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NUCLEI, PARTICLES, FIELDS, GRAVITATION, AND ASTROPHYSICS

Spinor Field Singular Functions in QED with Strong External Backgrounds

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Abstract—We construct and study singular functions in strong-field *QED* with two external electromagnetic fields that represent principally different types of external backgrounds, the first one belongs to the class of so-called t-potential electric steps (electric-like fields that are switched on and off at initial and final time instants), and the second one belongs to the class of so-called x-potential electric steps (time-independent electric-like fields of constant direction that are concentrated in a restricted spatial area). As the first background (T-constant electric field) is an uniform electric field which acts during a finite time interval T, whereas as the second background (L-constant electric field) is a constant electric field confined between two capacitor plates separated by a large distance L. For the both cases we find in- and out-solutions of the Dirac equation in terms of light cone variables. With the help of these solutions, we construct Fock-Schwinger proper-time integral representations for all the singular functions that provide nonperturbative (with respect to the external backgrounds) calculations of any transition amplitudes and mean values of any physical quantities. Considering calculations in the T-constant field and in the L-constant field as different regularizations of the corresponding calculations in the constant uniform electric field, we have demonstrated their equivalence for sufficiently large T and L.

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1. INTRODUCTION

Quantum electrodynamics (QED) describes perfectly processes with interacting charge particles and photons. QED with external electromagnetic field is a convenient model for treating processes with a small number of such particles against a background created by a huge number of photons, the totality of which, in certain circumstances, can be described semiclassically [1] and appears in the model as the external field. Thus in the model the electromagnetic field manifests itself as the external classical field and photons which are described in a purely quantum way. Such a model is called usually strong-field QED (SFQED). The external field in SFQED cannot be treated perturbatively and has to be taken into account exactly, whereas for processes with charged particles and the photons one can construct a perturbation theory. In such a perturbation theory there appear zero-order processes without the photons and higher order processes with the photons. An essential and nontrivial part of the SFQED is related to the zero-order processes. A particle production from the vacuum by strong electric-like external fields (the Schwinger effect [2] that attracts attention already for a long time, or the vacuum instability) is, in fact, a manifestation of the latter processes. For time-dependent external electric-like fields that are switched on and off at initial and final time instants a perturbation theory with respect to radiative corrections and with exact taking into account the interaction with strong external background was elaborated in [3]. The mentioned external fields of constant direction are called t-electric potential steps (t-steps in what follows). This perturbation theory uses essentially special sets of exact solutions of the Dirac equation with the corresponding t-steps (when such solutions can be found and all the calculations can be done analytically, we refer to these examples as exactly solvable cases). It includes a technics for calculating zero-order processes, modified Feynman rules for calculating scattering amplitudes with charge particles and photons, and a perturbation theory for calculating mean values. For simplicity, effects of particle creation are usually considered in uniform time-dependent external electric fields.

Approaches for treating quantum effects in SFQED with t-steps are not directly applicable to the SFQED with time-independent electric-like external fields of constant direction that are concentrated in restricted spatial area), so-called x-electric potential steps (x-steps in what follows). In [4] a nonperturbative approach for calculating zero-order processes in SFQED with x-steps was constructed. The corresponding technique is based on using special sets of exact solutions of the Dirac equation with x-step. These solutions are stationary plane waves with given longitudinal momenta p^{L} and p^{R} in macroscopic regions on the left of a x-step and on the right of a x-step, respectively (see examples in [4–6]). By analogy with SFQED with t-steps, one can construct a perturbation theory for SFQED with critical x-steps with respect to radiative corrections and with exact taking into account the interaction with strong field of x-steps.

Spinor field singular function with the corresponding external fields (generalizing well-known singular functions in standard QED, see e.g. [7]) are key elements in constructions of the perturbation theories in SFQED both with t-steps and x-steps. In these theories, amplitudes of transition processes and mean values of physical quantities are expressed via the causal propagator (in-out propagator) $S^c(X, X')$, the so-called in-in propagator $S^c_{\mathrm{in}}(X, X')$ and out-out propagator $S^c_{\mathrm{out}}(X, X')$. In turn, they are connected with in and out means as follows:

$$\begin{aligned} &S^c(X,X') \\ &= i\langle 0,\mathrm{out}|\hat{T}\hat{\Psi}(X)\hat{\Psi}^\dagger(X')\gamma^0|0,\mathrm{in}\rangle/\langle 0,\mathrm{out}|0,\mathrm{in}\rangle, \\ &S^c_{\mathrm{in}}(X,X') = i\langle 0,\mathrm{in}|\hat{T}\hat{\Psi}(X)\hat{\Psi}^\dagger(X')\gamma^0|0,\mathrm{in}\rangle, \\ &S^c_{\mathrm{out}}(X,X') = i\langle 0,\mathrm{out}|\hat{T}\hat{\Psi}(X)\hat{\Psi}^\dagger(X')\gamma^0|0,\mathrm{out}\rangle. \end{aligned} \tag{1}$$

Here $\hat{\Psi}(X)$ is the Heisenberg field operator satisfying the Dirac equation with the corresponding external field; $X = (X^\mu) = (t, \mathbf{r})$, $t = X^0$, $\mathbf{r} = (X^k)$, $x = X^1$, $\mu = 0, 1, \ldots, D$, $k = 1, \ldots, D$; T denotes the chronological ordering operation, and $|0, \mathrm{in}\rangle$ and $|0, \mathrm{out}\rangle$ are initial and final vacua. We note that in spite of the fact that the formal representations (1) hold true in SFQED both with t-steps and x-steps, in- and out-solutions are constructed differently, as well as creation and annihilation operators and the corresponding vacua.

The Dirac equation with an external electromagnetic field given by the potential $A_\mu(X)$ in d-dimensional space-time has the form ($\hbar = c = 1$):

$$(\gamma^\mu P_\mu - m)\psi(X) = 0, \quad P_\mu = i\partial_\mu - qA_\mu(X),$$

where $\psi(X)$ are $2^{[d/2]}$-component spinors and γ^μ—are Dirac matrices,

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \mathrm{diag}\,\underbrace{(1,-1,\ldots,-1)}_{d},$$

$$d = D + 1.$$

$q = -e$, $e > 0$, is the electron charge and m its mass.

Note that in the case of the vacuum instability all the singular functions (1) are different. Differences between the functions $S^c_{\mathrm{in}}(X, X')$ and $S^c_{\mathrm{out}}(X, X')$ and the causal propagator $S^c(X, X')$ are denoted by $S^p(X, X')$ and $S^{\bar{p}}(X, X')$,

$$\begin{aligned} S^p(X,X') &= S^c_{\mathrm{in}}(X,X') - S^c(X,X'), \\ S^{\bar{p}}(X,X') &= S^c_{\mathrm{out}}(X,X') - S^c(X,X'). \end{aligned} \tag{2}$$

The commutation function

$$\begin{aligned} S(X,X') &= i[\hat{\Psi}(X), \hat{\Psi}^\dagger(X')\gamma^0]_+, \\ S(X,X')|_{t=t'} &= i\gamma^0\delta(\mathbf{r} - \mathbf{r}'), \end{aligned} \tag{3}$$

is an important characteristic of the Dirac field. The form of such a function depends significantly on the structure of the external field. In [3] are formulated rules for constructing all the necessary singular functions as sums over corresponding exact solutions of the Dirac equation with t-steps. One can follow the same ideas for constructing singular functions in SFQED with x-steps.

