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MORSE THEORY FOR THE TRAVEL TIME BRACHISTOCHRONES IN STATIONARY SPACETIMES

FABIO GIANNONI, PAOLO PICCIONE, AND DANIEL V. TAUSK

Abstract. The travel time brachistochrone curves in a general relativistic framework are timelike curves, satisfying a suitable conservation law with respect to a an observer field, that are stationary points of the travel time functional. We develop a global variational theory for brachistochrones joining an event p and the worldline of an observer γ in a stationary spacetime $\mathcal{M}$.

1. Introduction: the General Relativistic Brachistochrone Problem.

In its original formulation, the brachistochrone problem is to determine the shape of a frictionless slide between two points in space in such a way that a particle, if released at rest from the first point and driven by the gravitational field, would reach the second point in the shortest time. This problem has several generalization; in this paper we study a general relativistic version of the problem, described below. We emphasize here that, due to the lack of a global notion of *time* in General Relativity, one has to distinguish between two different time minimizing problems: the so-called *travel time* brachistochrone and the *arrival time* brachistochrone. In the first case, one looks for trajectories of massive objects minimizing the time measured by a watch that travels with the object; in the second case one looks for trajectories minimizing the time measured by an observer placed at the final point. In this paper we will be concerned exclusively with the travel time brachistochrone problem; the arrival time brachistochrone are studied in [6]. The present article is a shortened version of the paper [7], which we will refer to for the details of some of the proofs omitted in the present article. We also refer to [7] for a detailed historical introduction to the brachistochrone problem; recent bibliography on the general relativistic brachistochrone problem can be found in [5, 6, 8, 9, 11, 17].

We recall briefly the formulation of the general relativistic (travel time) brachistochrone problem. Let $(\mathcal{M}, g)$ be a time oriented Lorentzian manifold, and and let Y be a fixed smooth future pointing timelike vector field on $\mathcal{M}$; denote by $\langle \cdot, \cdot \rangle$ the nondegenerate inner product induced by g on each tangent space. Let p be a fixed event in $\mathcal{M}$, γ a maximal integral curve of Y and let $k > 0$ be a fixed constant. A *travel time brachistochrone of energy k between p and γ* is a curve which is a stationary point of the functional $\sigma \mapsto \mathcal{T}_\sigma$, defined in the set of piecewise smooth timelike curves $\sigma : [0, 1] \to \mathcal{M}$ satisfying:

$$\sigma(0) = p, \quad \sigma(1) \in \mathrm{Im}(\gamma), \quad \langle \dot\sigma, \dot\sigma \rangle \equiv -\mathcal{T}_\sigma^2, \quad \langle \nabla_{\dot\sigma} \dot\sigma, Y \rangle = 0 \tag{1.1}$$

$$\langle \dot\sigma(0), Y\big(\sigma(0)\big) \rangle = -k \sqrt{-\langle \dot\sigma, \dot\sigma \rangle}. \tag{1.2}$$

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When Y is a Killing vector field, which is the case studied in the present paper, the two conditions $\langle \nabla_{\dot\sigma}\dot\sigma, Y\rangle = 0$ and (1.2) are equivalent to:

$$\langle \dot\sigma, Y\rangle = -k\mathcal{T}_\sigma. \tag{1.3}$$

There are several natural questions that one can ask about the solutions of this variational problem, for instance: how can one characterize the travel time brachistochrones in terms of differential equations? For which values of the initial velocity (or for which values of the *energy constant* k in (1.3) in the stationary case) do there exist travel time brachistochrones joining two given points in space? How many of them? Also, given a stationary path for the travel time, it is natural to ask about its minimality in terms of *conjugate points*. Global analysis on manifolds and critical point theory offer techniques that can be used in order to answer these questions. The purpose of this article is to present a complete variational theory for travel time brachistochrones under the assumption that $(\mathcal{M}, g)$ is a stationary Lorentzian manifold and that Y is a Killing vector field in $\mathcal{M}$. In particular, it will be developed a full-fledged infinite dimensional Morse theory for the critical points of the travel time. The general-relativistic brachistochrone problem in a stationary Lorentzian manifold is presented in a context of Global Analysis on infinite dimensional Hilbertian manifolds (Section 2). We then show that the travel time brachistochrones are smooth curves; they are characterized as the only solutions of a second order differential equation (formula (3.15) and Proposition 4.1). The brachistochrones are also characterized as local minimizers for the travel time, and, equivalently, as curves whose *spatial* part is a geodesic with respect to a suitable Riemannian structure on $\mathcal{M}$ (Proposition 4.5). We compute a second order variation formula for the travel time functional, which is characterized by a Morse Index Theorem (Theorem 7.12). This theorem relates the nature of a stationary point for the travel time with some metrical properties of $\mathcal{M}$ and with the *convexity* of the timelike curve γ representing the observer and measured by the second fundamental form of γ. Finally, under suitable completeness hypotheses for $\mathcal{M}$, we prove the global Morse relations for the travel time functional in a completion of the space $\mathcal{B}_{p,\gamma}(k)$ (Section 8); thanks to this relations one obtains estimates on the number of brachistochrones of fixed energy k between p and γ, according to the topology and the metric of $\mathcal{M}$. From a strictly mathematical point of view, the paper presents some technicalities that is worth discussing. The main difficulties in our variational problem are due to the presence of the double constraint given in (1.1), which are, respectively, quadratic and linear in the first derivative. Due to this kind of constraint, in order to put a differentiable structure on the set $\mathcal{B}_{p,\gamma}(k)$ of trial paths, one needs to consider a Hilbert space completion of $\mathcal{B}_{p,\gamma}(k)$ made in a Sobolev space of curves having at least the C^1-regularity, and thus one is forced to consider curves of class H^2 (see formula (2.13) and Proposition 2.1). However, the H^2-approach has the disadvantage of introducing new difficulties. In first place, the Riesz duality in the Hilbert spaces H^i involves products of functions and also their derivatives, resulting in lengthy and complicated calculations when using the Lagrange multipliers method. Moreover, the arrival time functional *does not* satisfy good compactness properties in the space of H^2-curves, like the Palais–Smale condition (see Appendix A), which is an essential tool for developing an infinite dimensional Morse Theory.

The problem of duality in Hilbert spaces of curves with *high* regularity is faced through the introduction of a suitable formalism based on the theory of *distribution* and *generalized functions*, whose technical details are worked out at the beginning of Section 3. As to the problem of lack of compactness for the travel time functional, the crucial observation here is that, if one is only interested in a local differentiable structure, then around each smooth (C^1) curve σ in $\mathcal{B}_{p,\gamma}(k)$ it can be defined a differentiable chart on the set of H^1-curves (see Proposition 2.5); observe that the solution to our variational problem are proven to be curves of class C^2. Moreover some technical difficulties can be removed using a suitable change of variable described in section 4, that allows to consider a variational problem with only one constraint. In such a way it is possible to prove the global Morse relations for the arrival time functional (Section 8).

2. **The Functional Spaces and the Variational Setup.** We will denote by $(\mathcal{M}, g)$ a stationary Lorentzian manifold, with g a Lorentzian metric tensor on $\mathcal{M}$, and Y is a smooth timelike Killing vector field on $\mathcal{M}$, which is assumed to be complete. The symbol $\langle \cdot, \cdot \rangle$ will denote the bilinear form induced by g on the tangent spaces of $\mathcal{M}$; the usual nabla symbol ∇ will denote the covariant derivative of the Levi–Civita connection of g. Given a smooth function ϕ on $\mathcal{M}$, for $q \in \mathcal{M}$ we denote by $\nabla\phi(q)$ the gradient of ϕ at q with respect to g, defined by $\langle \nabla\phi(q), \cdot \rangle = \mathrm{d}\phi(q)[\,\cdot\,]$; the Hessian $H^\phi(q)$ of ϕ at q is the symmetric bilinear form on $T_q\mathcal{M}$ given by $H^\phi(q)[v_1, v_2] = \langle \nabla_{v_1}\nabla\phi, v_2 \rangle$. We denote by $\psi : \mathcal{M} \times I\!R \to \mathcal{M}$ the flow of Y, i.e., for $q \in \mathcal{M}$ and $t \in I\!R$, $\psi(q, t)$ is the value $\gamma_q(t)$, where γ_q is the maximal integral line of Y satisfying $\gamma_q(0) = q$. The Killing property of Y will be used systematically in our computations through the following three facts: (1) the quantity $\langle Y, Y \rangle$ is constant along the flow lines of Y, (2) the differential $\mathrm{d}_x\psi(q, t_0) : T_q\mathcal{M} \to T_{\psi(q,t_0)}\mathcal{M}$ of the map $\psi(\cdot, t_0)$ is an isometry for all t_0, or, equivalently, for all t_0 the map $q \mapsto \psi(q, t)$ is a local isometry of $\mathcal{M}$, (3) $\langle \nabla_v Y, w \rangle = -\langle \nabla_w Y, v \rangle$ for all pair of vectors v and w; in particular, for all $v \in T\mathcal{M}$, we have $\langle \nabla_v Y, v \rangle = 0$. We set $m = \dim(\mathcal{M})$; let R be the *curvature tensor* of g:

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \tag{2.1}$$

for X, Y vector fields on $\mathcal{M}$. For $1 \le p \le +\infty$, $L^p([0,1], I\!R)$ will denote the space of p-integrable real functions; for $n \in I\!N$, $H^n([0,1], I\!R)$ will denote the Sobolev space of functions of class C^{n-1} having weak n-th derivative in $L^2([0,1], I\!R)$.

We introduce the auxiliary Riemannian metric g_{R} on $\mathcal{M}$, given by:

$$g_{\mathrm{R}}(p)[v_1, v_2] = \langle v_1, v_2 \rangle_{(\mathrm{R})} = \langle v_1, v_2 \rangle - 2\langle v_1, Y(q) \rangle \langle v_2, Y(q) \rangle \langle Y(q), Y(q) \rangle^{-1}, \tag{2.2}$$

for $q \in \mathcal{M}$ and $v_1,\, v_2 \in T_q\mathcal{M}$. It is easy to see that Y is Killing also in the metric g_{R}; moreover, the restriction of g and g_{R} on the orthocomplement of Y coincide.

We define the space $L^2([0,1], T\mathcal{M})$ of square integrable $T\mathcal{M}$-valued functions:

$$L^2([0,1], T\mathcal{M}) = \left\{ \zeta : [0,1] \to T\mathcal{M} \text{ measurable} : \int_0^1 \langle \zeta(t), \zeta(t) \rangle_{(\mathrm{R})} \,\mathrm{d}t < +\infty \right\}.$$

Let $\pi : T\mathcal{M} \to \mathcal{M}$ be the canonical projection. Given any curve $\sigma : I \subseteq I\!R \to A$, a *vector field along* σ is a map $\zeta : I \to T\mathcal{M}$ such that $\pi \circ \zeta = \sigma$. Let A be any open set of $\mathcal{M}$; the Sobolev space $H^1([0,1], A)$ is defined by:

$$H^1([0,1], A) = \left\{ \sigma : [0,1] \to A : \sigma \text{ absolutely continuous}, \dot\sigma \in L^2([0,1], T\mathcal{M}) \right\}.$$

For $A \subseteq \mathcal{M}$, the symbol $C^1([0,1], A)$ will denote the set of C^1-curves defined $[0,1]$ and with image in A; we also define the Sobolev space $H^2([0,1], A)$ as:

$$H^2([0,1], A) = \left\{ \sigma \in C^1([0,1], A) : \nabla_{\dot\sigma}\dot\sigma \in L^2([0,1], T\mathcal{M}) \right\}. \tag{2.3}$$

In the sequel, we will use the spaces $H^i([0,1],A)$, $i=1,2$, where A will be an open subset of $\mathcal{M}$ or $T\mathcal{M}$. If A is a smooth submanifold of $\mathcal{M}$, in particular if A is an open subset, then $H^i([0,1],A)$ has the structure of an infinite dimensional Hilbertian manifold, modeled on the Sobolev space $H^i([0,1],\mathbb{R}^m)$; for $\sigma \in H^i([0,1],A)$, the tangent space $T_\sigma H^i([0,1],A)$ can be identified with the Hilbert space:

$$T_\sigma H^i([0,1],A) = \big\{\zeta \in H^i([0,1],T\mathcal{M}) : \zeta \text{ vector field along } \sigma\big\}. \tag{2.4}$$

The inner product in $T_\sigma H^1([0,1],A)$ is given by:

$$\big\langle \zeta,\zeta\big\rangle_* = \int_0^1 \big(\big\langle \zeta,\zeta\big\rangle_{(\mathrm{R})} + \big\langle \nabla_{\dot\sigma}\zeta, \nabla_{\dot\sigma}\zeta\big\rangle_{(\mathrm{R})}\big)\,\mathrm{d}t, \tag{2.5}$$

while the inner product in $T_\sigma H^2([0,1],A)$ is given by:

$$\big\langle \zeta,\zeta\big\rangle_{**} = \int_0^1 \big(\big\langle \zeta,\zeta\big\rangle_{(\mathrm{R})} + \big\langle \nabla_{\dot\sigma}\zeta, \nabla_{\dot\sigma}\zeta\big\rangle_{(\mathrm{R})} + \big\langle \nabla^2_{\dot\sigma}\zeta, \nabla^2_{\dot\sigma}\zeta\big\rangle_{(\mathrm{R})}\big)\,\mathrm{d}t, \tag{2.6}$$

where $\nabla^2_{\dot\sigma}\zeta = \nabla_{\dot\sigma}(\nabla_{\dot\sigma}\zeta)$. Let $k \in \big]0, \sup_{\mathcal{M}} \big\langle Y(q),Y(q)\big\rangle\big[$ be fixed and define:

$$U_k = \big\{q \in \mathcal{M} : \big\langle Y(q),Y(q)\big\rangle + k^2 > 0\big\}. \tag{2.7}$$

Since $\big\langle Y,Y\big\rangle$ is constant on each integral line of Y, U_k is invariant by the flow of Y.

We will denote by p a fixed event of U_k and by $\gamma : \mathbb{R} \to U_k$ a given integral line of Y which does not pass through p. We introduce the spaces

$$\Omega^{(i)}_{p,\gamma} = \Omega^{(i)}_{p,\gamma}(U_k) = \big\{w \in H^i([0,1],U_k) : w(0) = p,\ w(1) \in \gamma(\mathbb{R})\big\}, \quad i = 1,2.$$

$\Omega^{(i)}_{p,\gamma}$ is a smooth submanifold of $H^i([0,1],U_k)$; for $w \in \Omega^{(i)}_{p,\gamma}$ $T_w\Omega^{(i)}_{p,\gamma}$ is given by:

$$T_w\Omega^{(i)}_{p,\gamma} = \big\{\zeta \in T_wH^i([0,1],U_k) : \zeta(0) = 0,\ \zeta(1) \in \mathbb{R}\cdot Y(w(1))\big\}. \tag{2.8}$$

For $w \in \Omega^{(i)}_{p,\gamma}$, $T_w\Omega^{(i)}_{p,\gamma}$ is a Hilbert space with respect to the inner products:

$$\big\langle \zeta,\zeta\big\rangle_1 = \int_0^1 \big\langle \nabla_{\dot w}\zeta, \nabla_{\dot w}\zeta\big\rangle_{(\mathrm{R})}\,\mathrm{d}t, \quad \zeta \in T_w\Omega^{(1)}_{p,\gamma}, \tag{2.9}$$

$$\big\langle \zeta,\zeta\big\rangle_2 = \int_0^1 \big(\big\langle \nabla_{\dot w}\zeta, \nabla_{\dot w}\zeta\big\rangle_{(\mathrm{R})} + \big\langle \nabla^2_{\dot w}\zeta, \nabla^2_{\dot w}\zeta\big\rangle_{(\mathrm{R})}\big)\,\mathrm{d}t, \quad \zeta \in T_w\Omega^{(2)}_{p,\gamma}. \tag{2.10}$$

Since $\zeta(0) = 0$ for all $\zeta \in T_w\Omega^{(i)}_{p,\gamma}$, then the inner products $\big\langle\cdot,\cdot\big\rangle_*$ and $\big\langle\cdot,\cdot\big\rangle_{**}$ of formulas (2.5) and (2.6) are equivalent in $T_w\Omega^{(2)}_{p,\gamma}$, respectively, to $\big\langle\cdot,\cdot\big\rangle_1$ and $\big\langle\cdot,\cdot\big\rangle_2$ of (2.9) and (2.10). Let F be the *action* functional on $\Omega^{(i)}_{p,\gamma}$:

$$F(\sigma) = \tfrac{1}{2}\int_0^1 \big\langle \dot\sigma,\dot\sigma\big\rangle\,\mathrm{d}t. \tag{2.11}$$

It is well known that F is smooth; for $\sigma \in \Omega^{(i)}_{p,\gamma}$ and $V \in T_\sigma\Omega^{(i)}_{p,\gamma}$, $\mathrm{d}F(\sigma)[V]$ is:

$$\mathrm{d}F(\sigma)[V] = \int_0^1 \big\langle \nabla_{\dot\sigma}V, \dot\sigma\big\rangle\,\mathrm{d}t. \tag{2.12}$$

For all $k \in \mathbb{R}^+$, we introduce the spaces $\mathcal{B}^{(i)}_{p,\gamma}(k)$, $i = 1,2$, by:

$$\mathcal{B}^{(i)}_{p,\gamma}(k) = \big\{\sigma \in \Omega^{(i)}_{p,\gamma} : \exists\, \mathcal{T}_\sigma \in \mathbb{R}^+ \text{ s.t. } \big\langle\dot\sigma, Y\big\rangle \equiv -k\,\mathcal{T}_\sigma,\ \big\langle\dot\sigma,\dot\sigma\big\rangle \equiv -{\mathcal{T}_\sigma}^2\big\}. \tag{2.13}$$

We define the *travel time functional* T on $\mathcal{B}^{(i)}_{p,\gamma}(k)$ by:

$$T(\sigma) = \mathcal{T}_\sigma. \tag{2.14}$$

We establish an infinite dimensional differentiable structure on $\mathcal{B}^{(2)}_{p,\gamma}(k)$ and $\mathcal{B}^{(1)}_{p,\gamma}(k)$.

