Self-duality and the holomorphic *ansatz* in a generalized BPS Skyrme model

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We propose a generalization of the Bogomol'ny–Prasad–Sommerfield Skyrme model [L. A. Ferreira, Exact self-duality in a modified Skyrme model, J. High Energy Phys. 07 (2017) 039] for simple compact Lie groups G that leads to Hermitian symmetric spaces. In such a theory, the Skyrme field takes its values in G, while the remaining fields correspond to the entries of a symmetric, positive, and invertible $\dim G \times \dim G$-dimensional matrix h. We also use the holomorphic map *Ansatz* between $S^2 \to G/H \otimes U(1)$ proposed in Ferreira and Livramento [Harmonic, holomorphic and rational maps from self-duality, arXiv:2412.02636] to study the self-dual sector of the theory, which generalizes the holomorphic *Ansatz* between $S^2 \to CP^N$ proposed in Ioannidou [Low-energy states in the SU(N) Skyrme models, in *International Meeting on Mathematical Methods in Modern Theoretical Physics (ISPM 98)* (1998), pp. 91–123, arXiv:hep-th/9811071]. This *Ansatz* is constructed using the fact that stable harmonic maps of the two S^2 spheres for compact Hermitian symmetric spaces are holomorphic or antiholomorphic [J. Eells and L. Lemaire, *Two Reports on Harmonic Maps* (World Scientific Publishing Company, Singapore, 1995)]. Apart from some special cases, the self-duality equations do not fix the matrix h entirely in terms of the Skyrme field, which is completely free, as it happens in the original self-dual Skyrme model for $G = SU(2)$. In general, the freedom of the h fields tend to grow with the dimension of G. The holomorphic *Ansatz* enable us to construct an infinite number of exact self-dual Skyrmions for each integer value of the topological charge and for each value of $N \geq 1$, in case of the CP^N, and for each values of $p, q \geq 1$ in case of $SU(p+q)/SU(p) \otimes SU(q) \otimes U(1)$.

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I. INTRODUCTION

The study of self-duality has shed light on the complex behavior of topological solutions in a wide variety of classical nonlinear field theories. The topological solitons are classified by a homotopic invariant quantity, the so-called topological charge, and self-duality can greatly facilitate the task of obtaining the topological solutions corresponding to the global energy minimizer [1]. This plays a fundamental role in the study of kinks and instantons in $(1+1)$ dimensions [1–4], vortex solutions in the Abelian Chern-Simons theory in $(2+1)$ dimensions [5], self-dual Skyrmions in $(3+1)$ dimensions [6,7], and in some non-Abelian gauge theories in $(3+1)$ dimensions, as the Yang-Mills-Higgs system [8].

The self-duality usually appears in models that possess two main ingredients. First, the static energy density of the model must be the sum of the squares of two objects A_α and $\tilde{A}_\alpha$ that depend on the fields and their first-order spacetime derivatives only, where the nature of the fields and the α index depend on each theory. Second, the topological charge density must be proportional to the contraction of such objects. It follows that the so-called self-duality equations $A_\alpha = \pm\tilde{A}_\alpha$ imply second-order differential Euler-Lagrange equations and also correspond to the global minimizer of the static energy, for each value of the topological charge (Q). The set of topological solutions of the self-duality equations is called the self-dual sector, which can be empty for some models, as is the case with the standard Skyrme model, as demonstrated in [9,10].

The standard Skyrme model is an effective classical field theory for the triplet of pions in $(3+1)$ dimensions in the low-energy regime [3,9,11–13]. The model is defined in terms of the $SU(2)$ Skyrme field U, which includes the three pion fields and is a map between two three-spheres. Its standard version contains only two terms in its action, one quadratic and the other quartic in the spacetime derivatives. The quartic term, or any other higher-order kinetic term, is essential to stabilize the Skyrmions under Derrick's scale argument [14]. This is still true even if any positive definite potential defined in terms of the Skyrme field is added since this $SU(2)$ field is scale invariant.

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Such a theory possesses a large number of modifications, some allowing the construction of electrically charged multi-Skyrmions. This is the case of the gauged version of the Skyrme model obtained by gauging the $U(1)$ subgroup of the $SU(2)$ global symmetry, associated with the generator of its Cartan subalgebra [15–19].

There are some modifications to the standard Skyrme model that lead to a nonempty self-dual sector. Notably, one of these modifications, the so-called Bogomol'ny–Prasad–Sommerfield (BPS) Skyrme model [6], can be directly derived from integral representations of the topological charge associated with the Skyrme field using ideas of self-duality seen in [1]. Such an approach spontaneously includes six extra fields corresponding to the entries of a symmetric, positive, and invertible 3×3 matrix h. The matrix h and its inverse appear contracted respectively to quadratic and quartic terms in the spacetime derivatives associated with the Skyrme field, i.e., the model is defined by

$$S_{\text{BPS}} = \int d^4x \left[\frac{m_0^2}{2} h_{ab} R_\mu^a R^{b,\mu} - \frac{1}{4e_0^2} h_{ab}^{-1} H_{\mu\nu}^a H^{b,\mu\nu}\right] \tag{1.1}$$

where m_0 is a coupling constant with dimension of mass, and e_0 is a dimensionless coupling constant. In addition, $R_\mu^a = i\widehat{\text{Tr}}(\partial_\mu U U^\dagger T_a)$, and $H_{\mu\nu}^a = \varepsilon_{abc} R_\mu^b R_\nu^c$, with T_a, $a = 1, 2, 3$, being the basis of the $SU(2)$ Lie algebra satisfying $[T_a, T_b] = i\varepsilon_{abc} T_c$, and $\widehat{\text{Tr}}(T_a T_b) = \delta_{ab}$. The standard Skyrme model is recovered by imposing $h = \mathbb{1}$.

It was demonstrated in [7] that self-duality equations of the BPS Skyrme model can be used to algebraically determine entierly the h matrix in terms of the matrix $\tau_{ab} \equiv R_i^a R_i^b$, with $a, b = 1, 2, 3$, in all regions where τ is nonsingular, while the Skyrme field is still completely free. This $SU(2)$ field is still completely free even at the points where τ is singular, but in this case some of the components of the matrix h are also free. The reason that leads to this freedom can be traced to the fact that the nine static Euler-Lagrange equations for the fields h and U are not all independent. In fact, the equations for the U field can be derived from the equations of the h fields when τ is nonsingular. The freedom of the Skyrme field leads to an infinite number of exact topological solutions to each value of the topological charge.

All the BPS solutions of the model (1.1) are scale independent due to the conformal invariance of the model in three spacial dimensions. This freedom of the shapes of topological solitons can improve the scope of physical application of the theory, especially if some extra term is added breaking the scale independence and selecting some specific form. By example, the scale dependence and the radial multisolitons configurations that live in the self-dual sector of the theory (1.1) are essential in one of its extensions, the false vacuum Skyrme model [20].

Self-duality can also play a fundamental role in models that are extensions of BPS theories, where the total static energy contains extra terms, even in nonperturbative approaches. On the one hand, self-duality can inspire the construction of *Ansätze* in quasi-self-dual models, where the extra terms weakly break the self-duality equations [21]. On the other hand, there are models that contain extra terms that do not break any of the self-duality equations, such as the false vacuum Skyrme model [20]. This is a powerful modification of the Skyrme model that leads to excellent classical results for the binding energy and radius of the nuclei. In fact, the results are such that for a list containing 256 nuclei with mass number $A \geq 12$, the root-mean-square deviation of the binding energy per nucleon and the root-mean-square radius, which are, respectively, of the order of 0.05 MeV and 0.04 fm, are of the same order as excellent fits based on phenomenological approaches.

The magic of the false vacuum BSP Skyrme model is that the BSP Skyrme term gives a massive contribution E_1 to the total nuclear mass, but the binding energy comes just from the extra terms, despite given a lower order contribution E_2 to the total mass. The BPS model is extended through the introduction of kinetic and potential terms for the baryonic density, which depends only on the Skyrme field, and a topological term that approximately reproduces the Coulomb interaction. The h fields are still being determined through the self-duality equations, since the additional terms do not depend on such fields. Curiously, Coleman's false vacuum argument [22–24] shows that the global minimizer of such a theory must have radial symmetry. This mathematical result reduces the three-dimensional static Euler-Lagrange equations to a single radial second order differential equation for a fractional power of the baryonic density. However, exploring the nature of h fields for generalizations of the BPS Skyrme model can help reveal the physical interpretation of such fields and if h can still be entirely determined in terms of the Skyrme fields for Lie groups other than $G = SU(2)$.

A self-dual modification of the Skyrme model proposed in [25,26], also known as the BPS Skyrme model, introduces a BPS sector without the need for additional fields beyond the pion degrees of freedom. In this construction, the standard Skyrme terms are replaced by a potential term together with a sextic term in the spacetime derivatives of the Skyrme field, the so-called BPS Skyrme term. Although it features zero binding energy at the classical level, the inclusion of Coulomb contributions and quantum corrections yields realistic binding energies for very heavy nuclei [27]. Furthermore, the model can be extended by adding the terms of the standard Skyrme theory with a small deformation parameter $\epsilon \ll 1$, bringing the theory close to the BPS limit and providing controlled corrections to the binding energies [28].

Extensions of such a BPS Skyrme model that contain the sextic term but do not include the h fields also play an important role in reducing Skyrme-type solutions to configurations with low binding energies [29–31]. The construction of Skyrme-type theories with a self-dual sector has also been obtained by the reduction of self-dual Yang-Mills theory in four dimensions to a Skyrme model coupled to a tower of vector mesons, leading to a reasonable description of the spectrum of light nuclei [32–35].

An important mathematical result that motivates us to generalize the BPS Skyrme model shows that for a simple compact Lie group G, it follows that $\pi_3(G) = \mathbb{Z}$, where $\pi_3(G)$ is the homotopy group of the mapping of G into a three-sphere. The topological charge associated with this map admits integral representation similar to the $G = SU(2)$ case. Let us consider now the cases where G leads to a Hermitian symmetric space $G/H \otimes U(1)$, where little group $H \otimes U(1)$ is a subgroup of G. Some examples of Lie groups that lead to a Hermitian symmetric space are $G = A_r, B_r, C_r, D_r, E_6, E_7$, and some counter-examples include the Lie groups $G = E_8, F_4, G_2$.

Another important mathematical result that can shed light on how we can study the self-duality in such model was derived by Eells and Lemaire [36]. It states that stable harmonic maps X from the two-sphere S^2 to compact Hermitian symmetric spaces are holomorphic or antiholomorphic. This laid the foundation for constructing the *Ansatz* holomorphic map *Ansatz* between $S^2 \to G/H \otimes U(1)$ proposed in [37] by Ferreira and Livramento. Although this *Ansatz* only works in certain specific representations of the Lie group G, as will be discussed later, it is major generalization of the holomorphic *Ansatz* between $S^2 \to CP^N$ proposed in [38] to the CP^N.

The main idea of this work is to first construct a generalized BPS Skyrme model with the Skyrme fields mapping the physical space to a simple compact Lie group G that leads to the Hermitian symmetric space $G/H \otimes U(1)$. In this case, as the indices of the rows and columns of the matrix h are contracted with each index of the generators of the Lie algebra $\mathcal{G}$ associated with G, in such a theory it becomes a $\dim \mathcal{G} \times \dim \mathcal{G}$ dimensional symmetric, invertible, and positive matrix. Therefore, the h matrix and the Skyrme fields can be written in terms of $\dim \mathcal{G}(\dim \mathcal{G} + 1)/2$ and $\dim \mathcal{G}$ independents fields, respectively.

Our second objective in this paper is to study the self-dual sector of such a theory through the holomorphic *Ansatz* between $S^2 \to G/H \otimes U(1)$ proposed in [37]. In particular, we want to determine whether the matrix h can still be entirely determined in terms of the Skyrme fields in the generalized BPS Skyrme model, similar to what happens in the case $G = SU(2)$, and whether U is still completely free. Despite the fact that the number of self-duality equations is, in principle, equal to the number of independent fields of the theory, the full determination of all the fields by the self-duality equations is not expected, since this does not happen even for the $G = SU(2)$ case, as discussed above. This *Ansatz* can drastically simplify the self-duality equations, aiding in our investigation of the self-dual sector and in the construction of exact BPS topological solutions.

A powerful holomorphic *Ansatz* for the standard Skyrme model was constructed for the $G = SU(2)$ case by Houghton *et al.* in [39] using harmonic maps from $S^3 \to S^3$. It is based on the rational map, which is a holomorphic function from $S^2 \to S^2$ [3,4,39,40]. Although solving all Euler-Lagrange equations only for $Q = \pm 1$, such *Ansatz* leads to a quite good approximation of the true topological solitons corresponding to the global energy minimizer, in the BPS Skyrme model (1.1), as the Skyrme field is completely free inside the self-dual sector, the rational map leads to an infinite number of exact solutions for each value of Q [7].

In interpreting the Skyrme model as a low-energy effective field theory of QCD in the limit where the number of colors is large, the number $N + 1$ of the Lie group $SU(N + 1)$ where the Skyrme field takes its values corresponds to the number of light quark flavors. The holomorphic *Ansatz* proposed by Houghton *et al.* in [39] for $G = SU(2)$ was generalized in [38] for $G = SU(N + 1)$ using harmonic maps from S^2 to $CP^N \cong SU(N + 1)/SU(N) \otimes U(1)$. As the other *Ansatzë* used to construct multi-Skyrmions for some values of N of the $G = SU(N + 1)$ case, the goal of this *Ansatz* is just give some approximation of the global energy minimizers. However, in general the energies obtained through such an *Ansatz* are marginally higher than the ones obtained through $SU(2)$ embeddings.

Through the holomorphic *Ansatz* we can construct an infinite number of exact topological solutions for each values of Q and N of the CP^N space. This set of self-dual solutions even includes field configurations based on the standard rational map *Ansatz*. We also obtain self-dual solutions within this *Ansatz* for the Hermitian symmetric space $SU(p + q)/SU(p) \otimes SU(q) \otimes U(1)$, which generalizes our results obtained for the CP^N. In this case, we also provide explicit solutions for any value of the topological charge. This study has the potential to reveal the nature of h fields in such models and how the role of self-duality manifests in determining the h and U fields.

The paper is organized as follows. In Sec. II, we construct the generalized BPS Skyrme model. Additionally, we obtain the self-dual equations and the expression for the topological charge and energy within the self-dual sector. In this section we also discuss an important symmetry of the model under the composition of parity and target space parity transformations. In Sec. III, we derive the Euler-Lagrange equations and demonstrate how they can be solved using the self-dual field configurations. In Sec. IV we construct our holomorphic *Ansatz* between $S^2 \to G/H \otimes U(1)$. We also obtain the explicit general form of the self-duality equations using the structure of the Hermitian symmetric space and our holomorphic *Ansatz*. In Secs. V and VI we study our holomorphic *Ansatz* in the

$G = SU(2)$ and $G = SU(N+1)$ cases, respectively. The case $SU(2)$ is done separately due to its peculiar structure, and in both cases we obtain exact multi-BPS Skyrmions for all integer values of the topological charge. In Sec. VII we obtain the self-dual equations inside the holomorphic *Ansatz* for the Hermitian symmetric space $SU(p+q)/SU(p)\otimes SU(q)\otimes U(1)$, and construct particular self-dual solutions for all integer values of the topological charge. In Sec. VIII we present our final considerations.