In the case of a constant uniform electromagnetic field the causal propagator of an electron $S^c(X, X')$ was found explicitly in the form of an integral over the Fock-Schwinger proper time many years ago [2]. This form is the basis of an effective action method; see [8] for a review. It is clear that a constant uniform electromagnetic field is an idealization which is useful for describing effects in slowly varying and weakly inhomogeneous fields. The case with a constant uniform electromagnetic field is seen as leading-order approximation of derivative expansion results from field-theoretic calculations [9, 10], that is, a locally constant field approximation (LCFA); e.g., see [11–17] and references therein.

A constant uniform electric field can be considered as the limit of a large duration of the T-constant electric field (a uniform electric field which acts during a time interval T) or as the limit of a large space scale of the L-constant electric field (a constant electric field confined between two capacitor plates separated by a distance L). SFQED in a T-constant electric field and L-constant electric field describes physically different problems. In this paper, we obtain an explicit form for all of the above singular functions and show that, in the limits $T, L \to \infty$, both approaches lead to the same results.

In this article we construct and study spinor singular functions in SFQED with T-constant electric field and in SFQED with L-constant electric field. To this end, in Section 2 we find in- and out-solutions of the

Dirac equation with T-constant electric field in terms of light cone variables. With the help of these solutions, we construct Fock-Schwinger proper-time integral representations for all the singular functions that provide nonperturbative (with respect to the external backgrounds) calculations of any transition amplitudes and mean values of any physical quantities. These representations are obtained for an arbitrary orientation of the external electric field, which nontrivially generalizes results of [18, 19]. In Section 3, we find appropriate sets of in- and out-solutions of the Dirac equation with L-constant electric field in terms of light cone variables. Using these sets, we construct Fock-Schwinger proper-time integral representations for the corresponding spinor singular functions. Obtained results are discussed in Discussion 4. Considering calculations in the T-constant field and in the L-constant field as different regularizations of the corresponding calculations in the constant uniform electric field, we have demonstrated their equivalence for sufficiently large T and L.

2. SINGULAR FUNCTIONS IN SFQED WITH T-CONSTANT ELECTRIC FIELD

2.1. In- and Out-Solutions

Here we consider the case of a t-step which is represented by T-constant electric field, which acts during a large time interval. The T-constant field is a possible regularization of a constant uniform electric field, lifted in the limit $T \to \infty$. For constructing spinor singular functions we need two complete sets of solutions to the Dirac equation, in-solutions $\{{}_\zeta\psi_n(x)\}$ and out-solutions $\{{}^\zeta\psi_n(x)\}$ with special asymptotics at $t \to -\infty$ and $t \to +\infty$ respectively. The subscript $\zeta = +$ corresponds to asymptotic electrons and $\zeta = -$ corresponds to asymptotic positrons. Since the explicit form of the sought solutions nontrivially depends on the orientation of the electric field relative to the x axis, we consider below the both possibilities separately.

We consider a constant electric field that has only one nonzero component E_x along the axis x. The field is given by a time-dependent electromagnetic potential $A_\mu(X)$,

$$A_\mu(X) = E_x t \delta_\mu^1, \quad E_x = \kappa E, \quad \kappa = \pm 1, \quad E > 0. \quad (4)$$

The case $\kappa = -1$ was considered in [18, 19]. We note that in the case $\kappa = +1$ the field direction coincides with the one used in the general formulation of SFQED with x-steps presented in [4].

Let us consider a complete set of solutions of the Dirac equation, having the following form:

$$\psi(X) = (\gamma P + m)\Phi(X), \quad X = (t, x, \mathbf{r}_\perp),$$
$$\Phi(X) = \phi(t,x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp)v_{\chi,\sigma},$$
$$\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp) = (2\pi)^{-(d-2)/2}\exp(i\mathbf{p}_\perp\mathbf{r}_\perp), \quad (5)$$
$$\mathbf{r}_\perp = (X^2, \dots, X^D), \quad \mathbf{p}_\perp = (p^2, \dots, p^D),$$
$$\gamma_\perp = (\gamma^2, \dots, \gamma^D),$$

where $v_{\chi,\sigma}$ with $\chi = \pm 1$ and $\sigma = (\chi_1, \chi_2, \dots, \chi_{[d/2]-1})$, $\sigma_j = \pm 1$, is a set of constant orthonormalized spinors satisfying the following equations:

$$\gamma^0\gamma^1 v_{\chi,\sigma} = \chi v_{\chi,\sigma}, \quad v^\dagger_{\chi,\sigma}v_{\chi',\sigma'} = \delta_{\chi,\chi'}\delta_{\sigma,\sigma'}.$$

In fact, functions (5) correspond to states with given momenta $\mathbf{p}_\perp$ in the perpendicular to the axis x direction. The quantum numbers χ and σ_j describe a spin polarization and provide a convenient parametrization of the solutions. Since in (1 + 1) and (2 + 1) dimensions ($d = 2, 3$) there are no any spin degrees of freedom, the quantum numbers a are absent. Note that in (2 + 1) dimensions, there are two nonequivalent representations for the γ matrices which correspond to different fermion species parametrized by $\chi = \pm 1$ respectively. In d dimensions, for any given momenta, there exist only $J_{(d)} = 2^{[d/2]-1}$ different spin states. Note that solutions (5), which differ only by values of χ are linearly dependent. Without loss of generality, we set $\chi = 1$ and introduce the notation $v_\sigma = v_{1,\sigma}$.

