Journal of Quantitative Spectroscopy & Radiative Transfer 307 (2023) 108661

Contents lists available at ScienceDirect

Journal of Quantitative Spectroscopy & Radiative Transfer

journal homepage: www.elsevier.com/locate/jqsrt

Optical forces and optical force partitions exerted on arbitrary sized spherical particles in the framework of generalized Lorenz–Mie theory

Gérard Gouesbet[a,*], V.S. De Angelis[b], Leonardo André Ambrosio[b]

[a] CORIA-UMR 6614- Normandie Université, CNRS-Université et INSA de Rouen, Campus Universitaire du Madrillet, Saint-Etienne-du Rouvray, 76800, France
[b] Department of Electrical and Computer Engineering, São Carlos School of Engineering, University of São Paulo, 400 Trabalhador São-carlense Ave., São Paulo, SP 13566-590, Brazil

ARTICLE INFO

Article history:
Received 24 January 2023
Revised 28 March 2023
Accepted 7 May 2023
Available online 14 May 2023

Keywords:
Generalized Lorenz–Mie theory
Optical forces
Scattering forces
Gradient forces
Non-standard forces
Arbitrary sized particles

ABSTRACT

This paper is the last paper (independently of possible future refinements and complements) aiming to the partition of optical forces in the framework of generalized Lorenz–Mie theory. After a paper devoted to forces exerted on quadrupoles, the present paper is devoted to forces exerted on arbitrary sized particles. These forces are expressed in terms of the beam shape coefficients which encode the structure of the illuminating beam and of Mie coefficients which encode the properties of the scatterer. The partition relies on a three-level categorization (mixing and recoil forces, gradient and non-gradient forces, scattering and non-standard forces) and on a two-level decomposition (K-forces with K being an integer ranging from 1 to ∞, and electric/magnetic/magnetoelectric forces).

1. Introduction

The present introduction is a summary of a previous introduction written at the occasion of a paper devoted to optical forces exerted on quadrupoles in the framework of the generalized Lorenz–Mie theory (GLMT). In the present introduction, we recall a few essential features and kindly ask the reader to report himself/herself to Gouesbet et al. [1] for more details.

Optical forces have been expressed in terms of beam shape coefficients (BSCs), which encode the structure of the illuminating beam, in a general off-axis configuration [2], following the restricted case of an on-axis configuration [3]. It is here recalled that these BSCs are obtained from the expressions of the radial electric and magnetic fields, e.g. Section 3.3.1. in Gouesbet and Gréhan [4]. Although these papers emphasized the case of illuminating Gaussian beams, the expressions provided for the optical forces were valid for arbitrary shaped beam illumination, see [5,6]. As discussed in Section 2, these papers already introduced the first-level categorization between mixing and recoil forces. The basic expressions of this first-level categorization will serve as starting bricks to develop the formalism presented in the following sections. Although the emphasis is made on homogeneous spherical particles, it must be noted that the GLMT formalism (*stricto sensu*, i.e. in the case of homogeneous spherical particles) for optical forces is valid as well to the cases of multilayered particles when the expressions of the BSCs are unchanged, requiring only to modify the expressions of the Mie coefficients [7,8], and to other kinds of particles leading to expressions which are formally identical to the ones of the GLMT *stricto sensu*, namely assemblies of spheres and aggregates [9–11] and spheres with an eccentrically located spherical inclusion [12–15]

Numerical evaluations of optical forces in the GLMT framework, possibly in relationship with experimental results, have afterward been provided in Ren et al. [16,17], Polaert et al. [18], Martinot-Lagarde et al. [19]. Optical torques in spherical coordinates have also been discussed by Polaert et al. [20]. Concerning complementary studies in spheroidal coordinates, for both optical forces and torques, the reader may refer to Xu et al. [21] and to Xu et al. [22]. Many other works from various worldwide authors contributed as well to the issue and have been recently quoted in a review paper with 284 references [23]. Many other examples may be found in Gouesbet [24] and Gouesbet [25], namely about 50 references in Gouesbet [24] for the period 2009–2013 and about 150 references in Gouesbet [25] for the period 2014–2018, concerning in particular (to cite a few topics) optical tweezers, stretching and deforming, transporting and sorting, binding, and pushing and pulling.

After Arthur Ashkin's work, compiled in Ashkin [26], it has been traditional to think of the optical forces in terms of a partition between gradient and scattering forces which may be viewed as a second-level categorization. However, although it may look strange

* Corresponding author.
E-mail address: gouesbet@coria.fr (G. Gouesbet).

https://doi.org/10.1016/j.jqsrt.2023.108661

for a theory completed in 1987 and 1988, a systematic study of optical force partition in GLMT, after a first occurrence in terms of gradient and scattering forces by Lock [27], only started recently in a systematic way in Gouesbet [28] when studying forces exerted on electric dipoles (in particular Rayleigh particles) where it has been uncovered that the categorization in terms of gradient and scattering forces actually generates two levels of categorization between gradient and non-gradient forces, non-gradient forces in turn being separated into scattering and non-standard forces. The results obtained in Gouesbet [28] required several papers to fully understand the situation, see a review in Section 10 of Gouesbet and Ambrosio [29]. These studies on electric dipoles have been completed by a study devoted to magnetodielectric dipoles [30,31], and by an extension to the case of quadrupoles [1]. The present paper is the last one of the series (notwithstanding possible future refinements and complements) and therefore deals with the case of arbitrary sized particles. It somehow completes and more important concludes the series.

Furthermore, beside the three-level categorization previously revealed, we must also introduce a parallel two-level decomposition of optical forces. The first decomposition concerns what we shall call K -forces or, in another language which may be convenient, forces of order K, which is a decomposition into an infinite number of forces (K from 1 to ∞). The second-level decomposition distinguishes between electric, magnetic and magnetoelectric forces.

The paper is organized as follows. Section 2 recalls a background on optical forces which will display the first-level categorization in terms of mixing and recoil forces as already displayed nearly four decades ago and which will serve to derive the expressions of K-forces in terms of BSCs and of Mie coefficients. Section 3 displays the expressions of K-forces. Section 4 will provide the interpretations of mixing forces, while Section 5 will provide the interpretations of recoil forces. Section 6 is a summary and a discussion of the results generated by the 3-level categorization and by the 2-level decomposition. Section 7 is a conclusion. Two appendices will refer to mixing gradient K-forces exhibited in Eq. (24) of Zheng et al. [32].

2. Mixing and recoil forces

We consider a Cartesian coordinate system O_Pxyz with a scatterer located at the origin O_P of the coordinates. The scatterer is illuminated by a structured beam encoded by the double set of BSCs $g_{n,TM}^m$ and $g_{n,TE}^m$ (TM standing for “Transverse Magnetic” and TE for “Transverse Electric”, n from 1 to infinity, and m from $-n$ to $+n$) with a time-dependence of the form $\exp(i\omega t)$ which is the usual convention in the GLMT framework. The axis O_Pz is traditionally chosen to define the direction of propagation of the beam. Spherical coordinates associated with the Cartesian coordinate system are denoted (r, θ, φ) as usual.

The distinction between mixing and recoil forces, already put forward nearly forty years ago [2,3], has been recalled, although with another language than the one used in the present paper, in a textbook [4]. Longitudinal mixing forces, corresponding to the forward momentum removed from the beam, are provided by the first term of Eq. (3.145) of [4], while recoil forces, corresponding to the forward momentum given by the scatterer to the scattered wave, is given in the second term. Transverse forces are similarly expressed by Eqs. (3.160) and (3.161) of [4]. It is traditional in GLMT to express these forces in terms of cross-sections denoted $\overline{F(\theta,\varphi)C_{ext}}$ (for the mixing forces) and $\overline{F(\theta,\varphi)C_{sca}}$ (for the recoil forces), in which $\overline{F(\theta,\varphi)}$ stands for $\overline{\cos\theta}$ for the longitudinal forces and $\overline{\sin\theta\cos\varphi}$, $\overline{\sin\theta\sin\varphi}$ for the transverse components (x- and y-components respectively). This notation is borrowed from van de Hulst [33]. The forces (cross-sections) $\mathbf{C} = C_{pr,i}$ ($i = x, y, z$) are then expressed, see Section 2.1 in Gouesbet et al. [1], as:

$$C_{pr,z} = \overline{\cos\theta}\ C_{ext} - \overline{\cos\theta}\ C_{sca} \tag{1}$$

$$C_{pr,x} = \overline{\sin\theta\ \ \cos\varphi}\ C_{ext} - \overline{\sin\theta\ \ \cos\varphi}\ C_{sca} \tag{2}$$

$$C_{pr,y} = \overline{\sin\theta\ \ \sin\varphi} C_{ext} - \overline{\sin\theta\ \ \sin\varphi} C_{sca} \tag{3}$$

in which we attached a privilege to the z-component because it is easier to evaluate than the other components. These equations express a first-level categorization between mixing forces with a subscript “ext” and recoil forces with a subscript “sca”. Let us comment the terminology used, referring ourselves to Eq. (1) (similar comments could be done for the two other equations). The subscript “ext” is motivated by the fact that the corresponding forces are obtained from a weighted integration of the extinction cross-section C_{ext} and the corresponding forces are then called extinction forces as well. However, because (i) the word “extinction” refers to “absorption plus scattering” and (ii) pure scattered fields are involved in the second term of the r.h.s. of the equation, we found the terminology of extinction forces confusing and eventually preferred the one of mixing forces. Similarly, the subscript “sca” is motivated by the fact that only scattered fields are involved in the evaluation of $\overline{\cos\theta\ C_{sca}}$ but this terminology should not be confusing up to the point that we would believe that these “sca” forces correspond to scattering forces. Instead, as we shall see, we shall find that recoil forces associated with the subscript “sca” contain gradient forces.

The relationship between forces $\mathbf{F}$ expressed as cross-sections $\mathbf{C}$ (i.e. in square meters) and actual forces expressed in newtons reads as $\mathbf{F}_{newton} = I_0\mathbf{C}/c$ in which c is the speed of light and in which I_0, namely the intensity of the incident light (for a plane wave) [33], may here be viewed as a normalization factor. Relying on Eq. (3.106), which defines a normalization factor as $E_0H_0^*/2 = 1$, and Eq. (3.144) of [4], and using $c = 1/\sqrt{\varepsilon\mu}$, with ε and μ being respectively the permittivity and the permeability of the host medium, we obtain:

$$\mathbf{F}_{newton} = \frac{1}{2}\varepsilon|E_0|^2\mathbf{C} \tag{4}$$

which is valid because the host medium is lossless.

In the present paper, the word “force” is conveniently used to denote actual forces in newtons or forces expressed as cross-sections, or it may as well be used to denote a force in the vectorial sense or to denote the components of a vectorial force, any possible ambiguity being removed by the context. The minus signs of the second members of the r.h.s. of Eqs. (1)–(3) are omitted in the sequel.

Considering the expressions for the forces explicitly given in subsections, 3.12.2 and 3.12.3 of [4], we may observe that forces are obtained by summing up an infinite number of subforces. From such expressions, we extract K-forces which are obtained from the general expressions of forces by extracting the terms involving the Mie coefficients a_K and b_K, and their products with Mie coefficients of order $(K+1)$. These K-forces are denoted by using a superscript K.

3. Forces of order *K* versus BSCs

Forces of order 1 and 2 have been discussed in Gouesbet et al. [1] and references therein. It is emphasized in Gouesbet et al. [1] that 1-forces and 2-forces, although exhibiting very strong similarities, exhibit as well a few differences. The most important difference is that the third-level categorization in which non-gradient terms are decomposed into scattering and non-standard forces rely, in the case of 1-forces, on perfectly discriminating definitions. Namely, for 1-forces, scattering forces are non-gradient forces which are proportional to the Poynting vector while non-standard forces are non-gradient forces which are not proportional to the Poynting vector. Such definitions do not however extend to K-forces, for $K > 1$. However, the three-level categorization has to be preserved for $K > 1$ for at least two reasons (i) scattering and non-standard forces may be defined on the basis of their structural properties and (ii) it provides a classification of optical forces valid whatever the value of K. Other reasons related to the coherency of the formalism will be notified when appropriate. In the present paper, only K-forces with $K > 1$ will be discussed (we shall omit in the sequel to repeat this condition). For 1-forces, the reader may refer to Gouesbet et al. [1] and references therein.

From Eq. (3.158) in Gouesbet and Gréhan [4], the mixing longitudinal K-force is found to read as:

$$\overline{\cos\theta C_{ext}^K} = \frac{\lambda^2}{\pi}\mathrm{Re}\left(a_K Z_E^K + b_K Z_H^K\right) \tag{5}$$

in which:

$$\begin{aligned} Z_E^K &= \frac{1}{K^2}\sum_{p=-K+1}^{K-1}\frac{(K+|p|)!}{(K-1-|p|)!}g_{K-1,TM}^{p*}g_{K,TM}^{p} \\ &+ \frac{1}{(K+1)^2}\sum_{p=-K}^{+K}\frac{(K+1+|p|)!}{(K-|p|)!}g_{K,TM}^{p}g_{K+1,TM}^{p*} \\ &- \frac{(2K+1)i}{K^2(K+1)^2}\sum_{p=-K}^{+K}p\frac{(K+|p|)!}{(K-|p|)!}g_{K,TM}^{p}g_{K,TE}^{p*} \end{aligned} \tag{6}$$

$$\begin{aligned} Z_H^K &= \frac{1}{K^2}\sum_{p=-K+1}^{K-1}\frac{(K+|p|)!}{(K-1-|p|)!}g_{K-1,TE}^{p*}g_{K,TE}^{p} \\ &+ \frac{1}{(K+1)^2}\sum_{p=-K}^{+K}\frac{(K+1+|p|)!}{(K-|p|)!}g_{K,TE}^{p}g_{K+1,TE}^{p*} \\ &+ \frac{(2K+1)i}{K^2(K+1)^2}\sum_{p=-K}^{+K}p\frac{(K+|p|)!}{(K-|p|)!}g_{K,TM}^{p*}g_{K,TE}^{p} \end{aligned} \tag{7}$$

in which the subscript E corresponds to an electric force associated with the Mie electric coefficient a_K, while the subscript H corresponds to a magnetic force associated with the Mie magnetic coefficient b_K. We shall later similarly use a subscript EH to denote magnetoelectric terms (and forces). The use of the letter Z recalls us that we are dealing with longitudinal forces. Furthermore, we begin our analysis with longitudinal terms (and forces) because they are simpler to investigate than transverse forces (for which, instead of Z, we shall use the letters X and Y for x- and y-components respectively). For $K = 2$, we recover Eqs. (78) and (79) of [1].