II. THE MODEL AND ITS CONSTRUCTION

Consider a simple compact Lie group G. It is known that the maps $S^3 \to G$ are classified by the integers since

$$\pi_3(G) = \mathbb{Z}. \tag{2.1}$$

The topological charge associated to such homotopy group is given by

$$Q = \frac{i}{48\pi^2\kappa}\int d^3x \varepsilon_{ijk}\mathrm{Tr}(R_i R_j R_k) \tag{2.2}$$

where

$$R_\mu \equiv i\partial_\mu U U^{-1} \equiv R_\mu^a T_a \tag{2.3}$$

with U being an element of the group G, and T_a, $a = 1, \ldots, \dim G$, being the generators of the corresponding compact simple Lie algebra

$$[T_a, T_b] = i f_{abc} T_c \tag{2.4}$$

and where we work with an orthogonal basis, i.e.,

$$\mathrm{Tr}(T_a T_b) = \kappa\delta_{ab} \tag{2.5}$$

with κ depending upon the representation where the trace is taken. We shall use a normalized trace defined as

$$\widehat{\mathrm{Tr}}(T_a T_b) \equiv \frac{1}{\kappa}\mathrm{Tr}(T_a T_b) = \delta_{ab}. \tag{2.6}$$

The quantities R_μ introduced in (2.3) satisfy by construction the Maurer-Cartan equation

$$\partial_\mu R_\nu - \partial_\nu R_\mu + i[R_\mu, R_\nu] = 0, \tag{2.7}$$

which allows us to split the topological charge (2.2) as

$$Q = \frac{1}{48\pi^2}\int d^3x \mathcal{A}_i^a \tilde{\mathcal{A}}_i^a \tag{2.8}$$

with

$$\mathcal{A}_i^a \equiv R_i^b k_{ba}; \qquad \tilde{\mathcal{A}}_i^a \equiv \frac{i}{2}k_{ab}^{-1}\varepsilon_{ijk}\widehat{\mathrm{Tr}}(T_b[R_j, R_k]) \tag{2.9}$$

where k_{ab} is some invertible matrix. Using the ideas of self-duality seen in [1], through this splitting we can introduce the self-duality equation as

$$\lambda\mathcal{A}_i^a = \tilde{\mathcal{A}}_i^a \quad \text{with} \quad \lambda = \pm me \tag{2.10}$$

or

$$\lambda R_i^b h_{ba} = \frac{i}{2}\varepsilon_{ijk}\widehat{\mathrm{Tr}}(T_a[R_j, R_k]) \tag{2.11}$$

where we have introduced a $\dim G \times \dim G$-dimensional matrix

$$h_{ab} = (kk^T)_{ab} = k_{ac}k_{bc}. \tag{2.12}$$

Due the fact the k is invertible and the definition (2.12), it so follows that the h matrix is invertible, symmetric, and positive. The fact that h is positive is less trivial, but consider real vector v and define $u \equiv k^T v$, which implies that $v = \vec{0} \Rightarrow u = \vec{0}$. Using the fact that k is invertible, we can write $v = k^{T-1}u$, and so $u = \vec{0} \Rightarrow v = \vec{0}$. Therefore, the fact that k is invertible implies that $u = \vec{0} \Leftrightarrow v = \vec{0}$. It so follows that for all nonvanishing real vector v we have $v^T h v = |k^T v|^2 = |u|^2 > 0$, and so $h = kk^T$ is a positive matrix.

The solutions of (2.9) solve the Euler-Lagrange equations associated to the following static energy functional constructed using ideas of self-duality [1]

$$\begin{aligned} E &= \frac{1}{2}\int d^3x \left[m^2(\mathcal{A}_i^a)^2 + \frac{1}{e^2}(\tilde{\mathcal{A}}_i^a)^2\right] \\ &= \frac{1}{2}\int d^3x \left[m^2 h_{ab}R_i^a R_i^b - \frac{1}{2e^2}h_{ab}^{-1}\widehat{\mathrm{Tr}}(T_a[R_j, R_k])\widehat{\mathrm{Tr}}(T_b[R_j, R_k])\right] \end{aligned} \tag{2.13}$$

which is the static energy of a generalized Skyrme model. The action associated to energy (2.13) that defines the generalized BPS Skyrme model is so given by

$$\begin{aligned} S &= \frac{1}{2}\int d^4x \left[m^2 h_{ab}R_\mu^a R^{b,\mu} + \frac{1}{2e^2}h_{ab}^{-1}\widehat{\mathrm{Tr}}(T_a[R_\mu, R_\nu])\widehat{\mathrm{Tr}}(T_b[R^\mu, R^\nu])\right] \\ &= \int d^4x \left[\frac{m^2}{2}h_{ab}R_\mu^a R^{b,\mu} - \frac{1}{4e^2}h_{ab}^{-1}H_{\mu\nu}^a H^{b,\mu\nu}\right] \end{aligned} \tag{2.14}$$

where we have defined [see Eq. (2.7)]

$$H^a_{\mu\nu} \equiv -i\widehat{\mathrm{Tr}}(T_a[R_\mu, R_\nu]) = \partial_\mu R^a_\nu - \partial_\nu R^a_\mu = f_{abc} R^b_\mu R^c_\nu. \tag{2.15}$$

We can write the energy (2.13) as

$$\begin{aligned} E &= \frac{1}{2e^2}\int d^3x[m^2e^2(\mathcal{A}^a_i)^2 + (\tilde{\mathcal{A}}^a_i)^2] \\ &= \frac{1}{2e^2}\int d^3x[\lambda\mathcal{A}^a_i - \tilde{\mathcal{A}}^a_i]^2 + \frac{\lambda}{e^2}\int d^3x \mathcal{A}^a_i\tilde{\mathcal{A}}^a_i \\ &= \frac{1}{2e^2}\int d^3x[\lambda\mathcal{A}^a_i - \tilde{\mathcal{A}}^a_i]^2 \\ &\quad + \mathrm{sign}(\lambda)48\pi^2\frac{m}{e}Q \geq \mathrm{sign}(\lambda)48\pi^2\frac{m}{e}Q, \end{aligned} \tag{2.16}$$

which corresponds to the usual BPS bound. When the self-duality (2.10) holds true the topological charge (2.8) can be written as

$$Q = \pm\frac{me}{48\pi^2}\int d^3x(\mathcal{A}^a_i)^2 \tag{2.17}$$

and so Q is positive for the plus sign ($\lambda > 0$) and negative otherwise ($\lambda < 0$), i.e.,

$$\mathrm{sign}(Q\lambda) = 1. \tag{2.18}$$

Then, using (2.10) and (2.18) the energy (2.16) of the self-dual solutions saturates the BPS bound, also given in (2.16), i.e., the energy becomes

$$E = 48\pi^2\frac{m}{e}|Q|. \tag{2.19}$$

Clearly, as usual, the self-dual energy (2.19) is proportional to the modulus of the topological charge. Contracting the self-duality equations (2.11) with R^c_i we get

$$\lambda\tau_{cb}h_{ba} = \sigma_{ca} \tag{2.20}$$

with

$$\tau_{ab} \equiv R^a_i R^b_i \tag{2.21}$$

and

$$\begin{aligned} \sigma_{ab} &\equiv \frac{i}{2}R^a_i\varepsilon_{ijk}\widehat{\mathrm{Tr}}(T_b[R_j, R_k]) \\ &= -\frac{1}{2}\varepsilon_{ijk}f_{bcd}R^a_iR^c_jR^d_k = -\frac{1}{2}\varepsilon_{ijk}R^a_iH^b_{jk}. \end{aligned} \tag{2.22}$$

Note that the self-dual equations (2.11) are labeled by one spatial index $i = 1, \ldots, 3$ and one algebra index $a = 1, \ldots, \dim G$, while, due to the contraction with R^c_i, the Eq. (2.20) are labeled by two algebraic indices $a, c = 1, \ldots, \dim G$. In Sec. III, we show that the $\dim G \times \dim G$ self-duality equations (2.20) are equivalent to the $\dim G \times 3$ self-duality equations (2.11).

From (2.2) and (2.22) the topological charge becomes

$$\begin{aligned} Q &= \frac{i}{96\pi^2}\int d^3x\varepsilon_{ijk}\widehat{\mathrm{Tr}}(R_i[R_j, R_k]) \\ &= -\frac{1}{96\pi^2}\int d^3x\varepsilon_{ijk}f_{abc}R^a_iR^b_jR^c_k \\ &= \frac{1}{48\pi^2}\int d^3x\sigma_{aa} \end{aligned} \tag{2.23}$$

Note that in the particular cases where τ is invertible we can write h in terms of the U fields only as

$$h = \frac{1}{\lambda}\tau^{-1}\sigma. \tag{2.24}$$

So, the self-duality equation is solved for any U-field configuration (as long as τ is invertible at least), and so the h fields are spectators in the sense they adjust themselves to that U-configuration to solve the self-duality equations. However, in the case τ is not invertible, the BPS Skyrmions need to be constructed by solving the self-duality equations (2.20).

Under space parity P transformations $(t, x_i) \to (t, -x_i)$ and under the target space parity P_U transformations $U \to U^{-1}$, where U can be any element of the target space G, we have the same transformations for the quantities $(\tau, \sigma, h, E) \to (\tau, -\sigma, -h, -E)$. Note that the way that h transforms under P and P_U can be derived from the way that σ and τ transform using the self-duality equations (2.20). Clearly, by the space parity transformations τ is a scalar, while E, σ, and the h fields are pseudoscalars. These two sets of transformations shows in particular that the energy E is invariant under the composition PP_U.

The fact that the h fields gets a minus sign in both transformations P and P_U leads to an important distinction of our theory and the standard Skyrme model, where by definition $h = \mathbb{1}$ does not transform. The energy of the standard Skyrme model E_{Sk} is also invariant under the composition PP_U, but this comes from the fact that E_{Sk} is also invariant under both P and P_U transformations separately.

The h fields plays the same role of the Wess-Zumino term with respect the P and P_U transformations. The Wess-Zumino term is introduced into the Skyrme model in [41] to break both invariances P and P_U while preserving the invariance PP_g. This is fundamental in the interpretation of the Skyrme model as an effective theory in the low energy regime, where just the composition PP_U must be a symmetry of the action. In fact, for three flavors the Skyrme field takes its values in the $SU(3)$ Lie group and can be written in terms of an octet formed by pions, kaons, and eta mesons [42–45]. The P_U invariance would forbid, for example, the process $K^+K^- \to \pi^+\pi^-\pi^0$, where K^+ is the kaon, K^- the antikaon, and (π^+, π^-, π^0) corresponds to the three pions with electrical charges $+e, -e, 0$, respectively, where e is the electric charge of the proton. However, this process can be observed experimentally and is allowed in QCD by the non-Abelian anomaly.

III. THE EULER-LAGRANGE EQUATIONS

The Euler-Lagrange equations associated to the Skyrme field and the action (2.14) are

$$\partial_\mu(-\lambda^2 h_{ab} R^{b,\mu} + f_{cba} R^c_\nu h^{-1}_{bd} H^{d,\mu\nu}) - f_{cba}[-\lambda^2 h_{bd} R^d_\mu R^{c,\mu} + h^{-1}_{bd} H^{d,\mu\nu} \partial_\mu R^c_\nu] = 0. \quad (3.1)$$

Its static version is given by

$$\partial_i(\lambda^2 h_{ab} R^b_i + f_{cba} R^c_j h^{-1}_{bd} H^d_{ij}) - f_{cba}[\lambda^2 h_{bd} R^d_i R^c_i + h^{-1}_{bd} H^d_{ij} \partial_i R^c_j] = 0. \quad (3.2)$$

The Euler-Lagrange equations associated to the h_{ab} fields and the action (2.14) are

$$\lambda^2 R^a_\mu R^{b,\mu} + \frac{1}{2} h^{-1}_{ac} h^{-1}_{bd} H^c_{\mu\nu} H^{d,\mu\nu} = 0. \quad (3.3)$$

Let us introduce

$$S^{(\pm),a}_i \equiv |\lambda| R^a_i \pm \frac{1}{2} \varepsilon_{ijk} h^{-1}_{ac} H^c_{jk}, \quad (3.4)$$

which satisfies by construction

$$S^{(+),a}_i S^{(-),b}_i = B_{ab} + A_{ab} \quad (3.5)$$

where

$$B_{ab} \equiv m^2 e^2 R^a_i R^b_i - \frac{1}{2} h^{-1}_{ac} h^{-1}_{bd} H^c_{ij} H^d_{ij};$$
$$A_{ab} \equiv |\lambda|[(\sigma h^{-1})_{ab} - (\sigma h^{-1})_{ba}]. \quad (3.6)$$

The Eq. (3.5) splits $S^{(+),a}_i S^{(-),b}_i$ into its symmetric and antisymmetric parts B_{ab} and A_{ab}, respectively. The static version of (3.3) becomes

$$B_{ab} = 0 \Leftrightarrow S^{(+),a}_i S^{(-),b}_i = A_{ab}. \quad (3.7)$$

On the other hand, the self-duality equations (2.11) can be written as

$$\mathrm{sign}(\lambda) = \pm 1 \Rightarrow S^{(\pm),a}_i = 0, \quad (3.8)$$

which implies

$$S^{(+),a}_i S^{(-),b}_i = 0. \quad (3.9)$$

Let us show that the dim $G \times$ dim G self-duality equations (2.20) are equivalent to the dim $G \times 3$ self-duality equations (2.11), which can also be written as (3.8). The Eq. (2.20) is obtained in Sec. II from (2.11). Now, let us prove that (2.20) implies (2.11). In particular, the self-duality equations are also solutions of the static Euler-Lagrange equation associated with the h field (3.7), i.e., $B_{ab} = 0$. On the other hand, from (2.20), we obtain that σh^{-1} is symmetric, and due to (3.6), we have $A_{ab} = 0$. Therefore, the rhs of (3.5) vanishes, reducing this equation to (3.9). The definition (3.4) leads to

$$S^{(+),a}_i + S^{(-),a}_i = 2|\lambda| R^a_i \quad (3.10)$$

Contracting (3.10) with $S^{(\pm),b}_i$ and using (3.9) we obtain

$$S^{(\pm),a}_i S^{(\pm),b}_i = 2|\lambda| R^a_i S^{(\pm),b}_i = 2|\lambda|[|\lambda|\tau_{ab} \mp (\sigma h^{-1})_{ab}] = 2|\lambda|[|\lambda| \mp \lambda]\tau_{ab} \quad (3.11)$$

where we use (2.20), which implies the self-duality equations (3.8), completing the proof.

