, $\mathcal{L}(\mathbb{Y},\mathbb{X})$ and $\mathcal{L}(\mathbb{Y})$ stand respectively for the class of bounded linear operators which go from $\mathbb{Y}$ into $\mathbb{X}$ and the class of bounded linear operators from $\mathbb{Y}$ into itself. We also require the following assumptions:

$\mathbf{H_2}$ The function $s \to T(s)y \in C([0,\infty),\mathbb{Y})$ for each $y \in \mathbb{Y}$ and there are positive constants $\widetilde{M}, \widetilde{\alpha}$ such that $\| T(s) \|_{\mathcal{L}(\mathbb{Y})} \leq \widetilde{M}e^{-\alpha s}$ for each $s \geq 0$. Moreover, the function $s \to AT(s)$ defined from $(0,\infty)$ into $\mathcal{L}(\mathbb{Y},\mathbb{X})$ is strongly measurable and there exist a non-decreasing function $H : [0,\infty) \to [0,\infty)$ and $\delta > 0$ with $e^{-\delta s}H(s) \in L^1([0,\infty))$ and such that $\| AT(s) \|_{\mathcal{L}(\mathbb{Y},\mathbb{X})} \leq e^{-\delta s}H(s)$ for every $s > 0$.

$\mathbf{H_3}$ The functions $f, g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ are continuous, $f(\cdot)$ is $\mathbb{Y}$-valued, $f : \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$ is continuous and there are a constant $L_f \in (0,1)$ and a continuous function $L_g : \mathbb{R} \to (0,\infty)$ such that

$$\begin{aligned} \|f(t,y_1) - f(t,y_2)\|_Y &\leq L_f\|y_1 - y_2\|, \qquad & t \in \mathbb{R},\ y_i \in \mathbb{X},\ i = 1,2, \\ \|g(t,y_1) - g(t,y_2)\| &\leq L_g(t)\|y_1 - y_2\|, \qquad & t \in \mathbb{R},\ y_i \in \mathbb{X},\ i = 1,2. \end{aligned}$$

$\mathbf{H_4}$ The functions $\gamma_i : \mathbb{R} \to [0,\infty)$ for $i = 1,2$, are nondecreasing, continuously differentiable with uniformly bounded derivatives on $\mathbb{R}$, and for each $u \in AP(\mathbb{X})$ the function $u(\gamma_i(\cdot)) \in AP(\mathbb{X})$ and

$$\limsup_{r\to\infty} \left(\frac{\mid \gamma_i(-r) \mid + \mid \gamma_i(r) \mid}{r} \right) < \infty.$$

Remark 3 *Note that the assumption $\mathbf{H_2}$ is achieved in many cases, see for instance Lunardi [20], and the section devoted to applications in Rankin [26].*

Remark 4 *Note also that if $\gamma_i(t) = t - p$, ($p \in \mathbb{R}$), or $\gamma_i(t) \in AP(\mathbb{R})$, then the assumption $\mathbf{H_2}$ is achieved.*

Lemma 3 *Assume γ_i for $i = 1,2$ verifies $\mathbf{H_2}$. If $u \in PAP(\mathbb{X})$, then $u(j_i(\cdot)) \in PAP(\mathbb{X})$ for $i = 1,2$.*

Proof: Consider the decomposition $u = v + w$, where $v \in AP(\mathbb{X})$ and $w \in PAP(\mathbb{X})$. Obviously, $v(j_i(\cdot)) \in AP(\mathbb{X})$. On the other hand, for $r > 0$ one can easily see that

$$\begin{aligned} &\frac{1}{2r}\int_{-r}^{r} \| w(\gamma_i(t)) \| \, dt \\ &\quad \leq \frac{1}{2r}\int_{\gamma_i(-r)}^{\gamma_i(r)} \| w(t) \| \, \gamma_i'(t)dt \\ &\quad \leq \frac{1}{2r}\int_{-|\gamma_i(-r)|}^{|\gamma_i(r)|} \| w(t) \| \, \gamma_i'(t)dt \\ &\quad \leq \frac{\sup_{t\in\mathbb{R}} \gamma_i'(t)}{2r}\int_{-|\gamma_i(-r)|-|\gamma_i(r)|}^{|\gamma_i(-r)|+|\gamma_i(r)|} \| w(t) \| \, dt \\ &\quad \leq \frac{\mid \gamma_i(-r) \mid + \mid \gamma_i(r) \mid}{r} \, \frac{\sup_{t\in\mathbb{R}} \gamma_i'(t)}{2(\mid \gamma_i(-r) \mid + \mid \gamma_i(r) \mid)}\int_{-|\gamma_i(-r)|-|\gamma_i(r)|}^{|\gamma_i(r)|+|\gamma_i(r)|} \| w(t) \| \, dt, \end{aligned}$$