From Eq. (3.155) in Gouesbet and Gréhan [4], the recoil longitudinal K-force is found to read as:

$$\overline{\cos\theta C_{sca}^K} = \frac{-2\lambda^2}{\pi}\mathrm{Re}\left(ia_K b_K^* Z_{EH}^{KK} + a_K a_{K+1}^* Z_E^{KK+1} + b_K b_{K+1}^* Z_H^{KK+1}\right) \tag{8}$$

in which:

$$Z_{EH}^{KK} = \frac{2K+1}{K^2(K+1)^2}\sum_{p=-K}^{+K}p\frac{(K+|p|)!}{(K-|p|)!}g_{K,TM}^{p}g_{K,TE}^{p*} \tag{9}$$

$$Z_E^{KK+1} = \frac{-1}{(K+1)^2}\sum_{p=-K}^{+K}\frac{(K+1+|p|)!}{(K-|p|)!}g_{K,TM}^{p}g_{K+1,TM}^{p*} \tag{10}$$

$$Z_H^{KK+1} = \frac{-1}{(K+1)^2}\sum_{p=-K}^{+K}\frac{(K+1+|p|)!}{(K-|p|)!}g_{K,TE}^{p}g_{K+1,TE}^{p*} \tag{11}$$

For $K = 2$, we recover Eqs. (81)–(84) of [1]. From general equations available from [1,2] and [4], and which are not repeated in the present paper, we similarly obtain the mixing x-transverse K-force reading as:

$$\overline{\sin\theta\cos\varphi C_{ext}^K} = \frac{\lambda^2}{2\pi}\mathrm{Re}\left(a_K X_E^K + b_K X_H^K\right) \tag{12}$$

in which:

$$X_E^K = X_E^{\alpha K} + X_E^{\beta K} + iX_E^{\gamma K} \tag{13}$$

$$X_H^K = X_H^{\alpha K} + X_H^{\beta K} + iX_H^{\gamma K} \tag{14}$$

in which:

$$\begin{aligned} X_E^{\alpha K} &= \frac{1}{K^2}\sum_{p=1}^{K-1}\frac{(K-1+p)!}{(K-1-p)!}\left(g_{K,TM}^{p-1}g_{K-1,TM}^{p*} + g_{K-1,TM}^{-p*}g_{K,TM}^{-p+1}\right) \\ &+ \frac{1}{(K+1)^2}\sum_{p=1}^{K-1}\frac{(K+p)!}{(K-p)!}\left(g_{K+1,TM}^{p-1*}g_{K,TM}^{p} + g_{K,TM}^{-p}g_{K+1,TM}^{-p+1*}\right) \\ &+ \frac{(2K)!}{(K+1)^2}\left(g_{K+1,TM}^{K-1*}g_{K,TM}^{K} + g_{K,TM}^{-K}g_{K+1,TM}^{-K+1*}\right) \end{aligned} \tag{15}$$

$$\begin{aligned} X_H^{\alpha K} &= \frac{1}{K^2}\sum_{p=1}^{K-1}\frac{(K-1+p)!}{(K-1-p)!}\left(g_{K,TE}^{p-1}g_{K-1,TE}^{p*} + g_{K-1,TE}^{-p*}g_{K,TE}^{-p+1}\right) \\ &+ \frac{1}{(K+1)^2}\sum_{p=1}^{K-1}\frac{(K+p)!}{(K-p)!}\left(g_{K+1,TE}^{p-1*}g_{K,TE}^{p} + g_{K,TE}^{-p}g_{K+1,TE}^{-p+1*}\right) \\ &+ \frac{(2K)!}{(K+1)^2}\left(g_{K+1,TE}^{K-1*}g_{K,TE}^{K} + g_{K,TE}^{-K}g_{K+1,TE}^{-K+1*}\right) \end{aligned} \tag{16}$$

$$\begin{aligned} X_E^{\beta K} &= \frac{-1}{K^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K-1,TM}^{p-1*}g_{K,TM}^{p} + g_{K,TM}^{-p}g_{K-1,TM}^{-p+1*}\right) \\ &- \frac{1}{(K+1)^2}\sum_{p=1}^{K}\frac{(K+1+p)!}{(K+1-p)!}\left(g_{K,TM}^{p-1}g_{K+1,TM}^{p*} + g_{K+1,TM}^{-p*}g_{K,TM}^{-p+1}\right) \\ &- \frac{(2K+2)!}{(K+1)^2}\left(g_{K,TM}^{K}g_{K+1,TM}^{K+1*} + g_{K+1,TM}^{-K-1*}g_{K,TM}^{-K}\right) \end{aligned} \tag{17}$$

$$\begin{aligned} X_H^{\beta K} &= \frac{-1}{K^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K-1,TE}^{p-1*}g_{K,TE}^{p} + g_{K,TE}^{-p}g_{K-1,TE}^{-p+1*}\right) \\ &- \frac{1}{(K+1)^2}\sum_{p=1}^{K}\frac{(K+1+p)!}{(K+1-p)!}\left(g_{K,TE}^{p-1}g_{K+1,TE}^{p*} + g_{K+1,TE}^{-p*}g_{K,TE}^{-p+1}\right) \\ &- \frac{(2K+2)!}{(K+1)^2}\left(g_{K,TE}^{K}g_{K+1,TE}^{K+1*} + g_{K+1,TE}^{-K-1*}g_{K,TE}^{-K}\right) \end{aligned} \tag{18}$$

$$\begin{aligned} X_E^{\gamma K} &= \frac{2K+1}{K^2(K+1)^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K,TM}^{-p}g_{K,TE}^{-p+1*} + g_{K,TE}^{-p*}g_{K,TM}^{-p+1}\right. \\ &\left. - g_{K,TM}^{p-1}g_{K,TE}^{p*} - g_{K,TE}^{p-1*}g_{K,TM}^{p}\right) \end{aligned} \tag{19}$$

$$\begin{aligned} X_H^{\gamma K} &= \frac{2K+1}{K^2(K+1)^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K,TM}^{p-1*}g_{K,TE}^{p} + g_{K,TE}^{p-1}g_{K,TM}^{p*}\right. \\ &\left. - g_{K,TM}^{-p*}g_{K,TE}^{-p+1} - g_{K,TE}^{-p}g_{K,TM}^{-p+1*}\right) \end{aligned} \tag{20}$$

For $K = 2$, we recover Eqs. (85)–(93) of [1]. Next, the recoil x-transverse K-force is found to read as:

$$\overline{\sin\theta\cos\varphi C_{sca}^K} = \frac{\lambda^2}{\pi}\mathrm{Re}\left(ia_K b_K^* X_{EH}^{KK} + a_K a_{K+1}^* X_E^{KK+1} + b_K b_{K+1}^* X_H^{KK+1}\right) \tag{21}$$

in which:

$$X_{EH}^{KK} = X_{EH}^{\gamma KK} \quad (22)$$

$$X_{E}^{KK+1} = X_{E}^{\alpha KK+1} + X_{E}^{\beta KK+1} \quad (23)$$

$$X_{H}^{KK+1} = X_{H}^{\alpha KK+1} + X_{H}^{\beta KK+1} \quad (24)$$

in which:

$$X_{E}^{\alpha KK+1} = \frac{1}{(K+1)^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K+1,TM}^{p-1*}g_{K,TM}^{p} + g_{K,TM}^{-p}g_{K+1,TM}^{-p+1*}\right) \quad (25)$$

$$X_{H}^{\alpha KK+1} = \frac{1}{(K+1)^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K+1,TE}^{p-1*}g_{K,TE}^{p} + g_{K,TE}^{-p}g_{K+1,TE}^{-p+1*}\right) \quad (26)$$

$$X_{E}^{\beta KK+1} = \frac{-1}{(K+1)^2}\sum_{p=1}^{K+1}\frac{(K+1+p)!}{(K+1-p)!}\left(g_{K,TM}^{p-1}g_{K+1,TM}^{p*} + g_{K+1,TM}^{-p*}g_{K,TM}^{-p+1}\right) \quad (27)$$

$$X_{H}^{\beta KK+1} = \frac{-1}{(K+1)^2}\sum_{p=1}^{K+1}\frac{(K+1+p)!}{(K+1-p)!}\left(g_{K,TE}^{p-1}g_{K+1,TE}^{p*} + g_{K+1,TE}^{-p*}g_{K,TE}^{-p+1}\right) \quad (28)$$

$$X_{EH}^{\gamma KK} = \frac{2K+1}{K^2(K+1)^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K,TE}^{-p*}g_{K,TM}^{-p+1} - g_{K,TM}^{p-1}g_{K,TE}^{p*} + g_{K,TM}^{-p}g_{K,TE}^{-p+1*} - g_{K,TE}^{p-1*}g_{K,TM}^{p}\right) \quad (29)$$

For $K = 2$, we recover Eqs. (94)–(97) of [1]. Next, the mixing y-transverse K-force is found to read as:

$$\overline{\sin\theta\sin\varphi}C_{ext}^{K} = \frac{\lambda^2}{2\pi}\mathrm{Im}\left(a_K Y_E^K + b_K Y_H^K\right) \quad (30)$$

in which:

$$Y_E^K = Y_E^{\alpha K} + Y_E^{\beta K} + iY_E^{\gamma K} \quad (31)$$

$$Y_H^K = Y_H^{\alpha K} + Y_H^{\beta K} + iY_H^{\gamma K} \quad (32)$$

in which:

$$Y_E^{\alpha K} = \frac{1}{K^2}\sum_{p=1}^{K-1}\frac{(K-1+p)!}{(K-1-p)!}\left(g_{K,TM}^{p-1}g_{K-1,TM}^{p*} - g_{K-1,TM}^{-p*}g_{K,TM}^{-p+1}\right) + \frac{1}{(K+1)^2}\sum_{p=1}^{K-1}\frac{(K+p)!}{(K-p)!}\left(g_{K,TM}^{-p}g_{K+1,TM}^{-p+1*} - g_{K+1,TM}^{p-1*}g_{K,TM}^{p}\right) + \frac{(2K)!}{(K+1)^2}\left(g_{K,TM}^{-K}g_{K+1,TM}^{-K+1*} - g_{K+1,TM}^{K-1*}g_{K,TM}^{K}\right) \quad (33)$$

$$Y_H^{\alpha K} = \frac{1}{K^2}\sum_{p=1}^{K-1}\frac{(K-1+p)!}{(K-1-p)!}\left(g_{K,TE}^{p-1}g_{K-1,TE}^{p*} - g_{K-1,TE}^{-p*}g_{K,TE}^{-p+1}\right) + \frac{1}{(K+1)^2}\sum_{p=1}^{K-1}\frac{(K+p)!}{(K-p)!}\left(g_{K,TE}^{-p}g_{K+1,TE}^{-p+1*} - g_{K+1,TE}^{p-1*}g_{K,TE}^{p}\right) + \frac{(2K)!}{(K+1)^2}\left(g_{K,TE}^{-K}g_{K+1,TE}^{-K+1*} - g_{K+1,TE}^{K-1*}g_{K,TE}^{K}\right) \quad (34)$$

$$Y_E^{\beta K} = \frac{-1}{K^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K,TM}^{-p}g_{K-1,TM}^{-p+1*} - g_{K-1,TM}^{p-1*}g_{K,TM}^{p}\right) - \frac{1}{(K+1)^2}\sum_{p=1}^{K}\frac{(K+1+p)!}{(K+1-p)!}\left(g_{K,TM}^{p-1}g_{K+1,TM}^{p*} - g_{K+1,TM}^{-p*}g_{K,TM}^{-p+1}\right) - \frac{(2K+2)!}{(K+1)^2}\left(g_{K,TM}^{K}g_{K+1,TM}^{K+1*} - g_{K+1,TM}^{-K-1*}g_{K,TM}^{-K}\right) \quad (35)$$

$$Y_H^{\beta K} = \frac{-1}{K^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K,TE}^{-p}g_{K-1,TE}^{-p+1*} - g_{K-1,TE}^{p-1*}g_{K,TE}^{p}\right) - \frac{1}{(K+1)^2}\sum_{p=1}^{K}\frac{(K+1+p)!}{(K+1-p)!}\left(g_{K,TE}^{p-1}g_{K+1,TE}^{p*} - g_{K+1,TE}^{-p*}g_{K,TE}^{-p+1}\right) - \frac{(2K+2)!}{(K+1)^2}\left(g_{K,TE}^{K}g_{K+1,TE}^{K+1*} - g_{K+1,TE}^{-K-1*}g_{K,TE}^{-K}\right) \quad (36)$$

$$Y_E^{\gamma K} = \frac{2K+1}{K^2(K+1)^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K,TE}^{p-1*}g_{K,TM}^{p} + g_{K,TM}^{-p}g_{K,TE}^{-p+1*} - g_{K,TM}^{p-1}g_{K,TE}^{p*} - g_{K,TE}^{-p*}g_{K,TM}^{-p+1}\right) \quad (37)$$

$$Y_H^{\gamma K} = \frac{2K+1}{K^2(K+1)^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K,TE}^{p-1}g_{K,TM}^{p*} + g_{K,TM}^{-p*}g_{K,TE}^{-p+1} - g_{K,TM}^{p-1*}g_{K,TE}^{p} - g_{K,TE}^{-p}g_{K,TM}^{-p+1*}\right) \quad (38)$$

For $K = 2$, we recover Eqs. (98)–(106) of [1]. Finally, the recoil y-transverse K-force is found to read as:

$$\overline{\sin\theta\sin\varphi}C_{sca}^{K} = \frac{\lambda^2}{\pi}\mathrm{Im}\left(ia_K b_K^* Y_{EH}^{KK} + a_K a_{K+1}^* Y_E^{KK+1} + b_K b_{K+1}^* Y_H^{KK+1}\right) \quad (39)$$

in which:

$$Y_{EH}^{KK} = Y_{EH}^{\gamma KK} \quad (40)$$

$$Y_E^{KK+1} = Y_E^{\alpha KK+1} + Y_E^{\beta KK+1} \quad (41)$$

$$Y_H^{KK+1} = Y_H^{\alpha KK+1} + Y_H^{\beta KK+1} \quad (42)$$

in which:

$$Y_E^{\alpha KK+1} = \frac{1}{(K+1)^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K,TM}^{-p}g_{K+1,TM}^{-p+1*} - g_{K+1,TM}^{p-1*}g_{K,TM}^{p}\right) \quad (43)$$

$$Y_H^{\alpha KK+1} = \frac{1}{(K+1)^2}\sum_{p=1}^{K}\frac{(K+p)!}{(K-p)!}\left(g_{K,TE}^{-p}g_{K+1,TE}^{-p+1*} - g_{K+1,TE}^{p-1*}g_{K,TE}^{p}\right) \quad (44)$$

$$Y_E^{\beta KK+1} = \frac{1}{(K+1)^2}\sum_{p=1}^{K+1}\frac{(K+1+p)!}{(K+1-p)!}\left(g_{K+1,TM}^{-p*}g_{K,TM}^{-p+1} - g_{K,TM}^{p-1}g_{K+1,TM}^{p*}\right) \quad (45)$$

$$Y_H^{\beta KK+1} = \frac{1}{(K+1)^2}\sum_{p=1}^{K+1}\frac{(K+1+p)!}{(K+1-p)!}\left(g_{K+1,TE}^{-p*}g_{K,TE}^{-p+1}\right.$$

$$- g_{K,TE}^{p-1} g_{K+1,TE}^{p*}\Big) \tag{46}$$

$$Y_{EH}^{\gamma KK} = \frac{2K+1}{K^2(K+1)^2} \sum_{p=1}^{K} \frac{(K+p)!}{(K-p)!} \Big(g_{K,TE}^{p-1*} g_{K,TM}^{p} + g_{K,TM}^{-p} g_{K,TE}^{-p+1*} - g_{K,TM}^{p-1} g_{K,TE}^{p*} - g_{K,TE}^{-p*} g_{K,TM}^{-p+1} \Big) \tag{47}$$

For $K = 2$, we recover Eqs. (107)–(110) of [1].