Now, let us explicitly show that the Euler-Lagrange equation for the U field (3.2) is implied by the self-duality equations (2.11). Using (2.15) we also can write (2.11) as

$$\varepsilon_{ijk} \partial_j R^a_k = \frac{1}{2} \varepsilon_{ijk} H^a_{jk} = -\lambda R^b_i h_{ba} \Rightarrow H^a_{ij} = -\lambda \varepsilon_{ijk} R^b_k h_{ba}. \quad (3.12)$$

Consequently, $\partial_i(h_{ab} R^a_i) = -\lambda^{-1} \varepsilon_{ijk} \partial_i \partial_j R^b_k = 0$. Using this expression, and defining L_a as the lhs of (3.2), we obtain

$$L_a \equiv \partial_i(\lambda^2 h_{ab} R^b_i + f_{cba} R^c_j h^{-1}_{bd} H^d_{ij}) - f_{cba}[\lambda^2 h_{bd} R^d_i R^c_i + h^{-1}_{bd} H^d_{ij} \partial_i R^c_j] = f_{cba} R^c_j \partial_i(h^{-1}_{bd} H^d_{ij}) - f_{cba} \lambda^2 h_{bd} R^d_i R^c_i. \quad (3.13)$$

However, (3.12) implies $R^c_j \partial_i(h^{-1}_{bd} H^d_{ij}) = -\lambda R^c_j \varepsilon_{ijk} \partial_i (h^{-1}_{bd} R^l_k h_{ld}) = -\lambda R^c_j \varepsilon_{ijk} \partial_i R^b_k = \lambda R^c_i \varepsilon_{ijk} \partial_j R^b_k = -\lambda^2 R^c_i R^d_i h_{db}$. Then, the first and second terms on the lhs of (3.13) are equal, and we can write it as

$$L_a = 2 f_{cba} \lambda R^c_i \varepsilon_{ijk} \partial_j R^b_k = -2i\lambda \widehat{\mathrm{Tr}}(C T_a) \quad (3.14)$$

where we use $f_{cba} = -i\widehat{\mathrm{Tr}}([T_c, T_b] T_a)$ and

$$C \equiv \varepsilon_{ijk}[R_i, \partial_j R_k] = \varepsilon_{ijk}(\partial_j [R_i, R_k] - [\partial_j R_i, R_k]) = -i\varepsilon_{ijk} \partial_j(\partial_i R_k - \partial_k R_i) - \varepsilon_{ijk}[\partial_j R_i, R_k] = -\varepsilon_{ijk}[\partial_j R_i, R_k] = -\varepsilon_{ijk}[R_i, \partial_j R_k] = -C. \quad (3.15)$$

Therefore, $C = 0$ leading due to (3.14) to $L_a = 0$, which corresponds to the static Euler-Lagrange equations for the Skyrme field (3.2).

In the self-dual sector, the Eq. (2.11) implies that σh^{-1} is symmetric. Therefore, $A_{ab} = 0$, and using (3.9) we obtain that the self-duality equations (3.8) imply the static Euler-Lagrange equations for the h field (3.7). However, the converse does not seem to hold true in general. In fact, in any domain $\mathcal{D} \subset S^3$ where σh^{-1} is not a symmetric matrix, Eq. (3.7) gives $S_i^{(+),a} S_i^{(-),b} \neq 0$. Therefore, in this domain we cannot have self-dual solutions, which satisfies (3.9). Additionally, note that this argument does not depend on whether τ is invertible.

In the case of $G = SU(2)$ we can treat the Maurer-Cartan components R_i^a as 3×3 matrix with the following ordering of rows and columns $R_{ia} \equiv R_i^a$, $i = 1, 2, 3$ and $a = 1, 2, 3$. Therefore, $\varepsilon_{ijk} R_i^a R_j^b R_k^c = \varepsilon_{abc} \varepsilon_{ijk} R_{i1} R_{j2} R_{k3} = \varepsilon_{abc} \det R$, and using (2.22), we obtain

$$\sigma_{ab} = -\frac{1}{2} \varepsilon_{ijk} \varepsilon_{bcd} R_i^a R_j^c R_k^d = -\delta_{ab} \det R. \tag{3.16}$$

The $G = SU(2)$ case is very special since both h^{-1} and σ are symmetric by construction, and the structure constant reduces to the Levi-Civita symbol. Using these two properties, it was demonstrated in [7] that the self-duality equations (3.8) are a consequence of the static Euler-Lagrange equations associated with the h fields. In particular, this implies that the static sector is equivalent to the self-dual sector for $G = SU(2)$.

IV. THE HOLOMORPHIC *ANSATZ*

Let us consider a compact simple Lie group G, and let ψ denote its highest positive root. This root can be write in terms of the simple roots α_a, $a = 1, 2, 3, \ldots, \text{rank}\, G$, as $\psi = \sum_{a=1}^{\text{rank}\, G} n_a \alpha_a$, where n_a's are positive integers. The irreducible compact Hermitian symmetric spaces, as defined in (see [46]), correspond to those cases where the expansion of ψ in terms of the simple roots presents at least one coefficient n_a as equals to unity, i.e.,

$$\psi = \alpha_* + \sum_{a=1, a\neq *}^{\text{rank}\, G} n_a \alpha_a \tag{4.1}$$

where α_* denote the simple root that appears only once in the expansion ($n_* = 1$).

Let us denote λ_* the fundamental weight of G, which is not orthogonal to α_*, i.e.,

$$\frac{2\lambda_* \cdot \alpha_*}{\alpha_*^2} = 1; \qquad \frac{2\lambda_* \cdot \alpha_a}{\alpha_a^2} = 0; \quad \text{for } a \neq *. \tag{4.2}$$

The Hermitian symmetric spaces are characterized by the $U(1)$ factor in the little group, and the involutive automorphism σ ($\sigma^2 = 1$), defining the symmetric space structure is inner and constructed from the generator Λ of the $U(1)$ subgroup, i.e.,

$$\sigma(T) \equiv e^{i\pi\Lambda} T e^{-i\pi\Lambda}; \quad \Lambda \equiv \frac{2\lambda_* \cdot H}{\alpha_*^2}; \text{ for any } T \in \mathcal{G} \tag{4.3}$$

where we choose to work in the Cartan-Weyl basis and H_i, $i = 1, 2, 3, \ldots, \text{rank} G$, are the generators of the Cartan subalgebra of $\mathcal{G}$. Denoting E_α as the step operator associated to the root α of $\mathcal{G}$, the Killing form of $\mathcal{G}$ becomes

$$\text{Tr}(H_i H_j) = \delta_{ij}, \quad \text{Tr}(H_i E_\alpha) = 0, \quad \text{Tr}(E_\alpha E_\beta) = \frac{2}{\alpha^2} \delta_{\alpha+\beta,0}. \tag{4.4}$$

The relations (4.3) and $[H_i, E_\alpha] = \alpha_i E_\alpha$, where the index i denotes the component of the root α, leads to $[\Lambda, E_\alpha] = \frac{2\lambda_* \cdot \alpha}{\alpha_*^2} E_\alpha$. Expanding the root through $\alpha = m_* \alpha_* + \sum_{a\neq *}^{\text{rank}\, G} m_{\alpha_a} \alpha_a$, where m_{α_a} are integers and m_* can take the values $-1, 0, 1$ due to (4.1), and using (4.2) we obtain $[\Lambda, E_\alpha] = n_{\alpha_a *} E_\alpha$. Consequently, the step operators $E_{\pm\alpha}$ (anti)commute with $e^{i\pi\Lambda}$ for $n_{\alpha_a *} = 0$ ($n_{\alpha_a *} = \pm 1$). Denoting γ as any positive root of G that does not contain α_* in its expansion in terms of simple roots, and α_κ as the remaining positive roots, we get from (4.3) that

$$\sigma(H_i) = H_i; \quad \sigma(E_{\pm\gamma}) = E_{\pm\gamma}; \quad \sigma(E_{\pm\alpha_\kappa}) = -E_{\pm\alpha_\kappa}. \tag{4.5}$$

Therefore, the Lie algebra $\mathcal{G}$ of G breaks in even and odd subalgebras under the involutive automorphism (4.3), denoted, respectively, by $\mathcal{K}$ and $\mathcal{P}$, i.e.,

$$\mathcal{G} = \mathcal{P} + \mathcal{K} \text{ with } \sigma(P) = -P \quad \sigma(K) = KP \in \mathcal{P}; \quad K \in \mathcal{K}. \tag{4.6}$$

Note that Λ and $E_{\pm\gamma}$ belong to the even subgroup $\mathcal{K}$, and Λ generates an $U(1)_\Lambda$ invariant subalgebra of it. Consequently, we can write $\mathcal{K} = \mathcal{H} \oplus \Lambda$, and we obtain the irreducible compact Hermitian symmetric space $G/H \otimes U(1)_\Lambda$. The subgroup H is generated by $H_a \equiv \frac{2\alpha_a \cdot H}{\alpha_a^2}$, with $\alpha_a \neq \alpha_*$, $(E_\gamma + E_{-\gamma})$, and $i(E_\gamma - E_{-\gamma})$. The odd subgroup is generated by $E_{\pm\alpha_\kappa}$, as defined in (4.5), where $\kappa = 1, 2, \ldots, \frac{\dim \mathcal{P}}{2}$.

The Hermitian symmetric space has the form of a coset G/K, where K is the little group $K = H \otimes U(1)$ and we get the usual algebraic structure of a symmetric space

$$[\mathcal{G}, \mathcal{G}] \subset \mathcal{G} \qquad [\mathcal{G}, \mathcal{P}] \subset \mathcal{P} \qquad [\mathcal{P}, \mathcal{P}] \subset \mathcal{G}. \tag{4.7}$$

The Hermitian character of such symmetric spaces is that $\mathcal{P}$ is even dimensional and it is split by Λ into two parts according its eigenvalues

$$\mathcal{P} = \mathcal{P}_+ + \mathcal{P}_- \qquad [\Lambda, P_\pm] = \pm P_\pm \qquad P_\pm \in \mathcal{P}_\pm. \tag{4.8}$$

The generators of $\mathcal{P}_+$ and $\mathcal{P}_-$ are, respectively, E_{α_κ} and $E_{-\alpha_\kappa}$, with $\kappa = 1, 2, \ldots, \frac{\dim \mathcal{P}}{2}$. It turns out that $\mathcal{P}_-$ is like the Hermitian conjugate of $\mathcal{P}_+$, and so both spaces have the same dimension, i.e., $\dim \mathcal{P}_+ = \dim \mathcal{P}_- = \frac{\dim \mathcal{P}}{2}$. Therefore, Λ not only provides the automorphism σ, but it also provides a gradation of the Lie algebra $\mathcal{G}$ into subspaces of grades 0 and ± 1. Since there are no subspaces of grades ± 2, it turns out that $\mathcal{P}_\pm$ are Abelian. So we have

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K} \qquad [\mathcal{K}, \mathcal{P}_\pm] \subset \mathcal{P}_\pm \qquad [\mathcal{P}_+, \mathcal{P}_+] = [\mathcal{P}_-, \mathcal{P}_-] = 0 \qquad [\mathcal{P}_+, \mathcal{P}_-] \subset \mathcal{K}. \tag{4.9}$$

The compact irreducible Hermitian symmetric spaces are

$$\begin{aligned} &SU(p+q)/SU(p) \otimes SU(q) \otimes U(1); \quad && SO(2N)/SU(N) \otimes U(1); \\ &SO(N+2)/SO(N) \otimes U(1); && Sp(N)/SU(N) \otimes U(1); \\ &E_6/SO(10) \otimes U(1); && E_7/E_6 \otimes U(1). \end{aligned} \tag{4.10}$$

The trace form is invariant under the automorphism σ, i.e., $\mathrm{Tr}(\sigma(T)\sigma(T')) = \mathrm{Tr}(TT')$. Therefore, the even and odd generators are orthogonal

$$\mathrm{Tr}(\mathcal{P}\mathcal{K}) = 0. \tag{4.11}$$

In addition one has

$$0 = \mathrm{Tr}(\Lambda[\mathcal{P}_\pm, \mathcal{P}_\pm]) = \mathrm{Tr}(\mathcal{P}_\pm[\Lambda, \mathcal{P}_\pm]) = \pm \mathrm{Tr}(\mathcal{P}_\pm \mathcal{P}_\pm) \tag{4.12}$$

and so

$$\mathrm{Tr}(\mathcal{P}_+\mathcal{P}_+) = \mathrm{Tr}(\mathcal{P}_-\mathcal{P}_-) = 0. \tag{4.13}$$

The even subalgebra $\mathcal{K}$ has the form $\mathcal{K} = \mathcal{H} \oplus \Lambda$. If $\mathcal{H}$ is simple or even semisimple [no $U(1)$ factors], then it is true that any of its elements can be written as the commutator of some other two, i.e., $\mathcal{H} = [\mathcal{H}', \mathcal{H}'']$. Then it follows that

$$\mathrm{Tr}(\Lambda\mathcal{H}) = \mathrm{Tr}(\Lambda[\mathcal{H}', \mathcal{H}'']) = \mathrm{Tr}([\Lambda, \mathcal{H}']\mathcal{H}'') = 0 \tag{4.14}$$

and so

$$\mathrm{Tr}(\Lambda\mathcal{H}) = 0. \tag{4.15}$$

One nice thing about symmetric spaces (not only Hermitian) is that one can parametrize it quite easily. Given a matrix of the group G one may construct the so-called principal variable

$$X(g) = g\sigma(g)^{-1} \quad \text{and so} \quad X(gk) = X(g) \quad \text{and} \quad \sigma(X) = X^{-1} \tag{4.16}$$

with k being any element of the K subgroup and g is any element of G. Therefore, $X(g)$ parametrizes the coset space (symmetric space) G/K. The three dimensional space $\mathbb{R}^3$ can be foliated with spheres with center at the origin, being useful introduce the spherical coordinates (r, θ, φ). We stereographically project the spheres on a plane with the infinity identified to a point, i.e., the Riemann sphere. The maps from that sphere to the Hermitian symmetric space are labeled by integers. Let z and $\bar{z}$ be the complex coordinates on that plane introduced by the coordinate system $(r, z, \bar{z})$ defined by $(z = z_1 + iz_2)$

$$x_1 = r\frac{i(\bar{z} - z)}{1 + |z|^2}; \quad x_2 = r\frac{z + \bar{z}}{1 + |z|^2}; \quad x_3 = r\frac{(-1 + |z|^2)}{1 + |z|^2}, \tag{4.17}$$

which have the metric

$$ds^2 = dr^2 + \frac{4r^2}{(1 + |z|^2)^2} dz d\bar{z}. \tag{4.18}$$

A powerful *Ansatz* for the Skyrme field U was proposed by L. A. Ferreira and L. R. Livramento in [37]. First, it considers the rational map for the group elements U of the group G as

$$U = g e^{if(r)\Lambda} g^{-1} = e^{if(r) g\Lambda g^{-1}} \qquad \sigma(g) = g^{-1} \tag{4.19}$$

where $f(r)$ is a radial profile function, Λ is defined in (4.3) and g is an element of the compact Lie group G, and it depends only z and $\bar{z}$, i.e., $g = g(z, \bar{z})$. Clearly, as Λ commutes with the elements of the even subgroup $K = H \otimes U(1)_\Lambda$, it so follows that $gk\Lambda(gk)^{-1} = g\Lambda g^{-1}$, with $k \in K$. Consequently, the term $g\Lambda g^{-1}$ depends only on the fields parametrizing the cosets in $G/H \otimes U(1)_\Lambda$, which are parametrized by a principal variable of the form (4.16). In fact, we can take g as a principal variable (4.16), i.e., $g(w) = w\sigma(w)^{-1}$, with w being any element of G depending only on z and $\bar{z}$. The definition (4.16) implies that $\sigma(g) = g^{-1}$, and so the principal variable X is reduced to $X(g) = g\sigma(g)^{-1} = g^2$.