Proposition 2.1. *$\mathcal{B}^{(2)}_{p,\gamma}(k)$ is a smooth submanifold of $\Omega^{(2)}_{p,\gamma}$. For $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$, the tangent space $T_\sigma\mathcal{B}^{(2)}_{p,\gamma}(k)$ can be identified with the Hilbert subspace of $T_\sigma\Omega^{(2)}_{p,\gamma}$:*

$$T_\sigma\mathcal{B}^{(2)}_{p,\gamma}(k) = \big\{\zeta \in T_\sigma\Omega^{(2)}_{p,\gamma} \exists\, C_\zeta \in \mathbb{R} \text{ such that } \big\langle \nabla_{\dot\sigma}\zeta, Y\big\rangle - \big\langle \zeta, \nabla_{\dot\sigma}Y\big\rangle \equiv C_\zeta \text{ and } \big\langle \nabla_{\dot\sigma}\zeta, \dot\sigma\big\rangle \equiv \mathcal{T}_\sigma C_\zeta k^{-1}\big\}. \tag{2.15}$$

Proof. For $\sigma \in \Omega^{(2)}_{p,\gamma}$, the maps $\langle\dot\sigma, Y\rangle$, $\langle\dot\sigma, Y\rangle^2$ and $\langle\dot\sigma,\dot\sigma\rangle$ are in $H^1([0,1],\mathbb{R})$. Let $k \in \mathbb{R}^+$ be a fixed constant. We consider the following map:

$$\mathcal{F} : \Omega^{(2)}_{p,\gamma} \to H^1([0,1],\mathbb{R}) \times H^1([0,1],\mathbb{R}) \tag{2.16}$$

$$\mathcal{F}(\sigma) = (\langle\dot\sigma, Y\rangle, \langle\dot\sigma, Y\rangle^2 + k^2\langle\dot\sigma,\dot\sigma\rangle). \tag{2.17}$$

$\mathcal{F}$ is a smooth map and $\mathcal{B}^{(2)}_{p,\gamma}(k) = \mathcal{F}^{-1}(\mathcal{C}^- \times \{0\})$, where

$$\mathcal{C} = \{h \in H^1([0,1],\mathbb{R}) : h \equiv h_0 \text{ (const.) a. e.}\}, \quad \mathcal{C}^- = \{h \in \mathcal{C} : h < 0 \text{ a. e.}\}.$$

The proof of the Proposition is obtained using the Inverse Mapping Theorem (see [12]) after showing that $\mathcal{F}$ is *transversal* to $\mathcal{C}^- \times \{0\}$. Details can be found in [7]. □

In some parts of the paper (see Section 7) we will need to consider variations of curves in $\mathcal{B}^{(2)}_{p,\gamma}(k)$ by curves σ satisfying the conditions (1.2) and (1.3), but not necessarily with endpoints in p and γ. For this reason, for $i = 1, 2$ we introduce:

$$\mathcal{B}^{(i)}_p(k) = \bigcup_{\gamma\subset U_k} \mathcal{B}^{(i)}_{p,\gamma}(k), \quad \text{and} \quad \mathcal{B}^{(i)}(k) = \bigcup_{p,\gamma\subset U_k} \mathcal{B}^{(i)}_{p,\gamma}(k), \tag{2.18}$$

where the unions in (2.18) are taken over all γ's that are integral lines of Y having image in U_k. Using the same argument of Proposition 2.1, it is a easy to prove that $\mathcal{B}^{(2)}_p(k)$ and $\mathcal{B}^{(2)}(k)$ are smooth Hilbert submanifolds of $H^2([0,1],U_k)$ and that, for $\sigma \in \mathcal{B}^{(2)}_p(k)$ or $\sigma \in \mathcal{B}^{(2)}(k)$, the tangent spaces $T_\sigma\mathcal{B}^{(2)}_p(k)$ and $T_\sigma\mathcal{B}^{(2)}(k)$ are given by:

$$T_\sigma\mathcal{B}^{(2)}_p(k) = \{\zeta \in T_\sigma H^2([0,1],U_k) : \zeta(0) = 0,\ \exists\, C_\zeta \in \mathbb{R} \text{ such that } \langle\nabla_{\dot\sigma}\zeta, Y\rangle - \langle\zeta, \nabla_{\dot\sigma}Y\rangle \equiv C_\zeta \text{ and } \langle\nabla_{\dot\sigma}\zeta,\dot\sigma\rangle \equiv \tfrac{\mathcal{T}_\sigma C_\zeta}{k}\}.$$

$$T_\sigma\mathcal{B}^{(2)}(k) = \{\zeta \in T_\sigma H^2([0,1],U_k) : \exists\, C_\zeta \in \mathbb{R} \text{ such that } \langle\nabla_{\dot\sigma}\zeta, Y\rangle - \langle\zeta, \nabla_{\dot\sigma}Y\rangle \equiv C_\zeta \text{ and } \langle\nabla_{\dot\sigma}\zeta,\dot\sigma\rangle \equiv \tfrac{\mathcal{T}_\sigma C_\zeta}{k}\}.$$

Corollary 2.2. *The differential* $\mathrm{d}T(\sigma)[\zeta]$ *of* T *on* $\mathcal{B}^{(2)}_{p,\gamma}(k)$ *is:*

$$\mathrm{d}T(\sigma)[\zeta] = -k^{-1}C_\zeta. \tag{2.19}$$

Proof. Since $\mathcal{B}^{(2)}_{p,\gamma}(k)$ is a smooth submanifold of $\Omega^{(2)}_{p,\gamma}$, then the restriction of the action functional F to $\mathcal{B}^{(2)}_{p,\gamma}(k)$ is smooth. For $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$, we have:

$$F(\sigma) = -\tfrac{1}{2}\mathcal{T}_\sigma^{\,2} < 0, \tag{2.20}$$

hence $T(\sigma) = \sqrt{-2F(\sigma)}$ is also smooth. Equality (2.19) follows easily by differentiating $\mathcal{T}_\sigma = -k^{-1}\langle\dot\sigma, Y\rangle$ and using $C_\zeta = \langle\nabla_{\dot\sigma}\zeta, Y\rangle - \langle\zeta, \nabla_{\dot\sigma}Y\rangle$. □

Definition 2.3. A *brachistochrone* of energy k between p and γ is a stationary point for the travel time functional T on $\mathcal{B}^{(2)}_{p,\gamma}(k)$. A brachistochrone curve σ is said to be *minimal* if σ is a minimum point for T on $\mathcal{B}^{(2)}_{p,\gamma}(k)$.

Corollary 2.4. *A curve* $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ *is a brachistochrone of energy* k *between* p *and* γ *iff for every* $\zeta \in T_\sigma\mathcal{B}^{(2)}_{p,\gamma}(k)$ *it is* $C_\zeta = \langle\nabla_{\dot\sigma}\zeta, Y\rangle - \langle\zeta, \nabla_{\dot\sigma}Y\rangle = 0$. □

We summarize the main properties of the set $\mathcal{B}^{(1)}_{p,\gamma}(k)$ as follows:

Proposition 2.5. *The set* $\mathcal{B}^{(1)}_{p,\gamma}(k)$ *is a metric space with the metric induced by* $H^1([0,1],\mathcal{M})$. *The inclusion* $\iota : \mathcal{B}^{(2)}_{p,\gamma}(k) \to \mathcal{B}^{(1)}_{p,\gamma}(k)$ *is continuous and it has dense image. If* $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ *is a map of class* C^1, *then there exists a neighborhood* $\mathcal{V}_\sigma$ *of* σ *in* $\mathcal{B}^{(1)}_{p,\gamma}(k)$ *that has the structure of an infinite dimensional Hilbertian manifold. In particular,* $\mathcal{B}^{(1)}_{p,\gamma}(k)$ *has a dense open subset that is a smooth Hilbert manifold. If*

$\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ is a curve of class C^1, then, for all $\sigma_1 \in \mathcal{V}_\sigma$, the tangent space $T_{\sigma_1}\mathcal{V}_\sigma$ can be identified with the Hilbert subspace of $T_{\sigma_1}\Omega^{(1)}_{p,\gamma}$ given by:

$$\begin{aligned} T_{\sigma_1}\mathcal{V}_\sigma &= \big\{\zeta \in T_\sigma\Omega^{(1)}_{p,\gamma} : \exists\, C_\zeta \in \mathbb{R} \text{ such that} \\ &\big\langle \nabla_{\dot\sigma}\zeta, Y\big\rangle - \big\langle \zeta, \nabla_{\dot\sigma} Y\big\rangle \equiv C_\zeta \text{ and } \big\langle \nabla_{\dot\sigma}\zeta, \dot\sigma\big\rangle \equiv k^{-1}\mathcal{T}_\sigma C_\zeta\big\}. \end{aligned} \tag{2.21}$$

The restriction of the travel time T to each neighborhood of the form $\mathcal{V}_\sigma$, for some $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ of class C^1, is smooth, and the same result of Corollary 2.2 holds. □

Definition 2.6. A curve $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ is said to be a *regular* point of $\mathcal{B}^{(1)}_{p,\gamma}(k)$ if $\mathcal{B}^{(1)}_{p,\gamma}(k)$ has the structure of a smooth Hilbert manifold in a neighborhood $\mathcal{V}_\sigma$ of σ. By Proposition 2.5, every curve $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ of class C^1 is a regular point of $\mathcal{B}^{(1)}_{p,\gamma}(k)$.
A *critical point* of T in $\mathcal{B}^{(1)}_{p,\gamma}(k)$ is a regular point σ of $\mathcal{B}^{(1)}_{p,\gamma}(k)$ such that $\mathrm{d}T(\sigma) = 0$ in $T_\sigma\mathcal{V}_\sigma$.

In analogy with Proposition 2.5, we observe that if σ is a regular point in $\mathcal{B}^{(1)}_p(k)$ or in $\mathcal{B}^{(1)}(k)$, then these two sets have the structure of smooth manifolds around σ. Their tangent spaces are given by:

$$\begin{aligned} T_\sigma\mathcal{B}^{(1)}_p(k) = \big\{\zeta \in T_\sigma H^1([0,1],U_k)&: \zeta(0)=0,\ \exists\, C_\zeta \in \mathbb{R} \text{ such that} \\ \big\langle \nabla_{\dot\sigma}\zeta, Y\big\rangle - \big\langle \zeta, \nabla_{\dot\sigma} Y\big\rangle &\equiv C_\zeta \text{ and } \big\langle \nabla_{\dot\sigma}\zeta, \dot\sigma\big\rangle \equiv k^{-1}\mathcal{T}_\sigma C_\zeta\big\}. \end{aligned} \tag{2.22}$$

$$\begin{aligned} T_\sigma\mathcal{B}^{(1)}(k) = \big\{\zeta \in T_\sigma H^1([0,1],U_k)&: \exists\, C_\zeta \in \mathbb{R} \text{ such that} \\ \big\langle \nabla_{\dot\sigma}\zeta, Y\big\rangle - \big\langle \zeta, \nabla_{\dot\sigma} Y\big\rangle &\equiv C_\zeta \text{ and } \big\langle \nabla_{\dot\sigma}\zeta, \dot\sigma\big\rangle \equiv k^{-1}\mathcal{T}_\sigma C_\zeta\big\}. \end{aligned} \tag{2.23}$$

3. **The First Variation of the Travel Time.** In this Section we use the Lagrange multiplier technique to derive a system of differential equation satisfied by the brachistochrones, and to extend the variational principle proven in [8]. To this aim, we need a global Banach differentiable structure on our set of maps, and for this reason we will work in the space $\mathcal{B}^{(2)}_{p,\gamma}(k)$ rather than $\mathcal{B}^{(1)}_{p,\gamma}(k)$ (see Proposition 2.5). This approach has the unpleasant drawback of making our notations and calculations much heavier then one would expect. This is due to the fact that the duality in the Sobolev spaces H^1 and $\tilde H^1$, which are the natural images for the map $\mathcal{F}$ defined by (2.16), involves also products of the first derivatives of the maps and of the Lagrange multipliers, resulting in very lengthy formulas that make it a complicated task to determine an explicit form of the Euler–Lagrange equation satisfied by the critical points of our functional. To overcome this difficulty, we use the formalism of *generalized functions* and *distributions* on Sobolev spaces. Unfortunately, our problem does not fit perfectly in the theory of distributions on Sobolev spaces presented in standard textbooks, and we are forced to develop our own theory from scratch. The first part of this section is devoted to this aim, and it is of rather technical nature. The proofs of the statements in this section will be omitted; details can be found in [7]. In the notation of Section 2, we want to extremize the action functional $F(\sigma) = \frac{1}{2}\int_0^1 \big\langle \dot\sigma, \dot\sigma\big\rangle\,\mathrm{d}s$ in the space of all curves $\sigma \in \Omega^{(2)}_{p,\gamma}$ subject to the constraint $\mathcal{F}(\sigma) \in \mathcal{C}^- \times \{0\}$. Then, $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ is a solution to our variational problem iff there exists $\Lambda \in \big(\tilde H^1([0,1],\mathbb{R}) \times H^1([0,1],\mathbb{R})\big)^*$ such that

$$\mathrm{d}F(\sigma) - \Lambda \circ (\Pi \circ \mathrm{d}\mathcal{F}(\sigma)) : T_\sigma\Omega^{(2)}_{p,\gamma} \longrightarrow \mathbb{R} \tag{3.1}$$

vanishes identically. In this case, Λ is unique, and it is the *Lagrange multiplier* of σ. A Lagrangian multiplier for our variational problem is of the form $\Lambda = (\lambda, \mu)$, where $\lambda \in \tilde{H}^1([0,1], I\!R)^*$ and $\mu \in H^1([0,1], I\!R)^*$; here the $*$ means the dual space in the sense of Banach spaces. Observe that $\tilde{H}^1([0,1], I\!R)^*$ can be identified with the closed subspace of $H^1([0,1], I\!R)^*$ consisting of functionals vanishing on constant functions. It is convenient to write the duality in the spaces $H^1([0,1], I\!R)$ and $\tilde{H}^1([0,1], I\!R)$ in the form $\lambda(f) = \int \lambda f$, $\mu(f) = \int \mu f$, where λ and μ are seen as *generalized functions*. Let I denote the interval $[0,1]$; for each $i \in I\!N$, let D^i be the dual space $H^i(I, I\!R)^*$. If $\pi_E : E \to \mathcal{M}$ is any fiber bundle over $\mathcal{M}$ with projection π_E, given $\sigma \in H^i(I, \mathcal{M})$, let $H^i(I, \sigma, E)$ denote the set of maps $\omega \in H^i(I, E)$ such that $\pi_E \circ \omega = \sigma$. We will consider in particular the tangent bundle $T\mathcal{M}$ and the cotangent bundle $T\mathcal{M}^*$ with their canonical projections onto $\mathcal{M}$. Finally, we denote by D^i_σ the dual space $H^i(I, \sigma, T\mathcal{M}^*)^*$. We remark that there are canonical inclusions $D^i \subset D^{i+1}$ and $D^i_\sigma \subset D^{i+1}_\sigma$ given by restriction. Keeping in mind the Riesz representation theorem, we define $H^0 = D^0 = L^2(I, I\!R)$ and $D^0_\sigma = L^2(I, \sigma, T\mathcal{M})$. Consider the following operations in the spaces D^i, D^i_σ, H^i:

(a) For $\lambda \in D^i$, $i \geq 1$, $f \in H^i(I, I\!R)$, $(\lambda f) \in D^i$ is $(\lambda f)(\phi) = \lambda(f\phi)$.
(b) For $\lambda \in D^i$, $i \geq 1$, and $V \in H^i(I, \sigma, T\mathcal{M})$, $(\lambda V) \in D^i_\sigma$ is defined by $(\lambda V)(\alpha) = \lambda(\alpha(V))$, for $\alpha \in H^i(I, \sigma, T\mathcal{M}^*)$.
(c) For $f \in H^i(I, I\!R)$, $i \geq 1$, and $\nu \in D^i_\sigma$, $(f\nu) \in D^i_\sigma$ is defined by $(f\nu)(\alpha) = \nu(f\alpha)$, for $\alpha \in H^i(I, \sigma, T\mathcal{M}^*)$.
(d) For $V \in H^i(I, \sigma, T\mathcal{M})$ and $\nu \in D^i_\sigma$, $i \geq 1$, the inner product $\langle \nu, V \rangle \equiv \langle V, \nu \rangle \in D^i$ is defined by $\langle \nu, V \rangle(\phi) = \nu(\langle \phi V, \cdot \rangle)$ for $\phi \in H^i(I, I\!R)$.
(e) For $\lambda \in D^i$, $i \geq 0$, we define $\int_I \lambda = \lambda(1) \in I\!R$.
(f) For $\lambda \in D^i$, $i \geq 0$, let $\tilde{\lambda} \in D^{i+1}$ be defined by $\tilde{\lambda}(\phi) = -\lambda(\phi')$, $\phi \in H^{i+1}(I, I\!R)$. This is well defined because the map $\phi \mapsto \phi'$ from $H^{i+1}(I, I\!R)$ to $H^i(I, I\!R)$ is linear and continuous.
(g) For $\nu \in D^i_\sigma$, $i \geq 0$, the element $\tilde{\nu} \in D^{i+1}_\sigma$ is defined by $\tilde{\nu}(\alpha) = -\nu(\nabla_{\dot\sigma} \alpha)$, where $\alpha \in H^{i+1}(I, \sigma, T\mathcal{M}^*)$.

For $t_0 \in I$, we denote by $\delta_{t_0} \in D^1$ the *Dirac delta* at t_0, which is the element defined by $\delta_{t_0}(\phi) = \phi(t_0)$ for all $\phi \in H^i(I, I\!R)$; moreover, for $A \in T_{\sigma(t_0)}\mathcal{M}$, $\delta^A_{t_0} \in D^i_\sigma$ will denote the element defined by $\delta^A_{t_0}(\alpha) = \alpha(t_0)(A)$.