4. Interpretations of mixing forces

4.1. Mixing forces in the z-direction

Eq. 5 is the sum of an electric force $\overline{\cos\theta C_{ext,E}^K}$ associated with the electric Mie coefficient a_K and of a magnetic force $\overline{\cos\theta C_{ext,H}^K}$ associated with the magnetic Mie coefficient b_K. Both of these forces may be decomposed into two subforces. The decomposition of the electric force $\overline{\cos\theta C_{ext,E}^K}$ reads as:

$$\overline{\cos\theta C_{ext,E}^K} = \overline{\cos\theta C_{ext,E}^{KR}} + \overline{\cos\theta C_{ext,E}^{KI}} \tag{48}$$

in which:

$$\overline{\cos\theta C_{ext,E}^{KI}} = \frac{-\lambda^2}{\pi} \mathrm{Im}(a_K)\mathrm{Im}(Z_E^K) \tag{49}$$

$$\overline{\cos\theta C_{ext,E}^{KR}} = \frac{\lambda^2}{\pi} \mathrm{Re}(a_K)\mathrm{Re}(Z_E^K) \tag{50}$$

in which the subscripts I and R refer to the imaginary and real parts respectively of the BSC-dependent terms. In the present case, the same subscripts correspond as well to the imaginary and real parts respectively of the Mie coefficients, but this feature is accidental and is not to be considered as a rule, as later counter-examples will show. For $K = 1$ and 2, $\overline{\cos\theta C_{ext,E}^{KI}}$ is a mixing gradient force [1]. We shall argue in Section 4.4 that the same is true for $K > 2$. For $K = 1$ and 2 again, $\overline{\cos\theta C_{ext,E}^{KR}}$ is the sum of a mixing scattering force and of a mixing non-standard force [1]. Let us carry the same decomposition for $K > 2$ before commenting and justifying. We then have, adding a supercript S standing for "Scattering" and NS standing for "Non-Standard":

$$\overline{\cos\theta C_{ext,E}^{KRS}} = \frac{\lambda^2}{\pi} \mathrm{Re}(a_K)\mathrm{Re}(Z_E^{KS}) \tag{51}$$

$$\overline{\cos\theta C_{ext,E}^{KRNS}} = \frac{\lambda^2}{\pi} \mathrm{Re}(a_K)\mathrm{Re}(Z_E^{KNS}) \tag{52}$$

in which:

$$Z_E^{KS} = -\frac{(2K+1)i}{K^2(K+1)^2} \sum_{p=-K}^{+K} p \frac{(K+|p|)!}{(K-|p|)!} g_{K,TM}^{p} g_{K,TE}^{p*} \tag{53}$$

$$Z_E^{KNS} = \frac{1}{K^2} \sum_{p=-K+1}^{K-1} \frac{(K+|p|)!}{(K-1-|p|)!} g_{K-1,TM}^{p*} g_{K,TM}^{p} + \frac{1}{(K+1)^2} \sum_{p=-K}^{+K} \frac{(K+1+|p|)!}{(K-|p|)!} g_{K,TM}^{p} g_{K+1,TM}^{p*} \tag{54}$$

For $K = 2$, we recover Eq. (117)–(120) of [1]. As previously recalled, for $K = 1$ this decomposition has a strong physical meaning since the force generated by Z_E^{KS} is proportional to the z -component of the Poynting vector while Z_E^{KNS} generates a non-gradient force which is not proportional to this z-component, a meaning which does not propagate to the case $K > 1$. However, structural differences, valid for $K = 1$, do propagate to $K > 1$, and serve as justifications to distinguish between scattering and non-standard forces. First, mixing scattering forces couple TM and TE partial waves while non-standard forces are generated by $TM - TM$ couplings (it will be $TE - TE$ in the case of magnetic forces). Second, all subscripts of the scattering force are equal to K while non-standard forces contain subscripts of different orders ($K-1$, K, $K+1$). In summary, the discrimination between scattering forces and non-standard forces for $K > 1$ relies on (i) the fact that it already exists in the case of 1-forces where it is the consequence of precise and accurate definitions of scattering and non-scattering forces and (ii) on structural differences which are valid whatever K and therefore provide a unified scheme of discrimination. Section 6 will furthermore show how much the distinction between scattering and non-standard forces introduces a deep coherency in the optical force partition. Another issue is that the non-standard 1-forces provide a contribution to spin-curl forces in the context of the dipole theory of forces [34–38], and in the context of GLMT, e.g. [31,39,40], and references therein. The physical signification of non-standard forces for K>1 then raises a question which is however outside of the scope of the present paper although it will have to be investigated.

Similarly, the decomposition of the magnetic force $\overline{\cos\theta C_{ext,H}^K}$ reads as:

$$\overline{\cos\theta C_{ext,H}^K} = \overline{\cos\theta C_{ext,H}^{KR}} + \overline{\cos\theta C_{ext,H}^{KI}} \tag{55}$$

in which:

$$\overline{\cos\theta C_{ext,H}^{KI}} = \frac{-\lambda^2}{\pi} \mathrm{Im}(b_K)\mathrm{Im}(Z_H^K) \tag{56}$$

$$\overline{\cos\theta C_{ext,H}^{KR}} = \frac{\lambda^2}{\pi} \mathrm{Re}(b_K)\mathrm{Re}(Z_H^K) \tag{57}$$

in which $\overline{\cos\theta C_{ext,H}^{KI}}$ will be shown to be a mixing gradient force (Section 4.4) while $\overline{\cos\theta C_{ext,H}^{KR}}$ is the summation of a scattering and of a non-standard force according to:

$$\overline{\cos\theta C_{ext,H}^{KRS}} = \frac{\lambda^2}{\pi} \mathrm{Re}(b_K)\mathrm{Re}(Z_H^{KS}) \tag{58}$$

$$\overline{\cos\theta C_{ext,H}^{KRNS}} = \frac{\lambda^2}{\pi} \mathrm{Re}(b_K)\mathrm{Re}(Z_H^{KNS}) \tag{59}$$

in which:

$$Z_H^{KS} = \frac{(2K+1)i}{K^2(K+1)^2} \sum_{p=-K}^{+K} p \frac{(K+|p|)!}{(K-|p|)!} g_{K,TM}^{p*} g_{K,TE}^{p} \tag{60}$$

$$Z_H^{KNS} = \frac{1}{K^2} \sum_{p=-K+1}^{K-1} \frac{(K+|p|)!}{(K-1-|p|)!} g_{K-1,TE}^{p*} g_{K,TE}^{p} + \frac{1}{(K+1)^2} \sum_{p=-K}^{+K} \frac{(K+1+|p|)!}{(K-|p|)!} g_{K,TE}^{p} g_{K+1,TE}^{p*} \tag{61}$$

For $K = 2$, we recover Eqs. (121)–(125) of [1].

4.2. Mixing forces in the x-direction

Similarly as for the z-direction, Eq. (12) is the sum of an electric force $\overline{\sin\theta\cos\varphi C_{ext,E}^K}$ and of a magnetic force $\overline{\sin\theta\cos\varphi C_{ext,H}^K}$. The electric force is decomposed in two subforces reading as:

$$\overline{\sin\theta\cos\varphi C_{ext,E}^{KR}} = \frac{\lambda^2}{2\pi} \mathrm{Re}(a_K)\mathrm{Re}(X_E^K) \tag{62}$$

$$\overline{\sin\theta\cos\varphi C_{ext,E}^{KI}} = \frac{-\lambda^2}{2\pi} \mathrm{Im}(a_K)\mathrm{Im}(X_E^K) \tag{63}$$

As previously, $\overline{\sin\theta\cos\varphi C^{KI}_{ext,E}}$ must be a gradient force (Section 4.4) while $\overline{\sin\theta\cos\varphi C^{KR}_{ext,E}}$ is decomposed as the sum of a scattering force and of a non-standard force according to:

$$\overline{\sin\theta\cos\varphi C^{KRS}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(a_K)\mathrm{Re}\left(iX_E^{\gamma K}\right) \tag{64}$$

$$\overline{\sin\theta\cos\varphi C^{KRNS}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(a_K)\mathrm{Re}\left(X_E^{\alpha K} + X_E^{\beta K}\right) \tag{65}$$

with the same justifications as for the z-component, as can be observed by examining the structure of $X_E^{\gamma K}$ to be compared with the structure of $(X_E^{\alpha K} + X_E^{\beta K})$. The magnetic force $\overline{\sin\theta\cos\varphi C^{K}_{ext,H}}$ is similarly decomposed as:

$$\overline{\sin\theta\cos\varphi C^{KR}_{ext,H}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(b_K)\mathrm{Re}\left(X_H^{K}\right) \tag{66}$$

$$\overline{\sin\theta\cos\varphi C^{KI}_{ext,H}} = \frac{-\lambda^2}{2\pi}\mathrm{Im}(b_K)\mathrm{Im}\left(X_H^{K}\right) \tag{67}$$

Once more, we anticipate that $\overline{\sin\theta\cos\varphi C^{KI}_{ext,H}}$ is a gradient force (Section 4.4) while $\overline{\sin\theta\cos\varphi C^{KR}_{ext,H}}$ is decomposed as the sum of a scattering force and of a non-standard force according to:

$$\overline{\sin\theta\cos\varphi C^{KRS}_{ext,H}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(b_K)\mathrm{Re}\left(iX_H^{\gamma K}\right) \tag{68}$$

$$\overline{\sin\theta\cos\varphi C^{KRNS}_{ext,H}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(b_K)\mathrm{Re}\left(X_H^{\alpha K} + X_H^{\beta K}\right) \tag{69}$$

with again the same justifications as previously. For $K = 2$, we recover the results of Section 6.1.2 of [1].

4.3. Mixing forces in the y-direction

From Eq. (30), we similarly have:

$$\overline{\sin\theta\sin\varphi C^{KR}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Im}(a_K)\mathrm{Re}(Y_E^{K}) \tag{70}$$

$$\overline{\sin\theta\sin\varphi C^{KI}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(a_K)\mathrm{Im}(Y_E^{K}) \tag{71}$$

in which, noting an interchange between the roles of the superscripts R and I, $\overline{\sin\theta\sin\varphi C^{KR}_{ext,E}}$ is a gradient force (Section 4.4) while $\overline{\sin\theta\sin\varphi C^{KI}_{ext,E}}$ is decomposed as:

$$\overline{\sin\theta\sin\varphi C^{KIS}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(a_K)\mathrm{Im}(iY_E^{\gamma K}) \tag{72}$$

$$\overline{\sin\theta\sin\varphi C^{KINS}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(a_K)\mathrm{Im}(Y_E^{\alpha K} + Y_E^{\beta K}) \tag{73}$$

Also, we have:

$$\overline{\sin\theta\sin\varphi C^{KR}_{ext,H}} = \frac{\lambda^2}{2\pi}\mathrm{Im}(b_K)\mathrm{Re}(Y_H^{K}) \tag{74}$$

$$\overline{\sin\theta\sin\varphi C^{KI}_{ext,H}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(b_K)\mathrm{Im}(Y_H^{K}) \tag{75}$$

in which $\overline{\sin\theta\sin\varphi C^{KR}_{ext,H}}$ is a gradient force (Section 4.4) while $\overline{\sin\theta\sin\varphi C^{KI}_{ext,H}}$ may be decomposed as:

$$\overline{\sin\theta\sin\varphi C^{KIS}_{ext,H}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(b_K)\mathrm{Im}(iY_H^{\gamma K}) \tag{76}$$

$$\overline{\sin\theta\sin\varphi C^{KINS}_{ext,H}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(b_K)\mathrm{Im}(Y_H^{\alpha K} + Y_H^{\beta K}) \tag{77}$$

For $K = 2$, we recover the results of Section 6.1.3 in Gouesbet et al. [1].

4.4. Mixing gradient forces

In the three previous subsections above, we announced that three electric forces, namely $\overline{\cos\theta C^{KI}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{KI}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{KR}_{ext,E}}$, and three magnetic forces, namely $\overline{\cos\theta C^{KI}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{KI}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{KR}_{ext,H}}$ are gradient forces. It is the great merit of Zheng et al. [32] to have demonstrated that these forces are indeed mixing gradient forces. For this, they rely on an arXiv paper by Jiang et al.[41], see as well [42]. In these works, the optical forces are deduced by using the Maxwell stress tensor, leading to the first categorization in terms of mixing/recoil forces omitting however to mention that such a categorization was already available nearly four decades ago. Other similar omissions seem to have been the motivation for a criticism by Nieto-Vesperinas [43]. Nevertheless, the expressions by Zheng et al. [32] of the mixing gradient forces, obtained by coupling field multiple derivatives and an angular spectrum decomposition, represent a genuine advance in the field of optical forces. Furthermore, the agreement between their results and ours is a corroboration of the validity of the results concerning mixing gradient forces. They also obtained expressions for recoil gradient forces but failed to see that recoil gradient forces may be readily expressed in terms of mixing gradient forces with simple formulas, as we shall demonstrate below.

The mixing gradient electric forces $\overline{\cos\theta C^{KI}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{KI}_{ext,E}}$ and $\overline{\sin\theta\sin\varphi C^{KR}_{ext,E}}$ above (renamed $\overline{\cos\theta C^{KIG}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{KIG}_{ext,E}}$ and $\overline{\sin\theta\sin\varphi C^{KRG}_{ext,E}}$ with "G" standing for "Gradient") may indeed be shown to be equivalent to the corresponding forces expressed in Eq. (24) of Zheng et al. [32]. However, rather than using the traditional BSCs of GLMT, Zheng et al. introduced so-called partial wave expansion coefficients (PWECs). The translation between PWECs and BSCs is provided in Appendix A. The translation between the electric forces of the present section and those of Eq. (24) in Zheng et al. [32] is provided in Appendix B which discusses as well the case of the mixing gradient magnetic forces (renamed $\overline{\cos\theta C^{KIG}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{KIG}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{KRG}_{ext,H}}$). For $K = 2$, see Section 6.1.4 of [1].