and hence $\frac{1}{2r}\int_{-r}^{r} \| w(\gamma_i(t)) \| dt$ converges to zero as $r \to \infty$. This completes the proof. ■

Lemma 4 *Let $u \in PAP(\mathbb{Y})$. If v is the function defined by*

$$v(t) = \int_{-\infty}^{t} AT(t-s)u(s)ds,$$

then $v \in PAP(\mathbb{X})$.

Proof: For $r > 0$ we get

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r} \| \int_{-\infty}^{t} AT(t-s)u(s)ds \| dt \\
&\quad \leq \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{t} \| AT(t-s) \|_{\mathcal{L}(\mathbb{Y},\mathbb{X})} \| u(s) \|_{Y} dsdt \\
&\quad \leq \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{t} e^{-\delta(t-s)}H(t-s) \| u(s) \|_{\mathbb{Y}} dsdt \\
&\quad \leq \frac{1}{2r}\int_{-\infty}^{-r}\int_{-r}^{r} e^{-\delta(t-s)}H(t-s) \| u(s) \|_{\mathbb{Y}} dtds \\
&\quad\quad + \frac{1}{2r}\int_{-r}^{r}\int_{s}^{r} e^{-\delta(t-s)}H(t-s) \| u(s) \|_{\mathbb{Y}} dtds \\
&\quad \leq \frac{\| u \|_{\mathbb{Y},\infty}}{2r}\int_{-\infty}^{-r} H(-r-s)\int_{-r-s}^{r-s} e^{-\delta\xi}d\xi ds + \frac{\eta}{2r}\int_{-r}^{r} \| u(s) \| dt \\
&\quad \leq \frac{\eta \| u \|_{\mathbb{Y},\infty}}{2r\delta} + \frac{\eta}{2r}\int_{-r}^{r} \| u(s) \| dt,
\end{aligned}$$

where $\eta = \int_0^\infty e^{-\delta s}H(s)ds$ and $\| u \|_{\mathbb{Y},\infty} = \sup_s (\| u(s) \|_{\mathbb{Y}})$.

The latter inequality proves our claim. ■

Definition 8 *A function $u \in BC(\mathbb{R}, \mathbb{X})$ is a mild pseudo almost periodic solution of neutral system (9), if the function $s \to AT(t-s)f(s, u(\gamma_1(s)))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and*

$$\begin{aligned}
u(t) &= -f(t, u(\gamma_1(t))) - \int_{-\infty}^{t} AT(t-s)f(s, u(\gamma_1(s)))ds \\
&\quad + \int_{-\infty}^{t} T(t-s)g(s, u(\gamma_2(t)))ds, \quad t \in \mathbb{R}.
\end{aligned}$$

We now prove the main results of this section.

Theorem 5 *Under assumptions $\mathbf{H_2}$, $\mathbf{H_3}$, and $\mathbf{H_4}$, the system (9)-(10) has a unique pseudo almost periodic solution whenever*

$$\Theta = \left[L_f\left(1 + \sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\delta(t-s)}H(t-s)ds\right) + M\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-w(t-s)}L_g(s)ds\right] < 1.$$

Proof: In $PAP(\mathbb{X})$ define the operator $\Gamma : PAP(\mathbb{X}) \to C(\mathbb{R}, \mathbb{X})$ by setting

$$\begin{aligned} \Gamma u(t) &= -f(t, u(\gamma_1(t)) - \int_{-\infty}^{t} AT(t-s) f(s, u(\gamma_1(s)) ds \\ &\quad + \int_{-\infty}^{t} T(t-s) g(s, u(\gamma_2(s)) ds, \qquad t \in \mathbb{R}. \end{aligned}$$