For $V \in H^i(I, \sigma, T\mathcal{M})$, $t_o \in I$ and $A \in T_{\sigma(t_0)}\mathcal{M}$, we have:

$$\langle V, \delta^A_{t_0} \rangle = \langle V(t_0), A \rangle \, \delta_{t_0}. \tag{3.2}$$

Namely, using property (d) above, for $\phi \in H^i(I, I\!R)$ we have:

$$\langle V, \delta^A_{t_0} \rangle(\phi) = \delta^A_{t_0}(\langle \phi V, \cdot \rangle) = \langle \phi(t_0)\, V(t_0), A \rangle = \phi(t_0) \langle V(t_0), A \rangle = \langle V(t_0), A \rangle \cdot \delta_{t_0}(\phi).$$

Proposition 3.1. *The following statements hold true:*

1. *for* $\lambda \in D^i$ *and* $\phi \in H^i(I, I\!R)$, $i \geq 1$, *it is* $\lambda(\phi) = \int_I \lambda\phi$*;*
2. *the dual space* $\tilde{H}^1(I, I\!R)^*$ *is identified with the closed subspace of* D^1 *consisting of elements* λ *satisfying* $\int_I (\lambda \cdot 1) = 0$*;*
3. *if* $\tilde{\lambda} = 0$*, then* $\lambda = 0$*;*
4. *for* $\nu \in D^i_\sigma$ *and* $V \in H^{i+1}(I, \sigma, T\mathcal{M})$, $i \geq 0$, *it is* $\int_I \langle \nu, \nabla_{\dot\sigma} V \rangle = -\int_I \langle \tilde{\nu}, V \rangle$*;*
5. *there exists a continuous linear injection of* $L^1(I, I\!R)$ *into* D^i, $i \geq 1$, *given by the map* $\lambda \in L^1(I, I\!R) \mapsto \hat{\lambda} \in D^i$, *where* $\hat{\lambda}(\phi) = \int_I \lambda(s)\phi(s)\, \mathrm{d}s$ *for all* $\phi \in H^i(I, I\!R)$*;*

6. *if* $L^1(I,\sigma,T\mathcal{M})$ *denotes the set of vector fields along* σ *whose Riemannian length (2.2) is Lebesgue integrable, then there is a continuous linear injection of* $L^1(I,\sigma,T\mathcal{M})$ *into* D^i_σ, $i \ge 1$, *given by* $\nu \in L^1(I,\sigma,T\mathcal{M}) \mapsto \hat{\nu} \in D^i_\sigma$, *where* $\hat{\nu}(\alpha) = \int_I \alpha(t)\nu(t)\,\mathrm{d}t$ *for* $\alpha \in H^i(I,\sigma,T\mathcal{M}^*)$;
7. *if* $\psi \in D^1$ *is such that* $\tilde{\psi} \in D^2$ *is also in* D^1 *(recall the inclusion* $D^1 \subset D^2$*), then* $\psi \in L^2(I,\mathbb{R})$*; similarly, if* $\psi, \tilde{\psi} \in D^1_\sigma$*, then* $\psi \in L^2(I,\sigma,T\mathcal{M})$;
8. *for* $\lambda \in D^i$ *and* $f \in H^{i+1}(I,\mathbb{R}) \subset H^i(I,\mathbb{R})$, $i \ge 0$, *it is* $\widetilde{(\lambda f)} = \tilde{\lambda} f + \lambda f'$;
9. *for* $\lambda \in D^i$ *and* $V \in H^{i+1}(I,\sigma,T\mathcal{M}) \subset H^i(I,\sigma,T\mathcal{M})$, $i \ge 0$, *it is* $\widetilde{(\lambda V)} = \tilde{\lambda} V + \lambda \nabla_{\dot{\sigma}} V$;
10. *for* $f \in H^1(I,\mathbb{R})$, *it is* $\tilde{f} = f(0)\,\delta_0 - f(1)\,\delta_1 + f'$;
11. *for* $V \in H^1(I,\sigma,T\mathcal{M})$, *it is* $\tilde{V} = \delta_0^{V(0)} - \delta_1^{V(1)} + \nabla_{\dot{\sigma}} V$.

Proof. See [7] □

We now present three preliminary results needed in the computation of the first variation for the travel time; their simple proof is omitted (see [7] for details):

Lemma 3.2. *Let* $\nu \in D^i_\sigma$ *and suppose that* $\int_I \langle V, \nu \rangle = 0$ *for all* $V \in H^i(I,\sigma,T\mathcal{M})$ *such that* $V(0) = 0$ *and* $V(1)$ *is parallel to* $Y(\sigma(1))$*. Then, we have* $\nu = \delta_0^A + \delta_1^B$ *for some* $A \in T_{\sigma(0)}\mathcal{M}$ *and* $B \in T_{\sigma(1)}\mathcal{M}$ *with* $\langle B, Y(\sigma(1)) \rangle = 0$. □

Lemma 3.3. *Let* $\lambda \in D^1$ *be fixed. If* $\tilde{\lambda} = c_0\,\delta_0 + c_1\,\delta_1$ *for some* $c_0, c_1 \in \mathbb{R}$*, then necessarily* $c_0 = -c_1$ *and* $\lambda \equiv c_0$ *is constant, i.e.,* $\lambda(\phi) = \int_I c_0\phi(t)\,\mathrm{d}t$ *for all* $\phi \in H^1(I,\mathbb{R})$. □

Lemma 3.4. *If* $\lambda \in L^1(I,\mathbb{R})$ *is such that* $\hat{\lambda} \in D^i$ *is of the form* $c_0\,\delta_0 + c_1\,\delta_1$ *for some* $c_0, c_1 \in \mathbb{R}$*, then* $\lambda \equiv 0$ *and* $c_0 = c_1 = 0$*. Similarly, if* $\nu \in L^1(I,\sigma,T\mathcal{M})$ *is such that* $\hat{\nu} \in D^i_\sigma$ *is of the form* $\delta_0^A + \delta_0^B$ *for some vectors* $A \in T_{\sigma(0)}\mathcal{M}$ *and* $B \in T_{\sigma(1)}\mathcal{M}$*, then* $\nu \equiv 0$ *and* $A = B = 0$. □

Recalling (3.1) and part 2 of Proposition 3.1, we now fix a curve $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$. From the definition (2.13) of $\mathcal{B}^{(2)}_{p,\gamma}(k)$ that there exists $\mathcal{T}_\sigma > 0$ such that:

$$\langle \dot{\sigma}, Y \rangle \equiv -k\,\mathcal{T}_\sigma, \quad \text{and} \quad \langle \dot{\sigma}, \dot{\sigma} \rangle = -{\mathcal{T}_\sigma}^2. \tag{3.3}$$

We assume that there exist $\lambda, \mu \in D^1$ (see part 2 of Proposition 3.1), with $\int_I \lambda = 0$, such that the equation:

$$\begin{aligned} 0 \;=\; & \int_I \langle \nabla_{\dot{\sigma}} V, \dot{\sigma} \rangle\,\mathrm{d}s - \lambda\big(\langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle\big) + \\ & -\mu\big(2\langle \dot{\sigma}, Y \rangle(\langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle) + 2k^2 \langle \nabla_{\dot{\sigma}} V, \dot{\sigma} \rangle\big) \end{aligned} \tag{3.4}$$

is satisfied for all $V \in T_\sigma\Omega^{(2)}_{p,\gamma}$. Using the formalism above we rewrite (3.4) as:

$$0 = \int_I \langle V, \lambda\,\nabla_{\dot{\sigma}} Y + 2\mu\langle \dot{\sigma}, Y \rangle\,\nabla_{\dot{\sigma}} Y \rangle + \int_I \langle \nabla_{\dot{\sigma}} V, \dot{\sigma} - \lambda\,Y - 2\mu\,\langle \dot{\sigma}, Y \rangle\,Y - 2\mu\,k^2\,\dot{\sigma} \rangle. \tag{3.5}$$

In the above formula, the "products" between the dual maps λ and μ with functions or vector fields along σ have to be interpreted in the sense of the operations (a)—(c) above; moreover, the inner product $\langle \cdot, \cdot \rangle$ in (3.5) is meant in the sense of (d). Observe that the elements:

$$\phi = \lambda\,\nabla_{\dot{\sigma}} Y + 2\mu\langle \dot{\sigma}, Y \rangle\,\nabla_{\dot{\sigma}} Y, \quad \text{and} \quad \psi = \dot{\sigma} - \lambda\,Y - 2\mu\langle \dot{\sigma}, Y \rangle\,Y - 2\mu k^2\,\dot{\sigma} \tag{3.6}$$

are in D^1_σ. We need the following *regularity* for the Lagrangian multipliers λ and μ:

Lemma 3.5. *The Lagrangian multipliers λ and μ are indeed L^2-functions, i.e., there exist $f_\lambda, f_\mu \in L^2(I, I\!R) \subset L^1(I, I\!R)$ such that $\lambda = \hat{f}_\lambda$ and $\mu = \hat{f}_\mu$.*

Proof. It follows from (3.5) and (3.6), Lemma 3.2 and part 7 of Proposition 3.1 (see [7] for details). □

We use the operation (f) to "integrate by parts" (3.5), and, keeping in mind parts 8 and 9 of Proposition 3.1, we obtain

$$\begin{aligned} 0 \;&=\; \int_I \big\langle V, \lambda\, \nabla_{\dot\sigma} Y + 2\mu \big\langle \dot\sigma, Y\big\rangle\, \nabla_{\dot\sigma} Y\big\rangle + + \int_I \big\langle V, 2\,\tilde\mu\, k^2 \dot\sigma + 2\mu\, k^2\, \nabla_{\dot\sigma}\dot\sigma\big\rangle, \\ -&\;\; \int_I \big\langle V, \tilde{\dot\sigma} - \tilde\lambda Y - \lambda\, \nabla_{\dot\sigma} Y - 2\tilde\mu \big\langle \dot\sigma, Y\big\rangle Y - 2\mu\big\langle \dot\sigma, Y\big\rangle \nabla_{\dot\sigma} Y\big\rangle + \end{aligned} \tag{3.7}$$

for all $V \in T_\sigma\Omega^{(2)}_{p,\gamma}$. We substitute $\tilde{\dot\sigma} = \nabla_{\dot\sigma}\dot\sigma + \delta_0^{\dot\sigma(0)} - \delta_1^{\dot\sigma(1)}$ in (3.7), and, from Lemma 3.2, we have:

$$\begin{aligned} \lambda\, \nabla_{\dot\sigma} Y + 4\mu\, \big\langle \dot\sigma, Y\big\rangle\, \nabla_{\dot\sigma} Y - \nabla_{\dot\sigma}\dot\sigma + \tilde\lambda\, Y + \lambda\, \nabla_{\dot\sigma} Y + 2\tilde\mu \big\langle \dot\sigma, Y\big\rangle\, Y + \\ + 2\tilde\mu\, k^2 \dot\sigma + 2\mu\, k^2\, \nabla_{\dot\sigma}\dot\sigma = \delta_0^A + \delta_0^{\dot\sigma(0)} + \delta_1^B - \delta_1^{\dot\sigma(1)}, \end{aligned} \tag{3.8}$$

for some $A \in T_{\sigma(0)}\mathcal{M}$ and some $B \in T_{\sigma(1)}\mathcal{M}$ such that $\big\langle B, Y(\sigma(1))\big\rangle = 0$.

Now, we multiply equation (3.8) by $\dot\sigma$, and since $\big\langle \nabla_{\dot\sigma} Y, \dot\sigma\big\rangle = \big\langle \nabla_{\dot\sigma}\dot\sigma, \dot\sigma\big\rangle = 0$ and $\big\langle \dot\sigma, Y\big\rangle = -k\mathcal{T}_\sigma$, $\big\langle \dot\sigma, \dot\sigma\big\rangle = -{\mathcal{T}_\sigma}^2$, using (3.2), we get:

$$\tilde\lambda\, k\, \mathcal{T}_\sigma = \big\langle \dot\sigma(0), A\big\rangle\, \delta_0 - {\mathcal{T}_\sigma}^2 \delta_0 + \big\langle \dot\sigma(1), B\big\rangle\, \delta_1 + T_\sigma^2 \delta_1. \tag{3.9}$$

This means that $\tilde\lambda$ is a linear combination of δ_0 and δ_1. By Lemma 3.3, λ is constant and $\big\langle \dot\sigma(0), A\big\rangle = -\big\langle \dot\sigma(1), B\big\rangle$. But λ constant and $\int_I \lambda = 0$ imply immediately:

$$\lambda = 0. \tag{3.10}$$

In particular, we have $\big\langle \dot\sigma(0), A\big\rangle = -\big\langle \dot\sigma(1), B\big\rangle = {\mathcal{T}_\sigma}^2$. We now substitute $\lambda = \tilde\lambda = 0$ in (3.8); multiplying the resulting equation by Y we obtain:

$$-2k\mathcal{T}_\sigma (\widetilde{\mu\big\langle Y, Y\big\rangle} + k^2 \tilde\mu) = (\big\langle Y(p), A\big\rangle - k\mathcal{T}_\sigma)\, \delta_0 + (\big\langle Y(\sigma(1)), B\big\rangle + k\mathcal{T}_\sigma)\, \delta_1. \tag{3.11}$$

Again, by Lemma 3.3, we have that $\big\langle Y(p), A\big\rangle = -\big\langle Y(\sigma(1)), B\big\rangle = 0$ and $\mu\,\big(\big\langle Y, Y\big\rangle + k^2\big) \equiv c$ for some $c \in I\!R$. Finally, by the equalities above we get $c = \frac{1}{2}$, and

$$\mu = \tfrac{1}{2}(k^2 + \big\langle Y, Y\big\rangle)^{-1}. \tag{3.12}$$

From (3.12) we compute easily:

$$\tilde\mu = -\big\langle \nabla_{\dot\sigma} Y, Y\big\rangle \big(\big\langle Y, Y\big\rangle + k^2\big)^{-2} + \mu(0)\, \delta_0 - \mu(1)\, \delta_1; \tag{3.13}$$

substituting (3.3), (3.10), (3.12) and (3.13) into (3.8) gives:

$$\begin{aligned} -\frac{\big\langle Y,Y\big\rangle}{\big\langle Y,Y\big\rangle + k^2}\, \nabla_{\dot\sigma}\dot\sigma - 2k^2 \frac{\big\langle \nabla_{\dot\sigma} Y, Y\big\rangle}{(\big\langle Y,Y\big\rangle + k^2)^2}\, \dot\sigma - \frac{2k\mathcal{T}_\sigma}{\big\langle Y,Y\big\rangle + k^2}\, \nabla_{\dot\sigma} Y + 2k\, \mathcal{T}_\sigma\, \frac{\big\langle \nabla_{\dot\sigma} Y, Y\big\rangle}{(\big\langle Y,Y\big\rangle + k^2)^2}\, Y = \\ = \delta_0^{A+\dot\sigma(0)} + \delta_1^{B-\dot\sigma(1)} - 2k\, (-\mathcal{T}_\sigma Y(p)\, \mu(0) + k\, \dot\sigma(0))\, \delta_0 + \\ + 2k\, (-\mathcal{T}_\sigma Y(\sigma(1))\, \mu(1) + k\, \dot\sigma(1))\, \delta_1. \end{aligned} \tag{3.14}$$

Observe that for $t_0 \in I$ and $v_0 \in T_{\sigma(t_0)}\mathcal{M}$, it is $v_0\, \delta_{t_0} = \delta_{t_0}^{v_0}$, hence, the second member of the equality (3.14) can be written as:

$$\begin{aligned} &\delta_0^{A_1} + \delta_1^{B_1}, \quad A_1 = A + \dot\sigma(0) - 2k\, (-\mathcal{T}_\sigma Y(p)\, \mu(0) + k\, \dot\sigma(0)), \\ &B_1 = B - \dot\sigma(1) + 2k\, (-\mathcal{T}_\sigma Y(\sigma(1))\, \mu(1) + k\, \dot\sigma(1)). \end{aligned}$$

Hence, by Lemma 3.4, the first member of the equality (3.14) is null, and also $A_1 = B_1 = 0$. Therefore, we obtain the following differential equation for σ:

$$\nabla_{\dot\sigma}\dot\sigma + 2k^2 \frac{\langle \nabla_{\dot\sigma} Y, Y\rangle}{\langle Y,Y\rangle\,(k^2+\langle Y,Y\rangle)}\,\dot\sigma + \frac{2k\,\mathcal{T}_\sigma}{\langle Y,Y\rangle}\,\nabla_{\dot\sigma} Y + -2k\,\mathcal{T}_\sigma \frac{\langle \nabla_{\dot\sigma} Y, Y\rangle}{\langle Y,Y\rangle\,(k^2+\langle Y,Y\rangle)}\,Y = 0. \tag{3.15}$$

We have proven the following:

Proposition 3.6. *Let $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$. Then, σ is a brachistochrone of energy k between p and γ iff σ is a curve of class C^2 and there exists $\mathcal{T}_\sigma > 0$ such that σ satisfies the differential equation (3.15).* □

Observe that any curve σ in $H^2(I,\mathcal{M})$ that satisfies (3.15) almost everywhere is automatically smooth. Recalling Definition 2.6, we have the following:

Proposition 3.7. *A curve σ is a brachistochrone of energy k between p and γ iff it is a critical point for T in $\mathcal{B}^{(1)}_{p,\gamma}(k)$.* □

4. **The Brachistochrone Differential Equation.** In this section we will take a closer look at the differential equation (3.15) and we will prove that it characterizes the brachistochrones between p and γ among all the curves in $\Omega^{(1)}_{p,\gamma}$ satisfying suitable initial conditions. Proposition 3.6 can be improved as follows:

Proposition 4.1. *A curve $\sigma \in \Omega^{(1)}_{p,\gamma}$ is a brachistochrone of energy k between p and γ iff σ is smooth and there exists a $\mathcal{T}_\sigma > 0$ such that σ satisfies (3.15), with initial velocity $\dot\sigma(0)$ satisfying:*

$$\langle \dot\sigma(0), \dot\sigma(0)\rangle = -{\mathcal{T}_\sigma}^2, \quad \textit{and} \quad \langle \dot\sigma(0), Y(p)\rangle = -k\,\mathcal{T}_\sigma. \tag{4.1}$$

Proof. From Proposition 3.6, all we need to prove is that any smooth curve $\sigma \in \Omega^{(1)}_{p,\gamma}$ that satisfies the differential equation (3.15) and whose initial velocity $\dot\sigma(0)$ satisfies (4.1) is in $\mathcal{B}^{(1)}_{p,\gamma}(k)$.