5. Interpretations of recoil forces

5.1. Recoil forces in the z-direction

The recoil force of Eq. (8) is the summation of a magnetoelectric force $\overline{\cos\theta C^{K}_{sca,EH}}$, of an electric force $\overline{\cos\theta C^{K}_{sca,E}}$ and of a magnetic force $\overline{\cos\theta C^{K}_{sca,H}}$. The magnetoelectric force may be decomposed in two terms according to:

$$\overline{\cos\theta C^{K}_{sca,EH}} = \overline{\cos\theta C^{KR}_{sca,EH}} + \overline{\cos\theta C^{KI}_{sca,EH}} \tag{78}$$

in which:

$$\overline{\cos\theta C^{KR}_{sca,EH}} = \frac{-2\lambda^2}{\pi}\mathrm{Re}(a_K b_K^*)\mathrm{Re}(iZ_{EH}^{KK}) \tag{79}$$

$$\overline{\cos\theta C^{KI}_{sca,EH}} = \frac{2\lambda^2}{\pi}\mathrm{Im}(a_K b_K^*)\mathrm{Im}(iZ_{EH}^{KK}) \tag{80}$$

Then, using Eqs. (9), (51), (53), (58), (60), and (79) we demonstrate that:

$$\begin{aligned}\overline{\cos\theta C^{KR}_{sca,EH}} &= 2\frac{\mathrm{Re}(a_K b_K^*)}{\mathrm{Re}(a_K)}\overline{\cos\theta C^{KRS}_{ext,E}} \\ &= 2\frac{\mathrm{Re}(a_K b_K^*)}{\mathrm{Re}(b_K)}\overline{\cos\theta C^{KRS}_{ext,H}}\end{aligned} \tag{81}$$

which is a generalization of Eqs. (35) and (141) of [1]. Eq. (81) shows that the longitudinal recoil magnetoelectric force $\overline{\cos\theta C^{KR}_{sca,EH}}$ (now renamed $\overline{\cos\theta C^{KRS}_{sca,EH}}$) is a recoil scattering magnetoelectric force which may be expressed in terms of longitudinal mixing scattering pure electric and pure magnetic forces. Such a relation (and many others of the same kind), exhibiting a relationship between recoil scattering forces and mixing scattering forces is another argument, valid as well for non-standard forces (e.g. an example in Eq. (89)), which supports the interest of the decomposition of non-gradient forces into scattering and non-standard forces. Concerning $\overline{\cos\theta C^{KI}_{sca,EH}}$ (now renamed $\overline{\cos\theta C^{KINS}_{sca,EH}}$), it is classified as being a non-standard force because (i) for $K=1$, it is indeed a non-standard force as being a non-gradient force which is not a scattering force and (ii) although the original definition of scattering force does not propagate from $K=1$ to $K>1$, there is a structural difference (i.e. the use of Re in Eq. (79) versus the use of Im in Eq. (80)) which does propagate from $K=1$ to $K>1$.

Next, the electric force $\overline{\cos\theta C^{K}_{sca,E}}$ may be decomposed into two subforces reading as:

$$\overline{\cos\theta C^{KR}_{sca,E}} = \frac{-2\lambda^2}{\pi}\mathrm{Re}(a_K a^*_{K+1})\mathrm{Re}\left(Z^{KK+1}_E\right) \tag{82}$$

$$\overline{\cos\theta C^{KI}_{sca,E}} = \frac{2\lambda^2}{\pi}\mathrm{Im}(a_K a^*_{K+1})\mathrm{Im}\left(Z^{KK+1}_E\right) \tag{83}$$

From Eq. (10), we deduce:

$$Z^{K-1K}_E = \frac{-1}{K^2}\sum_{p=-K+1}^{K-1}\frac{(K+|p|)!}{(K-1-|p|)!}g^p_{K-1,TM}g^{p*}_{K,TM} \tag{84}$$

Afterward, we use Eqs. (10), (54) and (84) to obtain:

$$Z^{KK+1}_E = -Z^{KNS}_E - Z^{K-1K*}_E \tag{85}$$

Inserting Eq. (85) into Eq. (82), we obtain:

$$\overline{\cos\theta C^{KR}_{sca,E}} = \frac{2\lambda^2}{\pi}\mathrm{Re}(a_K a^*_{K+1})\left[\mathrm{Re}(Z^{KNS}_E) + \mathrm{Re}(Z^{K-1K}_E)\right] \tag{86}$$

We may then use Eqs. (52) and (82), adapted from K to $(K-1)$, to obtain, from Eq. (86):

$$\overline{\cos\theta C^{KR}_{sca,E}} = 2\frac{\mathrm{Re}(a_K a^*_{K+1})}{\mathrm{Re}(a_K)}\overline{\cos\theta C^{KRNS}_{ext,E}} - \frac{\mathrm{Re}(a_K a^*_{K+1})}{\mathrm{Re}(a_{K-1}a^*_K)}\overline{\cos\theta C^{K-1R}_{sca,E}} \tag{87}$$

Assume, as a recurrence assumption, that $\overline{\cos\theta C^{K-1R}_{sca,E}}$ is a non-standard force (this is true for $K=2$), then $\overline{\cos\theta C^{KR}_{sca,E}}$ is a non-standard force. Also, this equation is not valid for $K=1$ since $\overline{\cos\theta C^{0R}_{sca,E}}$ does not exist, providing a supplementary reason (although not the most important one) to distinguish the cases $K=1$ and $K>1$. Clearly, we could obtain an equation which is valid as well for $K=1$ by using a multiplicative prefactor of the form $(1-\delta_{K-1,0})$ but even this process would emphasize that $K=1$ is a special value.

As a variant, let us return to Eq. (52) which expresses $\overline{\cos\theta C^{KRNS}_{ext,E}}$ versus Z^{KNS}_E. Using Eq. (85), $\overline{\cos\theta C^{KRNS}_{ext,E}}$ may be expressed as a summation of two non-standard forces, one of them reading as:

$$\overline{\cos\theta C^{KRNS2}_{ext,E}} = \frac{-\lambda^2}{\pi}\mathrm{Re}(a_K)\mathrm{Re}(Z^{KK+1}_E) \tag{88}$$

leading to:

$$\overline{\cos\theta C^{KR}_{sca,E}} = 2\frac{\mathrm{Re}(a_K a^*_{K+1})}{\mathrm{Re}(a_K)}\overline{\cos\theta C^{KRNS2}_{ext,E}} \tag{89}$$

which confirms that $\overline{\cos\theta C^{KR}_{sca,E}}$ (now renamed $\overline{\cos\theta C^{KRNS}_{sca,E}}$) is a non-standard force. We then have the case of a recoil non-standard electric force which may be expressed in terms of a mixing non-standard electric force. For $K=2$, we recover Eq. (198) of [1].

We now consider $\overline{\cos\theta C^{KI}_{sca,E}}$ of Eq. (83), and begin by elaborate a bit on the case $K=2$. It has been found, see Eqs. (179)–(180) in Gouesbet et al. [1], that $\overline{\cos\theta C^{2I}_{sca,E}}$ may be written as the summation of two forces according to:

$$\overline{\cos\theta C^{2I}_{sca,E}} = \overline{\cos\theta C^{2I\alpha}_{sca,E}} + \overline{\cos\theta C^{2I\beta}_{sca,E}} \tag{90}$$

in which:

$$\overline{\cos\theta C^{2I\alpha}_{sca,E}} = 2\mathrm{Im}(a_2 a^*_3)\sum_{j=1}^{2}\frac{\overline{\cos\theta C^{jI}_{ext,E}}}{\mathrm{Im}(a_j)} \tag{91}$$

$$\begin{aligned}\overline{\cos\theta C^{2I\beta}_{sca,E}} = &\frac{\lambda^2}{\pi}\mathrm{Im}(a_2 a^*_3)\mathrm{Im}\Big\{i[3\left(g^{-1}_{1,TM}g^{-1*}_{1,TE} - g^1_{1,TM}g^{1*}_{1,TE}\right)\\ &+\frac{\lambda^2}{\pi}\mathrm{Im}(a_2 a^*_3)\mathrm{Im}\Big[\frac{5i}{3}(g^{-1}_{2,TM}g^{-1*}_{2,TE} - g^1_{2,TM}g^{1*}_{2,TE})\\ &+\frac{40i}{3}\left(g^{-2}_{2,TM}g^{-2*}_{2,TE} - g^2_{2,TM}g^{2*}_{2,TE}\right)\Big]\Big\}\end{aligned} \tag{92}$$

Let us introduce:

$$\Delta Z^{KK+1}_E = \frac{-(2K+1)i}{K^2(K+1)^2}\sum_{p=-K}^{+K}p\frac{(K+|p|)!}{(K-|p|)!}g^p_{K,TM}g^{p*}_{K,TE} \tag{93}$$

Eq. (92) then becomes:

$$\overline{\cos\theta C^{2I\beta}_{sca,E}} = \frac{2\lambda^2}{\pi}\ \mathrm{Im}(a_2 a^*_3)\mathrm{Im}\sum_{j=1}^{2}\Delta Z^{jj+1}_E \tag{94}$$

In order to provide a demonstration by recurrence, we now assume that Eqs. (91) and (94) are valid for $(K-1)$-forces and demonstrate that they are valid for K-forces. We then have:

$$\overline{\cos\theta C^{K-1I\alpha}_{sca,E}} = 2\mathrm{Im}(a_{K-1}a^*_K)\sum_{j=1}^{K-1}\frac{\overline{\cos\theta C^{jI}_{ext,E}}}{\mathrm{Im}(a_j)} \tag{95}$$

$$\overline{\cos\theta C^{K-1I\beta}_{sca,E}} = \frac{2\lambda^2}{\pi}\mathrm{Im}(a_{K-1}a^*_K)\mathrm{Im}\sum_{j=1}^{K-1}\Delta Z^{jj+1}_E \tag{96}$$

Using Eqs. (6), (10), (84), and (93), we establish:

$$Z^{KK+1}_E = -Z^{K-1K*}_E - Z^K_E + \Delta Z^{KK+1}_E \tag{97}$$

which, using Eq. (83), leads to:

$$\begin{aligned}\overline{\cos\theta C^{KI}_{sca,E}} = &\frac{2\lambda^2}{\pi}\mathrm{Im}(a_K a^*_{K+1})\mathrm{Im}(\Delta Z^{KK+1}_E)\\ &-\frac{2\lambda^2}{\pi}\mathrm{Im}(a_K a^*_{K+1})\mathrm{Im}(Z^K_E)\\ &+\frac{2\lambda^2}{\pi}\mathrm{Im}(a_K a^*_{K+1})\mathrm{Im}(Z^{K-1K}_E)\end{aligned} \tag{98}$$

Let us first work out the last line of (98), denoted as III. From Eq. (83), we have:

$$\overline{\cos\theta C^{K-1I}_{sca,E}} = \frac{2\lambda^2}{\pi}\mathrm{Im}(a_{K-1}a^*_K)\mathrm{Im}(Z^{K-1K}_E) \tag{99}$$

leading to:

$$III = \frac{\mathrm{Im}(a_K a^*_{K+1})}{\mathrm{Im}(a_{K-1}a^*_K)}\overline{\cos\theta C^{K-1I}_{sca,E}} \tag{100}$$

But $\overline{\cos\theta C^{K-1I}_{sca,E}}$ can be decomposed according to:

$$\overline{\cos\theta C^{K-1I}_{sca,E}} = \overline{\cos\theta C^{K-1I\alpha}_{sca,E}} + \overline{\cos\theta C^{K-1I\beta}_{sca,E}} \tag{101}$$

We then decompose III into $III^\alpha + III^\beta$, and relying on Eqs. (100) and (101), we have:

$$III^\alpha = \frac{\mathrm{Im}(a_K a^*_{K+1})}{\mathrm{Im}(a_{K-1}a^*_K)}\overline{\cos\theta C^{K-1I\alpha}_{sca,E}} \tag{102}$$

$$III^\beta = \frac{\mathrm{Im}(a_K a^*_{K+1})}{\mathrm{Im}(a_{K-1}a^*_K)}\overline{\cos\theta C^{K-1I\beta}_{sca,E}} \tag{103}$$

We may now use the recurrence assumption of Eq. (95) to obtain:

$$III^{\alpha} = 2\mathrm{Im}(a_K a_{K+1}^*) \sum_{j=1}^{K-1} \frac{\overline{\cos\theta C_{ext,E}^{jI}}}{\mathrm{Im}(a_j)} \tag{104}$$

and the recurrence assumption of Eq. (96) to obtain:

$$III^{\beta} = \frac{2\lambda^2}{\pi}\mathrm{Im}(a_K a_{K+1}^*)\ \mathrm{Im}\sum_{j=1}^{K-1} \Delta Z_E^{jj+1} \tag{105}$$

Similarly as for Eq. (101), we decompose $\overline{\cos\theta C_{sca,E}^{KI}}$ as:

$$\overline{\cos\theta C_{sca,E}^{KI}} = \overline{\cos\theta C_{sca,E}^{KI\alpha}} + \overline{\cos\theta C_{sca,E}^{KI\beta}} \tag{106}$$

in which, using Eq. (98), we have:

$$\overline{\cos\theta C_{sca,E}^{KI\alpha}} = -\frac{2\lambda^2}{\pi}\mathrm{Im}(a_K a_{K+1}^*)\mathrm{Im}(Z_E^K) + III^{\alpha} \tag{107}$$

$$\overline{\cos\theta C_{sca,E}^{KI\beta}} = \frac{2\lambda^2}{\pi}\ \mathrm{Im}(a_K a_{K+1}^*)\mathrm{Im}(\Delta Z_E^{KK+1}) + III^{\beta} \tag{108}$$

which, using Eqs. (104) and (105) imply:

$$\overline{\cos\theta C_{sca,E}^{KI\alpha}} = 2\mathrm{Im}(a_K a_{K+1}^*) \sum_{j=1}^{K} \frac{\overline{\cos\theta C_{ext,E}^{jI}}}{\mathrm{Im}(a_j)} \tag{109}$$

in which $\overline{\cos\theta C_{ext,E}^{jI}}$ has been renamed $\overline{\cos\theta C_{ext,E}^{jIG}}$. Therefore $\overline{\cos\theta C_{sca,E}^{KI\alpha}}$, which may be renamed $\overline{\cos\theta C_{sca,E}^{KIG}}$, is a recoil gradient electric force, which is the gradient force contribution to $\overline{\cos\theta C_{sca,E}^{KI}}$, and may be expressed in terms of mixing gradient electric forces. Eq. (109) is the generalization of Eq. (179) in Gouesbet et al. [1].

The following equation is also implied:

$$\overline{\cos\theta C_{sca,E}^{KI\beta}} = \frac{2\lambda^2}{\pi}\ \mathrm{Im}(a_K a_{K+1}^*)\mathrm{Im}\sum_{j=1}^{K} \Delta Z_E^{jj+1} \tag{110}$$

in which $\overline{\cos\theta C_{sca,E}^{KI\beta}}$, renamed $\overline{\cos\theta C_{sca,E}^{KINS}}$, is the non-standard contribution to $\overline{\cos\theta C_{sca,E}^{KI}}$, by extension of the case $K = 1$, see conveniently Section 6 in Gouesbet et al. [1]. For $K = 2$, Eq. (110) is shown to be equivalent (after a small amount of computation) to Eq. (181) in Gouesbet et al. [1].