Scalar functions $\phi(t, x)$ satisfy the following equation:

$$\{\partial_t^2 - \partial_x^2 + 2ie\kappa Et\partial_x + [(eEt)^2 - ie\kappa + \mathbf{p}_\perp^2 + m^2]\}\phi(t,x) = 0. \quad (6)$$

We consider solutions of the Dirac equation with definite momenta,

$$\psi_n(X) = (\gamma P + m)\phi_n(t,x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp)v_\sigma,$$
$$\phi_n(t,x) = e^{ip_x x}\phi_n(t), \quad n = (p_x, \mathbf{p}_\perp, \sigma), \quad (7)$$
$$p_x = p^1, \quad \hat{p}_x\psi_n(X) = p_x\psi_n(X), \quad \hat{p}_x = -i\partial_x.$$

Asymptotics of solutions (7) for $\kappa = -1$ and $\kappa = +1$ at $t \to \pm\infty$ were studied in [19] and [20] respectively. Solutions

$$\begin{aligned} {}^{+}_{-}\phi_n(t,x) &= C_n e^{ip_x x} D_{-\rho}[\pm(1+i)\xi], \quad \kappa = +1, \\ {}^{-}_{+}\phi_n(t,x) &= \check{C}_n e^{ip_x x} D_{\rho-1}[\pm(1-i)\xi], \quad \kappa = +1, \\ {}^{-}_{+}\phi_n(t,x) &= C_n e^{ip_x x} D_{\rho-1}[\pm(1-i)\xi], \quad \kappa = -1, \\ {}^{+}_{-}\phi_n(t,x) &= \check{C}_n e^{ip_x x} D_{\rho-1}[\pm(1+i)\xi], \quad \kappa = -1, \end{aligned} \quad (8)$$

$$C_n = (4\pi eE)^{-1/2}e^{-\pi\lambda/8}, \quad \check{C}_n = (2\pi\lambda eE)^{-1/2}e^{-\pi\lambda/8},$$
$$\rho = \frac{i}{2}\lambda + \frac{\kappa+1}{2}, \quad \xi = \frac{eEt - \kappa p_x}{\sqrt{eE}}, \quad \lambda = \frac{\mathbf{p}_\perp^2 + m^2}{eE},$$

of Eq. (6) are used in constructing in-solutions and out-solutions, namely, the functions ${}_\zeta\phi_n(t, x)$ correspond to in-solutions $\{{}_\zeta\psi_n(X)\}$ whereas the functions

${}^{\zeta}\phi_n(t, x)$ correspond to out-solutions $\{{}^{\zeta}\psi_n(X)\}$. In Eqs. (8) the functions $D_\nu(x)$ are parabolic cylinder functions. The in- and out-solutions are orthonormal with respect to the standard inner product,

$$({}_{\zeta}\psi_n, {}_{\zeta}\psi_{n'}) = ({}^{\zeta}\psi_n, {}^{\zeta}\psi_{n'}) = \delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)\delta(p_x - p'_x)\delta_{\sigma,\sigma'},$$

$$(\psi, \psi') = \int \psi^\dagger(x)\psi'(x)d\mathbf{r}, \quad d\mathbf{r} = dx^1 \cdots dx^D.$$

It is also convenient to work with solutions of Eq. (6) that depend on the light-cone coordinates $x_\pm = t \pm x$. The solutions are parametrized by a set $n_- = (p_-, \mathbf{p}_\perp, \sigma)$ of quantum numbers and have the following form:

$$\begin{gathered}{}^{+\kappa}_{-\kappa}\phi_{n-}(t, x) = C_{n_-} \exp\left\{-ie\frac{\kappa E}{2}\left(\frac{1}{2}x_-^2 - x^2\right) - \frac{i}{2}p_- x_+ \right. \\ \left. + \frac{i}{2}[\kappa\lambda - 2i]\ln\left[\frac{\mp\pi_-}{\sqrt{eE}}\exp\left(-\frac{i\pi}{2}\theta(\kappa)\right)\right]\right\}, \\ C_{n_-} = (4\pi eE)^{-1/2} \\ \times\exp\left\{\frac{i}{4}[(2\lambda\log 2 + \pi)\kappa + \pi(1 + i\lambda)]\right\}, \\ \lambda = \frac{\mathbf{p}_\perp^2 + m^2}{eE}, \quad \pi_- = p_- + e\kappa E x_-. \end{gathered} \tag{9}$$

For the convenience, we have introduced here the notations:

$${}^{+\kappa}_{-\kappa}\phi = \begin{cases} {}^{+}_{-}\phi, & \kappa = +1 \\ {}^{-}_{+}\phi, & \kappa = -1, \end{cases}$$

where ${}^{\zeta}\phi$ and ${}_{\zeta}\phi$ are different sets of functions. Here p_- is momentum of the continuous spectrum and the eigenvalue of the operator $2i(\partial/\partial x_+)$. The signs $\pm\kappa$ assigned to the functions ${}^{+\kappa}_{-\kappa}\phi_{n_-}(t, x)$ are matched with those of the kinetic momentum π_- at $x_- \to \pm\infty$. In what follows we show that these states have needed for our constructions special asymptotics as $t \to \pm\infty$.

The sets of solutions (9) and (8) are related by an integral transformation

$$\begin{gathered}{}^{+\kappa}_{-\kappa}\phi_n(t, x) = (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty} M^*(p_x, p_-){}^{+\kappa}_{-\kappa}\phi_{n_-}(t, x)dp_-, \\ M(p_x, p_-) = \exp\left\{-\frac{i\kappa}{4eE}[(p_- + 2p_x)^2 - 2(p_x)^2]\right\}. \end{gathered} \tag{10}$$

The back transformation reads:

$${}^{+\kappa}_{-\kappa}\phi_{n_-}(t, x) = (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty} M(p_x, p_-){}^{+\kappa}_{-\kappa}\phi_n(t, x)dp_x. \tag{11}$$

We note that in the case of a constant uniform electric field transformations (10) and (11) were considered in [21].

As was mentioned above, the functions ${}^{+\kappa}\phi_n(t, x)$ correspond to out-solutions ${}^{+\kappa}\psi_n(X)$, whereas the functions ${}_{-\kappa}\phi_n(t, x)$ correspond to in-solutions ${}_{-\kappa}\psi_n(X)$. Transformations (11) allow one to construct in- and out-solutions ${}_{-\kappa}\psi_{n_-}(X)$ and ${}^{+\kappa}\psi_{n_-}(X)$ respectively with quantum numbers n_-:

$${}^{+\kappa}_{-\kappa}\psi_{n_-}(X) = (\gamma P + m){}^{+\kappa}_{-\kappa}\phi_{n_-}(t, x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp)v_\sigma.$$

At this stage, we introduce different sets of solutions:

$$\begin{gathered} \kappa = +1: \\ \begin{cases} {}_{+}\phi_{n_-}(t, x) = \theta(+\pi_-){}^{+}\phi_{n_-}(t, x)g({}^{+}|_{+}) \\ {}^{-}\phi_{n_-}(t, x) = \theta(-\pi_-){}_{-}\phi_{n_-}(t, x)g({}_{-}|^{-}), \end{cases} \\ \kappa = -1: \\ \begin{cases} {}^{+}\phi_{n_-}(t, x) = \theta(+\pi_-){}_{+}\phi_{n_-}(t, x)g({}_{+}|^{+}) \\ {}_{-}\phi_{n_-}(t, x) = \theta(-\pi_-){}^{-}\phi_{n_-}(t, x)g({}^{-}|_{-}), \end{cases} \end{gathered} \tag{12}$$

where $\theta(x)$ is the Heaviside step function. One can verify that the following relation holds true:

$$(2\pi eE)^{-1/2}\int_{-\infty}^{+\infty} M^*(p_x, p_-){}^{-\kappa}_{+\kappa}\phi_{n_-}(t, x)dp_- = {}^{-\kappa}_{+\kappa}\phi_n(t, x).$$