To this aim, it suffices to show that the functions $\eta(t) = \langle \dot\sigma(t), \dot\sigma(t)\rangle + {\mathcal{T}_\sigma}^2$ and $\theta(t) = \langle \dot\sigma(t), Y(\sigma(t))\rangle + k\,\mathcal{T}_\sigma$ are constant. If we multiply (3.15) by Y, we obtain $\langle \nabla_{\dot\sigma}\dot\sigma, Y\rangle + \frac{2k^2\langle \nabla_{\dot\sigma} Y, Y\rangle}{\langle Y,Y\rangle\,(k^2+\langle Y,Y\rangle)}\,\big(k\,\mathcal{T}_\sigma + \langle \dot\sigma, Y\rangle\big) = 0$, that can be written as:

$$\theta' + u\,\theta = 0, \quad u = \frac{2k^2\langle \nabla_{\dot\sigma} Y, Y\rangle}{\langle Y,Y\rangle\,(k^2+\langle Y,Y\rangle)}. \tag{4.2}$$

Since $\theta(0) = 0$, then, the uniqueness of the solution for equation (4.2) implies $\theta \equiv 0$. Arguing similarly, if we multiply (3.15) by $\dot\sigma$ we obtain $\eta \equiv 0$ and we are done. □

If q is any point in U_k, we denote by γ_q the maximal integral line of Y through q. Moreover, if $I = [a,b] \subseteq [0,1]$ is any interval, and if q_1, q_2 are any two points in U_k, we define $\mathcal{B}^{(1)}_{q_1,\gamma_{q_2}}(k, I)$ as the space of curves $\tau \in H^1(I, U_k)$ such that $\tau(a) = q_1$, $\tau(b) \in \gamma_{q_2}(\mathbb{R})$, and satisfying $\langle \dot\tau, Y\rangle \equiv -k\,\mathcal{T}_\tau$, $\langle \dot\tau, \dot\tau\rangle \equiv -\mathcal{T}_\tau^2$ for some $\mathcal{T}_\tau \in \mathbb{R}^+$.

Observe that if $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$, then, for every $I = [a,b] \subseteq [0,1]$, the restriction of σ to I is a curve in $\mathcal{B}^{(1)}_{\sigma(a),\gamma_{\sigma(b)}}(k, I)$.

Definition 4.2. A curve $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ is said to be a *local minimizer* for the travel time if, for all $0 \le a < b \le 1$ such that $b - a$ is sufficiently small, the restriction of σ to the interval $I = [a,b]$ is a minimum point for the travel time functional in the space $\mathcal{B}^{(1)}_{\sigma(a),\gamma_{\sigma(b)}}(k, I)$

Note that Definition 4.2 is essentially the definition of brachistochrones of energy k given in [8]. For curves that are local minimizers of the travel time, the differential equation (3.15) was established in [8] by means of a variational principle, that we can now state in a more complete form. We denote by Δ the smooth distribution on $\mathcal{M}$ given by the orthocomplement of the vector field Y. Observe that, since Y is timelike, the wrong way Schwartz's inequality implies that Δ is *spacelike*, i.e., the restriction of the Lorentzian metric g on Δ is positive definite. Let $\psi : \mathcal{M} \times I\!R \to \mathcal{M}$ be the flow of Y. Recall that, since Y is Killing, then $\psi(\cdot, t)$ is a local isometry for all $t \in I\!R$; moreover, it is easy to see that the distribution Δ is ψ-invariant, which means that $\psi_x(q, t_0)(\Delta_q) = \Delta_{\psi(q,t_0)}$, where $\psi_x(q, t_0)$ denotes the differential of the map $\psi(\cdot, t_0)$ at the point q. A function $\phi : \mathcal{M} \to I\!R$ is said to be Y-invariant if it is constant along the flow lines of Y; if ϕ is C^1, this amounts to saying that $\langle Y, \nabla\phi\rangle \equiv 0$. We define $\Omega^{(1)}_{p,\gamma}(\Delta)$ to be the subset of $\Omega^{(1)}_{p,\gamma}$ consisting of curves with tangent vector at each point lying in Δ:

$$\Omega^{(1)}_{p,\gamma}(\Delta) = \big\{ w \in \Omega^{(1)}_{p,\gamma} : \dot w(t) \in \Delta_{w(t)},\ \forall\, t \in [0, 1] \big\}. \tag{4.3}$$

We will call *horizontal* the curves in $\Omega^{(1)}_{p,\gamma}(\Delta)$. By the same arguments of Proposition 2.1, one checks immediately that, since $\langle Y, Y\rangle \neq 0$, $\Omega^{(1)}_{p,\gamma}(\Delta)$ is a smooth submanifold of $\Omega^{(1)}_{p,\gamma}$, and that, for $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$, $T_w\Omega^{(1)}_{p,\gamma}(\Delta)$ is given by:

$$T_w\Omega^{(1)}_{p,\gamma}(\Delta) = \big\{ V \in T_w\Omega^{(1)}_{p,\gamma} : \langle \nabla_{\dot w} V, Y\rangle - \langle V, \nabla_{\dot w} Y\rangle = 0 \big\}. \tag{4.4}$$

It will also be useful, as in the case of the spaces $\mathcal{B}^{(2)}_{p,\gamma}(k)$ and $\mathcal{B}^{(2)}_{p}(k)$ (see formula (2.18), to introduce the spaces $\Omega^{(1)}_{p}$ and $\Omega^{(1)}_{p}(\Delta)$, by:

$$\Omega^{(1)}_{p} = \textstyle\bigcup_{\gamma\subset U_k} \Omega^{(1)}_{p,\gamma}, \quad \text{and} \quad \Omega^{(1)}_{p}(\Delta) = \bigcup_{\gamma\subset U_k} \Omega^{(1)}_{p,\gamma}(\Delta). \tag{4.5}$$

We single out the following simple fact:

Lemma 4.3. *Let ϕ be a smooth Y-invariant positive function. Then, the functional*

$$E_\phi(w) = \tfrac12 \int_0^1 \phi(w)\, \langle \dot w, \dot w\rangle_{(R)}\, \mathrm{d}t \tag{4.6}$$

on $\Omega^{(1)}_{p,\gamma}$ and its restriction to $\Omega^{(1)}_{p,\gamma}(\Delta)$ have the same critical points. These critical points are geodesics in $\mathcal{M}$ with respect to the Riemannian metric $\phi \cdot g_{\mathrm R}$ that join p and γ and that are orthogonal to γ.

Proof. The critical points of E_ϕ in $\Omega^{(1)}_{p,\gamma}$ are precisely the geodesics in $\mathcal{M}$ with respect to $\phi\cdot g_{\mathrm R}$ that join p and γ and that are orthogonal to γ, i.e., $\langle \dot w(1), Y(w(1))\rangle_{(\mathrm R)} = 0$. Since ϕ is Y-invariant, then Y is Killing in the metric $\phi \cdot g_{\mathrm R}$, thus, for every such geodesic w, the quantity $\langle \dot w, Y\rangle_{(\mathrm R)} =$ is constant. Hence $\langle \dot w, Y\rangle_{(\mathrm R)} \equiv 0$ and w is horizontal. Therefore, the critical points of E_ϕ on $\Omega^{(1)}_{p,\gamma}$ belong to $\Omega^{(1)}_{p,\gamma}(\Delta)$, and clearly they are critical points of the restriction of E_ϕ to $\Omega^{(1)}_{p,\gamma}(\Delta)$.

Conversely, if w is a critical point of the restriction of E_ϕ to $\Omega^{(1)}_{p,\gamma}(\Delta)$, then the Gateaux derivative $\mathrm{d}E_\phi(w)[V]$ vanishes for all $V \in T_w\Omega^{(1)}_{p,\gamma}(\Delta)$. Let's define:

$$\mathbf{T}_w = \big\{ V \in T_w\Omega^{(1)}_{p,\gamma} : V = \tau \cdot Y,\ \text{for some } \tau \in H^1(I, I\!R) \text{ with } \tau(0) = \tau(1) = 0 \big\}.$$

Since Y is Killing in the metric $g_{\mathrm R}$, an easy calculation shows that for all $w \in \Omega^{(1)}_{p,\gamma}$, the Gateaux derivative $\mathrm{d}E_\phi(w)[V]$ vanishes for all $V \in \mathbf{T}_w$.

Moreover, for all $w \in \Omega^{(1)}_{p,\gamma}(\Delta)$ it is (see [8]) $T_w\Omega^{(1)}_{p,\gamma} = \mathbf{T}_w + T_w\Omega^{(1)}_{p,\gamma}(\Delta)$, which implies $\mathrm{d}E_\phi(w)[V] = 0$ for all $V \in T_w\Omega^{(1)}_{p,\gamma}$. This concludes the proof. ∎

The functional E_ϕ of (4.6) is called the *energy* functional of the metric $\phi \cdot g_{\mathrm{R}}$. The critical points of E_ϕ in $\Omega^{(1)}_{p,\gamma}$ (or equivalently in $\Omega^{(1)}_{p,\gamma}(\Delta)$, see [8]) are *horizontal geodesics* between p and γ with respect to the Riemannian metric $\phi \cdot g_{\mathrm{R}}$.

Let $\mathcal{D}$ be the map defined by $\mathcal{D}(\sigma) = w$, where:

$$\mathcal{D} : \Omega^{(1)}_{p,\gamma} \longrightarrow \Omega^{(1)}_{p,\gamma}(\Delta), \quad w(t) = \psi\big(\sigma(t), \mathtt{r}_\sigma(t)\big), \tag{4.7}$$

and $\mathtt{r}_\sigma$ is the unique solution on $[0, 1]$ of the Cauchy problem:

$$\mathtt{r}_\sigma{}' = -\big\langle \dot\sigma, Y\big\rangle \big\langle Y, Y\big\rangle^{-1}, \qquad \mathtt{r}_\sigma(0) = 0. \tag{4.8}$$

Using the Killing property of Y it is easily checked that $\mathcal{D}$ is well defined, i.e., the maximal solution of (4.8) is defined on the entire interval $[0, 1]$ and the corresponding curve w given by (4.7) is horizontal. Observe that, if $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$, then (4.8) gives:

$$\mathtt{r}_\sigma{}' = k\, \mathcal{T}_\sigma \big\langle Y, Y\big\rangle^{-1}. \tag{4.9}$$

In Section 7 we will need to use the differential $\mathrm{d}\mathcal{D}$ of $\mathcal{D}$ on brachistochrones:

Proposition 4.4. *The map $\mathcal{D}$ is smooth around the regular points of $\mathcal{B}^{(1)}_{p,\gamma}(k)$. If σ is a curve of class C^1 in $\mathcal{B}^{(1)}_{p,\gamma}(k)$ and $\zeta \in T_\sigma \mathcal{B}^{(1)}_{p,\gamma}(k)$, the differential $\mathrm{d}\mathcal{D}(\sigma)[\zeta]$ is:*

$$\mathrm{d}\mathcal{D}(\sigma)[\zeta] = \mathrm{d}_x\psi(\sigma, \mathtt{r}_\sigma)\,[\zeta + \tau_\zeta \cdot Y(\sigma)]\,, \tag{4.10}$$

where $\tau_\zeta : [0, 1] \to I\!R$ is the function:

$$\tau_\zeta(t) = -\int_0^t \big(C_\zeta \,\big\langle Y, Y\big\rangle + 2k\, \mathcal{T}_\sigma \big\langle \nabla_\zeta Y, Y\big\rangle\big)\big\langle Y, Y\big\rangle^{-2}\, \mathrm{d}r, \tag{4.11}$$

where C_ζ is the constant $\big\langle \nabla_{\dot\sigma}\zeta, Y\big\rangle - \big\langle \zeta, \nabla_{\dot\sigma} Y\big\rangle$. In particular, if σ is a brachistochrone, then τ_ζ takes the following form:

$$\tau_\zeta(t) = -2k\, \mathcal{T}_\sigma \int_0^t \big(\big\langle \nabla_\zeta Y, Y\big\rangle\big)\big\langle Y, Y\big\rangle^{-2}\, \mathrm{d}r. \tag{4.12}$$

Proof. The smooth dependence on σ of the solution $\mathtt{r}_\sigma$ of (4.8) proves that $\mathcal{D}$ is a smooth map. Formulas (4.10), (4.11) and (4.12) are easily obtained by differentiating (4.7) using (2.19), and keeping in mind that $\mathrm{d}_x\psi(\sigma, \mathtt{r}_\sigma)[Y(\sigma)] = Y(\psi(\sigma, \mathtt{r}_\sigma))$. In particular, formula (4.12) follows immediately from (4.11) and Corollary 2.4. □

Observe that formula (4.7) allows to extend the definition of the map $\mathcal{D}$ to the space $\mathcal{B}^{(1)}_p(k)$ and with values in $\Omega^{(1)}_p$; these spaces were defined in (2.18) and (4.5). Obviously, Proposition 4.4 remains true for the extension.

Proposition 4.5 (First Variational Principle for Brachistochrones)**.**
Let $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ be fixed. The following are equivalent:

1. *σ is a brachistochrone of energy k between p and γ;*
2. *σ is a local minimizer for the travel time;*
3. *$w = \mathcal{D}(\sigma) \in \Omega^{(1)}_{p,\gamma}(\Delta)$ is a horizontal geodesic between p and γ with respect to the Riemannian metric $\phi_k \cdot g_{\mathrm{R}}$, where:*

$$\phi_k = -\big\langle Y, Y\big\rangle \big(k^2 + \big\langle Y, Y\big\rangle\big)^{-1}. \tag{4.13}$$

Moreover, if one of the conditions above is satisfied, then $E_{\phi_k}(w) = \frac{1}{2}\mathcal{T}_\sigma{}^2$, where E_{ϕ_k} is the energy functional relative to the metric $\phi_k \cdot g_{\mathrm{R}}$, given by:

$$E_{\phi_k}(w) = \tfrac{1}{2}\int_0^1 \phi_k(w)\big\langle \dot w, \dot w\big\rangle_{(R)}\, \mathrm{d}t, \quad \forall\, w \in \Omega^{(1)}_{p,\gamma}. \tag{4.14}$$

Proof. The equivalence of conditions 1 and 2 follows from the fact that the brachistochrones of energy k between p and γ and the local minimizers for the travel time are characterized by the same differential equation (see Proposition 3.6 and Ref. [8, Definition 1.1, Corollary 3.2]). The equivalence of condition 2 and 3 is based on the fact that, for $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ and $w = \mathcal{D}(\sigma)$, using (3.3), (4.7) and (4.8), we get:

$$\phi_k(w)\big\langle \dot{w}, \dot{w}\big\rangle = \mathcal{T}_\sigma{}^2. \tag{4.15}$$

Observe that, since Y is Killing in the metric $\phi_k \cdot g_{\mathrm{R}}$, then a critical point of E_{ϕ_k} in $\Omega^{(1)}_{p,\gamma}(\Delta)$ is indeed a geodesic with respect to $\phi_k \cdot g_{\mathrm{R}}$ (see [8]). It follows that the quantity $\phi_k(w)\big\langle \dot{w}, \dot{w}\big\rangle$ is constant along each horizontal geodesic w.

Recalling (2.20), integrating formula (4.15) yields:

$$F = -E_{\phi_k} \circ \mathcal{D}. \tag{4.16}$$

From (4.15) it follows that σ is a local minimizer for the travel time iff w is a local minimizer for the energy functional E_{ϕ_k} in $\Omega^{(1)}_{p,\gamma}(\Delta)$, i.e., iff w is a horizontal geodesic between p and γ with respect to $\phi_k \cdot g_{\mathrm{R}}$.

The last statement of the thesis follows easily by integrating (4.15) over $[0,1]$. □

The result of Proposition 4.5 remains true for brachistochrones and horizontal geodesics with free endpoints in U_k.

5. **The Second Variation of the Travel Time.** Let M be a Banach manifold and $f : M \to \mathbb{R}$ be a smooth map. If $x_0 \in M$ is a critical point for f, i.e., $\mathrm{d}f(x_0) = 0$, denote by $H^f(x_0)$ the *Hessian* of f at x_0, which is a continuous symmetric bilinear form on $T_{x_0}M$ (see [3]).

It is easy to prove the following (see [7] for details):

Lemma 5.1. *Let M and N be Banach manifolds and $\mathcal{D} : M \to N$ be a smooth map; let $f : N \to \mathbb{R}$ be a smooth function. If $x_0 \in M$ is such that $\mathcal{D}(x_0)$ a critical point for f, then x_0 is a critical point for $f \circ \mathcal{D}$, and the Hessians $H^f(\mathcal{D}(x_0))$ and $H^{f\circ\mathcal{D}}(x_0)$ are related by:*

$$H^f(\mathcal{D}(x_0))\big[\mathrm{d}\mathcal{D}(x_0)[v], \mathrm{d}\mathcal{D}(x_0)[w]\big] = H^{f\circ\mathcal{D}}(x_0)[v,w], \tag{5.1}$$

for all $v, w \in T_{x_0}M$. □

From Lemma 5.1 and formula (4.16), setting $f = E_{\phi_k}$, it follows immediately:

Corollary 5.2 (Second order variational principle for brachistochrones)**.**
Let $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ be a brachistochrone and $w = \mathcal{D}(\sigma)$. Then, for all $\zeta_1, \zeta_2 \in T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$, we have:

$$H^F(\sigma)[\zeta_1,\zeta_2] = -H^{E_{\phi_k}}(w)\big[\mathrm{d}\mathcal{D}(w)[\zeta_1], \mathrm{d}\mathcal{D}(w)[\zeta_2]\big]. \quad \square \tag{5.2}$$

From (2.20) we obtain easily:

$$H^F(\sigma) = -\mathcal{T}_\sigma \cdot H^T(\sigma) \tag{5.3}$$

for all brachistochrone $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$. From (5.2) and (5.3) we obtain also:

$$H^T(\sigma)[\zeta_1,\zeta_2] = \mathcal{T}_\sigma{}^{-1} \cdot H^{E_{\phi_k}}(w)\big[\mathrm{d}\mathcal{D}(w)[\zeta_1], \mathrm{d}\mathcal{D}(w)[\zeta_2]\big], \tag{5.4}$$

for all brachistochrone $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ and all $\zeta_1, \zeta_2 \in T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$.

6. **The Riemannian Morse Index Theorem.** For Riemannian geodesics, the classical Morse Index Theorem (see for instance [14]) relates the index of the action functional with some geometrical properties of the geodesic. The main ingredients for the theory are given by the curvature tensor of the metric and the concepts of *Jacobi fields* and *conjugate* or *focal* points along a geodesic. In view to applications to the brachistochrone problem, in this section we quickly review some known results about the Morse Index Theorem for Riemannian geodesics joining a curve with a point, as presented, for instance, in [18]. Then, we prove a different version of this theorem in the case of an orthogonal geodesic between the integral line of a Killing vector field and a point.