The magnetic contribution $\overline{\cos\theta C_{sca,H}^{K}}$ is treated quite similarly as for the electric contribution $\overline{\cos\theta C_{sca,E}^{K}}$, with quite similar comments. To begin with, we decompose it into two subforces according to:

$$\overline{\cos\theta C_{sca,H}^{K}} = \overline{\cos\theta C_{sca,H}^{KR}} + \overline{\cos\theta C_{sca,H}^{KI}} \tag{111}$$

in which:

$$\overline{\cos\theta C_{sca,H}^{KR}} = \frac{-2\lambda^2}{\pi}\mathrm{Re}(b_K b_{K+1}^*)\mathrm{Re}(Z_H^{KK+1}) \tag{112}$$

$$\overline{\cos\theta C_{sca,H}^{KI}} = \frac{2\lambda^2}{\pi}\mathrm{Im}(b_K b_{K+1}^*)\mathrm{Im}(Z_H^{KK+1}) \tag{113}$$

Similarly as Eq. (87), we then obtain:

$$\overline{\cos\theta C_{sca,H}^{KR}} = 2\frac{\mathrm{Re}(b_K b_{K+1}^*)}{\mathrm{Re}(b_K)}\overline{\cos\theta C_{ext,H}^{KRNS}} - \frac{\mathrm{Re}(b_K b_{K+1}^*)}{\mathrm{Re}(b_{K-1} b_K^*)}\overline{\cos\theta C_{sca,H}^{K-1R}} \tag{114}$$

or, as a variant:

$$\overline{\cos\theta C_{sca,H}^{KR}} = 2\frac{\mathrm{Re}(b_K b_{K+1}^*)}{\mathrm{Re}(b_K)}\overline{\cos\theta C_{ext,H}^{KRNS2}} \tag{115}$$

in which:

$$\overline{\cos\theta C_{ext,H}^{KRNS2}} = \frac{-\lambda^2}{\pi}\mathrm{Re}(b_K)\mathrm{Re}(Z_H^{KK+1}) \tag{116}$$

Therefore, $\overline{\cos\theta C_{sca,H}^{KR}}$ (renamed $\overline{\cos\theta C_{sca,H}^{KRNS}}$) is a recoil non-standard magnetic force which is expressed in terms of a mixing non-standard magnetic force. For $\overline{\cos\theta C_{sca,H}^{KI}}$, proceeding again similarly as for the electric case, we first establish two recurrence assumptions reading as:

$$\overline{\cos\theta C_{sca,H}^{K-1I\alpha}} = 2\mathrm{Im}(b_{K-1} b_K^*) \sum_{j=1}^{K-1} \frac{\overline{\cos\theta C_{ext,H}^{jI}}}{\mathrm{Im}(b_j)} \tag{117}$$

$$\overline{\cos\theta C_{sca,H}^{K-1I\beta}} = \frac{2\lambda^2}{\pi}\mathrm{Im}(b_{K-1} b_K^*)\mathrm{Im}\sum_{j=1}^{K-1} \Delta Z_H^{jj+1} \tag{118}$$

in which:

$$\Delta Z_H^{KK+1} = \frac{-(2K+1)i}{K^2(K+1)^2}\sum_{p=-K}^{+K} p\frac{(K+|p|)!}{(K-|p|)!} g_{K,TE}^{p} g_{K,TM}^{p*} \tag{119}$$

and afterward establish by recurrence that:

$$\overline{\cos\theta C_{sca,H}^{KI\alpha}} = 2\mathrm{Im}(b_K b_{K+1}^*) \sum_{j=1}^{K} \frac{\overline{\cos\theta C_{ext,H}^{jI}}}{\mathrm{Im}(b_j)} \tag{120}$$

in which $\overline{\cos\theta C_{ext,H}^{jI}}$, which is a gradient force, has been renamed $\overline{\cos\theta C_{ext,H}^{jIG}}$. Therefore $\overline{\cos\theta C_{sca,H}^{KI\alpha}}$, which may be renamed $\overline{\cos\theta C_{sca,H}^{KIG}}$, is a recoil gradient magnetic force, which is the gradient force contribution to $\overline{\cos\theta C_{sca,H}^{KI}}$, and may be expressed in terms of mixing gradient magnetic forces. Eq. (120) is the generalization of Eq. (185) in Gouesbet et al. [1]. We similarly establish that:

$$\overline{\cos\theta C_{sca,H}^{KI\beta}} = \frac{2\lambda^2}{\pi}\ \mathrm{Im}(b_K b_{K+1}^*)\mathrm{Im}\sum_{j=1}^{K} \Delta Z_H^{jj+1} \tag{121}$$

in which $\overline{\cos\theta C_{sca,H}^{KI\beta}}$, renamed $\overline{\cos\theta C_{sca,H}^{KINS}}$, is the non-standard contribution to $\overline{\cos\theta C_{sca,H}^{KI}}$, by extension of the case $K = 1$, see conveniently Section 6 in Gouesbet et al. [1]. For $K = 2$, Eq. (121) is shown to be equivalent to Eq. (186) in Gouesbet et al. [1].

5.2. Recoil forces in the x-direction

The recoil force of Eq. (21) is the summation of a magnetoelectric force $\overline{\sin\theta\cos\varphi C_{sca,EH}^{K}}$, of an electric force $\overline{\sin\theta\cos\varphi C_{sca,E}^{K}}$ and of a magnetic force $\overline{\sin\theta\cos\varphi C_{sca,H}^{K}}$. The magnetoelectric force may be decomposed in two terms according to:

$$\overline{\sin\theta\cos\varphi C_{sca,EH}^{K}} = \overline{\sin\theta\cos\varphi C_{sca,EH}^{KR}} + \overline{\sin\theta\cos\varphi C_{sca,EH}^{KI}} \tag{122}$$

in which:

$$\overline{\sin\theta\cos\varphi C_{sca,EH}^{KR}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_K b_K^*)\mathrm{Re}(iX_{EH}^{KK}) \tag{123}$$

$$\overline{\sin\theta\cos\varphi C_{sca,EH}^{KI}} = \frac{-\lambda^2}{\pi}\mathrm{Im}(a_K b_K^*)\mathrm{Im}(iX_{EH}^{KK}) \tag{124}$$

From Eqs. (19), (20), (22) and (29), we have:

$$X_{EH}^{KK} = X_{EH}^{\gamma KK} = X_E^{\gamma K} = -X_H^{\gamma K*} \tag{125}$$

so that:

$$\overline{\sin\theta\cos\varphi C_{sca,EH}^{KR}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_K b_K^*)\mathrm{Re}(iX_E^{\gamma K}) = \frac{\lambda^2}{\pi}\mathrm{Re}(a_K b_K^*)\mathrm{Re}(iX_H^{\gamma K}) \tag{126}$$

We then use Eqs. (64) and (68) to establish:

$$\overline{\sin\theta\cos\varphi C_{sca,EH}^{KR}} = 2\frac{\mathrm{Re}(a_K b_K^*)}{\mathrm{Re}(a_K)}\overline{\sin\theta\cos\varphi C_{ext,E}^{KRS}}$$

$$= 2\frac{\mathrm{Re}(a_K b_K^*)}{\mathrm{Re}(b_K)}\overline{\sin\theta\cos\varphi C_{ext,H}^{KRS}} \quad (127)$$

Therefore, $\overline{\sin\theta\cos\varphi C_{sca,EH}^{KR}}$ (renamed $\overline{\sin\theta\cos\varphi C_{sca,EH}^{KRS}}$) is a recoil scattering magnetoelectric force which may be expressed in terms of mixing scattering pure electric and magnetic forces, while $\overline{\sin\theta\cos\varphi C_{sca,EH}^{KI}}$ (renamed $\overline{\sin\theta\cos\varphi C_{sca,EH}^{KINS}}$) is a recoil non-standard magnetoelectric force, with the same justification than the one used for $\overline{\cos\theta C_{sca,EH}^{KI}}$ of Eq. (80). Eq. (145) of [1] for $K=2$ is recovered from Eq. (127).

Next, the electric force $\overline{\sin\theta\cos\varphi C_{sca,E}^{K}}$ can be decomposed in two subforces according to:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{K}} = \overline{\sin\theta\cos\varphi C_{sca,E}^{KR}} + \overline{\sin\theta\cos\varphi C_{sca,E}^{KI}} \quad (128)$$

in which:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{KR}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_K a_{K+1}^*)\mathrm{Re}(X_E^{KK+1}) \quad (129)$$

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{KI}} = \frac{-\lambda^2}{\pi}\mathrm{Im}(a_K a_{K+1}^*)\mathrm{Im}(X_E^{KK+1}) \quad (130)$$

Using Eqs. (15), (17), (23), (25) and (27), we establish:

$$\mathrm{Re}(X_E^{KK+1}) = \mathrm{Re}\left(X_E^{\alpha K} + X_E^{\beta K} - X_E^{K-1K}\right) \quad (131)$$

so that $\overline{\sin\theta\cos\varphi C_{sca,E}^{KR}}$ of Eq. (129) becomes:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{KR}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_K a_{K+1}^*)\left[\mathrm{Re}(X_E^{\alpha K} + X_E^{\beta K}) - \mathrm{Re}(X_E^{K-1K})\right] \quad (132)$$

which, using Eqs. (62) and (65) leads to:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{KR}} = 2\frac{\mathrm{Re}(a_K a_{K+1}^*)}{\mathrm{Re}(a_K)}\overline{\sin\theta\cos\varphi C_{ext,E}^{KRNS}} - \frac{\mathrm{Re}(a_K a_{K+1}^*)}{\mathrm{Re}(a_{K-1} a_K^*)}\overline{\sin\theta\cos\varphi C_{sca,E}^{K-1R}} \quad (133)$$

Since $\overline{\sin\theta\cos\varphi C_{sca,E}^{K-1R}}$ is a non-standard force for $K=2$, e.g. Eq. (59) in Gouesbet et al. [1], it follows by recurrence that $\overline{\sin\theta\cos\varphi C_{sca,E}^{KR}}$ (renamed $\overline{\sin\theta\cos\varphi C_{sca,E}^{KRNS}}$) is a non-standard force whatever K. As a variant, let us return to Eq. (65) which expresses $\overline{\sin\theta\cos\varphi C_{ext,E}^{KRNS}}$ versus $\mathrm{Re}(X_E^{\alpha K} + X_E^{\beta K})$. Using Eq. (131), $\overline{\sin\theta\cos\varphi C_{ext,E}^{KRNS}}$ may be expressed as a summation of two non-standard forces, one of them reading as:

$$\overline{\sin\theta\cos\varphi C_{ext,E}^{KRNS2}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(a_K)\mathrm{Re}(X_E^{KK+1}) \quad (134)$$

leading to:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{KR}} = 2\frac{\mathrm{Re}(a_K a_{K+1}^*)}{\mathrm{Re}(a_K)}\overline{\sin\theta\cos\varphi C_{ext,E}^{KRNS2}} \quad (135)$$

which confirms that $\overline{\sin\theta\cos\varphi C_{sca,E}^{KR}}$ (now renamed $\overline{\sin\theta\cos\varphi C_{sca,E}^{KRNS}}$) is indeed a non-standard force. We then have the case of a recoil non-standard electric force which may be expressed in terms of a mixing non-standard electric force. Another variant for $K=2$ is available from Eq. (203) in Gouesbet et al. [1].

We now consider $\overline{\sin\theta\cos\varphi C_{sca,E}^{KI}}$. For $K=2$, we used the decomposition:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{2I}} = \overline{\sin\theta\cos\varphi C_{sca,E}^{2I\alpha}} + \overline{\sin\theta\cos\varphi C_{sca,E}^{2I\beta}} \quad (136)$$

in which, see Eq. (187) in Gouesbet et al. [1]:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{2I\alpha}} = 2\mathrm{Im}(a_2 a_3^*)\sum_{j=1}^{2}\frac{\overline{\sin\theta\cos\varphi C_{ext,E}^{jI}}}{\mathrm{Im}(a_j)} \quad (137)$$

For $\overline{\sin\theta\cos\varphi C_{sca,E}^{2,I\beta}}$, we used Eq. (83) of [30], and (144), (188) of [1] to establish:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{2I\beta}} = -\mathrm{Im}(a_2 a_3^*)\sum_{j=1}^{2}\frac{\overline{\sin\theta\cos\varphi C_{sca}^{jEHI}}}{\mathrm{Im}(a_j b_j^*)} \quad (138)$$

from which we deduce two assumptions ready for a demonstration by recurrence, reading as:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{K-1I\alpha}} = 2\mathrm{Im}(a_{k-1} a_K^*)\sum_{j=1}^{K-1}\frac{\overline{\sin\theta\cos\varphi C_{ext,E}^{jI}}}{\mathrm{Im}(a_j)} \quad (139)$$

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{K-1I\beta}} = -\mathrm{Im}(a_{K-1} a_K^*)\sum_{j=1}^{K-1}\frac{\overline{\sin\theta\cos\varphi C_{sca,EH}^{jI}}}{\mathrm{Im}(a_j b_j^*)} \quad (140)$$

and whose validity whatever K is now to be demonstrated. Using Eqs. (23), (25), (27) for X_E^{KK+1} and X_E^{K-1K}, Eq. (15) for $X_E^{\alpha K}$ and Eq. (17) for $X_E^{\beta K}$, we obtain:

$$\mathrm{Im}(X_E^{KK+1}) = \mathrm{Im}(X_E^{K-1K}) + \mathrm{Im}\left(X_E^{\alpha K} + X_E^{\beta K}\right) \quad (141)$$

which, using Eq. (13) may be rewritten as:

$$\mathrm{Im}(X_E^{KK+1}) = \mathrm{Im}(X_E^{K-1K}) + \mathrm{Im}(X_E^{K}) - \mathrm{Im}\left(iX_E^{\gamma K}\right) \quad (142)$$

Inserting Eq. (142) into Eq. (130), we have:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{KI}} = \frac{\lambda^2}{\pi}\mathrm{Im}(a_K a_{K+1}^*)\mathrm{Im}(iX_E^{\gamma K}) - \frac{\lambda^2}{\pi}\mathrm{Im}(a_K a_{K+1}^*)\mathrm{Im}(X_E^{K}) - \frac{\lambda^2}{\pi}\mathrm{Im}(a_K a_{K+1}^*)\mathrm{Im}(X_E^{K-1K}) \quad (143)$$

Let the last line of Eq. (143) be called III. Using Eq. (130), we obtain:

$$III = \frac{\mathrm{Im}(a_K a_{K+1}^*)}{\mathrm{Im}(a_{K-1} a_K^*)}\overline{\sin\theta\cos\varphi C_{sca,E}^{K-1I}} \quad (144)$$

which may be decomposed into two terms according to:

$$III = III^{\alpha} + III^{\beta} \quad (145)$$

in which:

$$III^{\alpha} = \frac{\mathrm{Im}(a_K a_{K+1}^*)}{\mathrm{Im}(a_{K-1} a_K^*)}\overline{\sin\theta\cos\varphi C_{sca,E}^{K-1I\alpha}} \quad (146)$$

$$III^{\beta} = \frac{\mathrm{Im}(a_K a_{K+1}^*)}{\mathrm{Im}(a_{K-1} a_K^*)}\overline{\sin\theta\cos\varphi C_{sca,E}^{K-1I\beta}} \quad (147)$$

We then decompose $\overline{\sin\theta\cos\varphi C_{sca,E}^{KI}}$ of Eq. (143) according to:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{KI}} = \overline{\sin\theta\cos\varphi C_{sca,E}^{KI\alpha}} + \overline{\sin\theta\cos\varphi C_{sca,E}^{KI\beta}} \quad (148)$$

in which:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{KI\alpha}} = -\frac{\lambda^2}{\pi}\mathrm{Im}(a_K a_{K+1}^*)\mathrm{Im}(X_E^{K}) + III^{\alpha} \quad (149)$$

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{KI\beta}} = \frac{\lambda^2}{\pi}\mathrm{Im}(a_K a_{K+1}^*)\mathrm{Im}(iX_E^{\gamma K}) + III^{\beta} \quad (150)$$

From Eqs. (63), (149) and the first recurrence assumption of Eq. (139), we then have:

$$\overline{\sin\theta\cos\varphi C_{sca,E}^{KI\alpha}} = 2\mathrm{Im}(a_K a_{K+1}^*)\sum_{j=1}^{K}\frac{\overline{\sin\theta\cos\varphi C_{ext,E}^{jI}}}{\mathrm{Im}(a_j)} \quad (151)$$

which is a recoil gradient electric force (renamed $\overline{\sin\theta\cos\varphi C_{sca,E}^{KIG}}$) which may be expressed in terms of mixing gradient electric forces $\overline{\sin\theta\cos\varphi C_{ext,E}^{jI}}$ (which have been renamed $\overline{\sin\theta\cos\varphi C_{ext,E}^{jIG}}$). For

$K=2$, see Eq. (187) in Gouesbet et al. [1] or, equivalently return to Eq. (137).