The functions ${}^{-\kappa}\phi_n(t, x)$ correspond to out-solutions ${}^{-\kappa}\psi_n(X)$, whereas the functions ${}_{+\kappa}\phi_n(t, x)$ correspond to in-solutions ${}_{+\kappa}\psi_n(X)$. Transformations (11) allow one to construct in and out-solutions ${}_{+\kappa}\psi_{n_-}(X)$ and ${}^{-\kappa}\psi_{n_-}(X)$ respectively with quantum numbers n_-:

$${}^{-\kappa}_{+\kappa}\psi_{n_-}(X) = (\gamma P + m){}^{-\kappa}_{+\kappa}\phi_{n_-}(t, x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp)v_\sigma.$$

Thus, we have constructed in- and out-solutions for two different directions of the electric field $\kappa = \pm 1$. There are exist mutual decompositions of these solutions:

$${}^{\zeta}\psi_{n_-}(X) = {}_{+}\psi_{n_-}(X)g({}_{+}|^{\zeta}) + {}_{-}\psi_{n_-}(X)g({}_{-}|^{\zeta}),$$

$${}_{\zeta}\psi_{n_-}(X) = {}^{+}\psi_{n_-}(X)g({}^{+}|_{\zeta}) + {}^{-}\psi_{n_-}(X)g({}^{-}|_{\zeta}).$$

Where coefficients g are:

$$\begin{gathered} g({}^{\zeta}|_{\zeta}) = \frac{\sqrt{\pi\lambda}\exp(-\pi\lambda/4)}{\Gamma(1 - i\zeta\lambda/2)}, \\ g({}^{-}|_{+}) = -g({}^{+}|_{-}) = \kappa\exp(-\pi\lambda/2). \end{gathered} \tag{13}$$

2.2. Proper-Time Representations

Using representations of singular functions as certain sums of solutions of the Dirac equation constructed in [3, 19], one can find their proper-time representations. Thus, proper-time representations for the singular functions $S^p(X, X')$ and $S^{\bar{p}}(X, X')$ follows from expressions:

$$
\begin{aligned}
&S^{p}(X,X') \\
&= i\int_{-\infty}^{\infty} dp_- \int_{\mathbb{R}^{d-2}} d\mathbf{p}_\perp \sum_{\sigma=\pm1} {}_{-}\psi_{n_-}(X)[g({}_{+}|^{-})g({}_{-}|^{-})^{-1}]^{\dagger}{}_{+}\bar{\psi}_{n_-}(X'), \\
&S^{\bar{p}}(X,X') \\
&= -i\int_{-\infty}^{\infty} dp_- \int_{\mathbb{R}^{d-2}} d\mathbf{p}_\perp \sum_{\sigma=\pm1} {}^{+}\psi_{n_-}(X)[g({}_{+}|^{+})^{-1}g({}_{+}|^{-})]^{-}\bar{\psi}_{n_-}(X').
\end{aligned}
\tag{14}
$$

Note that the singular functions (14) were denoted differently in [19], namely: as $-S^a(X, X')$ and $-S^p(X, X')$ respectively. Taking into account Eqs. (13) and (12), we obtain:

$$
\begin{aligned}
&S^{p}(X,X') = -\kappa\int_{-\infty}^{\infty} dp_- \theta(+\kappa\pi'_-)\,{}_{+}Y(X,X';p_-), \\
&S^{\bar{p}}(X,X') = +\kappa\int_{-\infty}^{\infty} dp_- \theta(-\kappa\pi'_-)\,{}^{+\kappa}Y(X,X';p_-), \\
&{}_{+}Y(X,X';p_-) \\
&= i\int_{\mathbb{R}^{d-2}} d\mathbf{p}_\perp \sum_{\sigma} {}_{+}\psi_{n_-}(X)\,{}_{+}\bar{\psi}_{n_-}(X'), \\
&{}^{+\kappa}Y(X,X';p_-) \\
&= i\int_{\mathbb{R}^{d-2}} d\mathbf{p}_\perp \sum_{\sigma} {}^{+\kappa}\psi_{n_-}(X)\,{}^{+\kappa}\bar{\psi}_{n_-}(X').
\end{aligned}
\tag{15}
$$

According to [19], the causal propagator and the commutation function can be represented as:

$$
\begin{aligned}
&S^{c}(X,X') = (\gamma P + m)\Delta^{c}(X,X'), \\
&\Delta^{c}(X,X') = \int_{\Gamma_c} f(X,X';s)ds,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&S(X,X') = (\gamma P + m)\Delta(X,X'), \\
&\Delta(X,X') = \mathrm{sgn}(t-t')\int_{\Gamma} f(X,X';s)ds,
\end{aligned}
\tag{17}
$$

where $\mathrm{sgn}(t-t') = \theta(t-t') - \theta(t'-t)$, and the function

$$
\begin{aligned}
&f(X,X';s) = \exp(-e\kappa E\gamma^0\gamma^1 s)f^{(0)}(X,X';s), \\
&f^{(0)}(X,X';s) = -\left(\frac{-i}{4\pi}\right)^{d/2}\frac{eE}{s^{(d-2)/2}\sinh(eEs)} \\
&\times\exp\left[-ism^2 + ie\Lambda + \frac{i}{4s}|\mathbf{r}_\perp - \mathbf{r}'_\perp| \right. \\
&\left. -\frac{i}{4}eE\coth(eEs)(y_0^2 - y_1^2)\right]
\end{aligned}
\tag{18}
$$

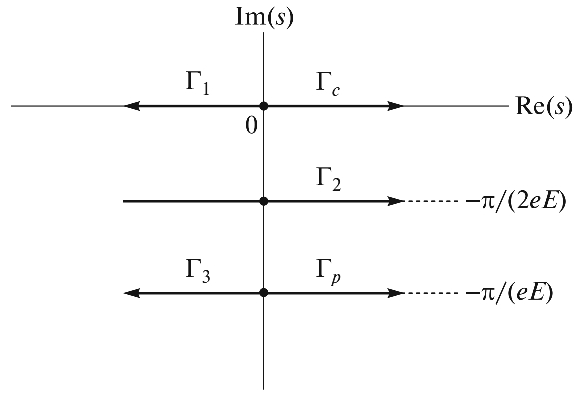

Fig. 1. Contours of integration Γ_1, Γ_2, Γ_3, Γ_c, Γ_p.

is the Fock-Schwinger kernel [2, 22]. Here and in what follows we use the four-vector $y_\mu = X_\mu - X'_\mu$, $y_0 = t - t'$, $y_1 = x' - x$.