In order to simplify the formulas, in this section we interchange the role of p and γ, that is, we consider curves starting at the curve γ and ending at the point p. Clearly, the final results (Theorems 6.6 and 6.7) will not be affected by this change. Moreover, all the results and the formulas of the previous sections remain true after changing the variable t with $1-t$ in the interval $[0,1]$, and, in particular, the role of the endpoints $t=0$ and $t=1$ will be interchanged. To avoid confusion, in this section we will use the symbols $\Omega^{(1)}_{\gamma,p}$ and $\Omega^{(1)}_{\gamma,p}(\Delta)$ to indicate the spaces of curves in U_k of class H^1 from γ to p. If we denote by $\mathcal{O}$ the *direction reversing map* for curves $w:[0,1]\to\mathcal{M}$, i.e., $\mathcal{O}(w)(t)=w(1-t)$, then clearly $\Omega^{(1)}_{\gamma,p}=\mathcal{O}(\Omega^{(1)}_{p,\gamma})$ and $\Omega^{(1)}_{\gamma,p}(\Delta)=\mathcal{O}(\Omega^{(1)}_{p,\gamma}(\Delta))$. Observe that, for all $i\in I\!N$, the restriction of $\mathcal{O}$ to the Sobolev manifold $H^i([0,1],\mathcal{M})$ is smooth, and its differential is formally given by:

$$\mathrm{d}\mathcal{O}[V](t)=V(1-t),\quad V\in H^i([0,1],T\mathcal{M}).$$

Observe also that the energy functional E_{ϕ_k} can be defined in $\Omega^{(1)}_{\gamma,p}$ by the same formula (4.14); obviously, a curve w is a critical point for E_{ϕ_k} in $\Omega^{(1)}_{p,\gamma}$ iff $\mathcal{O}(w)$ is a critical point for E_{ϕ_k} in $\Omega^{(1)}_{\gamma,p}$. In this case, we have:

$$H^{E_{\phi_k}}(w)[V,W]=H^{E_{\phi_k}}(\mathcal{O}(w))[\mathrm{d}\mathcal{O}[V],\mathrm{d}\mathcal{O}[W]],\quad \forall\,V,W\in T_w\Omega^{(1)}_{p,\gamma}. \tag{6.1}$$

By Lemma 4.3, we know that the critical points of the Riemannian energy functional E_{ϕ_k} corresponding to the metric $\phi_k\cdot g_{\mathrm{R}}$ on the spaces $\Omega^{(1)}_{\gamma,p}$ and $\Omega^{(1)}_{\gamma,p}(\Delta)$ are the same. However, given a horizontal geodesic w between p and γ, the Morse index of E_{ϕ_k} at w (see Definition 6.4) in the Hilbert manifold $\Omega^{(1)}_{\gamma,p}(\Delta)$ may be strictly less then the Morse index of E_{ϕ_k} at w in the manifold $\Omega^{(1)}_{\gamma,p}$. The purpose of this section is to prove that the two indices are indeed equal; we accomplish this result by proving an index theorem for the Morse index $m(w,E_{\phi_k})$ restricted the space $T_w\Omega^{(1)}_{\gamma,p}(\Delta)^\perp$, defined by:

$$T_w\Omega^{(1)}_{\gamma,p}(\Delta)^\perp=\big\{V\in T_w\Omega^{(1)}_{\gamma,p}(\Delta)\;\big|\;\phi_k(w)\cdot\big\langle V,\dot w\big\rangle_{(\mathrm{R})}\equiv C_V\ \text{(const.)}\big\}. \tag{6.2}$$

Observe that $T_w\Omega^{(1)}_{\gamma,p}(\Delta)^\perp$ is a (closed) Hilbert subspace of $T_w\Omega^{(1)}_{\gamma,p}(\Delta)$; moreover, if w is a horizontal geodesic with respect to the metric $\phi_k\cdot g_{\mathrm{R}}$ in $\Omega^{(1)}_{\gamma,p}$, then, for a vector field $V\in T_w\Omega^{(1)}_{\gamma,p}(\Delta)$ we have:

$$V\in T_w\Omega^{(1)}_{\gamma,p}(\Delta)^\perp\quad\Longleftrightarrow\quad\big\langle\nabla^{\{k\}}_{\dot w}V,\dot w\big\rangle_{(\mathrm{R})}=0, \tag{6.3}$$

where $\nabla^{\{k\}}$ is the covariant derivative of the Levi–Civita connection of $\phi_k\cdot g_{\mathrm{R}}$. Indeed, if $V\in T_w\Omega^{(1)}_{\gamma,p}(\Delta)^\perp$, then, since $\nabla^{\{k\}}_{\dot w}\dot w=0$,

$$0=\tfrac{\mathrm{d}}{\mathrm{d}t}\big[\phi_k(w)\cdot\big\langle V,\dot w\big\rangle_{(\mathrm{R})}\big]=\phi_k(w)\cdot\big\langle\nabla^{\{k\}}_{\dot w}V,\dot w\big\rangle_{(\mathrm{R})}.$$

On the other hand, if $0 = \phi_k(w) \cdot \langle \nabla^{\{k\}}_{\dot w} V, \dot w \rangle_{(\mathrm{R})} = \frac{\mathrm{d}}{\mathrm{d}t}\big[\phi_k(w) \cdot \langle V, \dot w\rangle_{(\mathrm{R})}\big]$, the quantity $\phi_k(w) \cdot \langle V, \dot w\rangle_{(\mathrm{R})}$ is constant and (6.3) is proven.

In particular, since $V(1) = 0$ (recall that we are considering curves ending at the fixed point p), if w is a horizontal geodesic and $V \in T_w\Omega^{(1)}_{\gamma,p}(\Delta)^\perp$, then $C_V = 0$. Hence, a vector field $V \in T_w\Omega^{(1)}_{\gamma,p}(\Delta)$ belongs to $T_w\Omega^{(1)}_{\gamma,p}(\Delta)^\perp$ iff it is everywhere perpendicular to w, which is the reason for the notation. We can easily write (6.3) in terms of the Lorentzian structure, by differentiating the above expression using the Lorentzian covariant derivative. Given a horizontal geodesic w and $V \in T_w\Omega^{(1)}_{\gamma,p}(\Delta)$, we have that $V \in T_w\Omega^{(1)}_{\gamma,p}(\Delta)^\perp$ iff the following equation holds:

$$\langle \nabla\phi_k(w), V\rangle \cdot \langle \dot w, \dot w\rangle + 2\,\phi_k(w) \cdot \langle \nabla_{\dot w} V, \dot w\rangle = 0. \tag{6.4}$$

We recall the basic facts concerning the Morse Index Theorem for Riemannian geodesics between a point and a curve, as it is presented, for instance, in Ref. [18].

Given a horizontal geodesic w in between p and γ with respect to the Riemannian metric $\phi_k \cdot g_{\mathrm{R}}$, let $\nabla^{\{k\}}$ and $R^{\{k\}}$ denote respectively the Levi–Civita connection and the curvature tensor (chosen with the same sign convention as in (2.1)) of the metric $\phi_k \cdot g_{\mathrm{R}}$, and let $\mathcal{J}^{\{k\}}_w$ be the finite dimensional vector space of all the Jacobi fields J along w with respect to $\phi_k \cdot g_{\mathrm{R}}$, i.e., all smooth vector fields satisfying the second order differential equation:

$$\nabla^{\{k\}}_{\dot w}\nabla^{\{k\}}_{\dot w} J - R^{\{k\}}(\dot w, J)\,\dot w = 0. \tag{6.5}$$

We recall that, in analogy with the Riemannian case, given a submanifold Σ of $\mathcal{M}$ whose tangent bundle $T\Sigma$ is non degenerate, i.e., the restriction of g to the tangent space $T_q\Sigma$ is non degenerate for all $q \in \Sigma$, one can define the *second fundamental form* S^Σ (also known as the *shape tensor* of Σ) as follows. For each $q \in \Sigma$ and each vector $n \in T_q\Sigma^\perp$, the second fundamental form of Σ in the direction of n is the bilinear form $S^\Sigma_n : T_q\Sigma \times T_q\Sigma \to I\!R$ defined by:

$$S^\Sigma_n(v_1, v_2) = \langle n, \nabla_{v_1} V_2\rangle, \tag{6.6}$$

where V_2 is any smooth vector field on Σ that takes value v_2 at q. One can show that S^Σ_n is well defined (i.e., formula (6.6) does not indeed depend on the choice of the extension V_2 of v_2), and it is *symmetric* (see for instance [1] and [15]).

In the following, we will denote by S^γ the second fundamental form of the timelike submanifold $\gamma(I\!R)$ of $\mathcal{M}$.

Let $\mathcal{J}^{\{k\}}_w(\gamma)$ denote the subspace of $\mathcal{J}^{\{k\}}_w$ consisting of all γ-*Jacobi fields* i.e., all the Jacobi fields J along w satisfying:

1. $J(0) \parallel Y(w(0))$;
2. $\langle \nabla_{\dot w(0)} J, Y\rangle + S^\gamma_{\dot w(0)}(J(0), Y) = \langle \nabla_{\dot w(0)} J, Y\rangle + \langle \dot w(0), \nabla_{J(0)} Y\rangle = 0.$

Finally, for $t_0 \in \,]0,1]$, we denote by $\mathcal{J}^{\{k\}}_w(\gamma, t_0)$ the set of γ-Jacobi fields J along w that vanish at t_0:

(3) $J(t_0) = 0$.

A point $w(t_0)$ along w is said to be a γ-*focal point* if $\dim(\mathcal{J}^{\{k\}}_w(t_0)) > 0$; the *multiplicity* of the a γ-focal point $w(t_0)$ is the dimension of $\mathcal{J}^{\{k\}}_w(t_0)$ (which is clearly finite, because the Jacobi fields are solutions of a second order linear system of differential equations).

Remark 6.1. It is well known that the set of γ-focal points along every Riemannian geodesic is discrete, hence there is only a finite number of γ-focal points along each compact portion of a geodesic. A simple proof of this fact can be obtained by an argument sketched in [15, Ex. 8, p. 299]).

Equation (6.5) is obtained by linearizing the geodesic equation in the metric $\phi_k \cdot g_{\rm R}$; hence, it is satisfied by vector fields along w that correspond to variations w_s, $s \in \,]-\varepsilon, \varepsilon[$ for some $\varepsilon > 0$, of w consisting of geodesics. Loosely speaking, the arrow-head of J traces out infinitesimally close neighboring geodesics to w.

Using the Riemannian metric $\phi_k \cdot g_{\rm R}$, condition 2 can also be written as:

(2b) $\big\langle \nabla^{\{k\}}_{\dot w(0)} J, Y \big\rangle_{\rm (R)} + \big\langle \dot w(0), \nabla^{\{k\}}_{J(0)} Y \big\rangle_{\rm (R)} = 0.$

Remark 6.2. Observe that, since Y is Killing, we obtain easily that, if J satisfies the differential equation 6.5, then the condition $\big\langle \nabla_{\dot w} J, Y \big\rangle + \big\langle \dot w, \nabla_J Y \big\rangle = 0$ is satisfied identically on $[0,1]$ provided that it is satisfied at one single point $t_0 \in [0,1]$.

From Remark 6.2 and formula (4.4), we obtain immediately the following characterization of the γ-Jacobi fields along a horizontal geodesic w:

Lemma 6.3. *Let w be a horizontal geodesic in $\Omega^{(1)}_{\gamma,p}(\Delta)$ and W a Jacobi field along w. Then, W is a γ-Jacobi field iff $W \in T_w\Omega^{(1)}_{\gamma,p}(\Delta)$.* □

Given a horizontal geodesic w, we denote by $I^{\{k\}}$ the *index form* on $T_w\Omega^{(1)}_{\gamma,p}$, or more in general on $T_w H^1([0,1], \mathcal{M})$, given by the symmetric bilinear form:

$$I^{\{k\}}(V_1, V_2) = \int_0^1 \phi_k(w)\Big(\big\langle \nabla^{\{k\}}_{\dot w} V_1, \nabla^{\{k\}}_{\dot w} V_2 \big\rangle_{\rm (R)} + \big\langle R^{\{k\}}(\dot w, V_1)\,\dot w, V_2 \big\rangle_{\rm (R)}\Big)\,{\rm d}t. \tag{6.7}$$

The symmetry of $I^{\{k\}}$ follows easily from the symmetry properties of the curvature tensor $R^{\{k\}}$; moreover, from the fundamental Lemma of Calculus of Variations, a simple integration by parts in (6.7) shows that a vector field W along w is a Jacobi field iff $I^{\{k\}}(W,V) = 0$ for all smooth vector field V along w such that $V(0) = V(1) = 0$. Let us recall the following:

Definition 6.4. Let M be a Hilbert manifold, $f : M \to I\!R$ be a map of class C^2 x_0 a critical point for f in M and X a Hilbert subspace of $T_{x_0}M$. The Morse index $m(x_0, f, X)$ of f at x_0 in X is the dimension of a maximal subspace of X on which the Hessian $H^f(x_0)$ is *negative* definite. Whenever there is no danger of confusion, we will denote by $m(x_0, f) = m(x_0, f, T_{x_0}M)$ the Morse index of f at x_0 in the entire tangent space $T_{x_0}M$.

The *kernel* of $H^f(x_0)$, denoted by ${\rm Ker}\,\big(H^f(x_0)\big)$ is the Hilbert subspace of $T_{x_0}M$ consisting of vectors X such that $H^f(x_0)[X,Y] = 0$ for all $Y \in T_{x_0}M$.

Remark 6.5. Observe that, for all subspace $X \subset T_{x_0}M$, we have

$$m(x_0, f, X) \le m(x_0, f). \tag{6.8}$$

On the other hand, suppose that X is a closed subspace of $T_{x_0}M$ and that the restriction of $H^f(x_0)$ to X is nondegenerate. Let X_1 be the orthogonal space to X relatively to the bilinear form $H^f(x_0)$, which is the closed subspace of $T_{x_0}M$ defined by $X_1 = \big\{V_1 \in T_{x_0}M : H^f(x_0)[V, V_1] = 0 \ \forall\ V \in X\big\}$. If the restriction of $H^f(x_0)$ to X_1 is positive semidefinite, then $m(x_0, f, X) = m(x_0, f)$.

If w is a horizontal geodesic between p and γ with respect to the Riemannian metric $\phi_k \cdot g_{\rm R}$, or equivalently, w is a critical point for E_{ϕ_k} in $\Omega^{(1)}_{\gamma,p}$, then the Hessian $H^{E_{\phi_k}}(w)$ is computed easily in terms of the metric $\phi_k \cdot g_{\rm R}$ as:

$$H^{E_{\phi_k}}(w)[V,V] = I^{\{k\}}(V,V) - \phi_k(w(0))\big\langle \nabla^{\{k\}}_{V(0)} V, \dot w(0) \big\rangle_{\rm (R)}. \tag{6.9}$$

Since $V(0)$ is tangent to the curve γ and $\dot{w}(0)$ is orthogonal to γ, then the term $\phi_k(w(0))\big\langle\nabla^{\{k\}}_{V(0)}V,\dot{w}(0)\big\rangle_{(\mathrm{R})}$ is tensorial in V, i.e., it only depends on the value $V(0)$. This is precisely the second fundamental form of the curve γ in the direction of the normal vector $\dot{w}(0)$ with respect to the metric $\phi_k\cdot g_{\mathrm{R}}$.

Let's now pass to the study of the second variation of E_{ϕ_k} in terms of the Riemannian metric $\phi_k\cdot g_{\mathrm{R}}$. Using integration by parts in the Index formula (6.7), it is easy to see that the set of γ-Jacobi fields $\mathcal{J}^{\{k\}}_w(t_0)$ can be also described as the kernel of the Hessian $H^{E_{\phi_k}}(w)$ restricted to the interval $[0,t_0]$; in particular:

$$\mathcal{J}^{\{k\}}_w=\mathrm{Ker}\left(H^{E_{\phi_k}}(w)\right). \tag{6.10}$$

The *geometric index* $\mu^{\{k\}}(w)$ of the horizontal geodesic w is defined as:

$$\mu^{\{k\}}(w)=\textstyle\sum_{t_0\in]0,1]}\dim\big(\mathcal{J}^{\{k\}}_w(t_0)\big). \tag{6.11}$$

Recall from Remark 6.1 that the number of γ-focal points along w is finite, hence the sum in (6.11) is finite. The Morse Index Theorem says that, if p is not a γ-focal point along w, the Morse index $m(w,E_{\phi_k})$ of E_{ϕ_k} in the space $T_w\Omega^{(1)}_{\gamma,p}$ is given by the number of γ-focal points along w, counted with multiplicity:

Theorem 6.6. *Let w be a critical point of E_{ϕ_k} in $\Omega^{(1)}_{\gamma,p}$, i.e., a geodesic from γ to p in the metric $\phi_k\cdot g_{\mathrm{R}}$ that starts orthogonally to γ. Then, the Morse index $m(w,E_{\phi_k})$ is finite; moreover, if p is not a γ-focal point along w, we have:*

$$m(w,E_{\phi_k})=\mu^{\{k\}}(w).\quad\square \tag{6.12}$$

Theorem 6.6 is obtained as a special case of [18, Theorem 2.5]. Observe that Theorem 6.6 holds without any assumption that γ be the integral line of a Killing vector field. In the rest of this section we will prove that, given a horizontal geodesic w in $\Omega^{(1)}_{\gamma,p}$, then $m(w,E_{\phi_k})$ is equal to the Morse index $\bar{m}(w,E_{\phi_k})$ of the restriction of the Hessian $H^{E_{\phi_k}}$ on the space $T_w\Omega^{(1)}_{\gamma,p}(\Delta)^{\perp}$. Observe that, by (6.8), we have

$$\bar{m}(w,E_{\phi_k})=m(w,E_{\phi_k},T_w\Omega^{(1)}_{\gamma,p}(\Delta)^{\perp})\le m(w,E_{\phi_k}).$$

The desired result will follow immediately from our next theorem:

Theorem 6.7 (Second Morse Index Theorem for Horizontal Geodesics)**.** *Let $(\mathcal{M},\tilde{g})$ be a complete Riemannian manifold, Y a never vanishing complete Killing vector field on $\mathcal{M}$, $\gamma:\mathbb{R}\to\mathcal{M}$ an integral curve of Y, and $p\in\mathcal{M}$ be a point in $\mathcal{M}\setminus\gamma(\mathbb{R})$.*

Let $\tilde{\Delta}=Y^{\perp}$ be the orthogonal distribution to Y; moreover let $\Omega^{(1)}_{\gamma,p}$, $\Omega^{(1)}_{\gamma,p}(\tilde{\Delta})$ denote the spaces:

$$\Omega^{(1)}_{\gamma,p}=\big\{w\in H^1([0,1],\mathcal{M})\;\big|\;w(0)\in\gamma(\mathbb{R}),\ w(1)=p\big\},$$
$$\Omega^{(1)}_{\gamma,p}(\tilde{\Delta})=\big\{w\in\Omega^{(1)}_{\gamma,p}\;\big|\;\tilde{g}(\dot{w},Y)\equiv 0\big\};$$

and, for $w\in\Omega^{(1)}_{\gamma,p}(\Delta)$, let $T_w\Omega^{(1)}_{\gamma,p}$, $T_w\Omega^{(1)}_{\gamma,p}(\tilde{\Delta})$ and $T_w\Omega^{(1)}_{\gamma,p}(\tilde{\Delta})^{\perp}$ be defined in the obvious way (see formulas (2.8), (4.4) and (6.2)).