For $\overline{\sin\theta\cos\varphi C^{KI\beta}_{sca,E}}$, we use the fact that $iX^{\gamma K}_{E}=iX^{KK}_{EH}$ (see Eq. (125)), and afterward Eq. (124) to use the relationship between $\overline{\sin\theta\cos\varphi C^{KI}_{sca,EH}}$ and $\mathrm{Im}(iX^{KK}_{EH})$, and finally the second recurrence assumption of Eq. (140) to express $\overline{\sin\theta\cos\varphi C^{K-1I\beta}_{sca,E}}$ of III^{β} to obtain:

$$\overline{\sin\theta\cos\varphi C^{KI\beta}_{sca,E}}=-\mathrm{Im}(a_K a^*_{K+1})\sum_{j=1}^{K}\frac{\overline{\sin\theta\cos\varphi C^{jI}_{sca,EH}}}{\mathrm{Im}(a_j b^*_j)} \tag{152}$$

which is a recoil non-standard electric force (to be renamed $\overline{\sin\theta\cos\varphi C^{KINS}_{sca,E}}$) which may be expressed in terms of recoil non-standard magnetoelectric forces.

The treatment of the magnetic force $\overline{\sin\theta\cos\varphi C^{K}_{sca,H}}$ is exactly parallel to the one of the electric force $\overline{\sin\theta\cos\varphi C^{K}_{sca,E}}$ so that we shall now omit most details and focus on the results. To begin with, the magnetic force $\overline{\sin\theta\cos\varphi C^{K}_{sca,H}}$ is decomposed into two subforces according to:

$$\overline{\sin\theta\cos\varphi C^{K}_{sca,H}}=\overline{\sin\theta\cos\varphi C^{KR}_{sca,H}}+\overline{\sin\theta\cos\varphi C^{KI}_{sca,H}} \tag{153}$$

in which:

$$\overline{\sin\theta\cos\varphi C^{KR}_{sca,H}}=\frac{\lambda^2}{\pi}\mathrm{Re}(b_K b^*_{K+1})\mathrm{Re}(X^{KK+1}_{H}) \tag{154}$$

$$\overline{\sin\theta\cos\varphi C^{KI}_{sca,H}}=\frac{-\lambda^2}{\pi}\mathrm{Im}(b_K b^*_{K+1})\mathrm{Im}(X^{KK+1}_{H}) \tag{155}$$

We then establish:

$$\overline{\sin\theta\cos\varphi C^{KR}_{sca,H}}=2\frac{\mathrm{Re}(b_K b^*_{K+1})}{\mathrm{Re}(b_K)}\overline{\sin\theta\cos\varphi C^{KRNS}_{ext,H}}-\frac{\mathrm{Re}(b_K b^*_{K+1})}{\mathrm{Re}(b_{K-1}b^*_K)}\overline{\sin\theta\cos\varphi C^{K-1R}_{sca,H}} \tag{156}$$

to be compared with Eq. (133), from which we argue, similarly as for the corresponding electric force, that $\overline{\sin\theta\cos\varphi C^{KR}_{sca,H}}$ is a recoil non-standard magnetic force. As a variant, let us return to Eq. (69) which expresses $\overline{\sin\theta\cos\varphi C^{KRNS}_{ext,H}}$ versus $\mathrm{Re}(X^{\alpha K}_{H}+X^{\beta K}_{H})$. Using Eqs. (16) for $X^{\alpha K}_{H}$, (18) for $X^{\beta K}_{H}$ and Eqs. (24), (26), (28) for both X^{KK+1}_{H} and X^{K-1K}_{H}, we establish the magnetic counterpart of Eq. (131), namely:

$$\mathrm{Re}(X^{\alpha K}_{H}+X^{\beta K}_{H})=\mathrm{Re}\left(X^{KK+1}_{H}+X^{K-1K}_{H}\right) \tag{157}$$

so that $\overline{\sin\theta\cos\varphi C^{KRNS}_{ext,H}}$ may be expressed as a summation of two non-standard forces, one of them reading as:

$$\overline{\sin\theta\cos\varphi C^{KRNS2}_{ext,H}}=\frac{\lambda^2}{2\pi}\mathrm{Re}(b_K)\mathrm{Re}(X^{KK+1}_{H}) \tag{158}$$

leading to:

$$\overline{\sin\theta\cos\varphi C^{KR}_{sca,H}}=2\frac{\mathrm{Re}(b_K b^*_{K+1})}{\mathrm{Re}(b_K)}\overline{\sin\theta\cos\varphi C^{KRNS2}_{ext,H}} \tag{159}$$

which confirms that $\overline{\sin\theta\cos\varphi C^{KR}_{sca,H}}$ (now renamed $\overline{\sin\theta\cos\varphi C^{KRNS}_{sca,H}}$) is indeed a non-standard force. We then have the case of a recoil non-standard magnetic force which may be expressed in terms of a mixing non-standard magnetic force. Another variant for $K=2$ is available from Eq. (204) in Gouesbet et al. [1].

For $\overline{\sin\theta\cos\varphi C^{KI}_{sca,H}}$, we first demonstrate two recurrence assumptions similar to the ones of Eqs. (139) and (140), reading as:

$$\overline{\sin\theta\cos\varphi C^{K-1I\alpha}_{sca,H}}=2\mathrm{Im}(b_{k-1}b^*_K)\sum_{j=1}^{K-1}\frac{\overline{\sin\theta\cos\varphi C^{jI}_{ext,H}}}{\mathrm{Im}(b_j)} \tag{160}$$

$$\overline{\sin\theta\cos\varphi C^{K-1I\beta}_{sca,H}}=\mathrm{Im}(b_{K-1}b^*_K)\sum_{j=1}^{K-1}\frac{\overline{\sin\theta\cos\varphi C^{jI}_{sca,EH}}}{\mathrm{Im}(a_j b^*_j)} \tag{161}$$

which are used to establish:

$$\overline{\sin\theta\cos\varphi C^{KI\alpha}_{sca,H}}=2\mathrm{Im}(b_k b^*_{K+1})\sum_{j=1}^{K}\frac{\overline{\sin\theta\cos\varphi C^{jI}_{ext,H}}}{\mathrm{Im}(b_j)} \tag{162}$$

$$\overline{\sin\theta\cos\varphi C^{KI\beta}_{sca,H}}=\mathrm{Im}(b_K b^*_{K+1})\sum_{j=1}^{K}\frac{\overline{\sin\theta\cos\varphi C^{jI}_{sca,EH}}}{\mathrm{Im}(a_j b^*_j)} \tag{163}$$

Similarly as for the corresponding electric case, $\overline{\sin\theta\cos\varphi C^{KI\alpha}_{sca,H}}$ (renamed $\overline{\sin\theta\cos\varphi C^{KIG}_{sca,H}}$) is a recoil gradient magnetic force which may be expressed in terms of mixing gradient magnetic forces (which have been renamed as $\overline{\sin\theta\cos\varphi C^{jIG}_{ext,H}}$), while $\overline{\sin\theta\cos\varphi C^{KI\beta}_{sca,H}}$ (to be renamed $\overline{\sin\theta\cos\varphi C^{KINS}_{sca,H}}$) is a recoil non-standard magnetic force which may be expressed in terms of recoil non-standard magnetoelectric forces. For $K=2$, see Eq. (189) in Gouesbet et al. [1] for $\overline{\sin\theta\cos\varphi C^{2I\alpha}_{sca,H}}$ and Eq. (190) in Gouesbet et al. [1] for a variant equivalent to Eq. (163).

5.3. Recoil forces in the y-direction

Recoil forces in the y-direction are treated similarly as for the x-direction, with however an interchange between the subscripts R and I. We shall therefore do the economy of demonstrations to focus on the results. We begin with the recoil magnetoelectric force of Eq. (39) which is decomposed into two subforces according to:

$$\overline{\sin\theta\sin\varphi C^{K}_{sca,EH}}=\overline{\sin\theta\sin\varphi C^{KR}_{sca,EH}}+\overline{\sin\theta\sin\varphi C^{KI}_{sca,EH}} \tag{164}$$

in which:

$$\overline{\sin\theta\sin\varphi C^{KR}_{sca,EH}}=\frac{\lambda^2}{\pi}\mathrm{Im}(a_K b^*_K)\mathrm{Re}(iY^{KK}_{EH})=\frac{\lambda^2}{\pi}\mathrm{Im}(a_K b^*_K)\mathrm{Re}(iY^{\gamma KK}_{EH}) \tag{165}$$

$$\overline{\sin\theta\sin\varphi C^{KI}_{sca,EH}}=\frac{\lambda^2}{\pi}\mathrm{Re}(a_K b^*_K)\mathrm{Im}(iY^{KK}_{EH})=\frac{\lambda^2}{\pi}\mathrm{Re}(a_K b^*_K)\mathrm{Im}(iY^{\gamma KK}_{EH}) \tag{166}$$

in which we used Eq. (40). We then establish:

$$\overline{\sin\theta\sin\varphi C^{KI}_{sca,EH}}=2\frac{\mathrm{Re}(a_K b^*_K)}{\mathrm{Re}(a_K)}\overline{\sin\theta\sin\varphi C^{KIS}_{ext,E}}=2\frac{\mathrm{Re}(a_K b^*_K)}{\mathrm{Re}(b_K)}\overline{\sin\theta\sin\varphi C^{KIS}_{ext,H}} \tag{167}$$

to be compared with Eq. (127), and showing that $\overline{\sin\theta\sin\varphi C^{KI}_{sca,EH}}$ (renamed $\overline{\sin\theta\sin\varphi C^{KIS}_{sca,EH}}$) is a recoil scattering magnetoelectric force which may be expressed in terms of mixing scattering pure electric and magnetic forces, while $\overline{\sin\theta\sin\varphi C^{KR}_{sca,EH}}$ (renamed $\overline{\sin\theta\sin\varphi C^{KRNS}_{sca,EH}}$) is a recoil non-standard magnetoelectric force with the same justification than for $\overline{\sin\theta\cos\varphi C^{KI}_{sca,EH}}$. For $K=2$, we recover Eq. (149) of [1].

Concerning the recoil electric field $\overline{\sin\theta\sin\varphi C^{K}_{sca,E}}$ of Eq. (39), we decompose it in two subforces according to:

$$\overline{\sin\theta\sin\varphi C^{K}_{sca,E}}=\overline{\sin\theta\sin\varphi C^{KR}_{sca,E}}+\overline{\sin\theta\sin\varphi C^{KI}_{sca,E}} \tag{168}$$

in which:

$$\overline{\sin\theta\sin\varphi C^{KR}_{sca,E}}=\frac{\lambda^2}{\pi}\mathrm{Im}(a_K a^*_{K+1})\mathrm{Re}(Y^{KK+1}_{E}) \tag{169}$$

$$\overline{\sin\theta\sin\varphi C^{KI}_{sca,E}} = \frac{\lambda^2}{\pi}\mathrm{Re}(a_K a^*_{K+1})\mathrm{Im}(Y^{KK+1}_E) \tag{170}$$

We then establish:

$$\overline{\sin\theta\sin\varphi C^{KI}_{sca,E}} = 2\frac{\mathrm{Re}(a_K a^*_{K+1})}{\mathrm{Re}(a_K)}\overline{\sin\theta\sin\varphi C^{KINS}_{ext,E}} - \frac{\mathrm{Re}(a_K a^*_{K+1})}{\mathrm{Re}(a_{K-1} a^*_K)}\overline{\sin\theta\sin\varphi C^{K-1I}_{sca,E}} \tag{171}$$

which is to be compared with Eq. (156), from which we similarly argue that $\overline{\sin\theta\sin\varphi C^{KI}_{sca,E}}$ (to be renamed $\overline{\sin\theta\sin\varphi C^{KINS}_{sca,E}}$)is a recoil non-standard electric force. As a variant, let us return to Eq. (73) which expresses $\overline{\sin\theta\sin\varphi C^{KINS}_{ext,E}}$ versus $\mathrm{Im}(Y^{\alpha K}_E + Y^{\beta K}_E)$. Using Eqs. (33) for $Y^{\alpha K}_E$, (35) for $Y^{\beta K}_E$ and Eqs. (41), (43), (45) for both Y^{KK+1}_E and Y^{K-1K}_E, we establish the y-component electric counterpart of Eq. (157), so that $\overline{\sin\theta\sin\varphi C^{KINS}_{ext,E}}$ may be expressed as a summation of two non-standard forces, one of them reading as:

$$\overline{\sin\theta\sin\varphi C^{KINS2}_{ext,E}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(a_K)\mathrm{Im}(Y^{KK+1}_E) \tag{172}$$

leading to:

$$\overline{\sin\theta\sin\varphi C^{KI}_{sca,E}} = 2\frac{\mathrm{Re}(a_K a^*_{K+1})}{\mathrm{Re}(a_K)}\overline{\sin\theta\sin\varphi C^{KINS2}_{ext,E}} \tag{173}$$

which confirms that $\overline{\sin\theta\sin\varphi C^{KI}_{sca,E}}$ is indeed a non-standard force. We then have the case of a recoil non-standard electric force which may be expressed in terms of a mixing non-standard electric force. Another variant for $K = 2$ is available from Eq. (205) in Gouesbet et al. [1].

Next, similarly as for Eqs. (139) and (140), we establish two recurrence assumptions:

$$\overline{\sin\theta\sin\varphi C^{K-1R\alpha}_{sca,E}} = 2\mathrm{Im}(a_{K-1}a^*_K)\sum_{j=1}^{K-1}\frac{\overline{\sin\theta\sin\varphi C^{jR}_{ext,E}}}{\mathrm{Im}(a_j)} \tag{174}$$

$$\overline{\sin\theta\sin\varphi C^{K-1R\beta}_{sca,E}} = -\mathrm{Im}(a_{K-1}a^*_K)\sum_{j=1}^{K-1}\frac{\overline{\sin\theta\sin\varphi C^{jR}_{sca,EH}}}{\mathrm{Im}(a_j b^*_j)} \tag{175}$$

from which, similarly as for the x-component case, we establish:

$$\overline{\sin\theta\sin\varphi C^{KR\alpha}_{sca,E}} = 2\mathrm{Im}(a_K a^*_{K+1})\sum_{j=1}^{K}\frac{\overline{\sin\theta\sin\varphi C^{jR}_{ext,E}}}{\mathrm{Im}(a_j)} \tag{176}$$

$$\overline{\sin\theta\sin\varphi C^{KR\beta}_{sca,E}} = -\mathrm{Im}(a_K a^*_{K+1})\sum_{j=1}^{K}\frac{\overline{\sin\theta\sin\varphi C^{jR}_{sca,EH}}}{\mathrm{Im}(a_j b^*_j)} \tag{177}$$

so that $\overline{\sin\theta\sin\varphi C^{KR\alpha}_{sca,E}}$ (to be renamed $\overline{\sin\theta\sin\varphi C^{KRG}_{sca,E}}$) is a recoil gradient electric force which may be expressed in terms of mixing gradient electric forces, while $\overline{\sin\theta\sin\varphi C^{KR\beta}_{sca,E}}$ (to be renamed $\overline{\sin\theta\sin\varphi C^{KRNS}_{sca,E}}$) is a recoil non-standard electric force which may be expressed in terms of recoil non-standard magnetoelectric forces.