From the estimate

$$\begin{aligned} \| AT(t-s) f(s, u(\gamma_1(s))) \| &\leq \| AT(t-s) \|_{\mathcal{L}(\mathbb{Y},\mathbb{X})} \| f(s, u(\gamma_1(s))) \|_{\mathbb{Y}} \\ &\leq e^{-\delta(t-s)} H(t-s) \| f(s, u(\gamma_1(s))) \|_{\mathbb{Y}}, \end{aligned}$$

we infer the Γu is well defined, continuous and that $\Gamma u \in BC(\mathbb{R}, \mathbb{X})$. Moreover, from Lemmas (3) and (4) it follows that Γ is $PAP(\mathbb{X})$-valued. To complete the proof, we must show that Γ is a strict contraction. For $u, v \in PAP(\mathbb{X})$,

$$\begin{aligned} \| \Gamma u(t) - \Gamma v(t) \| &\leq L_f \| u - v \|_\infty + \int_{-\infty}^{t} L_f e^{-\delta(t-s)} H(t-s) \| u - v \|_\infty \, ds \\ &\quad + M \int_{-\infty}^{t} e^{-w(t-s)} L_g(s) \| u - v \|_\infty \, ds \\ &\leq L_f \left(1 + \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{-\delta(t-s)} H(t-s) ds \right) \| u - v \|_\infty \\ &\quad + M \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{-w(t-s)} L_g(s) ds \| u - v \|_\infty \\ &\leq \Theta \| u - v \|_\infty . \end{aligned}$$

From $\Theta < 1$ it follows that Γ is a strict contraction, and hence by the Banach fixed-point principle, there exists a unique mild solution to (7)-(10) which obviously is pseudo almost periodic. the proof is now complete. ■

5 Examples

In this section we provide with some examples to illustrate our previous abstract results.

5.1 Case of a First-Order Boundary-Value Problem

We first introduce the required technical background. For that, let $\mathbb{X} = L^2([0, \pi])$ and let A be the operator given by $Af = f''$ with domain

$$D(A) := \{ f \in L^2([0, \pi]) : f'' \in L^2([0, \pi]), \ f(0) = f(\pi) = 0 \}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ on $\mathbb{X}$. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$, and corresponding normalized eigenfunctions $z_n(\xi) := \sqrt{\dfrac{2}{\pi}} \sin(n\xi)$. In addition, the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{X}$;

(b) For $f \in \mathbb{X}$, $T(t)f = \sum_{n=1}^{\infty} e^{-n^2 t}\langle f, z_n\rangle z_n$ and $Af = -\sum_{n=1}^{\infty} n^2\langle f, z_n\rangle z_n$, for every $f \in D(A)$.

Moreover, it is possible to define fractional powers of A as closed linear operators (see [20], [24, Chapter 2]). In particular,

(c) For $f \in \mathbb{X}$ and $\alpha \in (0,1)$, $(-A)^{-\alpha} f = \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}}\langle f, z_n\rangle z_n$;

(d) The operator $(-A)^{\alpha} : D((-A)^{\alpha}) \subseteq \mathbb{X} \to \mathbb{X}$ is given by

$$(-A)^{\alpha} f = \sum_{n=1}^{\infty} n^{2\alpha}\langle f, z_n\rangle z_n, \quad \forall f \in D((-A)^{\alpha}),$$

where $D((-A)^{\alpha}) = \{f(\cdot) \in \mathbb{X} : \sum_{n=1}^{\infty} n^{2\alpha}\langle f, z_n\rangle z_n \in \mathbb{X}\}$.

Consider the first-order boundary value problem

$$\begin{aligned}
\frac{\partial}{\partial t}\left[u(t,\xi) + \int_{-p}^{0}\int_{0}^{\pi} b(s,\eta,\xi)u(t+s,\eta)d\eta ds\right] &= \frac{\partial^2}{\partial \xi^2}u(t,\xi) + a_0(\xi)u(t,\xi) \\
&\quad + \int_{-p}^{0} a(s)u(t+s,\xi)ds, \qquad (11) \\
u(t,0) &= u(t,\pi) = 0, \qquad (12)
\end{aligned}$$

for $t \in \mathbb{R}$ and $\xi \in I = [0,\pi]$.

Note that equations of type (11)-(12) arise for instance in control systems described by abstract retarded functional-differential equations with feedback control governed by proportional integro-differential law, see [12, Examples 4.2] for details.