Let $\tilde{E}$ denote the energy functional of the metric $\tilde{g}$ in the space $\Omega^{(1)}_{\gamma,p}$; let w be a critical point of $\tilde{E}$ in $\Omega^{(1)}_{\gamma,p}$ (or, equivalently, in $\Omega^{(1)}_{\gamma,p}(\tilde{\Delta})$), and let $H^{\tilde{E}}(w)$ be the Hessian of $\tilde{E}$ at w.

Then, if p is not a γ-focal point along w, the three indices are equal:

$$m(w,H^{\tilde{E}})=m(w,H^{\tilde{E}},T_w\Omega^{(1)}_{\gamma,p}(\tilde{\Delta}))=m(w,H^{\tilde{E}},T_w\Omega^{(1)}_{\gamma,p}(\tilde{\Delta})^{\perp}). \tag{6.13}$$

Proof. See [7]. □

Theorem 6.7 can be applied to the Riemannian metric $\tilde{g} = \phi_k \cdot g_{\mathrm{R}}$ defined in U_k. Recalling that $\bar{m}(w, E_{\phi_k})$ denotes the Morse Index of the restriction of the Hessian $H^{E_{\phi_k}}$ on the space $T_w\Omega^{(1)}_{\gamma,p}(\Delta)^{\perp}$, we have thus proven the equality:

$$\bar{m}(w, E_{\phi_k}) = m(w, E_{\phi_k}). \tag{6.14}$$

7. **The Index Theorem for Brachistochrones.** We want to study now the Morse index of the travel time functional at a given brachistochrone σ, which is defined as the index of the symmetric bilinear form $H^T(\sigma)$ (see Definition 6.4). In this section we extend the classical the Morse Theory for Riemannian geodesics, in order to obtain a version of this theorem for brachistochrones (Theorem 7.12), by introducing the concepts of b-Jacobi fields and b-focal points along a brachistochrone σ (see Definitions 7.1 and 7.6 below).

We now begin with the study of the Hessian of the travel time functional.

Let $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ be a brachistochrone, since $\mathcal{T}_\sigma > 0$, formula (5.3) tells us that:

$$m(\sigma, T) = m(\sigma, -F), \quad \text{and} \quad \mathrm{Ker}\left(H^T(\sigma)\right) = \mathrm{Ker}\left(H^F(\sigma)\right). \tag{7.1}$$

We emphasize that from now on we will consider brachistochrone curves whose endpoints may vary in the open set U_k, whereas the value of their energy constant k is a *fixed* positive number. Let $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ be a fixed brachistochrone, and, recalling the definition of the space $\mathcal{B}^{(1)}(k)$ given in (2.18), we consider a variation $\sigma_s \in \mathcal{B}^{(1)}(k)$ of σ, depending smoothly on the parameter $s \in \,]-\varepsilon, \varepsilon[$ and such that $\sigma_0 = \sigma$. Suppose that each curve σ_s is a brachistochrone of energy k between $\sigma_s(0)$ and $\gamma_{\sigma_s(1)}$, where $\gamma_{\sigma_s(1)}$ is the integral line of Y passing through $\sigma_s(1)$.

This means that each σ_s satisfies the differential equation (3.15) and with initial tangent vector $\dot{\sigma}_s(0)$ satisfying the two conditions:

$$\big\langle \dot{\sigma}_s(0), Y(\sigma_s(0))\big\rangle^2 + k^2\big\langle \dot{\sigma}_s(0), \dot{\sigma}_s(0)\big\rangle = 0, \quad \text{and} \quad \big\langle \dot{\sigma}_s(0), Y(\sigma_s(0))\big\rangle < 0. \tag{7.2}$$

Definition 7.1. Given a brachistochrone $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$, a vector field $V \in T_\sigma\mathcal{B}^{(1)}(k)$ is called a *b-Jacobi field* if there exists a variation $\sigma_s \in \mathcal{B}^{(1)}(k)$ of σ as above such that $V = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma_s$.

In other words, a b-Jacobi field along σ is a variational vector field corresponding to variations made of brachistochrones with the same energy constant and, possibly, with different endpoints. By definition, the b-Jacobi fields are characterized by the property of satisfying the linearized brachistochrone equation:

Proposition 7.2. *Let $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ be a brachistochrone of travel time $\mathcal{T}_\sigma$ and let $V \in T_\sigma\mathcal{B}^{(1)}(k)$ be a variational vector field along σ, with constant $C_V = \big\langle \nabla_{\dot{\sigma}}V, Y\big\rangle -$*

$\langle V, \nabla_{\dot\sigma} Y\rangle$. *If V is a b-Jacobi field then V satisfies the second order linear differential equation:*

$$\begin{aligned}
\nabla^2_{\dot\sigma} V - R(\dot\sigma, V)\,\dot\sigma + \tfrac{2k\,\mathcal{T}_\sigma}{\langle Y,Y\rangle^2}\big(\nabla_{\dot\sigma}\nabla_V Y - \langle Y,Y\rangle R(\dot\sigma, V)\,Y - 2\langle \nabla_V Y, Y\rangle \nabla_{\dot\sigma} Y\big) +&\\
-2\tfrac{C_V}{\langle Y,Y\rangle}\nabla_{\dot\sigma} Y + \tfrac{2k^2\dot\sigma - 2k\,\mathcal{T}_\sigma Y}{\langle Y,Y\rangle(k^2+\langle Y,Y\rangle)}\big(\langle \nabla_{\dot\sigma}\nabla_V Y, Y\rangle + \langle \nabla_V Y, \nabla_{\dot\sigma} Y\rangle\big) +&\quad (7.3)\\
+\tfrac{2k^2\dot\sigma - 2k\,\mathcal{T}_\sigma Y}{\langle Y,Y\rangle^2(k^2+\langle Y,Y\rangle)^2}\big(-4\langle \nabla_{\dot\sigma} Y, Y\rangle\langle Y,Y\rangle\langle \nabla_V Y, Y\rangle - 2k^2\langle \nabla_{\dot\sigma} Y, Y\rangle\langle \nabla_V Y, Y\rangle\big)&\\
+\tfrac{2\langle \nabla_{\dot\sigma} Y, Y\rangle}{\langle Y,Y\rangle(k^2+\langle Y,Y\rangle)}\big(C_V Y - k\,\mathcal{T}_\sigma \nabla_V Y + k^2\nabla_{\dot\sigma} V\big) = 0,&
\end{aligned}$$

and the initial condition:

$$-\mathcal{T}_\sigma C_V + k\langle \nabla_{\dot\sigma} V(0), \dot\sigma(0)\rangle = 0. \tag{7.4}$$

Proof. See [7]. □

A partial converse to Proposition 7.2 is provided by the following Proposition:

Proposition 7.3. *Let $\sigma \in \mathcal{B}^{(1)}(k)$ be a brachistochrone and suppose that V is a smooth vector field along σ satisfying the differential equation (7.3), the initial condition (7.4) and with $V(0) = 0$. Then, V is a b-Jacobi field along σ, i.e., there exists a variation σ_s of σ consisting of brachistochrones between p and γ_s, $s \in\,]-\varepsilon, \varepsilon\,[$, such that $V = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma_s$.*

Proof. Given a vector $v_0 \in T_p\mathcal{M}$ such that

$$\langle v_0, Y(p)\rangle^2 + k^2\langle v_0, v_0\rangle = 0, \quad \text{and} \quad \langle v_0, v_0\rangle < 0, \tag{7.5}$$

then there exists a unique brachistochrone $\sigma_{v_0} \in \mathcal{B}^{(1)}_p(k)$ and such that $\dot\sigma_{v_0}(0) = v_0$. This is obtained by solving the differential equation (3.15) with initial conditions $\sigma(0) = p$ and $\dot\sigma(0) = v_0$. Moreover, the map $v_0 \mapsto \sigma_{v_0} \in \mathcal{B}^{(1)}_p(k)$ is C^1, due to the regular dependence on the data of the solution of the differential equation (3.15). Let $S \subset T_p\mathcal{M}$ be the set of vectors v_0 satisfying the conditions (7.5); S is a submanifold of $T_p\mathcal{M}$. Indeed, the condition $\langle v_0, v_0\rangle < 0$ is open; moreover, the gradient of the smooth map $G : T_p\mathcal{M} \ni v_0 \mapsto \langle v_0, Y(p)\rangle^2 + k^2\langle v_0, v_0\rangle \in I\!R$ is easily computed as:

$$G'(v_0) = 2\langle v_0, Y(p)\rangle \cdot Y(p) + 2k^2 v_0. \tag{7.6}$$

Multiplying by $Y(p)$ we get $\langle G'(v_0), Y(p)\rangle = 2\langle v_0, Y(p)\rangle\big(\langle Y(p), Y(p)\rangle + k^2\big) \neq 0$, because both v_0 and $Y(p)$ are timelike, so $\langle v_0, Y(p)\rangle \neq 0$, and $\langle Y(p), Y(p)\rangle + k^2 > 0$ in U_k. This implies that $G' \neq 0$, hence $G^{-1}(0)$ is a smooth submanifold of $T_p\mathcal{M}$. Clearly, $\dot\sigma(0) \in S$. Let $v_0(s) : \,]-\varepsilon, \varepsilon[\, \to S$ be a smooth map such that $v_0(0) = \dot\sigma(0) \in S$ and $v_0'(0) = \nabla_{\dot\sigma(0)} V$. Observe that $\nabla_{\dot\sigma(0)} V$ belongs to $T_{\dot\sigma(0)} S$, because, from (7.6), we have $\langle G'(\dot\sigma(0)), \nabla_{\dot\sigma(0)} V\rangle = 2\langle \dot\sigma(0), Y(p)\rangle\langle Y(p), \nabla_{\dot\sigma(0)} V\rangle + 2k^2\langle \dot\sigma(0), \nabla_{\dot\sigma(0)} V\rangle$. Since $V(0) = 0$, then $C_V = \langle Y(p), \nabla_{\dot\sigma(0)} V\rangle$, and from (7.4) one gets $\langle G'(\dot\sigma(0)), \nabla_{\dot\sigma(0)} V\rangle = 2k\left(-\mathcal{T}_\sigma C_V + k\langle \dot\sigma(0), \nabla_{\dot\sigma(0)} V\rangle\right) = 0$. Hence, $\nabla_{\dot\sigma(0)} V \in T_{\dot\sigma(0)} S$ and the curve $v_0(s)$ is well defined. Now, for all $s \in\,]-\varepsilon, \varepsilon\,[$, let σ_s be the unique brachistochrone in $\mathcal{B}^{(1)}_p(k)$ satisfying $\dot\sigma_s(0) = v_0(s)$; clearly, $\sigma_0 = \sigma$, and σ_s is a smooth variation of σ. Observe that, since σ_0 is defined on the closed interval $[0, 1]$, then we can assume that also σ_s is defined on $[0, 1]$ for all s. In order to conclude the proof, we need to show that the variational field $\tilde V = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma_s$ coincides with

V. By Proposition 7.2, $\tilde V$ satisfies the second order differential equation (7.3), while V satisfies (7.3) by assumption, and $\tilde V(0) = V(0) = 0$, because we are fixing the initial point p. By uniqueness, in order to prove that $\tilde V = V$ along σ it suffices to show that $\nabla_{\dot\sigma(0)}\tilde V = \nabla_{\dot\sigma(0)} V$. This is easily established by the following calculation, that concludes the proof:

$$\nabla_{\dot\sigma(0)}\tilde V = \frac{\mathrm{D}}{\mathrm{d}t}\Big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\sigma_s = \frac{\mathrm{D}}{\mathrm{d}s}\Big|_{s=0}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\sigma_s = \frac{\mathrm{D}}{\mathrm{d}s}\Big|_{s=0}\dot\sigma_s(0) = v_0'(0) = \nabla_{\dot\sigma(0)}V. \quad \square$$

Corollary 7.4. *If σ is a brachistochrone and V is a b-Jacobi field along σ such that $V(0) = 0$, then $V \in T_\sigma\mathcal{B}_p^{(1)}(k)$.*

Proof. Following the proof of Proposition 7.3, V is the variational vector field corresponding to a variation $\sigma_s \in \mathcal{B}_p^{(1)}(k)$ of σ. $\square$

In general, it may not be true that a b-Jacobi field V along a brachistochrone σ satisfying $V(0) = 0$ and $V(1) \in I\!R \cdot Y(\sigma(1))$ is the variational vector field corresponding to a family of brachistochrones in $\mathcal{B}_{p,\gamma}^{(1)}(k)$. However, such vector fields belong to $T_\sigma\mathcal{B}_{p,\gamma}^{(1)}(k)$, and they are in the kernel of the Hessian $H^F(\sigma)$:

Corollary 7.5. *If $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ is a brachistochrone and V is a b-Jacobi field along σ such that $V(0) = 0$ and $V(1)$ is parallel to $Y(\sigma(1))$, then $V \in T_\sigma\mathcal{B}_{p,\gamma}^{(1)}(k)$, and $V \in \mathrm{Ker}\big(H^F(\sigma)\big)$.*

Proof. By Corollary 7.4, $V \in T_\sigma\mathcal{B}_p^{(1)}(k)$; the first part of the statement follows immediately by observing that a vector field $V \in T_\sigma\mathcal{B}_p^{(1)}(k)$ belongs to $T_\sigma\mathcal{B}_{p,\gamma}^{(1)}(k)$ iff $V(1)$ is parallel to $Y(\sigma(1))$ (see formulas (2.8), (2.21) and (2.22)).

To prove the second part of the thesis, we need to show that $H^F(\sigma)[V,W] = 0$ for all $W \in T_\sigma\mathcal{B}_{p,\gamma}^{(1)}(k)$. By Corollary 5.2, we have:

$$H^F(\sigma)[V,W] = -H^{E_{\phi_k}}(\mathcal{D}(\sigma))\big[\mathrm{d}\mathcal{D}(\sigma)[V], \mathrm{d}\mathcal{D}(\sigma)[W]\big], \tag{7.7}$$

hence, to conclude the proof it suffices to show that $\mathrm{d}\mathcal{D}(\sigma)[V]$ is in the kernel of the Hessian $H^{E_{\phi_k}}(\mathcal{D}(\sigma))$. By (6.10), this amounts to proving that $X = \mathrm{d}\mathcal{D}(\sigma)[V]$ is the variational vector field corresponding to a smooth variation w_s of $w = \mathcal{D}(\sigma)$ consisting of horizontal geodesics in the metric $\phi_k \cdot g_{\mathrm{R}}$ between p and some integral curve γ_s of Y lying in U_k. To see this, let σ_s be a smooth variation of σ consisting of brachistochrones in $\mathcal{B}_p^{(1)}(k)$ between p and some curve γ_s in U_k, and with variational vector field V. Such a variation exists by Proposition 7.3. Then, if we consider the curves $w_s = \mathcal{D}(\sigma_s)$, by part 3 of Proposition 4.5, each w_s is a horizontal geodesic between p and γ_s; by Proposition 4.4, w_s is a smooth variation of w. Finally:

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}w_s = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\mathcal{D}(\sigma_s) = \mathrm{d}\mathcal{D}(\sigma)[\tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma_s] = \mathrm{d}\mathcal{D}(\sigma)[V] = X. \quad \square$$

Definition 7.6. Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a brachistochrone. A point $\sigma(t_0)$ of σ is said to be a *b-focal point* if there exists a non zero b-Jacobi field V along $\sigma\big|_{[t_0,1]}$ that vanishes at t_0, that is, a non zero vector field V along σ for which the quantity $C_V = \big\langle \nabla_{\dot\sigma}V, Y\big\rangle - \big\langle V, \nabla_{\dot\sigma}Y\big\rangle$ is constant along σ, such that $V(t_0) = 0$, satisfying the differential equation (7.3) and the condition:

$$-\mathcal{T}_\sigma C_V + k\big\langle \nabla_{\dot\sigma}V(t_0), \dot\sigma(t_0)\big\rangle = 0. \tag{7.8}$$

In the above situation, we will also say that $\sigma(t_0)$ is *b-conjugate* to $\sigma(1) = p$ along σ.

For every $t_0 \in [0,1]$, the set $\mathcal{J}_\sigma(t_0)$ of vector fields V satisfying the above conditions in the interval $[t_0, 1]$ is a vector field; if $\sigma(t_0)$ is a b-focal point along σ, then *multiplicity* $\mu_\sigma(t_0)$ of $\sigma(t_0)$ is the dimension of $\mathcal{J}_\sigma(t_0)$. The *geometric index* $\mu(\sigma)$ of the brachistochrone σ is defined to be the (possibly infinite) number:

$$\mu(\sigma) = \sum_{t_0\in[0,1[} \mu_\sigma(t_0) \in I\!N \cup \{+\infty\}. \tag{7.9}$$

Observe that every vector field along $\sigma\big|_{[t_0,1]}$ which is solution of the linear differential equation (7.3) in the interval $[t_0, 1]$, can be extended to a vector field along σ satisfying the equation on the entire interval $[0, 1]$. Also, it follows easily from Propositions 4.1 and 7.3 that if the quantity $\big\langle\nabla_{\dot\sigma}V, Y\big\rangle - \big\langle V, \nabla_{\dot\sigma}Y\big\rangle$ is constant on $[t_0, 1]$ and if V satisfies (7.3) on $[0, 1]$, then $\big\langle\nabla_{\dot\sigma}V, Y\big\rangle - \big\langle V, \nabla_{\dot\sigma}Y\big\rangle$ is constant on $[0, 1]$. In particular, from Proposition 7.3 we have that $\sigma(t_0)$ is a b-focal point iff there exists a non trivial variation σ_s, $s \in\,]-\varepsilon, \varepsilon\,[$ of brachistochrones of energy k between $\sigma(t_0)$ and γ, depending smoothly on s, and such that $\sigma_0 = \sigma\big|_{[t_0,1]}$.