We finish with the magnetic forces $\overline{\sin\theta\sin\varphi C^{K}_{sca,H}}$ of Eq. (39) that we decompose again into two subforces according to:

$$\overline{\sin\theta\sin\varphi C^{K}_{sca,H}} = \overline{\sin\theta\sin\varphi C^{KR}_{sca,H}} + \overline{\sin\theta\sin\varphi C^{KI}_{sca,H}} \tag{178}$$

in which:

$$\overline{\sin\theta\sin\varphi C^{KR}_{sca,H}} = \frac{\lambda^2}{\pi}\mathrm{Im}(b_K b^*_{K+1})\mathrm{Re}(Y^{KK+1}_H) \tag{179}$$

$$\overline{\sin\theta\sin\varphi C^{KI}_{sca,H}} = \frac{\lambda^2}{\pi}\mathrm{Re}(b_K b^*_{K+1})\mathrm{Im}(Y^{KK+1}_H) \tag{180}$$

For $\overline{\sin\theta\sin\varphi C^{KR}_{sca,H}}$, instead of Eq. (171), we establish:

$$\overline{\sin\theta\sin\varphi C^{KI}_{sca,H}} = 2\frac{\mathrm{Re}(b_K b^*_{K+1})}{\mathrm{Re}(b_K)}\overline{\sin\theta\sin\varphi C^{KINS}_{ext,H}} - \frac{\mathrm{Re}(b_K b^*_{K+1})}{\mathrm{Re}(b_{K-1} b^*_K)}\overline{\sin\theta\sin\varphi C^{K-1I}_{sca,H}} \tag{181}$$

which is a recoil non-standard magnetic force. As a variant, let us return to Eq. (77) which expresses $\overline{\sin\theta\sin\varphi C^{KINS}_{ext,H}}$ versus $\mathrm{Im}(Y^{\alpha K}_H + Y^{\beta K}_H)$. Using Eqs. (34) for $Y^{\alpha K}_H$, (36) for $Y^{\beta K}_H$ and Eqs. (42), (44), (46) for both Y^{KK+1}_H and Y^{K-1K}_H, we establish the magnetic component of Eq. (173), so that $\overline{\sin\theta\sin\varphi C^{KINS}_{ext,H}}$ may be expressed as a summation of two non-standard forces, one of them reading as:

$$\overline{\sin\theta\sin\varphi C^{KINS2}_{ext,H}} = \frac{\lambda^2}{2\pi}\mathrm{Re}(b_K)\mathrm{Im}(Y^{KK+1}_H) \tag{182}$$

leading to:

$$\overline{\sin\theta\sin\varphi C^{KI}_{sca,H}} = 2\frac{\mathrm{Re}(b_K b^*_{K+1})}{\mathrm{Re}(b_K)}\overline{\sin\theta\sin\varphi C^{KINS2}_{ext,H}} \tag{183}$$

which confirms that $\overline{\sin\theta\sin\varphi C^{KI}_{sca,E}}$ (to be renamed $\overline{\sin\theta\sin\varphi C^{KINS}_{sca,E}}$) is indeed a non-standard force. We then have the case of a recoil non-standard magnetic force which may be expressed in terms of a mixing non-standard magnetic force. Another variant for $K = 2$ is available from Eq. (206) in Gouesbet et al. [1].

Also, instead of Eqs. (176) and (177), we obtain:

$$\overline{\sin\theta\sin\varphi C^{KR\alpha}_{sca,H}} = 2\mathrm{Im}(b_K b^*_{K+1})\sum_{j=1}^{K}\frac{\overline{\sin\theta\sin\varphi C^{jR}_{ext,H}}}{\mathrm{Im}(b_j)} \tag{184}$$

$$\overline{\sin\theta\sin\varphi C^{KR\beta}_{sca,H}} = \mathrm{Im}(b_K b^*_{K+1})\sum_{j=1}^{K}\frac{\overline{\sin\theta\sin\varphi C^{jR}_{sca,EH}}}{\mathrm{Im}(a_j b^*_j)} \tag{185}$$

Eq. (184) shows that $\overline{\sin\theta\sin\varphi C^{KR\alpha}_{sca,H}}$ (to be renamed $\overline{\sin\theta\sin\varphi C^{KRG}_{sca,H}}$) is a recoil gradient magnetic force which may be expressed in terms of mixing gradient magnetic forces, while $\overline{\sin\theta\sin\varphi C^{KR\beta}_{sca,H}}$ (to be renamed $\overline{\sin\theta\sin\varphi C^{KRNS}_{sca,H}}$) is a recoil non-standard magnetic force which may be expressed in terms of recoil non-standard magnetoelectric forces. For $K = 2$, Eq. (184) reduces to Eq. (193) of [1] while Eq. (185) is found to be equivalent to Eq. (194) of [1].

6. Summary of results and classifications

We now conveniently summarize the classification of K-forces ($K > 1$) developed in the present paper, based on a three-level categorization and a two-level decomposition. The classifications and comments are listed starting from the z-components which are the easiest to evaluate, followed by the x- and y-components. We have tried to discuss the various components following the order in which they occurred in the paper, excepted for some cases motivated by symmetry, aesthetic and convenience considerations. We shall consider (i) mixing forces, (ii) type-1 recoil forces which are not expressed in terms of mixing forces (recoil forces in their own right) and (iii) type-2 recoil forces which are expressed in terms of mixing forces (recoil forces mixing-force dependent). This classification is displayed at the best of our present knowledge and understanding, although it might have possibly to be modified by further investigations.

6.1. Mixing forces

The classification of mixing forces has been obtained as follows.

(i) Mixing gradient electric forces $\overline{\cos\theta C^{KIG}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{KIG}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{KRG}_{ext,E}}$, see Eqs. (49), (63), (70).
(ii) Mixing gradient magnetic forces $\overline{\cos\theta C^{KIG}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{KIG}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{KRG}_{ext,H}}$, see Eqs. (56), (67), (74).
(iii) Mixing scattering electric forces $\overline{\cos\theta C^{KRS}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{KRS}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{KIS}_{ext,E}}$, see Eqs. (51), (64), (72).
(iv) Mixing non-standard electric forces $\overline{\cos\theta C^{KRNS}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{KRNS}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{KINS}_{ext,E}}$, see Eqs. (52), (65), (73).
(v) Mixing scattering magnetic forces $\overline{\cos\theta C^{KRS}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{KRS}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{KIS}_{ext,H}}$, see Eqs. (58), (68), (76).
(vi) Mixing non-standard magnetic forces $\overline{\cos\theta C^{KRNS}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{KRNS}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{KINS}_{ext,H}}$, see Eqs. (59), (69), (77).

6.2. Type-1 recoil forces

(vii) Recoil non-standard magnetoelectric forces $\overline{\cos\theta C^{KINS}_{sca,EH}}$, $\overline{\sin\theta\cos\varphi C^{KINS}_{sca,EH}}$, $\overline{\sin\theta\sin\varphi C^{KRNS}_{sca,EH}}$, see Eqs. (80), (124), (165).
(viii) Recoil non-standard electric force $\overline{\cos\theta C^{KINS}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{KINS}_{sca,E}}$, $\overline{\sin\theta\sin\varphi C^{KRNS}_{sca,E}}$, see Eqs. (110), (152) (expressed in terms of recoil non-standard magnetoelectric forces), (177) (expressed as well in terms of recoil non-standard magnetoelectric forces).
(ix) Recoil non-standard magnetic forces $\overline{\cos\theta C^{KINS}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{KINS}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{KRNS}_{sca,H}}$, see Eqs. (121), (163) (expressed as well in terms of recoil non-standard magnetoelectric forces), (185) (expressed as well in terms of recoil non-standard magnetoelectric forces).

6.3. Type-2 recoil forces

(x) Recoil scattering magnetoelectric forces $\overline{\cos\theta C^{KRS}_{sca,EH}}$, $\overline{\sin\theta\cos\varphi C^{KRS}_{sca,EH}}$, $\overline{\sin\theta\sin\varphi C^{KIS}_{sca,EH}}$. These forces may be expressed in terms of mixing scattering pure electric and magnetic forces, see Eqs. (81), (127), (167).
(xi) Recoil non-standard electric forces $\overline{\cos\theta C^{KRNS}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{KRNS}_{sca,E}}$, $\overline{\sin\theta\sin\varphi C^{KINS}_{sca,E}}$. These forces may be expressed in terms of mixing non-standard electric forces, see Eqs. (89), (135), (173).
(xii) Recoil gradient electric forces $\overline{\cos\theta C^{KIG}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{KIG}_{sca,E}}$, $\overline{\sin\theta\sin\varphi C^{KRG}_{sca,E}}$. These forces may be expressed in terms of mixing gradient electric forces, see Eqs. (109), (151), (176).
(xiii) Recoil non-standard magnetic force $\overline{\cos\theta C^{KRNS}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{KRNS}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{KINS}_{sca,H}}$. These forces may be expressed in terms of mixing non-standard magnetic forces, see Eqs. (115), (159), (183).
(xiv) Recoil gradient magnetic forces $\overline{\cos\theta C^{KIG}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{KIG}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{KRG}_{sca,H}}$. These forces may be expressed in terms of mixing gradient magnetic forces, see Eqs. (120), (162), (184).

Let us remark, if applicable, that non-standard forces are expressed in terms of non-standard forces, that scattering forces are expressed in terms of scattering forces, and that gradient forces are expressed in terms of gradient forces. These features confirm by their coherency the interest of the categorization in terms of gradient, scattering and non-standard forces. Another remarkable feature is that recoil gradient forces may be expressed in terms of mixing gradient force.

Another presentation of these results (not present in our previous paper [1]), distinguishing between gradient, scattering and non-standard forces, may be more appealing to the reader. We then obtain the following partition:

6.4. Gradient forces

(i) Mixing electric forces $\overline{\cos\theta C^{KIG}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{KIG}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{KRG}_{ext,E}}$, see Eqs. (49), (63), (70) which are expressed using electric Mie coefficients a_K.
(ii) Mixing magnetic forces $\overline{\cos\theta C^{KIG}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{KIG}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{KRG}_{ext,H}}$, see Eqs. (56), (67), (74) which are expressed using magnetic Mie coefficients b_K.
(iii) Recoil electric forces $\overline{\cos\theta C^{KIG}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{KIG}_{sca,E}}$, $\overline{\sin\theta\sin\varphi C^{KRG}_{sca,E}}$. These forces may be expressed in terms of mixing gradient electric forces, see Eqs. (109), (151), (176), and involve a coupling between electric Mie coefficients a_K and a^*_{K+1}.
(iv) Recoil magnetic forces $\overline{\cos\theta C^{KIG}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{KIG}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{KRG}_{sca,H}}$. These forces may be expressed in terms of mixing gradient magnetic forces, see Eqs. (120), (162), (184), and involve a coupling between magnetic Mie coefficients b_K and b^*_{K+1}.

6.5. Scattering forces

(v) Mixing electric forces $\overline{\cos\theta C^{KRS}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{KRS}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{KIS}_{ext,E}}$, see Eqs. (51), (64), (72) which are expressed using electric Mie coefficients a_K.
(vi) Mixing magnetic forces $\overline{\cos\theta C^{KRS}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{KRS}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{KIS}_{ext,H}}$, see Eqs. (58), (68), (76) which are expressed using magnetic Mie coefficients b_K.
(vii) Recoil magnetoelectric forces $\overline{\cos\theta C^{KRS}_{sca,EH}}$, $\overline{\sin\theta\cos\varphi C^{KRS}_{sca,EH}}$, $\overline{\sin\theta\sin\varphi C^{KIS}_{sca,EH}}$. These forces may be expressed in terms of mixing scattering pure electric and magnetic forces, see Eqs. (81),(127), (167), and involve a coupling between electric Mie coefficients a_K and a^*_{K+1}.

6.6. Non-standard forces

(viii) Mixing electric forces $\overline{\cos\theta C^{KRNS}_{ext,E}}$, $\overline{\sin\theta\cos\varphi C^{KRNS}_{ext,E}}$, $\overline{\sin\theta\sin\varphi C^{KINS}_{ext,E}}$, see Eqs. (52), (65), (73) which are expressed using electric Mie coefficients a_K.
(ix) Mixing magnetic forces $\overline{\cos\theta C^{KRNS}_{ext,H}}$, $\overline{\sin\theta\cos\varphi C^{KRNS}_{ext,H}}$, $\overline{\sin\theta\sin\varphi C^{KINS}_{ext,H}}$, see Eqs. (59), (69), (77) which are expressed using magnetic Mie coefficients b_K.
(x) Recoil electric forces $\overline{\cos\theta C^{KRNS}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{KRNS}_{sca,E}}$, $\overline{\sin\theta\sin\varphi C^{KINS}_{sca,E}}$. These forces may be expressed in terms of mixing non-standard electric forces, see Eqs. (89), (135), (173) (Type-2 recoil forces) and $\overline{\cos\theta C^{KINS}_{sca,E}}$, $\overline{\sin\theta\cos\varphi C^{KINS}_{sca,E}}$, $\overline{\sin\theta\sin\varphi C^{KRNS}_{sca,E}}$, see Eqs. (110), (152), (177) (Type-1 recoil forces expressed in terms of recoil non-standard magnetoelectric forces, i.e. not expressed in terms of mixing forces). They involve a coupling between electric Mie coefficients a_K and a^*_{K+1}.
(xi) Recoil magnetic forces $\overline{\cos\theta C^{KRNS}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{KRNS}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{KINS}_{sca,H}}$. These forces may be expressed in terms of mixing non-standard magnetic forces, see Eqs. (115), (159), (183) (Type-2 recoil forces) and $\overline{\cos\theta C^{KINS}_{sca,H}}$, $\overline{\sin\theta\cos\varphi C^{KINS}_{sca,H}}$, $\overline{\sin\theta\sin\varphi C^{KRNS}_{sca,H}}$, see Eqs. (121), (163), (185) (Type-1 recoil forces expressed in terms of recoil non-standard magnetoelectric forces, i.e. not expressed in terms of mixing forces). They involve a coupling between magnetic Mie coefficients b_K and b^*_{K+1}.
(xii) Recoil magnetoelectric forces $\overline{\cos\theta C^{KINS}_{sca,EH}}$, $\overline{\sin\theta\cos\varphi C^{KINS}_{sca,EH}}$, $\overline{\sin\theta\sin\varphi C^{KRNS}_{sca,EH}}$, see Eqs. (80), (124), (165) which involve

a coupling $a_K b_K^*$ between electric and magnetic Mie coefficients of the same order.