The Maurer-Cartan form associated to g, i.e., $g^{-1}\partial_i g$ can be projected into even and odd subspaces through

$$g^{-1}\partial_i g = P_i + K_i, \quad P_i = \frac{1 - \sigma}{2} g^{-1}\partial_i g, \quad K_i = \frac{1 + \sigma}{2} g^{-1}\partial_i g. \tag{4.20}$$

Since we are dealing with Hermitian symmetric spaces we can use (4.8) to split P_i into the ± 1 subspaces, i.e.,

$$P_i = P_i^{(+)} + P_i^{(-)} \qquad [\Lambda, P_i^{(\pm)}] = \pm P_i^{(\pm)}. \tag{4.21}$$

The second key ingredient of the *Ansatz* proposed in [37] is the construction of the g elements. Let us introduce

$$S = \sum_\kappa w_\kappa E_{\alpha_\kappa}; \qquad [\Lambda, S] = S, \tag{4.22}$$

which lies on $\mathcal{P}_+$ by definition, and where w_κ are functionals of the fields parametrizing the Hermitian symmetric spaces $G/H \otimes U(1)_\Lambda$. Consequently,

$$S^\dagger = \sum_\kappa w_\kappa^* E_{-\alpha_\kappa}; \qquad [\Lambda, S^\dagger] = -S^\dagger. \tag{4.23}$$

First, S and $S^\dagger$ are even, respectively, holomorphic and antiholomorphic, i.e., $S = S(z)$ and $S^\dagger(\bar z)$, or vice versa. We can consider both cases by writing

$$S = S(\chi) \quad \text{and} \quad S^\dagger = S^\dagger(\bar\chi) \quad \text{with} \quad \chi = z, \bar z \tag{4.24}$$

where $\chi = z$ ($\chi = \bar z$) corresponds to a (anti)holomorphic matrix S. Second, the *Ansatz* works for representations of the Lie algebra of G where the matrix associated to S is nilpotent with index of nilpotency equal to two, and an eigenvector of the Hermitian matrix $SS^\dagger$ with a non-negative eigenvalue, i.e.,

$$S^2 = 0; \qquad (SS^\dagger)S = \omega S \tag{4.25}$$

where the eigenvalue ω is non-negative. From (4.25) we can obtain some useful relations, such as

$$\partial_i SS = \partial_i S^\dagger S^\dagger = 0; \qquad SS^\dagger \partial_i SS^\dagger S = \omega(\partial_i \omega S - S\partial_i S^\dagger S). \tag{4.26}$$

On the other hand, the Eq. (4.25) implies $S^{\dagger 2} = 0$ and $(S^\dagger S)S^\dagger = \omega S^\dagger$. Third, the *Ansatz* for $g = g(z, \bar z)$ corresponds to

$$g = \mathbb{1} + \frac{1}{\vartheta}\left[i(S + S^\dagger) - \frac{1}{\vartheta + 1}(SS^\dagger + S^\dagger S)\right]; \quad \vartheta \equiv \sqrt{1 + \omega}, \tag{4.27}$$

which is unitary and satisfies $\sigma(g) = g^{-1}$. Alternatively, g given by (4.27) can be also written as

$$g = e^{iS} e^{\varphi [S, S^\dagger]} e^{iS^\dagger} = e^{ia(S + S^\dagger)} \tag{4.28}$$

with $\varphi = \omega^{-1} \ln \sqrt{1 + \omega}$ and $a = \omega^{-\frac{1}{2}} \arcsin(\frac{\sqrt{\omega}}{\sqrt{1 + \omega}})$. Using also (4.20), (4.21), and (4.25) we get

$$P_i^{(+)} = \frac{i}{\vartheta}\partial_i S + \frac{i}{\vartheta^2(1 + \vartheta)}(S\partial_i S^\dagger S - 2\vartheta \partial_i \vartheta S),$$
$$P_i^{(-)} = \frac{i}{\vartheta}\partial_i S^\dagger + \frac{i}{\vartheta^2(1 + \vartheta)}(S^\dagger \partial_i SS^\dagger - 2\vartheta \partial_i \vartheta S^\dagger), \tag{4.29}$$

which satisfies $(P_i^{(+)})^\dagger = -P_i^{(-)}$. Let us introduce the Hermitian and invertible operator

$$\Omega \equiv \mathbb{1} + \frac{SS^\dagger + S^\dagger S}{1 + \vartheta} \Rightarrow \Omega^{-1} = \mathbb{1} - \frac{SS^\dagger + S^\dagger S}{\vartheta(1 + \vartheta)}. \tag{4.30}$$

Using also (4.25) and (4.26) it so follows that

$$\Omega P_i^{(+)} \Omega = i\partial_i S \overset{\dagger}{\Rightarrow} \Omega P_i^{(-)} \Omega = i\partial_i S^\dagger. \tag{4.31}$$

The details of these calculations are presented in [37].

An important consequence of (4.31) is that the holomorphic and the holomorphic *Ansatz* (4.24) are equivalent to

$$\partial_{\bar\chi} S = \partial_\chi S^\dagger = 0 \Leftrightarrow P_{\bar\chi}^{(+)} = P_\chi^{(-)} = 0 \quad \text{with}$$
$$\chi \equiv \begin{cases} z, & S = S(z) \\ \bar z, & S = S(\bar z) \end{cases}. \tag{4.32}$$

In addition, for later convenience we can also introduce the sign function

$$\eta \equiv \begin{cases} +1, & \chi = z \\ -1, & \chi = \bar z \end{cases}. \tag{4.33}$$

The relation (4.32) also shows that the terms $S\partial_\chi S^\dagger S$ and $S^\dagger \partial_{\bar\chi} SS^\dagger$ vanish, reducing $P_\chi^{(+)}$ and $P_{\bar\chi}^{(-)}$, which can be obtained from (4.29), to

$$P_\chi^{(+)} = i\frac{(1 + \vartheta)^2}{\vartheta}\partial_\chi\left(\frac{S}{(1 + \vartheta)^2}\right);$$
$$P_{\bar\chi}^{(-)} = i\frac{(1 + \vartheta)^2}{\vartheta}\partial_{\bar\chi}\left(\frac{S^\dagger}{(1 + \vartheta)^2}\right). \tag{4.34}$$

Using (4.3) and (4.27) we have that the Lie algebra element appearing in the *Ansatz* (4.19) is given by

$$g\Lambda g^{-1} = \Lambda - \frac{1}{(1 + \omega)}([S, S^\dagger] + i(S - S^\dagger)). \tag{4.35}$$

In particular, if the square of the $U(1)$ generator Λ can be written as $\Lambda^2 = c\Lambda + \frac{1}{4}(1 - c^2)\mathbb{1}$ in such a representation, where c is a real number, it so follows that the quantity

$$Z \equiv \frac{1 + c}{2}\mathbb{1} - g\Lambda g^{-1} \tag{4.36}$$

is a projector, i.e., $Z = Z^2$. This allow us to easily compute the exponential on the rhs of (4.19) reducing the rational map to

$$U = e^{if\frac{1+c}{2}\mathbb{1}}e^{-ifZ} = e^{if\frac{c+1}{2}}[\mathbb{1} + (e^{-if} - 1)Z]. \qquad (4.37)$$

Although this condition on Λ is not assumed in the above *Ansatz*, it appears in the parametrization of the Hermitian symmetric space $G/K = SU(p+q)/SU(p) \otimes SU(q) \otimes U(1)$ given in [37].

A. The self-dual sector

Using (4.20) and (4.21) and the fact that K_i commutes with Λ, the Maurer-Cartan form $R_i = i\partial_i UU^{-1}$ associated to the rational map (4.19) becomes

$$R_i = -V^{-1}\Sigma_i V \qquad (4.38)$$

with

$$V \equiv e^{-if\Lambda/2}g^{-1}; \qquad \Sigma_i \equiv \partial_i f\Lambda - 2\sin\frac{f}{2}\left(P_i^{(+)} - P_i^{(-)}\right). \qquad (4.39)$$

We now have that

$$R_i^b = \frac{1}{\kappa}\mathrm{Tr}(T_b R_i) = -\frac{1}{\kappa}\mathrm{Tr}(VT_bV^{-1}\Sigma_i) = -\frac{1}{\kappa}\mathrm{Tr}(T_c\Sigma_i)d_{cb}(V) \qquad (4.40)$$

where we have introduced the matrix for the group elements in the adjoint representation of G

$$gT_ag^{-1} = T_b d_{ba}(g). \qquad (4.41)$$

Similarly we have

$$\mathrm{Tr}(T_a[R_j, R_k]) = \mathrm{Tr}(VT_aV^{-1}[\Sigma_j, \Sigma_k]) = \mathrm{Tr}(T_c[\Sigma_j, \Sigma_k])d_{ca}(V). \qquad (4.42)$$

From (4.39) we note that Σ_i has components along the $U(1)$ generator Λ, and on the odd subspace $\mathcal{P}$. In fact, using (4.32) and the definition (4.39) we have

$$\Sigma_r = f'\Lambda; \quad \Sigma_\chi = -2\sin\frac{f}{2}P_\chi^{(+)}; \quad \Sigma_{\bar\chi} = 2\sin\frac{f}{2}P_{\bar\chi}^{(-)} \qquad (4.43)$$

where $P_\chi^{(+)}$ and $P_{\bar\chi}^{(-)}$ are given by (4.34).

It follows from (4.21) and (4.38) that $\varepsilon_{ijk}\mathrm{Tr}(R_iR_jR_k) = 12\varepsilon_{ijk}\partial_i f\sin^2\frac{f}{2}\mathrm{Tr}(P_j^{(+)}P_k^{(-)})$, which reduces the topological charge (2.2) to

$$Q = \frac{i}{4\pi^2\kappa}\int drdzd\bar{z}\partial_r f\sin^2\frac{f}{2}\mathrm{Tr}\left(P_z^{(+)}P_{\bar z}^{(-)} - P_{\bar z}^{(+)}P_z^{(-)}\right) = \frac{1}{2\pi}[f(r) - \sin f(r)]_{r=0}^{r=\infty}Q_{\mathrm{top}} \qquad (4.44)$$

with $Q_{\mathrm{top}} \equiv \frac{i}{4\pi\kappa}\int dzd\bar{z}\mathrm{Tr}(P_z^{(+)}P_{\bar z}^{(-)} - P_{\bar z}^{(+)}P_z^{(-)}) = \eta\frac{i}{4\pi\kappa}\int dzd\bar{z}\mathrm{Tr}(P_\chi^{(+)}P_{\bar\chi}^{(-)})$, where we use (4.32) and (4.33).

Using (4.38) and (4.43) the self-duality equations (2.11) can be written as

$$\lambda\tilde\tau_{cb}\tilde h_{ba} = \tilde\sigma_{ca} \qquad (4.45)$$

where we have introduced the matrices

$$\tilde h_{ab} \equiv d_{ac}(V)h_{cd}d_{db}^{-1}(V); \quad \tilde\tau_{ab} \equiv d_{ac}(V)\tau_{cd}d_{db}^{-1}(V); \quad \tilde\sigma_{ab} \equiv d_{ac}(V)\sigma_{cd}d_{db}^{-1}(V). \qquad (4.46)$$

The adjoint representation of a compact simple Lie group is unitary and real, and so d is an orthogonal matrix, i.e., $d^T = d^{-1}$. Therefore, $\tilde h_{ab}$ and $\tilde\tau_{ab}$ are still symmetric. In addition, we have $\tilde\tau_{ab} = \Sigma_i^a\Sigma_i^b$ and $\tilde\sigma_{ab} = -\frac{i}{2}\widehat{\mathrm{Tr}}(T_a\Sigma_i)\times\varepsilon_{ijk}\widehat{\mathrm{Tr}}(T_b[\Sigma_j, \Sigma_k])$. Using $i\varepsilon_{ijk}\frac{\partial r}{\partial x^i}\frac{\partial\chi}{\partial x^j}\frac{\partial\bar\chi}{\partial x^k} = \eta\frac{(1+|z|^2)^2}{2r^2}$ and (4.43) we obtain

$$\tilde\sigma_{ab} \equiv y(\Gamma^{ab}_{r\chi\bar\chi} + \Gamma^{ab}_{\bar\chi r\chi} + \Gamma^{ab}_{\chi\bar\chi r})$$

with $y \equiv 2\eta f'\sin^2\frac{f}{2}\frac{(1+|z|^2)^2}{r^2}$ and $\Gamma^{ab}_{\alpha\beta\gamma} \equiv -(4f'\sin^2\frac{f}{2})^{-1}\times\widehat{\mathrm{Tr}}(T_a\Sigma_\alpha)\widehat{\mathrm{Tr}}(T_b[\Sigma_\beta, \Sigma_\gamma])$, which is antisymmetric under the exchange of its last two indices β and γ and has the following components

$$\Gamma^{ab}_{r\chi\bar\chi} = \widehat{\mathrm{Tr}}(T_a\Lambda)\widehat{\mathrm{Tr}}\left(T_b\left[P_\chi^{(+)}, P_{\bar\chi}^{(-)}\right]\right) \quad \text{that vanishes if } T_a \neq \Lambda \quad \text{or} \quad T_b \in \mathcal{P}$$

$$\Gamma^{ab}_{\bar\chi r\chi} = \widehat{\mathrm{Tr}}\left(T_aP_{\bar\chi}^{(-)}\right)\widehat{\mathrm{Tr}}\left(T_bP_\chi^{(+)}\right) \quad \text{that vanishes if } T_a \notin \mathcal{P}_+ \quad \text{or} \quad T_b \notin \mathcal{P}_-$$

$$\Gamma^{ab}_{\chi\bar\chi r} = \widehat{\mathrm{Tr}}\left(T_aP_\chi^{(+)}\right)\widehat{\mathrm{Tr}}\left(T_bP_{\bar\chi}^{(-)}\right) \quad \text{that vanishes if } T_a \notin \mathcal{P}_- \quad \text{or} \quad T_b \notin \mathcal{P}_+$$

where we use (4.21) and the cyclic property of the trace. The components of the matrices $\tilde{\tau}$ and $\tilde{\sigma}$ become

$$\tilde{\tau}_{\mathcal{H}a}=0;\qquad \tilde{\tau}_{\Lambda a}=\begin{cases} f'^2[\widehat{\mathrm{Tr}}(\Lambda^2)]^2, & T_a=\Lambda \\ 0, & \text{otherwise}\end{cases}$$
$$\tilde{\tau}_{\mathcal{P}_+a}=\begin{cases} (-\eta f')^{-1}y\widehat{\mathrm{Tr}}\left(\mathcal{P}_+P_{\bar{\chi}}^{(-)}\right)\widehat{\mathrm{Tr}}\left(T_aP_{\chi}^{(+)}\right), & T_a\in\mathcal{P}_- \\ 0, & \text{otherwise}\end{cases}$$
$$\tilde{\tau}_{\mathcal{P}_-a}=\begin{cases} (-\eta f')^{-1}y\widehat{\mathrm{Tr}}\left(\mathcal{P}_-P_{\chi}^{(+)}\right)\widehat{\mathrm{Tr}}\left(T_aP_{\bar{\chi}}^{(-)}\right), & T_a\in\mathcal{P}_+ \\ 0, & \text{otherwise}\end{cases}\tag{4.47}$$

and

$$\tilde{\sigma}_{a\mathcal{H}}=\begin{cases} y\widehat{\mathrm{Tr}}(\Lambda^2)\widehat{\mathrm{Tr}}\left(\mathcal{H}\left[P_{\chi}^{(+)},P_{\bar{\chi}}^{(-)}\right]\right), & T_a=\Lambda \\ 0, & \text{otherwise}\end{cases}$$
$$\tilde{\sigma}_{\Lambda\Lambda}=y\widehat{\mathrm{Tr}}(\Lambda^2)\widehat{\mathrm{Tr}}\left(P_{\chi}^{(+)}P_{\bar{\chi}}^{(-)}\right)$$
$$\tilde{\sigma}_{\mathcal{P}_+\mathcal{P}_-}=\tilde{\sigma}_{\mathcal{P}_-\mathcal{P}_+}=y\widehat{\mathrm{Tr}}(\mathcal{P}_+P_{\bar{\chi}}^{(-)})\widehat{\mathrm{Tr}}(\mathcal{P}_-P_{\chi}^{(+)})\overset{(4.47)}{=}-\eta f'\tilde{\tau}_{\mathcal{P}_+\mathcal{P}_-}$$
$$\tilde{\sigma}_{\mathcal{H}b}=\tilde{\sigma}_{\mathcal{P}\Lambda}=\tilde{\sigma}_{\mathcal{P}_+\mathcal{P}_+}=\tilde{\sigma}_{\mathcal{P}_-\mathcal{P}_-}=0.\tag{4.48}$$