To study (11)-(12) we put $\mathbb{X} = L^2([0,\pi])$ and $\mathcal{B} = C([-p,0];\mathbb{X})$. In addition to that we suppose that the functions a, a_0, a_1 are continuous and that the following holds:

(i) The functions $b(\cdot)$, $\frac{\partial^i}{\partial \zeta^i} b(\tau,\eta,\zeta)$, $i = 1,2$ are (Lebesgue) measurable, $b(\tau,\eta,\pi) = 0$, $b(\tau,\eta,0) = 0$ for every (τ,η) and

$$N_1 := \max\{\int_0^{\pi}\int_{-p}^{0}\int_0^{\pi}\left(\frac{\partial^i}{\partial \zeta^i} b(\tau,\eta,\zeta)\right)^2 d\eta d\tau d\zeta : i = 0,1,2\} < \infty.$$

Define f, $g : C([-p,0];\mathbb{X})$ by setting

$$\begin{aligned}
f(t,\psi)(\xi) &:= \int_{-p}^{0}\int_{0}^{\pi} b(s,\eta,\xi)\psi(s,\eta)d\eta ds \\
g(t,\psi)(\xi) &:= a_0(\xi)u(t,\xi) + \int_{-p}^{0} a(s)\psi(s,\xi)ds.
\end{aligned}$$

In view of the above, it is clear that the system (11)-(12) can be rewritten as an abstract system of the form (1). By a straightforward estimation that uses **(i)** one can show that f has values in $D(A)$ and that $f(t,\cdot) : C([-p,0];\mathbb{X}) \to [D(A)]$ is a bounded linear operator with $\| Af(t,\cdot) \| \leq (N_1 p)^{\frac{1}{2}}$ for each $t \in \mathbb{R}$. Furthermore, g is a bounded linear operator on $\mathbb{X}$ with

$$\| g(t,\cdot) \| \leq \| a_0 \|_\infty + \sqrt{p \left(\int_{-p}^{0} a^2(s) ds \right)}$$

for every $t \in \mathbb{R}$.

The next result is a consequence of Theorem 4.

Proposition 1 *Under the previous assumptions, system (11)-(12) has a unique pseudo almost periodic solution whenever*

$$\left[2\sqrt{N_1 p} + \| a_0 \|_\infty + \sqrt{p \left(\int_{-p}^{0} a^2(s) ds \right)} \right] < 1.$$

5.2 Reaction-Diffusion Equations with Delay

Many different types of differential equations, reaction diffusion equation with delay, wave equations, age-dependent population equations among others, can be described as abstract semilinear functional differential equations, see Wu [31] and the extensive bibliography in this book. Now, we apply our previous results to study the abstract system of the form

$$x'(t) = Ax(t) + g(t, x(t-p)),$$

where A is the infinitesimal generator of an uniformly exponentially stable semigroup of linear operators $(T(t))_{t\geq 0}$ on a Banach $\mathbb{X}$. This system is motivated by the reaction-diffusion equation given by

$$u'(t,\xi) = \frac{\partial^2}{\partial \xi^2} u(t,\xi) + g(t, u(t-p,\xi)), \tag{13}$$

$$u(t,0) = u(t,\pi) = 0, \tag{14}$$

$$u(\tau,\xi) = \varphi(\tau,\xi), \quad \tau \in [-p,0], \xi \in [0,\pi]. \tag{15}$$

The next result is a simple consequence of Theorem 4. Here, we assume that M, w are positive constants such that

$$\| T(t) \| \leq M e^{-wt}, \quad t \geq 0.$$

Proposition 2 *Assume that $g : \mathbb{R} \times C([-p,0];\mathbb{X}) \to \mathbb{X}$ is continuous and the existence of a positive and integrable function $L_g : \mathbb{R} \to \mathbb{R}$ such that*

$$\|g(t,\psi_1) - g(t,\psi_2)\| \leq L_g(t) \|\psi_1 - \psi_2\|_\infty ,$$

for every $t \in \mathbb{R}$ and all $\psi_1, \psi_2 \in C([-p,0];\mathbb{X})$.

If $M \sup_{t\in\mathbb{R}} \left(\int_{-\infty}^{t} e^{-w(t-s)} L_g(s) ds \right) < 1$, then there exists a pseudo almost periodic mild solution to the problem (13)-(5.9)-(15).

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