We now want to relate the b-focal points along a brachistochrone σ with the γ-focal points along the corresponding Riemannian geodesic $w = \mathcal{D}(\sigma)$ (Theorem 7.12 below). Given a horizontal geodesic w, a Jacobi field along w is a (smooth) vector field J along w satisfying the differential equation (6.5). From (2.8) and (4.4), such a vector field J belongs to the tangent space $T_w\Omega^{(1)}_{\gamma,p}(\Delta)$ iff $J(1) = 0$, $J(0) \in I\!R \cdot Y(w(0))$ (recall that we are considering curves w starting on γ and arriving at p), and $\big\langle\nabla_{\dot w}J, Y\big\rangle + \big\langle\dot w, \nabla_J Y\big\rangle \equiv 0$. Recalling Remark 6.2, this last equality is satisfied identically on $[0, 1]$ provided that it is satisfied at some point $t_0 \in [0, 1]$. Hence, recalling the definitions 1, 2 and 3 of page 711 and Remark 6.2, we have that the set of Jacobi fields in $T_w\Omega^{(1)}_{\gamma,p}(\Delta)$ coincides with the finite dimensional vector space $\mathcal{J}^{\{k\}}_w(\gamma, 0)$: $\mathcal{J}^{\{k\}}_w \cap T_w\Omega^{(1)}_{\gamma,p}(\Delta) = \mathcal{J}^{\{k\}}_w(\gamma, 0)$. We introduce the map:

$$\mathcal{G} : \Omega^{(1)}_{p,\gamma} \longrightarrow \Omega^{(1)}_{p,\gamma}, \quad \mathcal{G}(w)(t) = \psi(w(t), h_w(t)), \tag{7.10}$$

where $h_w(t) = -k\int_0^t \sqrt{\phi_k(w(0))\big\langle\dot w(0), \dot w(0)\big\rangle_{(\mathrm{R})}}\big\langle Y, Y\big\rangle^{-1}\,\mathrm{d}r$. As in the case of the map $\mathcal{D}$, it is easy to see that $\mathcal{G}$ is smooth; moreover, using (4.15) one checks that it is a left-inverse for $\mathcal{D}$ in $\mathcal{B}^{(1)}_{p,\gamma}(k)$, i.e., for all $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$, we have:

$$\mathcal{G}(\mathcal{D}(\sigma)) = \sigma. \tag{7.11}$$

Proposition 7.7. *Let σ be a brachistochrone and $w = \mathcal{O}(\mathcal{D}(\sigma))$. If $J \in \mathcal{J}^{\{k\}}_w(\gamma, 1)$, then there exists $V \in \mathcal{J}_\sigma(0)$ a b-Jacobi field along σ such that $\mathrm{d}\mathcal{O} \circ \mathrm{d}\mathcal{D}(\sigma)[V] = J$.*

Proof. Let $s \in\,]-\varepsilon, \varepsilon\,[$ and w_s be a smooth variation of w consisting of horizontal geodesics and such that $J = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} w_s$. Let $\sigma_s = \mathcal{G}(\mathcal{O}(w_s)) \in \mathcal{B}^{(1)}_p(k)$; since $\mathcal{G}$ is smooth, then σ_s is a smooth variation of σ. Moreover, $\mathcal{O}(\mathcal{D}(\sigma_s)) = w_s$, and since w_s is a horizontal geodesic, by Proposition 4.5, σ_s is a brachistochrone in $\mathcal{B}^{(1)}_p(k)$ for all s. By Definition 7.1, $V = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma_s$ is a b-Jacobi field in $\mathcal{J}_\sigma(0)$. Note that $V(0) = 0$ because $\sigma_s(0) = p$ for all s. It is easily computed: $\mathrm{d}\mathcal{O} \circ \mathrm{d}\mathcal{D}(\sigma)[V] = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\mathcal{O}(\mathcal{D}(\sigma_s)) = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} w_s = J$, which concludes the proof. □

Proposition 7.7 gives the surjectivity of the map $\mathrm{d}\mathcal{O}\circ\mathrm{d}\mathcal{D}(\sigma)$ restricted to $\mathcal{J}_\sigma(0)$ and $\mathcal{J}^{\{k\}}_w(\gamma, 0)$. The injectivity of $\mathrm{d}\mathcal{D}(\sigma)$ can be proven on the entire space $T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$:

Proposition 7.8. *For all $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$, $\mathrm{d}\mathcal{D}(\sigma) : T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k) \to T_{\mathcal{D}(\sigma)}\Omega^{(1)}_{p,\gamma}$ is an injective map.*

Proof. It suffices to prove that $\mathrm{d}\mathcal{D}(\sigma)$ has a left inverse $L : T_{\mathcal{D}(\sigma)}\Omega^{(1)}_{p,\gamma} \to T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$. Such a map L is given by the differential of the map $\mathcal{G}$ defined by (7.10). Indeed, by

(7.11), $\mathcal{G}\circ\mathcal{D}$ is the identity on $\mathcal{B}^{(1)}_{p,\gamma}(k)$,and by differentiating we have that $\mathrm{d}\mathcal{G}\circ\mathrm{d}\mathcal{D}(\sigma)$ is the identity on $T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$ for all $\sigma\in\mathcal{B}^{(1)}_{p,\gamma}(k)$. □

Proposition 7.9. *Let $\sigma\in\mathcal{B}^{(1)}_{p,\gamma}(k)$ be a brachistochrone and $w=\mathcal{D}(\sigma)$. Then, the image of the differential $\mathrm{d}\mathcal{D}(\sigma)$ in $T_w\Omega^{(1)}_{p,\gamma}$ is given by $T_w\Omega^{(1)}_{p,\gamma}(\Delta)^\perp$ (see (6.2)).*

Proof. We first show that $\mathrm{d}\mathcal{D}(\sigma)\subset T_w\Omega^{(1)}_{p,\gamma}(\Delta)^\perp$. To this end, let $\zeta\in T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$ be fixed; by (2.21) and Corollary 2.4, it satisfies $\langle\nabla_{\dot\sigma}\zeta,\dot\sigma\rangle\equiv 0$. Since $\mathcal{D}(\mathcal{B}^{(1)}_{p,\gamma}(k))\subset\Omega^{(1)}_{p,\gamma}(\Delta)$, then clearly $\mathrm{d}\mathcal{D}(T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k))\subset T_w\Omega^{(1)}_{p,\gamma}(\Delta)$. Moreover, let $V=\mathrm{d}\mathcal{D}(\sigma)[\zeta]$. For the inclusion $\mathrm{d}\mathcal{D}(T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k))\subset T_w\Omega^{(1)}_{p,\gamma}(\Delta)^\perp$ we need to show that (6.4) is satisfied. Using formulas (3.3), (4.7), (4.8), (4.10), (4.12) and (4.15), we compute:

$$\begin{aligned}&\langle\nabla\phi_k(w),V\rangle\langle\dot w,\dot w\rangle+2\,\phi_k(w)\cdot\langle\nabla_{\dot w}V,\dot w\rangle=\\&=-\frac{2k^2\mathcal{T}_\sigma{}^2}{\langle Y,Y\rangle\,(k^2+\langle Y,Y\rangle)}\langle\nabla_Y Y,\zeta\rangle+\frac{2k^2\mathcal{T}_\sigma{}^2}{\langle Y,Y\rangle\,(k^2+\langle Y,Y\rangle)}\langle\nabla_Y Y,\zeta\rangle=0.\end{aligned}\tag{7.12}$$

For the opposite inclusion, let V be fixed in $T_w\Omega^{(1)}_{p,\gamma}(\Delta)^\perp$ and let $w_s\in\Omega^{(1)}_p$ be a variation of w with variational vector field V such that $\langle\dot w_s,Y(w_s)\rangle\equiv 0$ and $\langle\dot w_s,\dot w_s\rangle\equiv c_s$ (constant). Such a variation exists, provided that we do not require the condition $w_s(1)\in\gamma(I\!R)$. For all s, define $\sigma_s=\mathcal{G}(w_s)$ where $\mathcal{G}$ is the map defined in (7.10). Then, σ_s is a variation of σ in $\mathcal{B}^{(1)}_p(k)$; if $\zeta=\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma_s\in T_\sigma\mathcal{B}^{(1)}_p(k)$ is the corresponding variational vector field, $\mathrm{d}\mathcal{D}(\sigma)[\zeta]=V$. To conclude, we need to show that $\zeta\in T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$, i.e., that $\zeta(1)$ is parallel to $Y(\sigma(1))$. This follows easily from the fact that $V(1)$ is a multiple of $Y(w(1))$ and from formula (4.10). □

In analogy with formula (4.10), for all $a\in[0,1[$ we can define a linear map $\mathcal{L}_a$ on the space of vector fields along $\sigma\big|_{[a,1]}$ satisfying the two conditions appearing in (2.21) on the interval $[a,1]$, and taking values in the space of vector fields along $w\big|_{[a,1]}$. The map $\mathcal{L}_a$ is given by:

$$\mathcal{L}_a[\zeta](r)=\mathrm{d}_x\psi(\sigma(r),t^a_\sigma(r))[\zeta(r)+\tau^a_\zeta\cdot Y(\sigma(r))],\tag{7.13}$$

$$\mathtt{r}_\sigma{}^a(r)=-\int_a^r\langle\dot\sigma,Y\rangle\langle Y,Y\rangle^{-1}\,\mathrm{d}u,\quad\text{and}\quad\tau^a_\zeta(r)=-\int_a^r\frac{C_\zeta\langle Y,Y\rangle+2k\,\mathcal{T}_\sigma\langle\nabla_\zeta Y,Y\rangle}{\langle Y,Y\rangle^2}\,\mathrm{d}u.$$

In particular, $\mathcal{L}_0=\mathrm{d}\mathcal{D}(\sigma)$; observe also that, of $\zeta(a)=0$, then $\mathcal{L}_a[\zeta](a)=0$.
The result of Propositions 7.7 and 7.8 can be extended immediately to the maps $\mathrm{d}\mathcal{O}\circ\mathcal{L}_{t_0}:\mathcal{J}_\sigma(t_0)\to\mathcal{J}^{\{k\}}_w(\gamma,t_0)$ for all $t_0\in[0,1[$:

Corollary 7.10. *Let $\sigma\in\mathcal{B}^{(1)}_{p,\gamma}(k)$ be a brachistochrone and $w=\mathcal{D}(\sigma)$ the corresponding geodesic in $\Omega^{(1)}_{p,\gamma}(\Delta)$. Then, for all $t_0\in[0,1[$, the linear map $\mathrm{d}\mathcal{O}\circ\mathcal{L}_{t_0}$ gives an isomorphism of the vector spaces of Jacobi fields $\mathcal{J}_\sigma(t_0)$ and $\mathcal{J}^{\{k\}}_w(\gamma,t_0)$.*

Proof. The proofs of Propositions 7.7 and 7.8 can be repeated *verbatim*, by replacing the initial point p with the point $\sigma(t_0)$. □

The kernel of the Hessian $H^F(\sigma)$ in $T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$ consists precisely of b-Jacobi fields.

Proposition 7.11. *Let σ be a brachistochrone. A vector field $V\in T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$ is a b-Jacobi field along σ iff $V\in\operatorname{Ker}\big(H^F(\sigma)\big)$ in $T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$.*

Proof. Corollary 7.5 proves that any b-Jacobi field along σ is in the kernel of $H^F(\sigma)$. Conversely, let σ be a fixed brachistochrone and $\zeta \in \mathrm{Ker}\,\big(H^F(\sigma)\big)$. From Corollary 7.10, it suffices to prove that the vector field $J = \mathrm{d}\mathcal{O} \circ \mathrm{d}\mathcal{D}(\sigma)[\zeta]$ is a γ-Jacobi field with respect to the Riemannian metric $\phi_k \cdot g_{\mathrm{R}}$ along the geodesic $w = \mathcal{O}(\mathcal{D}(\sigma))$. Moreover, since $J \in T_w\Omega^{(1)}_{\gamma,p}(\Delta)$, from Lemma 6.3 it suffices to show that J is a Jacobi field along w, i.e., that it satisfies equation (6.5). Observe that, by Proposition 7.9, J is in $T_w\Omega^{(1)}_{\gamma,p}(\Delta)^\perp$, hence it satisfies the two equations:

$$\big\langle \nabla^{\{k\}}_{\dot w} J, Y\big\rangle_{(\mathrm{R})} - \big\langle J, \nabla^{\{k\}}_{\dot w} Y\big\rangle_{(\mathrm{R})} = 0, \quad \big\langle J, \dot w\big\rangle_{(\mathrm{R})} = \big\langle \nabla^{\{k\}}_{\dot w} J, \dot w\big\rangle_{(\mathrm{R})} = 0. \tag{7.14}$$

To prove that J is Jacobi, let $V \in C^\infty_o([0,1], T\mathcal{M})$ be any smooth vector field along w vanishing at the endpoints. We set: $W = V - \mu \cdot Y - \lambda \cdot \dot w$, where λ and μ are functions in $H^1([0,1], I\!R)$ to be determined in such a way that the resulting vector field W belongs to $T_w\Omega^{(1)}_{\gamma,p}(\Delta)$. This condition is satisfied by setting:

$$\mu(t) = -\int_t^1 \phi_k(w) \cdot \frac{\big\langle \nabla^{\{k\}}_{\dot w} V, Y\big\rangle_{(\mathrm{R})} + \big\langle \dot w, \nabla^{\{k\}}_{V} Y\big\rangle_{(\mathrm{R})}}{\big\langle Y, Y\big\rangle_{(\mathrm{R})}}\,\mathrm{d}r, \quad \lambda(t) = -\int_t^1 \frac{\big\langle \nabla^{\{k\}}_{\dot w} V, \dot w\big\rangle_{(\mathrm{R})}}{\big\langle \dot w, \dot w\big\rangle_{(\mathrm{R})}}\,\mathrm{d}r. \tag{7.15}$$

Observe that, since w is a geodesic with respect to $\phi_k \cdot g_{\mathrm{R}}$ one has:

$$\lambda(0) = \lambda(1) = \mu(1) = 0. \tag{7.16}$$

Arguing as in the proof of Theorem 6.7 since Y is Killing in the metric $\phi_k \cdot g_{\mathrm{R}}$, then its restriction to w is a Jacobi field:

$$\nabla^{\{k\}}_{\dot w} \nabla^{\{k\}}_{\dot w} Y = R^{\{k\}}(\dot w, Y)\,\dot w. \tag{7.17}$$

Recalling (6.9), keeping in mind (7.16) and $V(0) = V(1) = 0$, we have:

$$\begin{aligned} H^{E_{\phi_k}}(w)[J, W] = {} & I^{\{k\}}(J, V) - I^{\{k\}}(J, \lambda \cdot \dot w) - I^{\{k\}}(J, \mu \cdot Y) \\ & - \mu(1)\,\phi_k(w(1))\,\big\langle \nabla^{\{k\}}_{J(1)} Y, \dot w(1)\big\rangle_{(\mathrm{R})}. \end{aligned} \tag{7.18}$$

From (6.7), the second equation of (7.14) and the anti-symmetry of the curvature tensor $R^{\{k\}}$, the term $I^{\{k\}}(J, \lambda \cdot \dot w)$ is easily seen to vanish:

$$I^{\{k\}}(J, \lambda \cdot \dot w) = \int_0^1 \phi_k(w)\big(\lambda' \langle \nabla^{\{k\}}_{\dot w} J, \dot w\rangle_{(\mathrm{R})} + \lambda\,\big\langle R^{\{k\}}(\dot w, J)\,\dot w, \dot w\big\rangle_{(\mathrm{R})}\big)\,\mathrm{d}t = 0. \tag{7.19}$$

From (6.7), integrating by parts and using formulas (7.14), (7.17) and the symmetry of the curvature tensor $R^{\{k\}}$, one proves the following equality (see [7]):

$$I^{\{k\}}(J, \mu \cdot Y) = -\mu(1) \cdot \phi_k(w(1)) \cdot \big\langle \dot w(1), \nabla^{\{k\}}_{J(1)} Y\big\rangle_{(\mathrm{R})}. \tag{7.20}$$

Finally, from (7.18), (7.19) and (7.20), we have proven the equality:

$$I^{\{k\}}(J, V) = H^{E_{\phi_k}}(w)[J, W].$$

Since $W \in T_w\Omega^{(1)}_{\gamma,p}(\Delta)$, then W is in the image of $\mathrm{d}\mathcal{O} \circ \mathrm{d}\mathcal{D}$, say $W = \mathrm{d}\mathcal{O} \circ \mathrm{d}\mathcal{D}(\sigma)[\zeta_1]$ for some $\zeta_1 \in T_\sigma\mathcal{B}^{(1)}_{p,\gamma}(k)$. Since $\zeta \in \mathrm{Ker}\,\big(H^F(\sigma)\big)$ and $J = \mathrm{d}\mathcal{O} \circ \mathrm{d}\mathcal{D}(\sigma)[\zeta]$, then, by Corollary 5.2 and formula (6.1), it is $H^{E_{\phi_k}}(w)[J, W] = -H^F(\sigma)[\zeta, \zeta_1] = 0$, and, in particular, $I^{\{k\}}(J, V) = 0$. Hence, we have that $I^{\{k\}}(J, V) = 0$ for all smooth vector field along w vanishing at the endpoints, and this implies that J is a Jacobi field, concluding the proof. $\square$

Theorem 7.12. *Let $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ be a brachistochrone and $w = \mathcal{O}(\mathcal{D}(\sigma)) \in \Omega^{(1)}_{\gamma,p}$ the corresponding horizontal geodesic. Then, a point $\sigma(t_0)$ is a b-focal point along σ iff $w(t_0)$ is a γ-focal point along w, in which case the two focal points have the same multiplicity. In particular, we have*

$$\mu(\sigma) = \mu^{\{k\}}(w). \tag{7.21}$$

Moreover, if p is not a b-focal point along σ, then the Morse index $m(\sigma, T)$ is equal to the geometric index $\mu(\sigma)$ of σ:

$$m(\sigma, T) = \mu(\sigma). \tag{7.22}$$

Proof. By Corollary 7.10, for all $t_0 \in [0, 1[$ we have $\dim\big(\mathcal{J}^{\{k\}}_w(t_0)\big) = \mu_\sigma(t_0)$. This implies that $\sigma(t_0)$ is a b-focal point along σ iff $w(t_0)$ is a γ-focal point; moreover, summing over all $t_0 \in [0, 1[$, we obtain (7.21). From Corollary 5.2, (5.4) and (6.1),

$$m(\sigma, T) = m(\sigma, -F); \tag{7.23}$$

from (7.7) and Propositions 7.8 and 7.9 we obtain:

$$m(\sigma, -F) = \bar{m}(w, E_{\psi_k}); \tag{7.24}$$

finally, from (6.14) we have the equality:

$$\bar{m}(w, E_{\phi_k}) = m(w, E_{\phi_k}). \tag{7.25}$$

If p is not a γ-focal point along w, or equivalently if p is not a b-focal point along σ, then, Theorem 6.6 implies:

$$m(w, E_{\phi_k}) = \mu^{\{k\}}(w); \tag{7.26}$$

the equality (7.22) follows at once from (7.21) and (7.23)—(7.26). ☐

Corollary 7.13. *Let $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ be a brachistochrone. Then, σ is never a local maximum for T.* ☐

From the equality (7.22) we get that, if $\mu(\sigma) = 0$, then the Morse index of the travel time vanishes at σ, hence σ is a local minimum for T. Therefore, we have:

Corollary 7.14. *Let $\sigma \in \mathcal{B}^{(1)}_{p,\gamma}(k)$ be a brachistochrone and $w = \mathcal{O}(\mathcal{D}(\sigma))$. Suppose that there are no γ-focal points along w. Then, σ is a local minimum for the arrival time functional T.* ☐

8. **The Global Morse Relations.** In this section we will use the infinite dimensional Morse theory to prove some equalities relating the differential structure of the travel time brachistochrone problem and the topological structure carried by the set of continuous paths joining p and γ in U_k. We will omit some of the proofs that are analogous to the proof of similar results in [4]. In this section, we will make the following assumptions:

1. the vector field Y is complete in U_k, i.e., its integral lines are defined over the entire real line;
2. $\gamma : \mathbb{R} \to U_k$ is an integral line of Y without self-intersection;
3. p is an event in U_k;
4. k^2 is a regular value for the function $-\langle Y, Y \rangle$;
5. $\overline{U}_k = U_k \bigcup \partial U_k$ is complete with respect to the Riemannian metric (2.2);
6. the function $-\langle Y, Y \rangle$ is *bounded away from* 0 in U_k, i.e., there exists a positive constant ν such that $-\langle Y, Y \rangle \geq \nu > 0$ in U_k;

7. p and γ are not b-conjugate, i.e., for any brachistochrone σ of energy k in $\mathcal{B}^{(1)}_{p,\gamma}(k)$, the points $\sigma(0) = p$ and $\sigma(1)$ are not b-conjugate along σ.