The index $\mathcal{H}$ in the row or column indices of the matrices $\tilde{\tau}$ and $\tilde{\sigma}$ represents any index a that labels the generators T_a of the subalgebra $\mathcal{H}$, and so on. A crucial consequence of $\tilde{\tau}_{a\mathcal{H}}=\tilde{\sigma}_{a\mathcal{H}}=0$ for all $a=1,\ldots,\dim G$ is that none of the self-dual equations (4.45) depends on the $\tilde{h}_{\mathcal{H}\mathcal{H}}$ fields. Then, if $\mathcal{H}\neq\varnothing$, it follows that $\tilde{h}_{\mathcal{H}\mathcal{H}}$ is totally undetermined and $\tilde{\tau}$ is not invertible. Otherwise, we could apply $\tilde{\tau}^{-1}$ to the self-dual equations (4.45) fixing $\tilde{h}$ entirely.

The self-duality equations associated with the row index $\mathcal{H}$ of $\tilde{\tau}$, given by $\lambda\tilde{\tau}_{\mathcal{H}b}\tilde{h}_{ba}=\tilde{\sigma}_{\mathcal{H}a}$, are automatically satisfied by (4.47) and (4.43), and the remaining equations are reduced to

$$\tilde{\sigma}_{\Lambda b}=\lambda\tilde{\tau}_{\Lambda\Lambda}\tilde{h}_{\Lambda b}\Rightarrow\tilde{h}_{\Lambda b}=\frac{\tilde{\sigma}_{\Lambda b}}{\lambda\tilde{\tau}_{\Lambda\Lambda}},\tag{4.49}$$

$$\begin{aligned}0&=\tilde{\tau}_{\mathcal{P}_+\mathcal{P}_-}\tilde{h}_{\mathcal{P}_-\mathcal{P}_+}=\tilde{\tau}_{\mathcal{P}_+\mathcal{P}_-}\tilde{h}_{\mathcal{P}_-\mathcal{H}}\\&=\tilde{\tau}_{\mathcal{P}_-\mathcal{P}_+}\tilde{h}_{\mathcal{P}_+\mathcal{P}_-}=\tilde{\tau}_{\mathcal{P}_-\mathcal{P}_+}\tilde{h}_{\mathcal{P}_+\mathcal{H}},\end{aligned}\tag{4.50}$$

$$0=\tilde{\tau}_{\mathcal{P}_-\mathcal{P}_+}(\tilde{h}_{\mathcal{P}_+\mathcal{P}_+}+\eta\lambda^{-1}f'\mathbb{1})=\tilde{\tau}_{\mathcal{P}_+\mathcal{P}_-}(\tilde{h}_{\mathcal{P}_-\mathcal{P}_-}+\eta\lambda^{-1}f'\mathbb{1}).\tag{4.51}$$

Note that there is an implicit sum over the line index of the $\tilde{h}$ matrix, leading to a linear system to the $\tilde{h}_{ab}$ fields. However, in (4.49) this sum is performed over a single generator, which corresponds to the $U(1)$ generator Λ. This is a consequence of the fact that $\tilde{\tau}_{\Lambda\Lambda}$ is the only nonvanishing component of $\tilde{\tau}_{\Lambda a}$ given in (4.47). Therefore, $\tilde{h}_{\Lambda a}$ is fully determined by

$$\tilde{h}_{\Lambda\Lambda}=\alpha\eta\mathrm{Tr}(P_{\chi}^{(+)}P_{\bar{\chi}}^{(-)});$$
$$\tilde{h}_{\Lambda\mathcal{H}}=\alpha\eta\mathrm{Tr}(\mathcal{H}[P_{\chi}^{(+)},P_{\bar{\chi}}^{(-)}]);\qquad \tilde{h}_{\Lambda\mathcal{P}_\pm}=0$$
$$\text{with}\quad \alpha\equiv\frac{2\sin^2\frac{f}{2}}{\lambda f'\mathrm{Tr}(\Lambda^2)}\frac{(1+|z|^2)^2}{r^2}.\tag{4.52}$$

Note that we can replace the modified trace $\widehat{\mathrm{Tr}}$ defined in (2.6) by the usual trace Tr since the κ factor cancels in the self-duality equations (4.45).

Clearly, if the matrix $\tilde{\tau}_{\mathcal{P}_-\mathcal{P}_+}$ is invertible, then (4.50) and (4.51) lead to

$$\tilde{h}_{\mathcal{P}_\pm\mathcal{H}}=0\qquad \tilde{h}_{\mathcal{P}\mathcal{P}}=-\eta\lambda^{-1}f'\mathbb{1}\tag{4.53}$$

However, if $\tilde{\tau}_{\mathcal{P}_-\mathcal{P}_+}$ is not invertible, the fields (4.52) and (4.53) are still a particular self-dual solution of (4.45). In any case, using (4.47), the self-duality equation for the fields $\tilde{h}_{\mathcal{P}_\pm\mathcal{H}}$ of (4.50) can be written as

$$\mathrm{Tr}(\mathcal{P}_-P_{\chi}^{(+)})\tilde{h}_{\mathcal{P}_-\mathcal{H}}=\mathrm{Tr}(\mathcal{P}_+P_{\bar{\chi}}^{(-)})\tilde{h}_{\mathcal{P}_+\mathcal{H}}=0.\tag{4.54}$$

There is no other self-duality equation that depends on the fields $\tilde{h}_{\mathcal{P}_\pm\mathcal{H}}$. Thus, there are just $\dim\mathcal{H}$ equations to fix the components $\tilde{h}_{\mathcal{P}_+\mathcal{H}}$, and there is an independent set of $\dim\mathcal{H}$ equations to fix the components $\tilde{h}_{\mathcal{P}_-\mathcal{H}}$. Therefore, there are at least $2\dim\mathcal{H}(\dim\mathcal{P}_+-1)$ components of $\tilde{h}_{\mathcal{H}\mathcal{P}}$ free. On the other hand, using also (4.32), we obtain a set of $\dim\mathcal{P}_+$ equations to the fields $\tilde{h}_{\mathcal{P}_-\mathcal{P}_+}$ given by

$$\mathrm{Tr}(\mathcal{P}_-P_{\chi}^{(+)})\tilde{h}_{\mathcal{P}_-\mathcal{P}_+}=0.\tag{4.55}$$

There is another set of linear equations given by $\mathrm{Tr}(\mathcal{P}_+P_{\bar{\chi}}^{(-)})\tilde{h}_{\mathcal{P}_+\mathcal{P}_-}=0$ that comes from (4.50) but corresponds to the complex conjugate of (4.55). Finally, using (4.47) we can write (4.51) as

$$\mathrm{Tr}(\mathcal{P}_+P_{\bar{\chi}}^{(-)})\tilde{h}_{\mathcal{P}_+\mathcal{P}_+}=-\eta\lambda^{-1}f'\mathrm{Tr}(\mathcal{P}_+P_{\bar{\chi}}^{(-)}),\tag{4.56}$$

$$\mathrm{Tr}(\mathcal{P}_-P_{\chi}^{(+)})\tilde{h}_{\mathcal{P}_-\mathcal{P}_-}=-\eta\lambda^{-1}f'\mathrm{Tr}(\mathcal{P}_-P_{\chi}^{(+)}).\tag{4.57}$$

Consequently, there are $\dim\mathcal{P}_+$ equations to fix the $\dim\mathcal{P}_+(\dim\mathcal{P}_++1)/2$ fields $\tilde{h}_{\mathcal{P}_+\mathcal{P}_+}$, and the same follows for the fields $\tilde{h}_{\mathcal{P}_-\mathcal{P}_-}$. Since $\tilde{h}_{\mathcal{P}\mathcal{P}}$ has $\dim\mathcal{P}$ diagonal elements and forms itself a symmetric matrix, then the relations (4.55)–(4.57) together compose a set of $3\dim\mathcal{P}_+$ equations that contain $\frac{\dim\mathcal{P}(\dim\mathcal{P}+1)}{2}=\dim\mathcal{P}_+(2\dim\mathcal{P}_++1)$ components of the $\tilde{h}$ fields.

Such facts lead to the freedom of at least $2\dim\mathcal{P}_+(\dim\mathcal{P}_+-1)$ components of the $\tilde{h}_{\mathcal{P}\mathcal{P}}$ matrix.

The above arguments show that $\dim\mathcal{P}_+=1$ is a necessary condition for the fields $\tilde{h}_{\mathcal{P}\mathcal{P}}$ to be fully determined by the self-duality equations (4.45). On the other hand, we also show above that if $\tilde{\tau}_{\mathcal{P}_-\mathcal{P}_+}$ is invertible, then $\tilde{h}_{\mathcal{P}\mathcal{P}}$ must be fully determined by (4.53). Consequently, $\dim\mathcal{P}_+=1$ is also a necessary condition for the $\tilde{\tau}_{\mathcal{P}_-\mathcal{P}_+}$ matrix to be invertible. In particular, for the $G=SU(2)$ case $\tilde{\tau}_{\mathcal{P}_-\mathcal{P}_+}$ is a real-value function.

V. THE $SU(2)/U(1)$ HERMITIAN SYMMETRIC SPACE

In this case we have the symmetric space $SU(2)/U(1)$ and so

$$\Lambda=T_3;\quad \mathcal{H}=\varnothing;\quad \mathcal{P}_+=\{T_+\};\quad \mathcal{P}_-=\{T_-\} \tag{5.1}$$

with

$$[T_3,T_\pm]=\pm T_\pm;\qquad [T_+,T_-]=2T_3. \tag{5.2}$$

The quantity g is

$$g=\frac{1}{\sqrt{1+|u|^2}}\begin{pmatrix}1 & iu\\ i\bar{u} & 1\end{pmatrix} \tag{5.3}$$

with $S=u$ and $S^\dagger=\bar{u}$ and

$$g^{-1}\partial_i g=\frac{1}{1+|u|^2}[i(\partial_i uT_+ +\partial_i\bar{u}T_-)+(u\partial_i\bar{u}-\bar{u}\partial_i u)T_3]. \tag{5.4}$$

Then, the quantities K_i, $P_i^{(+)}$ and $P_i^{(-)}$ introduced in (4.20) and (4.21) become

$$K_i=\frac{u\partial_i\bar{u}-\bar{u}\partial_i u}{1+|u|^2}T_3;\; P_i^{(+)}=\frac{i\partial_i u}{1+|u|^2}T_+;\; P_i^{(-)}=\frac{i\partial_i\bar{u}}{1+|u|^2}T_-. \tag{5.5}$$

Note that (anti)holomorphic *Ansatz* $S=S(\chi)$ implies $u=u(\chi)$. We shall use the trace form in the doublet representation where [see Eq. (2.5)]

$$\kappa=\frac{1}{2};\qquad \mathrm{Tr}(T_+T_-)=1. \tag{5.6}$$

Clearly, since there is no generator of the subalgebra $\mathcal{H}$, the self-dual equations (4.54) are trivial. Using (5.1) and (5.5) it follows that the components $\tilde{h}_{\Lambda\Lambda}$, $\tilde{h}_{\Lambda\mathcal{P}_\pm}$, $\tilde{h}_{\mathcal{P}_\pm\mathcal{P}_\pm}$, and $\tilde{h}_{\mathcal{P}_\pm\mathcal{P}_\mp}$, given in (4.52) and (4.55)–(4.57), become

$$\tilde{h}_{\Lambda\Lambda}=-\eta\frac{4\sin^2(\frac{f}{2})}{\lambda f' r^2}\frac{(1+|z|^2)^2}{(1+|u|^2)^2}u'\bar{u}';$$
$$\tilde{h}_{T_+T_+}=\tilde{h}_{T_-T_-}=-\eta\frac{f'}{\lambda}, \tag{5.7}$$

$$\tilde{h}_{\Lambda T_\pm}=\tilde{h}_{T_\pm\Lambda}=\tilde{h}_{T_+T_-}=\tilde{h}_{T_-T_+}=0. \tag{5.8}$$

Therefore, the $\tilde{h}$ fields form the diagonal matrix

$$\tilde{h}=-\eta\frac{f'}{\lambda}\mathrm{diag.}\left(1,1,\frac{4\sin^2(\frac{f}{2})}{r^2f'^2}\frac{(1+|z|^2)^2}{(1+|u|^2)^2}u'\bar{u}'\right), \tag{5.9}$$

which is fully determined in terms of the fields $f,u,\bar{u}$, which remains totally free. Note that due to (4.46) the eigenvalues of the $\tilde{h}$ matrix are the same as eigenvalues of the h matrix, which is non-negative. It so follows from (5.9) that the profile function f must be a monotonic function and

$$\mathrm{sign}(f'\lambda)=-\eta, \tag{5.10}$$

which due to (2.18) also implies that $\mathrm{sign}(f'Q)=-\eta$.