The function $f(X, X'; s)$ satisfies the following differential equation and the initial conditions:

$$
\begin{aligned}
&-i\frac{d}{ds}f(X,X';s) = (P^2 - m^2 + ieE\gamma^0\gamma^1)f(X,X';s), \\
&\lim_{s\to\pm0} f(X,X';s) = \pm i\delta(X - X').
\end{aligned}
$$

Note that only term Λ in Eq. (18) is a gauge dependent quantity which can be represented as an integral along a line the points X and X',

$$
\Lambda = -\int_{X'}^{X} A_\mu(\tilde{X})d\tilde{X}^\mu. \tag{19}
$$

In the gauge under consideration, the electromagnetic potential $A_\mu(X)$ is given by Eq. (4), such that we have $\Lambda = \kappa E y_1(t + t')/2$.

The integration contours of the above integrals are shown on Figs. 1 and 2. The contours Γ_c and Γ_1 are placed just below the real axis everywhere outside of the origin. The function $f^{(0)}(X, X'; s)$ has two singular points, on the complex plane between the contours $\Gamma_c - \Gamma_1$ and $\Gamma_p - \Gamma_3$. Namely, they are situated at the imaginary axis: $s_0 = 0$ and $eEs_1 = -i\pi$.

The contour Γ connects the points $s = +0$ and $s = e^{-i\pi}0$, it is situated in the lower part of the complex plane s in a small enough neighborhood of the point $s = 0$. Note that the integral over the contour Γ in Eq. (17) can be related to an integral over a contour $\Gamma_c - \Gamma_2 - \Gamma_1$. It can be seen that the kernel $f(X, X'; s)$ has no other peculiarities in a sufficiently small neighborhood of the point $s = 0$. Taking this into account and closing the integration contour $\Gamma_c - \Gamma_2 - \Gamma_1$ as $\mathrm{Re}s \to \pm\infty$, one can transform it into the contour Γ.

We note that representation (16) has the Schwinger form [2]. Representation (17) has an universal structure inherent to the proper-time representation for the

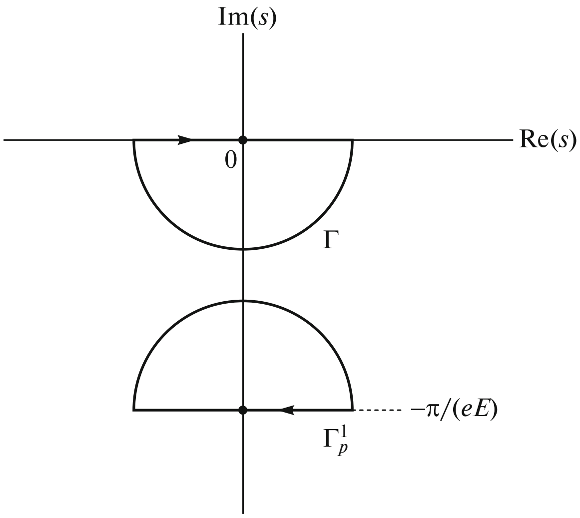

Fig. 2. Contours of integration Γ, Γ_p^1.

commutation function (see [18]). This implies that integral (17) satisfies the Dirac equation and the standard equal-time initial condition $S(X, X')|_{t=t'} = i\gamma^0\delta(\mathbf{r} - \mathbf{r}')$. In turn this proves the completeness of both sets $\{_\pm\psi_{n\_}(x)\}$ and $\{^\pm\psi_{n\_}(x)\}$ on the t = const hyperplane.

Following the procedure presented in [19] and taking into account that $\mathbf{Ey} = \kappa E y^1$, we can represent singular functions (15) as proper-time integrals:

$$\begin{aligned} &S^p(X,X') = (\gamma P + m)\Delta^p(X,X'),\\ &S^{\bar{p}}(X,X') = (\gamma P + m)\Delta^{\bar{p}}(X,X'),\\ &-\Delta^p(X,X') = \int_{\Gamma_p} f(X,X';s)ds + \theta(\mathbf{Ey})\int_{\Gamma_p^1} f(X,X';s)ds\\ &-\Delta^{\bar{p}}(X,X') = \int_{\Gamma_p} f(X,X';s)ds + \theta(-\mathbf{Ey})\int_{\Gamma_p^1} f(X,X';s)ds. \end{aligned} \tag{20}$$

Here the integration contour Γ_p^1 (with its radius tending to zero) connects the points $s = e^{-i\pi}0 - i\pi/(eE)$ and $s = +0 - i\pi/(eE)$. Note that the integral over the contour Γ_p^1 in Eq. (20) can be related to an integral over the contour $\Gamma_2 + \Gamma_3 - \Gamma_p$. Closing the integration contour $\Gamma_2 + \Gamma_3 - \Gamma_p$ as $\mathrm{Re}\,s \to \pm\infty$, one can transform it into the contour Γ_p^1.

Using Eq. (20), one obtains proper-time representations for singular functions $S^c_{\text{in/out}}(X, X')$,

$$S^c_{\text{in/out}}(X,X') = S^{p/\bar{p}}(X,X') + S^c(X,X'). \tag{21}$$

Representations (16), (17), and (20) can be easily written in a covariant form by using the field tensor $F_{\mu\nu}$ (see, e.g., [18, 19]). For example, in the case $d = 4$, one obtains:

$$\begin{aligned} &f(X,X';s) = \exp\left(-\frac{e}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}s\right)f^{(0)}(X,X';s),\\ &f^{(0)}(X,X';s) = \frac{e^2EB\exp(-ie\Lambda')}{(4\pi)^2\sinh(eEs)\sin(eBs)}\\ &\times\exp\left[-im^2s - i\frac{1}{4}yqF\coth(qFs)y\right],\\ &\Lambda' = -\int_{X'}^{X}(A^E_\mu(\tilde{E}) + A^B_\mu(\tilde{X}))d\tilde{X}^\mu, \end{aligned} \tag{22}$$

where E and B are electric and magnetic eigenvalues of the field tensor $F_{\mu\nu}$, $A^E_\mu + A^B_\mu$ are potentials of electric (E) and magnetic (B) components, respectively, and the integral is taken along the line.

As it follows from Eqs. (18) and (20), for the functions $S^{p,\bar{p}}(X, X')$ (and for $S^c_{\text{in/out}}(X, X')$ as well), the change $E_x = E \to E_x = -E$ is equivalent to the change $x \to -x$, $x' \to -x'$, and $\gamma^1 \to -\gamma^1$. Wherein the projection $\mathbf{Ey}/E = (E_x/E)y^1$ of the displacement vector $\mathbf{y} = (y^2, \ldots, y^D)$ on the field direction and the function $f(X, X'; s)$ do not change, this implies that the mean current of created particles remains directed along the electric field.

Note that the integration contours in the proper-time representations of the causal propagator and the commutation function are insensitive to the direction of the electric field. However, the integration contours in the proper-time representations for singular functions $S^{p,\bar{p}}(X, X')$ do depend on the projection $\mathbf{Ey}$. It is natural since namely these singular functions determine the influence of the external electric field on electric currents of created particles. Such an observation was not possible to extract from representations obtained in [18, 19] for particular choice of the coordinate system.