7. Conclusion

The present paper discussed the partition of optical forces exerted by EM arbitrary shaped beams on arbitrary sized particles in the framework of GLMT. It complements and concludes a series of papers which previously considered successively the case of electric dipoles (in particular Rayleigh particles), of magnetoelectric dipoles and of quadrupoles. The partition first relies on a first-level categorization between mixing and recoil forces already published nearly four decades ago in early works devoted to GLMT. A second-level categorization distinguishes gradient forces and non-gradient forces. Although non-gradient forces are usually named scattering forces, we rely on the existence of non-standard forces uncovered in Gouesbet [28] (where they were called axicon forces in an inappropriate way) to introduce a third-level categorization in terms of scattering and non-standard forces. A parallel two-level decomposition distinguishes between (i) K-forces, K from 1 to ∞ and (ii) electric, magnetic and magnetoelectric forces. All the forces in the different partitions are expressed in terms of BSCs which encode the description of the illuminating beam and of Mie coefficients which encode the properties of the scatterer. One of the most appealing results is that most of the recoil forces may be expressed in terms of mixing forces. In particular, all recoil gradient forces may be expressed in terms of mixing gradient forces. Furthermore, the reader which would be content with a decomposition between conservative (gradient) forces and non-conservative forces would simply obtain these non-conservative forces by summing up the scattering and the non-standard forces.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

National Council for Scientific and Technological Development (CNPq) (426990/2018-8, 307898:2018-0, 140270/2022-1), São Paulo Research Foundation (2021/06121-0, 2020/05280-5).

Appendix A. Translation between PWECs and BSCs

The relationships between expansion coefficients used in Zheng et al. [32], denoted PWECs, and the traditional BSCs of the GLMT are established in Appendix A of [1]. We then obtained:

$$q_{mn} = i(-1)^{n+m+1}(-1)^{(m-|m|)/2} \sqrt{\frac{2n+1}{n(n+1)}} \frac{\sqrt{(n-m)!(n+m)!}}{(n-|m|)!} g_{n,TE}^{m} \quad (186)$$

$$p_{mn} = (-1)^{n+m}(-1)^{(m-|m|)/2} \sqrt{\frac{2n+1}{n(n+1)}} \frac{\sqrt{(n-m)!(n+m)!}}{(n-|m|)!} g_{n,TM}^{m} \quad (187)$$

It may be interesting to separate the cases $m \geq 0$ and $m < 0$. We then obtain:

$$q_{mn} = i(-1)^{n+m+1} \sqrt{\frac{2n+1}{n(n+1)}} \sqrt{\frac{(n+m)!}{(n-m)!}} g_{n,TE}^{m} \text{ for } m \geq 0 \quad (188)$$

$$q_{mn} = i(-1)^{n+1} \sqrt{\frac{2n+1}{n(n+1)}} \sqrt{\frac{(n-m)!}{(n+m)!}} g_{n,TE}^{m} \text{ for } m < 0 \quad (189)$$

$$p_{mn} = (-1)^{n+m} \sqrt{\frac{2n+1}{n(n+1)}} \sqrt{\frac{(n+m)!}{(n-m)!}} g_{n,TM}^{m} \text{ for } m \geq 0 \quad (190)$$

$$p_{mn} = (-1)^{n} \sqrt{\frac{2n+1}{n(n+1)}} \sqrt{\frac{(n-m)!}{(n+m)!}} g_{n,TM}^{m} \text{ for } m < 0 \quad (191)$$

Appendix B. Comparing mixing gradient forces

According to Eq. (24) in Zheng et al. [32], the mixing gradient electric force reads as:

$$\mathbf{F}_E^{K,mix,G} = -2\pi\varepsilon \frac{|E_0|^2}{k^2} [\mathrm{Im}(a_2)\mathrm{Im}(\mathbf{A}_K^* + \mathbf{A}_{K-1} + \mathbf{U}_K)] \quad (192)$$

The longitudinal component then becomes:

$$F_{E,z}^{K,mix,G} = -2\pi\varepsilon \frac{|E_0|^2}{k^2} [\mathrm{Im}(a_K)\mathrm{Im}(A_{Kz}^* + A_{K-1z} + U_{Kz})] \quad (193)$$

From Eqs. (9), (10) in Zheng et al. [32], we may evaluate A_{Kz}^*, A_{K-1z}, and U_{Kz} in terms of Zheng's expansion coefficients, and thereafter use Appendix A to express the results in terms of BSCs. After a bit of straightforward (but tedious) calculations, we then obtain:

$$A_{Kz}^* + A_{K-1z} + U_{Kz} = -Z_E^K \quad (194)$$

Therefore, Eq. (193) leads to:

$$F_{E,z}^{2,mix,G} = +2\pi\varepsilon \frac{|E_0|^2}{k^2} \mathrm{Im}(a_2)\mathrm{Im}(Z_E^K) \quad (195)$$

to be compared to Eq. (49) which is conveniently repeated below:

$$\overline{\cos\theta} C_{ext,E}^{KI} = \frac{-\lambda^2}{\pi} \mathrm{Im}(a_K)\mathrm{Im}(Z_E^K) \quad (196)$$

This comparison is sufficient to demonstrate that $\overline{\cos\theta} C_{ext,E}^{KI}$ is indeed a mixing gradient force. For a better agreement, we convert cross-sections (forces expressed in square meters) to genuine forces in newtons, relying on Eq. (4) to obtain:

$$\mathbf{F}_{ext,E}^{KI} = -2\pi\varepsilon \frac{|E_0|^2}{k^2} \mathrm{Im}(a_2)\mathrm{Im}(Z_E^2) \quad (197)$$

which differs from Eq. (195) by a sign difference. This sign difference is due to the fact that the time-harmonic convention used by Zheng et al. [32] is of the form $\exp(-i\omega t)$ in contrast with the time convention in GLMT which is of the form $\exp(i\omega t)$, implying that we have to change a_2 to a_2^*, i.e. $\mathrm{Im}(a_2)$ to $-\mathrm{Im}(a_2)$.

We may similarly deal with the transverse electric components $\overline{\sin\theta\cos\varphi} C_{ext,E}^{KI}$ and $\overline{\sin\theta\sin\varphi} C_{ext,E}^{KR}$, and with the magnetic components $\overline{\cos\theta} C_{ext,H}^{KI}$, $\overline{\sin\theta\cos\varphi} C_{ext,H}^{KI}$, $\overline{\sin\theta\sin\varphi} C_{ext,H}^{KR}$ (although it is more expedient to invoke a duality between electric and magnetic components, as already discussed in Gouesbet et al. [30]), to demonstrate that these forces are mixing gradient forces as well.

References

[1] Gouesbet G, Angelis VD, Ambrosio L. Optical forces and optical force categorizations exerted on quadrupoles in the framework of generalized Lorenz–Mie theory. J Quant Spectrosc Radiat Transf 2023;298. Article 108487

[2] Gouesbet G, Maheu B, Gréhan G. Light scattering from a sphere arbitrarily located in a Gaussian beam, using a Bromwich formulation. J Opt Soc Am A 1988;5(9):1427–43.

[3] Gouesbet G, Gréhan G, Maheu B. Scattering of a Gaussian beam by a Mie scatter center, using a Bromwich formalism. J Opt (Paris) 1985;16(2):83–93. Republished in selected papers on light scattering SPIE Milestone series, Vol 951, 1988, edited by M Kerker

[4] Gouesbet G, Gréhan G. Generalized Lorenz–Mie theories. 2nd ed. Springer International Publishing AG; 2017.
[5] Maheu B, Gouesbet G, Gréhan G. A concise presentation of the generalized Lorenz–Mie theory for arbitrary location of the scatterer in an arbitrary incident profile. J Opt (Paris) 1988;19(2):59–67.
[6] Gouesbet G, Gréhan G, Maheu B. Combustion measurements. In: Chigier N, editor. Generalized Lorenz–Mie theory and applications to optical sizing. Hemisphere Publishing Corporation, New-York, USA; 1991. p. 339–84.
[7] Onofri F, Gréhan G, Gouesbet G. Electromagnetic scattering from a multilayered sphere located in an arbitrary beam. Appl Opt 1995;34(30):7113–24.
[8] Wu Z, Guo L, Ren K, Gouesbet G, Gréhan G. Improved algorithms for electromagnetic scattering of plane waves and shaped beams by multilayered spheres. Appl Opt 1997;36(21):5188–98.
[9] Gouesbet G, Gréhan G. Generalized Lorenz–Mie theory for assemblies of spheres and aggregates. J Opt A 1999;1(6):706–12.
[10] Briard P, Wang J, Han Y. Shaped beam scattering by an aggregate of particles using generalized Lorenz–Mie theory. Opt Commun 2016;365:186–93.
[11] Briard P, Han Y, Chen Z, Cai X, Wang J. Scattering of aggregated particles illuminated by a zeroth-order Bessel beam. Opt Commun 2017;391:42–7.
[12] Gouesbet G, Gréhan G. Generalized Lorenz–Mie theory for a sphere with an eccentrically located spherical inclusion. J Mod Opt 2000;47(5):821–37.
[13] Han G, Han Y. Radiation force on a sphere with an eccentric inclusion illuminated by a laser beam. Acta Phys Sin 2009;58(9):6167–73.
[14] Wang J, Gouesbet G, Han Y, Gréhan G. Study of scattering from a sphere with an eccentrically located spherical inclusion by generalized Lorenz–Mie theory: internal and external field distributions. J Opt Soc Am A 2011;28(1):24–39.
[15] Wang J, Gouesbet G, Gréhan G, Saengkaew S. Morphology-dependent resonances in an eccentrically layered sphere illuminated by a tightly focused off-axis Gaussian beam. J Opt Soc Am A 2011;28(9):1849–59.
[16] Ren K, Gréhan G, Gouesbet G. Radiation pressure forces exerted on a particle arbitrarily located in a Gaussian beam by using the generalized Lorenz–Mie theory, and associated resonance effects. Opt Commun 1994;108(4–6):343–54.
[17] Ren K, Gréhan G, Gouesbet G. Prediction of reverse radiation pressure by generalized Lorenz–Mie theory. Appl Opt 1996;35(15):2702–10.
[18] Polaert H, Gréhan G, Gouesbet G. Improved standard beams with applications to reverse radiation pressure. Appl Opt 1998;37(12):2435–40.
[19] Martinot-Lagarde G, Pouligny B, Angelova M, Gréhan G, Gouesbet G. Trapping and levitation of a dielectric sphere with off-centred Gaussian beams. II. GLMT-analysis. Pure Appl Opt 1995;4(5):571–85.
[20] Polaert H, Gréhan G, Gouesbet G. Forces and torques exerted on a multilayered spherical particle by a focused Gaussian beam. Opt Commun 1998;155(1–3):169–79.
[21] Xu F, Ren K, Gouesbet G, Cai X, Gréhan G. Theoretical prediction of radiation pressure force exerted on a spheroid by an arbitrarily shaped beam. Phys Rev E 2007;75:1–14. Art 026613
[22] Xu F, Lock J, Gouesbet G, Tropea C. Radiation torque exerted on a spheroid: analytical solution. Phys Rev A 2008;78:1–17. Art 013843
[23] Gouesbet G. Generalized Lorenz–Mie theories and mechanical effects of laser light, on the occasion of Arthur Ashkin's receipt of the 2018 Nobel prize in physics for his pioneering work in optical levitation and manipulation: a review. J Quant Spectrosc Radiat Transf 2019;225:258–77.
[24] Gouesbet G. Latest achievements in generalized Lorenz–Mietheories: a commented reference database. Ann Phys 2014;526(11–12):461–89.
[25] Gouesbet G. T-matrix methods for electromagnetic structured beams: a commented reference database for the period 2014–2018. J Quant Spectrosc Radiat Transf 2019;230:247–81.
[26] Ashkin A. Optical trapping and manipulations of neutral particles using lasers: a reprint volume with commentaries. World Scientific, Singapore; 2006.
[27] Lock J. Calculation of the radiation trapping force for laser tweezers by use of generalized Lorenz–Mie theory. II. On-axis trapping force. Appl Opt 2004;43(12):2545–54.
[28] Gouesbet G. Gradient, scattering and other kinds of longitudinal optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz–Mie theory. J Quant Spectrosc Radiat Transf 2020;246. Article 106913
[29] Gouesbet G, Ambrosio L. Rayleigh limit of the generalized Lorenz–Mietheory for on-axis beams and its relationship with the dipole theory of forces. Part I. Non dark axisymmetric beams of the first kind, with the example of Gaussian beams. J Quant Spectrosc Radiat Transf 2021;266. Article 107569
[30] Gouesbet G, de Angelis V, Ambrosio L. Optical forces and optical force categorizations on small magnetodielectric particles in the framework of generalized Lorenz–Mie theory. J Quant Spectrosc Radiat Transf 2022;279. Article 108046
[31] Ambrosio L, de Angelis V, Gouesbet G. The generalized Lorenz–Mie theory and its identification with the dipole theory of forces for particles with electric and magnetic properties. J Quant Spectrosc Radiat Transf 2022;281. Article 108104
[32] Zheng H, Yu X, Lu W, Ng J, Lin Z. GCforce: Decomposition of optical force into gradient and scattering parts. Comput Phys Commun 2019;237:188–98.
[33] van de Hulst H. Light scattering by small particles. Wiley, New York; 1957.
[34] Chaumet P, Nieto-Vesperinas M. Time-averaged total force on a dipolar sphere in an electromagnetic field. Opt Lett 2000;25(15):1065–7.
[35] Albaladejo S, Marqués M, Laroche M, Saenz J. Scattering forces from the curl of the spin angular momentum of a light field. Phys Rev Lett 2009;102. Article 113602
[36] Ruffner D, Grier D. Comment on "scattering forces from the curl of the spin angular momentum of a light field". Phys Rev Lett 2013;111(5). doi:10.1103/PhysRevLett.111.059301. Article 059301
[37] Marqués M, Saenz J. Marqués and Saenz reply. Phys Rev Lett 2013;111(5). doi:10.1103/PhysRevLett.111.059302. Article 059302
[38] Marago O, Jones P, Gucciardi P, Volpe G, Ferrari A. Optical trapping and manipulation of nanostructures. Nat Nanotechnol 2013;8(11):807–19.
[39] Ambrosio L, Gouesbet G. On the Rayleigh limit of the generalized Lorenz–Mie theory and its formal identification with the dipole theory of forces. I. The longitudinal case. J Quant Spectrosc Radiat Transf 2021;262. Article 107531
[40] Ambrosio L, Gouesbet G. On the Rayleigh limit of the generalized Lorenz–Mie theory and its formal identification with the dipole theory of forces. II. The transverse case. J Quant Spectrosc Radiat Transf 2021;266. Article 107591
[41] Jiang Y., Chen H., Chen J., Ng J., Lin Z.. Universal relationships between optical forces/torque and orbital versus spin momentum/angular momentum of light. arXiv:1511085462017a;.
[42] Jiang Y., Chen J., Ng J., Lin Z.. Decomposition of optical force into conservative and nonconservative components. arXiv:160405138v22017b.
[43] Nieto-Vesperinas M.. Comment on "Poynting vector, orbital and spin momentum and spin momentum and angular momentum versus optical force and torque on arbitrary particle in generic optical fields". arXiv:160506041v1[physicsoptics]2016;.