For the function $u(\chi)$ to be a well-defined map between two-spheres it has to be a ratio of two polynomials $p(\chi)$ and $q(\chi)$ without common roots, i.e., the so-called rational map *Ansatz* [3,39,40,47]

$$u(\chi)=\frac{p(\chi)}{q(\chi)}. \tag{5.11}$$

The topological degree of the u map is equal to the highest degree among the polynomials $p(\chi)$ and $q(\chi)$ and can be written in the integral representation as

$$\deg\, u=\int\frac{idzd\bar{z}}{2\pi(1+|z|^2)^2}\left(\frac{1+|z|^2}{1+|u|^2}\left|\frac{du}{dz}\right|\right)^2$$
$$=\max\{\deg\, p,\deg\, q\}. \tag{5.12}$$

Therefore, using (5.5) and (5.12) the topological charge (4.44) becomes

$$Q=\eta\frac{[f-\sin f]_{r=\infty}^{r=0}}{2\pi}\deg\, u. \tag{5.13}$$

VI. THE $SU(N+1)/SU(N)\otimes U(1)$ HERMITIAN SYMMETRIC SPACE

For the Hermitian symmetric space $CP^N=SU(N+1)/SU(N)\otimes U(1)$ we choose $\alpha_*=\alpha_N$, which implies $\lambda_*=\lambda_N$, and work with the fundamental $(N+1)\times(N+1)$ representation of $SU(N+1)$ (see Fig. 1). The S matrix is is parametrized by N complex scalar fields $u_a=u_a(\chi)$, with $a=1,\ldots,N$, corresponding with the components of

$A_r \equiv SU(r+1)$

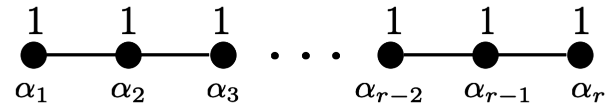

FIG. 1. Dynkin diagrams of the simple Lie algebra A_r. The α_a's below the spots label the simple roots, and the numbers above correspond to the integers m_a in the expansion of the highest root $\psi = \sum_{a=1}^{r} m_a \alpha_a$, while the black spots correspond to $m_a = 1$.

$u^T = (u_1, \ldots, u_N)$. The Λ and S matrices defined, respectively, by (4.3) and (4.22) are given by

$$\Lambda = \frac{1}{N+1}\begin{pmatrix} \mathbb{1}_{N\times N} & 0 \\ 0 & -N \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} O_{N\times N} & u \\ O_{1\times N} & 0 \end{pmatrix} \tag{6.1}$$

where $O_{1\times N}$ is a $1\times N$ zero matrix, and so on. The S matrix (6.1) satisfies (4.25) with $\omega = u^\dagger u$ and so the g elements (4.27) that parametrize the CP^N are given by the unitary matrix

$$g = \frac{1}{\vartheta}\begin{pmatrix} \Delta & iu \\ iu^\dagger & 1 \end{pmatrix} \tag{6.2}$$

where Δ is a $N\times N$ Hermitian matrix defined by

$$\Delta \equiv \vartheta \mathbb{1}_{N\times N} + (1-\vartheta)T_u; \quad \text{with} \quad \vartheta \equiv \sqrt{1+u^\dagger u} \tag{6.3}$$

where $T_u \equiv \frac{u\otimes\bar{u}}{u^\dagger u}$ is a projector, i.e., $T_u^2 = T_u$. Note that due to (6.3) u is an eigenvector of Δ with eigenvalue $+1$, i.e., $\Delta u = u$, which also implies that $u^\dagger \Delta = u^\dagger$ and $\Delta^{-1} u = u$.

The two sets of N Abelian matrices $P_+^a \in \mathcal{P}_+$ and $P_-^a \in \mathcal{P}_-$ associated with (6.2) are given by

$$(P_+^a)_{bc} \equiv \delta_{ba}\delta_{c(N+1)}, \qquad (P_-^a)_{bc} = (P_+^{a\dagger})_{bc} = \delta_{b(N+1)}\delta_{ca} \tag{6.4}$$

with $b, c = 1, \ldots, N+1$, and $a = 1, \ldots, N$. The two sets of N generators T^{2a-1} and T^{2a}, with $a = 1, \ldots, N$, are related to $P_\pm^a$ through $P_\pm^a = T^{2a-1} \pm iT^{2a}$, and satisfy the orthogonality relation (2.5) with $\kappa = \frac{1}{2}$. Therefore, together with (6.2) such a generator satisfies (4.8), and in addition we have

$$\mathrm{Tr}(P_\pm^a P_\pm^b) = 0; \qquad \mathrm{Tr}(P_\pm^a P_\mp^b) = \delta_{ab}; \qquad [\Lambda, P_\pm^a] = \pm P_\pm^a. \tag{6.5}$$

Note that the last equation of (6.5) corresponds to (4.8).[1]

The $N^2 - 1$ generator of the group $\mathcal{H} = SU(N)$ can be broken in three set of generators. The first two sets $\mathcal{H}_R$ and $\mathcal{H}_I$ contain $\frac{N(N-1)}{2}$ generators each and can be labeled by the pair nm, with $n = 1, \ldots, N$ and $m = 1, \ldots, n-1$, i.e. $m < n$. The third set $\mathcal{H}_s$ contains $N-1$ generators and is labeled by the index $s = 1, \ldots, N-1$. The generators of such sets can be written, respectively, for all $a, b = 1, \ldots, N$, as

$$\begin{aligned} (H_R^{nm})_{ab} &= \frac{1}{2}(\delta_{an}\delta_{bm} + \delta_{am}\delta_{bn}), \\ (H_I^{nm})_{ab} &= -\frac{i}{2}(\delta_{an}\delta_{bm} - \delta_{am}\delta_{bn}), \\ (H_s)_{ab} &= \frac{1}{2}(\delta_{as}\delta_{bs} - \delta_{a(s+1)}\delta_{b(s+1)}). \end{aligned} \tag{6.6}$$

Note that H_R^{nm}, H_I^{nm}, and H_s are extensions of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$, respectively, and for such basis $\kappa = \frac{1}{2}$. It so follows that

$$\begin{aligned} \dim \mathcal{H} &= N^2 - 1; \\ \dim \mathcal{P} &= 2\dim \mathcal{P}_+ = 2\dim \mathcal{P}_- = 2N. \end{aligned} \tag{6.7}$$

In particular, for the $SU(2)$ case ($N = 1$) we have $\mathcal{H} = \varnothing$ [see Eq. (5.1)].

Due to (6.2) and (6.4) the quantities K_i and $P_i^{(\pm)}$ defined in (4.20) and (4.21) so become

$$K_i = -\frac{\partial_i \vartheta}{\vartheta}\mathbb{1} + \frac{1}{\vartheta^2}\begin{pmatrix} \Delta\partial_i\Delta + u\otimes\partial_i\bar{u} & 0 \\ 0 & u^\dagger\partial_i u \end{pmatrix}, \tag{6.8}$$

$$P_i^{(+)} = \frac{i(\Delta\partial_i u)_a}{\vartheta^2}P_+^a; \qquad P_i^{(-)} = \frac{i(\partial_i u^\dagger \Delta)_a}{\vartheta^2}P_-^a \tag{6.9}$$

where there is an implicit sum over the index a. In addition, the topological charge (4.44) becomes

$$Q = \eta \frac{[f - \sin f]_{r=\infty}^{r=0}}{2\pi}\int \frac{|\Delta u'|^2}{(1+|u|^2)^2}\frac{idz\wedge d\bar{z}}{2\pi}. \tag{6.10}$$

Using (6.1), (6.4), (6.6), and (6.9) the fields $\tilde{h}_{\Lambda\mathcal{P}_\pm}$ and $\tilde{h}_{\Lambda\mathcal{H}}$ fixed through (4.52) become

$$\begin{aligned} &\tilde{h}_{\Lambda\Lambda} = \beta|\Delta u'|^2; \quad \tilde{h}_{\Lambda\mathcal{P}_+^b} = \tilde{h}_{\Lambda\mathcal{P}_-^b} = 0; \\ &\tilde{h}_{\Lambda H_s} = \frac{1}{2}\beta(M_{ss} - M_{(s+1)(s+1)}), \\ &\tilde{h}_{\Lambda H_R^{nm}} = \frac{1}{2}\beta(M_{nm} + M_{mn}); \quad \tilde{h}_{\Lambda H_I^{nm}} = \frac{1}{2}\beta i(M_{nm} - M_{mn}) \end{aligned} \tag{6.11}$$

[1]Clearly, the Eq. (4.8) is invariant by transformations $P_\pm^a \to \alpha P_\pm^a$ with α being any complex number. Therefore, the only matrix proportional to Λ that satisfies (4.8) is itself, which is given by (6.2).

where

$$\beta \equiv -\frac{\eta}{\lambda f'}\frac{(N+1)}{N}\frac{2\sin^2(\frac{f}{2})}{r^2}\frac{(1+z\bar{z})^2}{(1+u^\dagger u)^2};$$
$$M_{nm} \equiv (\Delta u')_n (u'^\dagger \Delta)_m. \tag{6.12}$$

On the other hand, the self-dual equations for the fields $\tilde{h}_{\mathcal{P}_\pm \mathcal{H}}$ and $\tilde{h}_{\mathcal{P}_\pm \mathcal{P}_\mp}$, as given, respectively, in (4.54) and (4.55), are reduced to

$$(u'^\dagger \Delta)_a \tilde{h}_{P^a_+ \mathcal{H}} = (\Delta u')_a \tilde{h}_{P^a_- \mathcal{H}} = (\Delta u')_a \tilde{h}_{P^a_- P^b_+} = 0 \tag{6.13}$$

while the self-dual equations for the fields $\tilde{h}_{\mathcal{P}_+\mathcal{P}_+}$ and $\tilde{h}_{\mathcal{P}_-\mathcal{P}_-}$, as given, respectively, in (4.56) and (4.57), are reduced to

$$(\Delta u')_a \tilde{h}_{P^a_- P^b_-} + \eta \frac{f'}{\lambda}(\Delta u')_b = (u'^\dagger \Delta)_a \tilde{h}_{P^a_+ P^b_+} + \eta \frac{f'}{\lambda}(u'^\dagger \Delta)_b = 0. \tag{6.14}$$

Using (6.7), the first and second equations of (6.13), from the left to the right, form each a set of dim $\mathcal{H} = N^2 - 1$ linear equations for the $N(N^2-1)$ fields $\tilde{h}_{P^a_+ \mathcal{H}}$ and the $N(N^2-1)$ fields $\tilde{h}_{P^a_- \mathcal{H}}$, respectively. Since $\tilde{h}$ is symmetric, the third equation of (6.13), from the left to the right, forms a set of dim $\mathcal{P}_+ = N$ linear equations for the N^2 fields $\tilde{h}_{P^a_- P^b_+}$. Finally, the first and second equation of (6.14), from the left to the right, forms each a set of dim $\mathcal{P}_+ = N$ linear equations for the $N(N+1)/2$ fields $\tilde{h}_{P^a_+ P^b_+}$ and the $N(N+1)/2$ fields $\tilde{h}_{P^a_- P^b_-}$, respectively.

Consequently, only for $N = 1$ do we have enough equations to determine such a components of the $\tilde{h}$ fields. However, for such a case dim $\mathcal{H} = 0$ and dim $\mathcal{P}_+ = 1$. Thus, although we have four equations and only three independent fields in $\tilde{h}_{\mathcal{P}\mathcal{P}}$, it so follows that the third and fourth equations of (6.13), from the left to the right, which correspond to one equation each, become equivalent. Thus, the $\tilde{h}$ fields are totally determined in terms of the fields $f, u, \bar{u}$, which remains totally free, as we shown in Sec. V.

A. An explicit example: Exact generalized Skyrmions for each integer value of Q on the CP^N spaces

To construct an explicit self-dual configuration consider the *Ansatz* where all the u fields are equal to the same holomorphic rational map $u_1(\chi) = p(\chi)/q(\chi)$ between the Riemann spheres S^2 [see Eq. (5.11)], the $\tilde{h}_{\mathcal{P}\mathcal{P}}$-fields forms a diagonal matrix, and $\tilde{h}_{\mathcal{P}\mathcal{H}} = 0$, i.e.,

$$u_a = u_1 = \frac{p(\chi)}{q(\chi)}; \quad \tilde{h}_{\mathcal{P}\mathcal{H}} = \tilde{h}_{P^a_\pm P^b_\mp} = 0; \quad \tilde{h}_{P^a_\pm P^b_\pm} = \delta_{ab} \tilde{h}_{P^a_\pm P^b_\pm} \tag{6.15}$$

for $a, b = 1, \ldots, N$ and where there is no implicit sum over a or b. It so follows that all self-dual equations given in (6.13) are automatically satisfied, while (6.14) imposes that $\tilde{h}_{\mathcal{P}\mathcal{P}}$ must be

$$\tilde{h}_{\mathcal{P}\mathcal{P}} = -\eta \frac{f'}{\lambda} \mathbb{1}_{2N\times 2N}. \tag{6.16}$$

On the other hand, using the *Ansatz* (6.15) and the definitions (6.3) and (6.12) we obtain $(\Delta u')_n = u'_1$, with $n = 1, \ldots, N$, which due to (6.15) implies $M_{nm} = u'_1 \bar{u}'_1$. Therefore, the fields given in (4.55) are reduced to

$$\tilde{h}_{\Lambda\Lambda} = -\frac{\eta}{\lambda f'}\frac{2(N+1)\sin^2(\frac{f}{2})}{r^2}\frac{(1+z\bar{z})^2}{(1+N|u_1|^2)^2}\bar{u}'_1 u'_1,$$
$$\tilde{h}_{\Lambda \mathcal{P}_\pm} = \tilde{h}_{\Lambda \mathcal{H}_I} = \tilde{h}_{\Lambda \mathcal{H}_s} = 0; \qquad \tilde{h}_{\Lambda \mathcal{H}_R^{nm}} = \frac{1}{N}\tilde{h}_{\Lambda\Lambda} \tag{6.17}$$

where we used $u'^\dagger \Delta^2 u' = \sum_{n=1}^N M_{nn} = N\bar{u}'_1 u'_1$. Therefore, the only nonvanishing components of $\tilde{h}_{\Lambda\mathcal{H}}$ are the $\tilde{h}_{\Lambda\mathcal{H}_R}$ fields, which in turn form a column with all components equal to $N^{-1}\tilde{h}_{\Lambda\Lambda}$. An interesting consequence is that the nonsingular $\tilde{h}$ matrix inside the *Ansatz* (6.15) is nondiagonal for $N > 1$. On the other hand, the field configuration (6.15)–(6.17) is a clear generalization of the (5.9), as obtained for $N = 1$ in Sec. V. In fact, the $\tilde{h}$ matrix has the explicit form

	$\mathcal{H}_R$	$\mathcal{H}_I$	$\mathcal{H}_s$	$\mathcal{P}_+$	$\mathcal{P}_-$	Λ
$\mathcal{H}_R$				0	0	$\tilde{h}_{\mathcal{H}_R\Lambda}$
$\mathcal{H}_I$				0	0	0
$\mathcal{H}_s$				0	0	0
$\mathcal{P}_+$	0	0	0	$\tilde{h}_{\mathcal{P}_+\mathcal{P}_+}$	0	0
$\mathcal{P}_-$	0	0	0	0	$\tilde{h}_{\mathcal{P}_+\mathcal{P}_+}$	0
Λ	$\tilde{h}_{\Lambda\mathcal{H}_R}$	0	0	0	0	$\tilde{h}_{\Lambda\Lambda}$

where the blank spaces are the free $\tilde{h}_{\mathcal{H}\mathcal{H}}$ components, the zeros clearly are null matrices, $\tilde{h}_{\mathcal{P}_\pm\mathcal{P}_\pm} = -\eta\lambda^{-1} f' \mathbb{1}_{N\times N}$, and $\tilde{h}_{\mathcal{H}_R\Lambda}$ is a dim $\mathcal{H}_R \times 1$ matrix with all the components equal to $N^{-1}\tilde{h}_{\Lambda\Lambda}$. On the other hand, the topological charge (6.10) is reduced to

$$Q = \eta \frac{[f - \sin f]^{r=0}_{r=\infty}}{2\pi} \deg u_1 \tag{6.18}$$

Let us consider the boundary conditions for the perfil function $f(0) = 2\pi m$ and $f(\infty) = 0$ for some sign function $\eta' = 1$, and $f(0) = 0$ and $f(\infty) = 2\pi m$ for $\eta' = -1$, where m is any positive integer. Clearly, these boundary conditions ensures that Q is an integer. The algebraic degree of the rational map, which corresponds to the highest degree among the polynomials $p(\chi)$ and $q(\chi)$, denoted by a positive integer n, is equal to the topological degree of the map $u_1(\chi)$. Thus, the topological charge (6.18) inside the rational map *Ansatz* becomes

$$Q = \eta'\eta m n, \tag{6.19}$$

which due to (2.18) implies $\mathrm{sign}(\lambda) = \eta'\eta$.