3. SINGULAR FUNCTIONS IN SFQED WITH L-CONSTANT ELECTRIC FIELD

3.1. In- and Out-Solutions

Here we construct spinor singular functions in SFQED with L-constant electric field. The field has only one nonzero component E_x along the axis x,

$$E_x(x) = \begin{cases} 0, & x \in (-\infty, -L/2] \cup [L/2, \infty) \\ E, & x \in S_{\text{int}} = (-L/2, L/2), \quad L > 0. \end{cases}$$

We assume that corresponding potential step is sufficiently large, $eEL \gg 2m$ (i.e., it is critical). In this case the field $E_x(x)$ and leading contributions to vacuum mean values can be considered as macroscopic physical quantities. In this sense the L-constant electric field is weakly inhomogeneous and characteristics of the vacuum instability have some universal features (see [15]). This effect of particle creation is due to the extensive Klein zone. We stress that in the limit $L \to \infty$ the L-constant electric field is one of a possible regularization for a constant uniform electric field.

Some characteristics of the vacuum instability, in particular, deformations of spinor singular functions in SFQED with L-constant electric field for large L, can be approximately calculated in SFQED with a constant uniform electric field. The latter problem itself is important and its solution in the framework of the consistent QFT formulation [4] is given below, see also results obtained in [5] for the L-constant electric field.

We chose the following electromagnetic potentials to describe the constant uniform electric field E directed along the axis x:

$$A_0(X) = -Ex, \quad A_k(X) = 0. \tag{23}$$

Let us consider a complete set of stationary solutions of the Dirac equation with electromagnetic field (23), having the following form:

$$\begin{gathered}\psi_{n_0}(X) = (\gamma P + m)\Phi_{n_0}(X),\\ \Phi_{n_0}(X) = \varphi_{n_0}(t,x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp)v_{\chi,\sigma}, \quad n_0 = (p_0, \mathbf{p}_\perp, \sigma),\\ \varphi_{n_0}(t,x) = (2\pi)^{-1/2}\exp(-ip_0 t)\varphi_{n_0}(x),\end{gathered} \tag{24}$$

where $\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp)$ and $v_{\chi,\sigma}$ are given by Eq. (5). Using reasons presented in Section 2, we choose $\chi = 1$. The scalar functions $\varphi_{n_0}(x)$ obey the second-order differential equation:

$$\begin{gathered}\{\hat{p}_x^2 - iU'(x) - [p_0 - U(x)]^2\\ + \mathbf{p}_\perp^2 + m^2\}\varphi_{n_0}(x) = 0, \quad U(x) = -eA_0(x).\end{gathered} \tag{25}$$

Solutions of the Dirac equation with well-defined left and right asymptotics we denote as ${}_\zeta\psi_{n_0}(X)$ and ${}^\zeta\psi_{n_0}(X)$,

$$\hat{p}_x\, {}_\zeta\psi_{n_0}(X) = p^{\mathrm{L}}\, {}_\zeta\psi_{n_0}(X), \quad x \to -\infty, \quad \zeta = \mathrm{sgn}(p^{\mathrm{L}}),$$

$$\hat{p}_x\, {}^\zeta\psi_{n_0}(X) = p^{\mathrm{R}}\, {}^\zeta\psi_{n_0}(X), \quad x \to +\infty, \quad \zeta = \mathrm{sgn}(p^{\mathrm{R}}).$$

The solutions ${}_\zeta\psi_{n_0}(X)$ and ${}^\zeta\psi_{n_0}(X)$ describe asymptotically particles with given momenta p^{L} as $x \to -\infty$ and p^{R} as $x \to +\infty$ respectively. One can see that the solutions ${}_\zeta\psi_{n_0}(X)$ and ${}^\zeta\psi_{n_0}(X)$ have form (24) with functions $\varphi_{n_0}(x)$ denoted here as ${}_\zeta\varphi_{n_0}(x)$ or ${}^\zeta\varphi_{n_0}(x)$ respectively. The latter functions have the following asymptotics:

$${}_\zeta\varphi_{n_0}(x) = {}_\zeta C\exp[ip^{\mathrm{L}}x], \quad x \to -\infty,$$

$${}^\zeta\varphi_{n_0}(x) = {}^\zeta C\exp[ip^{\mathrm{R}}x], \quad x \to +\infty.$$

Here ${}_\zeta C$ and ${}^\zeta C$ are normalization constants.

The solutions ${}_\zeta\psi_{n_0}(X)$ and ${}^\zeta\psi_{n_0}(X)$ satisfy the following orthonormality relations on $x = $ const hyperplane:

$$\begin{gathered}({}_\zeta\psi_{n_0}, {}_{\zeta'}\psi_{n_0'})_x = \zeta\delta_{\zeta,\zeta'}\delta_{n_0,n_0'},\\ ({}^\zeta\psi_{n_0}, {}^{\zeta'}\psi_{n_0'})_x = -\zeta\delta_{\zeta,\zeta'}\delta_{n_0,n_0'},\\ (\psi,\psi')_x = \int \psi^\dagger(X)\gamma^0\gamma^1\psi'(X)dtd\mathbf{r}_\perp,\\ \delta_{n_0,n_0'} = \sigma_{\sigma,\sigma'}\delta(p_0 - p_0')\delta(\mathbf{p}_\perp - \mathbf{p}_\perp').\end{gathered} \tag{26}$$

Their mutual decompositions have the form:

$$\begin{gathered}{}^\zeta\psi_{n_0}(X) = {}_+\psi_{n_0}(X)\tilde{g}({}_+|^\zeta) - {}_-\psi_{n_0}(X)\tilde{g}({}_-|^\zeta),\\ {}_\zeta\psi_{n_0}(X) = {}^-\psi_{n_0}(X)\tilde{g}({}^-|_\zeta) - {}^+\psi_{n_0}(X)\tilde{g}({}^+|_\zeta),\end{gathered} \tag{27}$$

where expansion coefficients are defined from the relations:

$$\begin{gathered}({}_\zeta\psi_{n_0}, {}^{\zeta'}\psi_{n_0'}(X))_x = \tilde{g}({}_\zeta|^{\zeta'})\delta_{n_0,n_0'},\\ \tilde{g}({}^{\zeta'}|_\zeta) = \tilde{g}({}_\zeta|^{\zeta'})^*.\end{gathered}$$

We note that coefficients $\tilde{g}$ differ from the g that appear in SFQED with t-steps; see Section 2. The coefficients $\tilde{g}$ satisfy the following unitary relations:

$$\begin{gathered}|\tilde{g}({}_-|^+)|^2 = |\tilde{g}({}_+|^-)|^2,\\ |\tilde{g}({}_+|^+)|^2 = |\tilde{g}({}_-|^-)|^2,\\ \frac{\tilde{g}({}_+|^-)}{\tilde{g}({}_-|^-)} = \frac{\tilde{g}({}^+|_-)}{\tilde{g}({}^+|_+)},\\ |\tilde{g}({}_+|^-)|^2 - |\tilde{g}({}_+|^+)|^2 = 1.\end{gathered} \tag{28}$$

Now we return to solving Eq. (25). It can be rewritten as follows:

$$\begin{gathered}\left[\frac{d^2}{d\xi^2} + \xi^2 + i - \lambda\right]\varphi_{n_0}(x) = 0,\\ \xi = \frac{eEx - p_0}{\sqrt{eE}}, \quad \lambda = \frac{\mathbf{p}_\perp^2 + m^2}{eE}.\end{gathered} \tag{29}$$

The general solution of Eq. (29) is completely determined by an appropriate pair of linearly independent Weber parabolic cylinder functions (WPCFs), either $D_\rho[(1 - i)\xi]$ and $D_{-1-\rho}[(1 + i)\xi]$ or $D_\rho[-(1 - i)\xi]$ and $D_{-1-\rho}[-(1 + i)\xi]$, where $\rho = -i\lambda/2 - 1$.