We denote by $B_{p,\gamma}(k)$ the set of brachistochrones in $\mathcal{B}^{(1)}_{p,\gamma}(k)$; moreover, let $\mathcal{C}^0_{p,\gamma}$ denote the set of continuous paths joining p and γ in U_k endowed with the topology of the uniform convergence.

Theorem 8.1. *Under the assumptions 1—7 above, given any coefficient field $\mathcal{K}$, the following equality between formal power series in the variable $\lambda \in \mathcal{K}$ holds true:*

$$\sum_{\sigma\in B_{p,\gamma}(k)} \lambda^{\mu(\sigma)} = \sum_{i=1}^{\infty} \dim\left(H_i(\mathcal{C}^0_{p,\gamma},\mathcal{K})\right)\, \lambda^i + (1+\lambda)\,Q(\lambda), \tag{8.1}$$

where $\mu(\sigma)$ is the geometric index of the brachistochrone σ, $H_i(\mathcal{C}^0_{p,\gamma},\mathcal{K})$ is the i-th homology vector space of $\mathcal{C}^0_{p,\gamma}$ with coefficients in $\mathcal{K}$, and Q is a formal power series in λ with coefficients in $\mathbb{N}\bigcup\{+\infty\}$.

Remark 8.2. If the open set U_k is contractible, then also the space $\mathcal{C}^0_{p,\gamma}$ is contractible, and thus, for every field $\mathcal{K}$, its homology spaces $H_i(\mathcal{C}^0_{p,\gamma},\mathcal{K})$ vanish for all $i > 0$ and $H_0(\mathcal{C}^0_{p,\gamma},\mathcal{K}) \simeq \mathcal{K}$. In this case, under the assumptions 1—7 above, formula (8.1) becomes:

$$\sum_{\sigma\in B_{p,\gamma}(k)} \lambda^{\mu(\sigma)} = 1 + (1+\lambda)Q(\lambda). \tag{8.2}$$

Setting $\lambda = 1$ in (8.2), we get immediately that the number of travel time brachistochrones of energy k between p and γ is either infinite (if $Q(1) = +\infty$) or *odd* (if $Q(1) < +\infty$). On the other hand, if U_k is not contractible, then, since γ is contractible in U_k (because of the injectivity of γ), then there are infinitely many indices i such that $\dim(H_i(\mathcal{C}^0_{p,\gamma},\mathcal{K})) > 0$ (see [19]). Hence, if U_k is not contractible, then there exist infinitely many brachistochrones of energy k between p and γ in U_k.

In order to prove Theorem 8.1, we will use the functional E_{ϕ_k} (defined in (4.6)) in the space $\Omega^{(1)}_{p,\gamma} = \Omega^{(1)}_{p,\gamma}(U_k)$, and ϕ_k is the function defined in (4.13). Since U_k is open, we need to use a *penalization* argument, as follows. Let us consider the function: $\Psi_k = \langle Y, Y\rangle + k^2$; It is $\partial U_k = \Psi_k^{-1}(0)$, moreover $\Psi_k(q) > 0$ iff $q \in U_k$. By assumption 4, the Riemannian gradient $\nabla^{(\mathrm{R})}\Psi_k$ does not vanish on ∂U_k, where $\nabla^{(\mathrm{R})}$ denotes the gradient with respect to the Riemannian metric (2.2).

We define a family χ_ε of real functions of class C^2, for $\varepsilon > 0$:

$$\chi(s) = e^s - (1 + s + \tfrac{s^2}{2}), \quad \chi_\varepsilon(s) = \begin{cases} \chi(s - \frac{1}{\varepsilon}), & \text{if } s \ge \frac{1}{\varepsilon};\\ 0, & \text{if } s < \frac{1}{\varepsilon}.\end{cases} \tag{8.3}$$

Finally, for all $\varepsilon \in]0,1]$, we define the *penalized* functional:

$$E_\varepsilon(w) = E_{\phi_k}(w) + \int_0^1 \chi_\varepsilon\left(\tfrac{1}{\Psi_k(w)^2}\right)\,\mathrm{d}t. \tag{8.4}$$

For all $\varepsilon > 0$, E_ε is a functional of class C^2 on $\Omega^{(1)}_{p,\gamma}$, which satisfies good *compactness properties*, as it will be discussed below. By the completeness of $\overline{U}_k$, it is not too difficult to prove that, for every $c \in \mathbb{R}$, the sublevel $E^c_\varepsilon = \{w \in \Omega^{(1)}_{p,\gamma}(U_k) : E_\varepsilon(w) \le c\}$ is a complete metric subspace of $\Omega^{(1)}_{p,\gamma}(U_k)$, with respect to the metric induced by the Hilbert structure (2.5). Moreover, using the same techniques employed in [5], one proves the following two facts. First, E_ε satisfies the *Palais–Smale condition* (see for instance [10]) at every level $c \in \mathbb{R}$, i.e., every sequence $\{w_n\}$ in E^c_ε such that $\mathrm{d}E_\varepsilon(w_n)$ tends to 0 as $n \to \infty$, has a convergent subsequence in E^c_ε. Second, for all $c \in \mathbb{R}$ there exists $\delta(c) > 0$ and $\varepsilon(c) \in]0,1]$ such that, for all $\varepsilon \in]0,\varepsilon(c)]$ and for all critical point w_ε of E_ε in $\Omega^{(1)}_{p,\gamma}(U_k)$ with $E_\varepsilon(w_\varepsilon) \le c$, then w_ε is also a critical point for E_{ϕ_k}, and $\Psi_k(w_\varepsilon(t)) \ge \delta(c)$ for all $t \in [0,1]$. In particular, it follows that if c is a regular value for E_{ϕ_k}, using (8.3) and (8.4) we obtain the existence of a number $\varepsilon'(c) \in]0, \varepsilon(c)]$ such that, for all $\varepsilon \in]0, \varepsilon'(c)]$, c is a regular value also for the functional E_ε, and a curve $w \in \Omega^{(1)}_{p,\gamma}$ is a critical point for E_ε iff it is a critical point for E_{ϕ_k} (with $E_\varepsilon(w) = E_{\phi_k}(w)$) and $m(w, E_\varepsilon) = m(w, E_{\phi_k})$, where $m(z,G)$ denotes the *Morse Index* of the functional G at the critical point z.

Then, using assumption 7, for all $\varepsilon \in]0, \varepsilon'(c)]$, every critical point w of E_ε in E^c_ε is *nondegenerate*, which allows to obtain the Morse Relations in E^c_ε (see Ref. [13]):

Proposition 8.3. *If c is a regular value for E_{ϕ_k}, then there exists $\varepsilon'(c) \in]0,1]$ such that, for every $\varepsilon \in]0, \varepsilon'(c)]$, we have:*

$$\sum\nolimits_{w\in\mathcal{G}^c_{p,\gamma}} \lambda^{m(w,E_{\phi_k})} = \sum\nolimits_{i=0}^{\infty} \dim\left(H_i(E^c_\varepsilon, \mathcal{K})\right)\lambda^i + (1+\lambda)\,Q_c(\lambda), \tag{8.5}$$

where $\mathcal{G}^c_{p,\gamma}$ is the set of horizontal geodesics between p and γ with energy less than or equal to c and $Q_c(\lambda)$ is a polynomial in the variable λ with coefficients in $\mathbb{N}$. □

We recall that, given a topological pair (A,B), i.e., a topological space A and a subspace $B \subset A$ with the induced topology, we say that B is a *weak deformation retract* of A if there exists a continuous map $H : A\times[0,1] \to A$ such that: (1) $H(\cdot,0)$ is the identity map of A, (2) $H(B,s) \subset B$ for all $s \in [0,1]$, (3) $H(A,1) \subset B$. Given a topological pair (A,B), we denote by $P_\lambda(A,B)$ the Poincaré series of (A,B) in the variable λ, which is given by: $P_\lambda(A,B;\mathcal{K}) = \sum_{i=0}^{\infty} \dim\left(H_i(A,B;\mathcal{K})\right)\lambda^i$, where $H_i(A,B;\mathcal{K})$ is the i-th relative homology space of (A,B) with coefficients in the field $\mathcal{K}$. Now, for $\delta > 0$, we denote by $\Omega^{(1)}_{p,\gamma}(\delta)$ the set:

$$\Omega^{(1)}_{p,\gamma}(\delta) = \left\{ w \in \Omega^{(1)}_{p,\gamma} : \Psi_k(w(t)) \ge \delta \ \forall\, t \in [0,1] \right\}.$$

Using the results of Ref. [4], we can prove that if c is a regular value of E_{ϕ_k}, there exists $\delta_0 = \delta_0(c) > 0$ and $\varepsilon_0 = \varepsilon_0(c)$ such that, for all $\delta \in]0, \delta_0]$ and for all $\varepsilon \in]0, \varepsilon_0]$, the following two statements hold:

$$\Omega^{(1)}_{p,\gamma}(\delta) \cap E^c_{\phi_k} \text{ is a weak deformation retract of } E^c_{\phi_k}, \tag{8.6}$$

$$\Omega^{(1)}_{p,\gamma}(\delta) \cap E^c_{\varepsilon} \text{ is a weak deformation retract of } E^c_{\varepsilon}. \tag{8.7}$$

Observe that, if ε is sufficiently small, we have $\Omega^{(1)}_{p,\gamma}(\delta) \cap E^c_{\phi_k} = \Omega^{(1)}_{p,\gamma}(\delta) \cap E^c_\varepsilon$. Then, using standard techniques in Algebraic Topology, from (8.6) and (8.7) we deduce easily that, if c_1 and c_2 are critical values of E_{ϕ_k}, with $c_1 < c_2$, then there exists $\varepsilon_0 \in]0,1]$ such that, for all $\varepsilon \in]0, \varepsilon_0]$, the following identities between Poincaré series hold:

$$P_\lambda(E^{c_2}_\varepsilon;\mathcal{K}) = P_\lambda(E^{c_2}_{\phi_k};\mathcal{K}), \quad \text{and} \quad P_\lambda(E^{c_2}_\varepsilon, E^{c_1}_\varepsilon;\mathcal{K}) = P_\lambda(E^{c_2}_{\phi_k}, E^{c_1}_{\phi_k};\mathcal{K}).$$

Using the above identities and the same technique of [4, Theorem 1.6], one passes to the limit as $c \to +\infty$ in (8.5), obtaining the Morse relations for the functional E_{ϕ_k} in $\Omega^{(1)}_{p,\gamma}(U_k)$:

Theorem 8.4. *Under assumptions 1—7, for all coefficient field $\mathcal{K}$, we have:*

$$\sum\nolimits_{w\in\mathcal{G}_{p,\gamma}} \lambda^{m(w,E_{\phi_k})} = \sum\nolimits_{i=0}^{\infty} \dim\left(H_i(\Omega^{(1)}_{p,\gamma}(U_k);\mathcal{K})\right)\lambda^i + (1+\lambda)\,Q(\lambda), \tag{8.8}$$

where $\mathcal{G}_{p,\gamma}$ is the set of all horizontal geodesics between p and γ and $Q(\lambda)$ is a formal power series in λ (depending on the choice of $\mathcal{K}$) with coefficients in $\mathbb{N} \cup \{+\infty\}$. □

Proof of Theorem 8.1. By Proposition 4.5 and Lemma 4.3, $w \in \mathcal{G}_{p,\gamma}$ iff $w = \mathcal{D}(\sigma)$, where $\mathcal{D}$ is the deformation map of (4.7) and σ is a travel time brachistochrone of energy k between p and γ. By the first part of Theorem 7.12, the hypothesis 7 implies that every $w \in \mathcal{G}_{p,\gamma}$ is a nondegenerate critical point of E_{ϕ_k} in $\Omega^{(1)}_{p,\gamma}(U_k)$. Moreover, by Theorem 6.6, we have $m(w, E_{\phi_k}) = \mu^{\{k\}}$, while, by Theorem 7.12, it is $\mu^{\{k\}}(w) = \mu(\sigma)$. Then, formula (8.8) can be written as:

$$\sum_{\sigma\in B_{p,\gamma}(k)} \lambda^{\mu(\sigma)} = \sum_{i=0}^{\infty} \dim\left(H_i(\Omega^{(1)}_{p,\gamma}(U_k);\mathcal{K})\right)\lambda^i + (1+\lambda)\,Q(\lambda).$$

Finally, it is well known ([14, Theorem 17.1]) that $\Omega^{(1)}_{p,\gamma}(U_k)$ has the same homotopy type of $\mathcal{C}^0_{p,\gamma}(U_k)$, which concludes the proof. □

Appendix A. F **does not satisfy the Palais–Smale condition in** $\mathcal{B}^{(2)}_{p,\gamma}(k)$**.** Let's consider $\mathcal{M} = I\!R^3$ to be the flat 3-dimensional Minkowski spacetime, with metric $\langle\cdot,\cdot\rangle$ given by $\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$ and set $Y = \frac{\partial}{\partial z}$. Let $\langle\cdot,\cdot\rangle_o$ be the metric $\mathrm{d}x^2 + \mathrm{d}y^2$ in $I\!R^2$. We fix a point $p = (p_0, 0)$ in $\mathcal{M}$ and a curve $\gamma(r) = (p_1, r)$, where $x_0, x_1 \in I\!R^2$, and a real constant $k > -\langle Y, Y\rangle \equiv 1$. The set $\mathcal{B}^{(2)}_{p,\gamma}(k)$ consists of curves $\sigma(t) = (x(t), y(t), z(t))$ where $\mathbf{x}(t) = (x(t), y(t)) \in H^2([0,1], I\!R^2)$ is a curve that joins p_0 and p_1, $z \in H^2([0,1], I\!R)$, $z(0) = 0$, and there exists $\mathcal{T}_\sigma > 0$ such that $\dot z = k\,\mathcal{T}_\sigma$ and $\langle\dot{\mathbf{x}}, \dot{\mathbf{x}}\rangle_o - \dot z^2 = -{\mathcal{T}_\sigma}^2$, and so $\langle\dot{\mathbf{x}}, \dot{\mathbf{x}}\rangle_o = (k^2-1)\,{\mathcal{T}_\sigma}^2 > 0$ (constant). It is easy to see that the map $(\mathbf{x}, z) \mapsto \mathbf{x}$ gives a diffeomorphism of $\mathcal{B}^{(2)}_{p,\gamma}(k)$ and the Hilbert manifold:

$$\Omega^{(2)}_{c,p_0,p_1} = \left\{\mathbf{x} \in H^2([0,1], I\!R^2 : \mathbf{x}(0) = p_0,\ \mathbf{x}(1) = p_1,\ \langle\dot{\mathbf{x}}, \dot{\mathbf{x}}\rangle_o \equiv C_{\mathbf{x}} = \text{const.} > 0\right\};$$

moreover, the travel time functional T and the action functional F on $\mathcal{B}^{(2)}_{p,\gamma}(k)$ are transformed respectively into (constant multiples of) the Euclidean length functional L and the Euclidean energy functional E on $\Omega^{(2)}_{c,p_0,p_1}$: $L(\mathbf{x}) = \int_0^1 \langle\dot{\mathbf{x}}, \dot{\mathbf{x}}\rangle_o\,\mathrm{d}t$ and $E(\mathbf{x}) = \frac{1}{2}\int_0^1 \langle\dot{\mathbf{x}}, \dot{\mathbf{x}}\rangle_o^2\,\mathrm{d}t$. It is not hard to prove that the only critical point of L and E on $\Omega^{(2)}_{c,p_0,p_1}$ is the Euclidean geodesic, i.e., the straight segment, between p_0 and p_1 in $I\!R^2$. On the other hand, if $p_0 \neq p_1$, the manifold $\Omega^{(2)}_{c,p_0,p_1}$ is complete, and it is easy to see that its first homotopy group is infinite. Thus, if either L or E satisfied the Palais–Smale condition on $\Omega^{(2)}_{c,p_0,p_1}$, one would have infinitely many distinct geodesics between p_0 and p_1 in $I\!R^2$, which is clearly absurd. It follows that neither T nor F satisfies the Palais–Smale condition on $\mathcal{B}^{(2)}_{p,\gamma}(k)$. The same argument shows that neither T nor F satisfies the Palais–Smale condition in any set of curves satisfying a regularity stronger than C^1.

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Dipartimento di Matematica e Fisica,
Universitá di Camerino, Italy
E-mail address: giannoni@campus.unicam.it
URL: http://www2.unicam.it/~giannoni

Departamento de Matemática
Universidade de São Paulo, Brazil
piccione@ime.usp.br, tausk@ime.usp.br
http://www.ime.usp.br/~piccione, /~tausk