As the rational map and the profile function are still free, we have an infinite number of exact solutions for any integer value of the topological charge and for each value of N. By example, for the Hermitian symmetric space $SU(3)/SU(2)\otimes U(1)$ ($N=2$) let us consider the radial solutions $p(\chi)=\chi$ and $q(\chi)=\sqrt{N}$, i.e. $u_1 = \frac{\chi}{\sqrt{N}}$, which turn the topological charge and static energy densities, as well the $\tilde{h}$ fields, spherically symmetric. In such a case $\dim\mathcal{H}_s = \dim\mathcal{H}_R = \dim\mathcal{H}_I = 1$, $\dim\mathcal{P}_+ = \dim\mathcal{P}_- = 2$, and the $\tilde{h}$ matrix becomes

$$\tilde{h} = -\eta' f'\begin{pmatrix} & & & 0 & 0 & 0 & 0 & \frac{3\sin^2(\frac{f}{2})}{2f'^2\zeta^2} \\ & -\frac{\eta'}{f'}\tilde{h}_{\mathcal{H}\mathcal{H}} & & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{3\sin^2(\frac{f}{2})}{2f'^2\zeta^2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\sin^2(\frac{f}{2})}{f'^2\zeta^2} \end{pmatrix} \tag{6.20}$$

where we introduce the dimensionless radius $\zeta \equiv |\lambda| r$, and we use $\lambda = \eta'\eta|\lambda|$.

Now, let us choose $\tilde{h}_{\mathcal{H}\mathcal{H}} = -\eta' f' \mathbb{1}_{3\times 3}$ and take the example $f = 4m\arctan((\frac{a}{\zeta})^{\eta'})$, where a is and arbitrary positive dimensionless constant. This choice of the perfil function and the $\tilde{h}_{\mathcal{H}\mathcal{H}}$ terms preserves the positivity of the $\tilde{h}$ matrix, reducing (6.20) to the spherically symmetric form

$$\tilde{h} = \delta\begin{pmatrix} 1 & O_{1\times 6} & \frac{1}{2}\gamma \\ O_{6\times 1} & \mathbb{1}_{6\times 6} & O_{6\times 1} \\ \frac{1}{2}\gamma & O_{1\times 6} & \gamma \end{pmatrix} \tag{6.21}$$

with $\gamma \equiv \frac{3\sin^2(\frac{f}{2})}{f'^2\zeta^2} = \frac{(a^2+\zeta^2)^2}{4a^2m^2\zeta^2}\sin^2\left(2m\arctan((\frac{a}{\zeta})^{\eta})\right)$, $\delta \equiv \frac{4am|\lambda|}{a^2+\zeta^2}$, and where $O_{1\times 6}$ denotes a 1×6 zero matrix, and so on. The $\tilde{h}$ matrix has six eigenstates equal to δ and the other two are $\frac{1}{2}\delta(1+\gamma\pm\sqrt{1+2(-1+\gamma)\gamma})$. However, as we proof in the Appendix, for such a perfil function we have

$$g(\zeta) \equiv \frac{4\sin^2(\frac{f}{2})}{f'^2\zeta^2} = \frac{(a^2+\zeta^2)^2}{4m^2a^2\zeta^2}\sin^2 b \le 1;$$
$$b \equiv 2m\arctan\left(\left(\frac{a}{\zeta}\right)^{\eta'}\right), \tag{6.22}$$

which implies $0 \le \gamma = \frac{3}{4}g \le g \le 1$. Therefore, all eigenvalues of the $\tilde{h}$ matrix are non-negative. In particular, for the $Q = \eta'\eta$ topological solutions ($m=1$), we have $\gamma = \frac{3}{4}$ and $\delta = \frac{4a|\lambda|}{a^2+\zeta^2}$ and therefore the diagonal components of $\tilde{h}$ are equal and all nonvanishing $\tilde{h}$ fields fall asymptotically with $1/\zeta^2$.

VII. THE CASE OF $SU(p+q)/SU(p)\otimes SU(q)\otimes U(1)$

In the case of the Hermitian symmetric space $SU(p+q)/SU(p)\otimes SU(q)\otimes U(1)$, we choose $\alpha_* = \alpha_p$, which implies $\lambda_* = \lambda_p$, the fundamental $(p+q)\times(p+q)$ representation of $SU(p+q)$. The S matrix is parametrized by p complex scalar fields $u_a = u_a(\chi)$, with $a = 1,\ldots,p$, and q complex scalar fields $v_b = v_b(\chi)$, with $b = 1,\ldots,q$, corresponding with the components of $u^T = (u_1,\ldots,u_p)$ and $v^T = (v_1,\ldots,v_q)$. The Λ and S matrices defined, respectively, by (4.3) and (4.22) are given by

$$\Lambda = \frac{1}{p+q}\begin{pmatrix} q\mathbb{1}_{p\times p} & O_{p\times q} \\ O_{q\times p} & -p\mathbb{1}_{q\times q}\end{pmatrix} \quad \text{and}$$
$$S = \begin{pmatrix} O_{p\times p} & u\otimes v \\ O_{q\times p} & O_{q\times q}\end{pmatrix} \tag{7.1}$$

where $O_{p\times q}$ is a $p\times q$ zero matrix, and so on. We consider that both the fields u and v are (anti)holomorphic when S is (anti)holomorphic. The S matrix (6.1) satisfies (4.25) with $\omega = |u|^2|v|^2$ and so the g elements given in (4.27) become

$$g = \frac{1}{\vartheta}\begin{pmatrix} \Delta_u & iu\otimes v \\ i\bar{v}\otimes\bar{u} & \Delta_v^T\end{pmatrix}; \qquad \Delta_x \equiv \vartheta\mathbb{1} + (1-\vartheta)T_x \tag{7.2}$$

with $\vartheta = \sqrt{1+\omega}$, and where $T_x \equiv \frac{x\otimes\bar{x}}{x^\dagger x}$ is a projector, i.e., $T_x^2 = T_x$ and x is any complex vector. The operator Δ_x is Hermitian and inversible, its inverse corresponds to $\Delta_x^{-1} = \vartheta^{-1}(\mathbb{1}-(1-\vartheta)T_x)$ and its square to $\Delta_x^2 = (1+\omega)\mathbb{1} - \omega T_x$. The vector x is an eigenvector with eigenvalue $+1$ of both operators Δ_x and Δ_x^{-1}. It so follows that $\Delta u = u$, $\Delta_v^T\bar{v} = \bar{v}$, $\Delta_v^T\bar{v} = \bar{v}$, and $u^\dagger\Delta_u = u^\dagger$.

The Λ matrix (7.1) satisfies $\Lambda^2 = c\Lambda + \frac{1}{4}(1-c^2)$ with $c = \frac{q-p}{q+p}$, then the field U have the form given in (4.37) (see Sec. IV), i.e.,

$$U = e^{\frac{iqf(r)}{p+q}}[\mathbb{1} + (e^{-if(r)}-1)Z]; \quad \text{with}$$
$$Z = \frac{1}{\vartheta^2}\begin{pmatrix} \omega T_u & iu\otimes v \\ -i\bar{v}\otimes\bar{u} & \Delta_v^{T2}\end{pmatrix}. \tag{7.3}$$

The reduction to the CP^N case occurs by imposing $p = N$, $q = 1$ with $v = v_1$ implying $\Delta_v = 1$. In addition, the field v_1 can be absorbed in the field u through the transformation

$u \to u/v_1$, which reduces Δ_u to (6.3) and g to (6.2). This transformation is equivalent to setting $v = 1$.

The two sets of $p \times q$ Abelian generators P_+^{cd} and P_-^{cd}, with $c = 1, \ldots, p$ and $d = 1, \ldots, q$, of the $\mathcal{P}_+$ and $\mathcal{P}_-$ subalgebras are given by

$$(P_+^{cd})_{ab} \equiv \delta_{ac}\delta_{(d+p)b}, \qquad (P_-^{cd})_{ab} = \delta_{a(d+p)}\delta_{bc} \tag{7.4}$$

with $a, b = 1, \ldots, p+q$, which satisfies $\mathrm{Tr}(P_+^{cd}P_-^{c'd'}) = \delta_{cc'}\delta_{dd'}$, with $c' = 1, \ldots, p$ and $d' = 1, \ldots, q$. There are $p^2 - 1$ generators of the group $SU(p)$ and $q^2 - 1$ generators of the group $SU(q)$ associated with the $H = SU(p) \otimes SU(q)$ subgroup of $SU(p+q)$. For each of such a group we can break such a generator into three distinct types, similar to what we do in (6.6). In case of the $SU(p)$ group, the first two sets $\mathcal{H}_{R_p}$ and $\mathcal{H}_{I_p}$ contain $\frac{p(p-1)}{2}$ generators each and can be labeled by the pair nm, with $n = 1, \ldots, p$ and $m = 1, \ldots, p-1$, i.e., $p < n$. The third set $\mathcal{H}_{s_p}$ contains $p - 1$ generators and is labeled by the index $s = 1, \ldots, p-1$. The same follows for the $SU(q)$ group by changing $p \to q$ and changing the indices $m \to k$, $n \to l$, and $s \to r$. The generators of the $SU(p)$ and $SU(q)$ groups are given by

$$\begin{aligned}
(H_{R_p}^{nm})_{ab} &= \frac{1}{2}(\delta_{an}\delta_{bm} + \delta_{am}\delta_{bn});\\
(H_{R_q}^{lk})_{ab} &= \frac{1}{2}(\delta_{a(l+p)}\delta_{b(k+p)} + \delta_{a(k+p)}\delta_{b(l+p)})\\
(H_{I_p}^{nm})_{ab} &= -\frac{i}{2}(\delta_{an}\delta_{bm} - \delta_{am}\delta_{bn});\\
(H_{I_q}^{lk})_{ab} &= -\frac{i}{2}(\delta_{a(l+p)}\delta_{b(k+p)} - \delta_{a(k+p)}\delta_{b(l+p)})\\
(H_s)_{ab} &= \frac{1}{2}(\delta_{as}\delta_{bs} - \delta_{a(s+1)}\delta_{b(s+1)});\\
(H_r)_{ab} &= \frac{1}{2}(\delta_{a(r+p)}\delta_{b(r+p)} - \delta_{a(r+p+1)}\delta_{b(r+p+1)})\\
&\text{with} \quad a, b = 1, \ldots, p+q.
\end{aligned} \tag{7.5}$$

Note that for such a basis we have

$$\kappa = \frac{1}{2}. \tag{7.6}$$

Using (4.30) and (7.2) we get

$$\Omega^{-1} = \frac{1}{\vartheta}\begin{pmatrix} \Delta_u & O_{p\times q} \\ O_{q\times p} & \Delta_v^T \end{pmatrix}, \tag{7.7}$$

which together with (4.31) fixes $P_\chi^{(+)} = i\Omega^{-1}\partial_\chi S\Omega^{-1}$, $P_{\bar\chi}^{(-)} = i\Omega^{-1}\partial_{\bar\chi}S^\dagger\Omega^{-1}$ and the commutator $[P_\chi^{(+)}, P_{\bar\chi}^{(-)}]$ through

$$\begin{aligned}
P_\chi^{(+)} &= i\vartheta^{-2}\begin{pmatrix} O_{p\times p} & B \\ O_{q\times p} & O_{q\times q} \end{pmatrix};\\
P_{\bar\chi}^{(-)} &= i\vartheta^{-2}\begin{pmatrix} O_{p\times p} & O_{p\times q} \\ B^\dagger & O_{q\times q} \end{pmatrix}\\
\left[P_\chi^{(+)}, P_{\bar\chi}^{(-)}\right] &= -\frac{1}{\vartheta^4}\begin{pmatrix} BB^\dagger & O_{p\times q} \\ O_{q\times p} & -B^\dagger B \end{pmatrix};\\
&\text{with} \quad B \equiv \Delta_u\partial_\chi(u \otimes v)\Delta_v^T.
\end{aligned} \tag{7.8}$$

From (7.4) and (7.8) we can also write

$$P_\chi^{(+)} = i\vartheta^{-2}B_{cd}P_+^{cd}; \qquad P_{\bar\chi}^{(-)} = -(P_\chi^{(+)})^\dagger = i\vartheta^{-2}B_{dc}^\dagger P_-^{cd}. \tag{7.9}$$

On the other hand, using $\omega = \vartheta^2 - 1$, the definition of the operators Δ_x and T_x introduced in (7.2) we obtain

$$\begin{aligned}
B &= \vartheta\left[\partial_\chi(u \otimes v) - \frac{2\partial_\chi\vartheta}{1+\vartheta}u \otimes v\right];\\
B^\dagger &= \vartheta\left[\partial_{\bar\chi}(\bar v \otimes \bar u) - \frac{2\partial_{\bar\chi}\vartheta}{1+\vartheta}\bar v \otimes \bar u\right].
\end{aligned} \tag{7.10}$$