Using asymptotic expansions of WPCFs, one can classify by signs of the momenta $p^{\rm L}$ and $p^{\rm R}$. As a result, we obtain four sets of solutions of Eq. (29)

$$
\begin{aligned}
{}_{+}\varphi_{n_0}(x) &= {}_{+}CD_{-1-\rho}[-(1+i)\xi] \sim e^{-i\xi^2/2}, \\
&\qquad \xi \to -\infty, \\
{}_{-}\varphi_{n_0}(x) &= {}_{-}CD_{\rho}[-(1-i)\xi] \sim e^{i\xi^2/2}, \\
&\qquad \xi \to -\infty,
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
{}^{+}\varphi_{n_0}(x) &= {}^{+}CD_{\rho}[(1-i)\xi] \sim e^{i\xi^2/2}, \\
&\qquad \xi \to \infty, \\
{}^{-}\varphi_{n_0}(x) &= {}^{-}CD_{-1-\rho}[(1+i)\xi] \sim e^{-i\xi^2/2}, \\
&\qquad \xi \to \infty,
\end{aligned}
\tag{31}
$$

$$
{}^{-\zeta}C = {}_{\zeta}C = (eE)^{-1/2} e^{\pi\lambda/8} \left[\frac{\lambda}{2}(1+\zeta) + 1 - \zeta\right]^{-1/2}.
$$

This allows us to construct the corresponding Dirac spinors that are in- and out-solutions,

$$
\begin{aligned}
&\text{in-solutions: } {}_{-}\psi_{n_0}(X), \quad {}^{-}\psi_{n_0}(X), \\
&\text{out-solutions: } {}_{+}\psi_{n_0}(X), \quad {}^{+}\psi_{n_0}(X).
\end{aligned}
\tag{32}
$$

The coefficients $\tilde{g}$ have the form:

$$
\tilde{g}({}_{-}|^{+}) = \tilde{g}({}_{+}|^{-}) = e^{\pi\lambda/2}. \tag{33}
$$

According to rules of the general formulation (see [4]), differential mean numbers of created pairs have the form:

$$
N_{n_0}^{\rm cr} = |\tilde{g}({}_{+}|^{-})|^{-2} = e^{-\pi\lambda}.
$$

In the same manner, that was used in Section 2 to construct the spinor singular functions (1) and (2) we will construct complete sets of solutions of the Dirac equation with well-defined left and right asymptotics in terms of light-cone variables $x_{\pm} = t \pm x$. To this aim, we consider Dirac spinors $\tilde{\psi}_{n_-}(X)$ that are parametrized by quantum numbers $n_- = (p_-, p_\perp, \sigma)$. The spinors represented as:

$$
\begin{aligned}
\tilde{\psi}_{n_-}(X) &= (\gamma P + m)\Phi_{n_-}(X), \\
\Phi_{n_-}(X) &= \varphi_{n_-}(t,x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp)v_\sigma,
\end{aligned}
\tag{34}
$$

where functions $\varphi_{n_-}(t, x)$ satisfy the following equation:

$$
\begin{aligned}
&\{\hat{p}_x^2 - iU'(x) - [\hat{p}_0 - U(x)]^2 \\
&\quad + \mathbf{p}_\perp^2 + m^2\}\varphi_{n_-}(t,x) = 0.
\end{aligned}
\tag{35}
$$

Now we construct nonstationary solutions of Eq. (35). We note that this equation admits integrals of motion $\hat{Y}_\alpha$, $\alpha = 0, 1, 2, 3$ that are linear differential operators of the first order,

$$
\hat{Y}_0 = ie, \quad \hat{Y}_1 = \partial_t, \quad \hat{Y}_2 = \partial_x + ieEt,
$$

$$
\hat{Y}_3 = x\partial_t + t\partial_x + \frac{ieE}{2}(t^2 + x^2).
$$

The operators $\hat{Y}_\alpha$ form a four-dimensional Lie algebra $\mathscr{L}$ with the following nonzero commutation relations:

$$
[\hat{Y}_1, \hat{Y}_2] = E\hat{Y}_0, \quad [\hat{Y}_1, \hat{Y}_3] = \hat{Y}_2, \quad [\hat{Y}_2, \hat{Y}_3] = \hat{Y}_1.
$$

Equation (35) can be considered as an equation for the eigenfunctions of the Casimir operator $\hat{K} = 2E\hat{Y}_0\hat{Y}_3 - \hat{Y}_1^2 + \hat{Y}_2^2$,

$$
\hat{K}\varphi_{n_-}(t,x) = (\mathbf{p}_\perp^2 + m^2)\varphi_{n_-}(t,x), \quad [\hat{K}, \hat{Y}_a] = 0.
$$

This fact allows us to use a non-commutative integration method [23–25] to construct complete sets of solutions based on the symmetry of the equation. Namely, we define an irreducible λ-representation of Lie algebra in the space of functions of the variable $p_- \in (-\infty, +\infty)$ by the help of the operators $\ell_a(p_-, \partial_{p_-}, j)$, $\alpha = 0, 1, 2, 3; j \in (0, \infty)$,

$$
\ell_0(p_-, \partial_{p_-}, j) = ie,
$$

$$
\ell_1(p_-, \partial_{p_-}, j) = -eE\partial_{p_-} + \frac{i}{2}p_-,
$$

$$
\ell_2(p_-, \partial_{p_-}, j) = eE\partial_{p_-} + \frac{i}{2}p_-,
$$

$$
\ell_3(p_-, \partial_{p_-}, j) = -p_-\partial_{p_-} + ij - 1,
$$

$$
\begin{aligned}
&\ell_1^2(p_-, \partial_{p_-}, j) - \ell_2^2(p_-, \partial_{p_-}, j) \\
&- 2E\ell_0(p_-, \partial_{p_-}, j)\ell_3(p_-, \partial_{p_-}, j) = (2eE)j.
\end{aligned}
$$