Using (7.1) and (7.4)–(7.10) the fields $\tilde h_{\Lambda\mathcal{P}_\pm}$ and $\tilde h_{\Lambda\mathcal{H}}$ fixed through (4.52) become

$$\begin{aligned}
&\tilde h_{\Lambda\Lambda} = \beta\mathrm{Tr}(M); \qquad \tilde h_{\Lambda\mathcal{P}_\pm} = 0\\
&\tilde h_{\Lambda H_s} = \frac{1}{2}\beta(M_{ss} - M_{(s+1)(s+1)}); \qquad \tilde h_{\Lambda H_r} = \frac{1}{2}\beta(N_{rr} - N_{(r+1)(r+1)})\\
&\tilde h_{\Lambda H_{R_p}^{nm}} = \frac{1}{2}\beta(M_{nm} + M_{mn}); \qquad \tilde h_{\Lambda H_{R_q}^{lk}} = \frac{1}{2}\beta(N_{lk} + N_{kl})\\
&\tilde h_{\Lambda H_{I_p}^{nm}} = \frac{1}{2}\beta i(M_{nm} - M_{mn}); \qquad \tilde h_{\Lambda H_{I_q}^{lk}} = \frac{1}{2}\beta i(N_{lk} - N_{kl})
\end{aligned} \tag{7.11}$$

where

$$\begin{aligned}
&\beta \equiv -\eta\vartheta^{-4}\alpha = -\eta\frac{p+q}{qp}\frac{2\sin^2\frac{f}{2}}{\lambda f' r^2}\frac{(1+|z|^2)^2}{\vartheta^4};\\
&M \equiv BB^\dagger; \qquad N \equiv B^\dagger B
\end{aligned} \tag{7.12}$$

and the explicit form of M, N, and the trace $\mathrm{Tr}(M) = \mathrm{Tr}(N)$ corresponds to

$$M = \vartheta^2\left\{\partial_\chi\partial_{\bar\chi}(\omega T_u) - \frac{2}{1+\vartheta}[(\partial_\chi\vartheta\partial_{\bar\chi} + \partial_{\bar\chi}\vartheta\partial_\chi)(\omega T_u)] + 4\frac{\partial_\chi\vartheta\partial_{\bar\chi}\vartheta}{(1+\vartheta)^2}(\omega T_u)\right\},$$

$$N = \vartheta^2\left\{\partial_\chi\partial_{\bar\chi}(\omega T_v^T) - \frac{2}{1+\vartheta}[(\partial_\chi\vartheta\partial_{\bar\chi} + \partial_{\bar\chi}\vartheta\partial_\chi)(\omega T_v^T)] + 4\frac{\partial_\chi\vartheta\partial_{\bar\chi}\vartheta}{(1+\vartheta)^2}(\omega T_v^T)\right\},$$

$$\mathrm{Tr}(M) = 2\vartheta^2(\vartheta\partial_\chi\partial_{\bar\chi}\vartheta - \partial_\chi\vartheta\partial_{\bar\chi}\vartheta) = \vartheta^2\left[\partial_\chi\partial_{\bar\chi}\omega - \frac{\partial_\chi\omega\partial_{\bar\chi}\omega}{1+\omega}\right]. \tag{7.13}$$

On the other hand, the self-dual equations for the fields $\tilde h_{\mathcal{P}_\pm\mathcal{H}}$ and $\tilde h_{\mathcal{P}_\pm\mathcal{P}_\mp}$, as given, respectively, in (4.54) and (4.55), are reduced to

$$B_{cd}\tilde h_{P_-^{cd}\mathcal{H}} = B_{dc}^\dagger\tilde h_{P_+^{cd}\mathcal{H}} = B_{cd}\tilde h_{P_-^{cd}\mathcal{P}_+} = 0 \tag{7.14}$$

while the self-dual equations for the fields $\tilde h_{\mathcal{P}_+\mathcal{P}_+}$ and $\tilde h_{\mathcal{P}_-\mathcal{P}_-}$, as given, respectively, in (4.56) and (4.57), are reduced to

$$B_{dc}^\dagger\tilde h_{P_+^{cd}P_+^{nm}} + \eta\lambda^{-1}f'B_{mn}^\dagger = B_{cd}\tilde h_{P_-^{cd}P_-^{nm}} + \eta\lambda^{-1}f'B_{nm} = 0. \tag{7.15}$$

Using (7.6) and (7.8), which imply $\mathrm{Tr}(P_\chi^{(+)}P_{\bar\chi}^{(-)}) = -\vartheta^{-4}\mathrm{Tr}M$, and (7.13), the topological charge (4.44) becomes

$$Q = \frac{1}{2\pi}[f(r) - \sin f(r)]_{r=0}^{r=\infty}Q_{\mathrm{top}},$$
$$Q_{\mathrm{top}} = -\frac{i\eta}{2\pi}\int\frac{dzd\bar z}{\vartheta^4}\mathrm{Tr}M = -\frac{i\eta}{2\pi}\int\frac{dzd\bar z}{\vartheta^2}\left[\partial_\chi\partial_{\bar\chi}\omega - \frac{\partial_\chi\omega\partial_{\bar\chi}\omega}{1+\omega}\right]. \tag{7.16}$$

A. An explicit example: Exact generalized Skyrmions on the $SU(p+q)/SU(p)\otimes SU(q)\otimes U(1)$ spaces

To construct explicit self-dual configurations consider the *Ansatz* where all the component fields u are v are equal to the same holomorphic rational map $u_1(\chi) = p_u(\chi)/q_u(\chi)$ and $v_1(\chi) = p_v(\chi)/q_v(\chi)$, respectively, between the Riemann spheres S^2 [see Eq. (5.11)]. By definition, p_t and q_t, with $t = u$, v, does not share any common root, since u_1 and v_1 are rational maps. However, we also impose that p_u and q_v do not any share common root, and we impose the same restriction to p_v and q_u. Therefore, the product u_1v_1 is also a rational map. The *Ansatz* for the fields u, v, $\tilde h_{\mathcal{PH}}$, and $\tilde h_{\mathcal{PP}}$ is an generalization of (6.15) and (6.16), and is given by

$$u_c = u_1 = \frac{p_u(\chi)}{q_u(\chi)}; \qquad v_d = v_1 = \frac{p_v(\chi)}{q_v(\chi)};$$
$$\tilde h_{\mathcal{PH}} = \tilde h_{\mathcal{P}_\pm\mathcal{P}_\mp} = 0; \qquad \tilde h_{P_\pm^{cd}P_\pm^{nm}} = -\eta\frac{f'}{\lambda}\delta_{cn}\delta_{dm} \tag{7.17}$$

where $P_\pm^{cd}$ are defined in (7.4) and $c, n = 1, \ldots, p$ and $d, m = 1, \ldots, q$. For these indices, the *Ansatz* (7.17) implies $(T_u)_{an} = p^{-1}$ and $(T_v)_{dm} = q^{-1}$, leading due to (7.13) to $M_{cn} = M_{11}$ and $N_{dm} = N_{11}$, where

$$M_{11} = \frac{\mathrm{Tr}M}{p}, \quad N_{11} = \frac{\mathrm{Tr}M}{q}, \quad \mathrm{Tr}(M) = pq\left|\frac{d}{d\chi}(u_1v_1)\right|^2 \geq 0. \tag{7.18}$$

Therefore, the fields given in (7.11) become

$$\tilde h_{\Lambda\Lambda} = \beta\mathrm{Tr}M; \qquad \tilde h_{\Lambda H_{R_p}^{nm}} = \beta M_{11}; \qquad \tilde h_{\Lambda H_{R_q}^{lk}} = \beta N_{11},$$
$$\tilde h_{\Lambda\mathcal{P}_\pm} = \tilde h_{\Lambda H_s} = \tilde h_{\Lambda H_r} = \tilde h_{\Lambda H_{I_p}^{nm}} = \tilde h_{\Lambda H_{I_q}^{nm}} = 0 \tag{7.19}$$

while all the self-dual equations given in (6.13) and (6.14) are automatically satisfied by (7.17). Note that using (7.12) all the nonvanishing components of $\tilde h$ other than the free terms $\tilde h_{\mathcal{HH}}$ are non-negative if we impose the follow condition over the perfil function f

$$-\eta\,\mathrm{sign}\left(\frac{f'}{\lambda}\right) \geq 0 \Rightarrow \beta \geq 0. \tag{7.20}$$

Using $\omega = pq|u_1|^2|v_1|^2$, the topological charge (7.16) becomes

$$Q = \left[\frac{f(r) - \sin f(r)}{2\pi}\right]_{r=0}^{r=\infty}Q_{\mathrm{top}};$$
$$Q_{\mathrm{top}} = -\frac{i\eta}{2\pi}\int\frac{dzd\bar z|\frac{d}{d\chi}(\sqrt{pq}u_1v_1)|^2}{(1+|\sqrt{pq}u_1v_1|^2)^2}. \tag{7.21}$$

However, the Eq. (5.12) shows the integral representation of the degree of a rational map u. Such a degree is in particular invariant by a the multiplication $u \to cu$, $\forall\ c \in \mathbb{R}_+^*$. Therefore, Q_{top} given (7.21) corresponds with the integral representation of the rational map u_1v_1, i.e., $Q_{\mathrm{top}} = \deg(u_1v_1)$ and the topological charge becomes (7.21)

$$Q = \eta\left[\frac{f(r) - \sin f(r)}{2\pi}\right]_{r=0}^{r=\infty}\deg(u_1v_1). \tag{7.22}$$

Consequently, by choosing the degree of the rational map $u_1 v_1$ and the boundary conditions of the perfil function f we get an infinite number of exact self-dual solutions, given by (7.17) and (7.19), for the Hermitian symmetric space $SU(p+q)/SU(p) \otimes SU(q) \otimes U(1)$. The only restriction is to choose such a fields and the free term $\tilde{h}_{\mathcal{H}\mathcal{H}}$ such as that $\tilde{h}$ is non-negative, as it is done in the example given in Sec. VI.

VIII. CONCLUSION

In the self-dual sector of our generalization of the BPS Skyrme model for any compact Lie group G that leads to a Hermitian symmetric space, our holomorphic *Ansatz* shows that the full determination of the h fields in terms of the Skyrme fields happens only for some particular Lie groups. Although this characteristic of the BPS Skyrmions for the $G = SU(2)$ case is not a general feature of the generalized theory, this model possesses the main symmetries of the original BPS Skyrme model.

As in the original BPS Skyrme model, the h fields in our generalized BPS Skyrme model continue to play the same role as the Wess-Zumino term with respect to breaking the invariance by the parity and target space parity transformations P and P_g, respectively, while preserving the symmetry by the composition PP_g. These properties may shed light on the physical nature of the h fields, which may be related to the chiral anomaly.

Our holomorphic *Ansatz* simplifies drastically the self-dual equations. It leads directly to the determination of the components of $\tilde{h}_{\Lambda\Lambda}, \tilde{h}_{\Lambda\mathcal{P}_\pm}, \tilde{h}_{\Lambda\mathcal{H}}$ in terms of the Skyrme field, and leads to algebraic equations for the $\tilde{h}_{\mathcal{H}\mathcal{P}_\pm}$, $\tilde{h}_{\mathcal{P}_\pm\mathcal{P}_\pm}, \tilde{h}_{\mathcal{P}_\pm\mathcal{P}_\mp}$ components. However, there are at least a number of $\dim \mathcal{P}_+(2\dim \mathcal{P}_+ - 3)$ components of $\tilde{h}_{\mathcal{P}\mathcal{P}}$ and $2\dim \mathcal{H}(\dim \mathcal{P}_+ - 1)$ components of $\tilde{h}_{\mathcal{H}\mathcal{P}}$ totally free. Clearly, the freedom of the system grows with the dimension of Lie algebra $\mathcal{G}$. In fact, the $\tilde{h}$ fields can be entirely determined in terms of the Skyrme field inside the holomorphic *Ansatz* (4.19) only if $\mathcal{H} = \varnothing$ and $\dim \mathcal{P}_+ = 1$, which corresponds to $G = SU(2)$.

The generalized holomorphic *Ansatz* for $G = SU(N+1)$ leads to an infinite number of exact BPS Skyrmions for all integer values of the topological charge and for all $N \geq 1$. We also show how to construct a more restrictive *Ansatz* based on the usual rational map $S^2 \to S^2$, which fixes all components of the $\tilde{h}$ matrix except $\tilde{h}_{\mathcal{H}\mathcal{H}}$. Using this approach, we gave an example of $\tilde{h}$ matrix that leads to exact spherically symmetric BPS Skyrmions for all integer values of Q and N. The self-dual sector within the holomorphic *Ansatz* for the Hermitian symmetric space $SU(p+q)/SU(p) \otimes SU(q) \otimes U(1)$ is quite similar to the CP^N case, despite being a generalization. In fact, we can even obtain particular solutions for each value of the topological charge, where all the nondiagonal entries of the $\tilde{h}$ matrix vanish, expect the terms $\tilde{h}_{\Lambda\mathcal{H}^{nm}_{R_p}}$ and $\tilde{h}_{\Lambda\mathcal{H}^{nm}_{R_q}}$.

Our theory facilitates the construction of highly symmetric multi-BPS Skyrmions, and extensions of this model may have some important physical applications. One example is the generalization of the false vacuum Skyrme model to $G = SU(N+1)$. This is very promising since such a theory is strongly based on spherical symmetric multisolitions, which also appear in our generalized BPS Skyrme model. On the other hand, our holomorphic *Ansatz* for Hermitian symmetric spaces may be of great value in constructing multisolitons in a vast number of similar theories.

Extensions of the generalized BPS Skyrme model may break both the self-duality equations and the conformal invariance in three spatial dimensions. This can be achieved by introducing kinetic and potential terms for the h fields into the action, as done in the quasi-self-dual model proposed in [21] for $G = SU(2)$. This may result in the full determination of all fields of the model, as is the case in [21]. Another application of our work is the construction of a generalization of the $SU(2)$ false vacuum Skyrme model introduced in [20] to larger groups. These applications may shed light on the physical meaning of the h fields, which could depend on the type of extension.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: THE PROOF OF THE RELATION (6.22)

Let us introduce the non-negative real-valued function g as

$$g(\zeta) \equiv \frac{4\sin^2(\frac{f}{2})}{f'^2\zeta^2} = \frac{(a^2+\zeta^2)^2}{4m^2a^2\zeta^2}\sin^2 b;$$
$$b \equiv 2m \arctan\left(\left(\frac{a}{\zeta}\right)^{\eta'}\right) \tag{A1}$$

and let us proof that

$$g \leq 1. \tag{A2}$$

The function g and its first-order derivative are continuous and g satisfies $g(0) = g(\infty) = 1$. Therefore, the maximum of g must be at a critical point ζ_p or at $\zeta = 0, 1$. At $\zeta = a$ we have $g(a) = m^{-2}\sin^2(m\frac{\pi}{2}) \leq 1$. Clearly, for $m = 1$ we have $g = 1$, which satisfies (A2). From now on, we will study $g(\zeta)$ for $m \geq 2$ and for ζ lying on the interval $I \equiv (0, \infty)/\{a\}$. The critical points ζ_c in the interval I correspond to the solutions of

$$\sin^2 b_c = \eta \frac{2ma\zeta}{(\zeta^2 - a^2)} \sin(b_c)\cos(b_c) \qquad \text{(A3)}$$

where $b_c \equiv b|_{\zeta=\zeta_c}$. We can break the solutions of the Eq. (A3) into two types, corresponding to those cases where $\sin b_c = 0$ and $\sin b_c \neq 0$. Clearly, if the critical point satisfies $\sin b_c = 0$, which solves automatically (A3), we have $g(\zeta_c) = 0$. Otherwise, the Eq. (A3) is reduced to $\sin b_c = d\cos(b_c)$, with $d \equiv \frac{2ma\zeta}{(\zeta^2 - a^2)}$, which leads to

$$b_c = \arctan(d) + \pi n_c; \quad \forall \;\; n_c \in \mathbb{Z}. \qquad \text{(A4)}$$

For this case we have $\sin^2 b_c = \frac{d^2}{1+d^2}$, which reduces (A1) to

$$g_c = \left[1 + 4(m^2 - 1)\left(\frac{(\frac{\zeta}{a})}{(1 + \frac{\zeta}{a})^2}\right)^2\right]^{-1} \leq 1. \qquad \text{(A5)}$$

Therefore, in the interval I with $m \geq 2$ we have $g(\zeta) \leq 1$, completing the proof of (A2).

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