Integrating the set of equations

$$
[\hat{Y}_a + \ell_a(p_-, \partial_{p_-}, j)]\varphi_{n_-}(t,x) = 0
$$

together with Eq. (35), we obtain the algebraic equation $j = -\lambda/2$ and two complete sets ${}_{-}\varphi_{n_-}$ and ${}^{+}\varphi_{n_-}$ of solutions of the latter equation. These solutions are parametrized by a set of quantum numbers n_-:

$$
\begin{aligned}
{}^{+}_{-}\varphi_{n_-}(t,x) &= C'\exp\left[ie\frac{E}{2}\left(\frac{1}{2}x_-^2 - t^2\right)\right. \\
&\quad \left. - \frac{i}{2}(\lambda - 2i)\ln\frac{\pm i\pi_-}{\sqrt{eE}} - \frac{i}{2}p_-x_+\right], \\
\pi_- &= p_- + eEx_-.
\end{aligned}
\tag{36}
$$

Then the quantum number p_- is an eigenvalue of the symmetry operator $i(\hat{Y}_1 + \hat{Y}_2)$:

$$
i(\hat{Y}_1 + \hat{Y}_2){}^{+}_{-}\varphi_{n_-}(t,x) = p_-{}^{+}_{-}\varphi_{n_-}(t,x).
$$

In this case solutions of the Dirac equation have form (34) with functions (36),

$$
\begin{aligned}
{}^{+}_{-}\tilde{\psi}_{n_-}(X) &= (\gamma P + m){}^{+}_{-}\Phi_{n_-}(X), \\
{}^{+}_{-}\Phi_{n_-}(X) &= {}^{+}_{-}\varphi_{n_-}(t,x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp)v_\sigma.
\end{aligned}
\tag{37}
$$

It is useful to construct a direct and inverse integral transformation that relate functions (36) to functions (30) and (31). To this end we look for solutions of Eq. (25) in the form

$$ {}_{-}^{+}\varphi_{n_0}(t,x) = (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty}\tilde{M}^*(p_0,p_-)\,{}_{-}^{+}\varphi_{n_-}(t,x)dp_- \quad (38) $$

that also satisfy the equation

$$ \hat{p}_0\,{}_{-}^{+}\varphi_{n_0}(t,x) = p_0\,{}_{-}^{+}\varphi_{n_0}(t,x). \quad (39) $$

Substituting (38) into (39) with account taken of the condition

$$ \left[\partial_t - eE\partial_{p_-} + \frac{i}{2}p_-\right]\varphi_{n_-}(t,x) = 0, $$

we see that the function $\tilde{M}(p_0, p_-)$ must be a solution of the following equation:

$$ -i\left(-eE\partial_{p_-} + \frac{i}{2}p_-\right)\tilde{M}(p_0,p_-) = p_0\tilde{M}(p_0,p_-). $$

We choose its particular solution

$$ \tilde{M}(p_0,p_-) = \exp\frac{i}{4eE}(p_-^2 - 4p_-p_0 + 2p_0^2), $$

which satisfies the orthogonality relation

$$ \int_{-\infty}^{+\infty}\tilde{M}^*(p_0,p_-)\tilde{M}(p_0,p'_-)dp_0 = 2\pi eE(p_- - p'_-). \quad (40) $$

The inverse transformation reads:

$$ \begin{aligned} &{}_{-}^{+}\varphi_{n_-}(t,x) \\ &= (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty}\tilde{M}(p_0,p_-)\,{}_{-}^{+}\varphi_{n_0}(t,x)dp_0. \end{aligned} \quad (41) $$

To make transformation (38) and (41) consistent, we have to

$$ C' = 2^{i\lambda/4}e^{\pi\lambda/4}(4\pi eE)^{-1/2} $$

in Eqs. (30), (31).

The inverse transformation (41) shows that the functions ${}_{-}^{+}\varphi_{n_-}(t, x)$ are expressed via functions ${}_{-}^{+}\varphi_{n_0}(t, x)$ with well defined left and right asymptotics.

The transformations (38) and (41) imply similar relations for solutions of the Dirac equation,

$$ \begin{aligned} {}_{-}^{+}\psi_{n_0}(X) &= (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty}\tilde{M}^*(p_0,p_-)\,{}_{-}^{+}\tilde{\psi}_{n_-}(X)dp_-, \\ {}_{-}^{+}\tilde{\psi}_{n_-}(X) &= (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty}\tilde{M}(p_0,p_-)\,{}_{-}^{+}\psi_{n_0}(X)dp_0, \end{aligned} \quad (42) $$

where ${}_{-}^{+}\psi_{n_0}(X)$ are given by Eq. (24) with the functions $\varphi_{n_0}(x)$ denoted as ${}_{-}^{+}\varphi_{n_0}(x)$ and ${}_{-}^{+}\tilde{\psi}_{n_-}(X)$ are given by Eq. (37). As it follows from the second transformation in Eq. (42) and from relation (40) spinors ${}_{-}^{+}\tilde{\psi}_{n_-}(X)$ satisfy orthonormality relations on the hyperplane x = const,

$$ ({}_{-}^{+}\tilde{\psi}_{n_-}, {}_{-}^{+}\tilde{\psi}_{n'_-})_x = -\delta_{n_-,n'_-}. $$

According to Eq. (32), function ${}^{+}\tilde{\psi}_{n_-}(X)$ describes out-solution and function ${}_{-}\tilde{\psi}_{n_-}(X)$ describes in-solution. Coefficients $\tilde{g}({}_{-}|^{+})$ given by Eq. (33) do not depend on p_0 or p_- and are the same for ${}_{-}^{+}\tilde{\psi}_{n_-}$.

One can form two complete and orthonormal sets of solutions of the Dirac equation: $\{{}_{\pm}\tilde{\psi}_{n_-}(x)\}$ and $\{{}^{\pm}\tilde{\psi}_{n_-}(x))\}$ using Eq. (37); additional sets of solutions we choose as follows:

$$ \begin{aligned} {}^{-}\tilde{\psi}_{n_-}(X) &= -\theta(\pi_-)\,{}_{-}\tilde{\psi}_{n_-}(X)\tilde{g}({}_{-}|^{-}), \\ {}_{+}\tilde{\psi}_{n_-}(X) &= -\theta(-\pi_-)\,{}^{+}\tilde{\psi}_{n_-}(X)\tilde{g}({}^{+}|_{+}). \end{aligned} \quad (43) $$

They can be represented as:

$$ \begin{aligned} {}_{+}^{-}\tilde{\psi}_{n_-}(X) &= (\gamma P + m)\,{}_{+}^{-}\Phi_{n_-}(X), \\ {}_{+}^{-}\Phi_{n_-}(X) &= {}_{+}^{-}\varphi_{n_-}(t,x)\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp)v_{\chi,\sigma}, \\ {}_{+}^{-}\varphi_{n_-}(t,x) &= (2\pi)^{-1/2}\exp(-ip_0t)\,{}_{+}^{-}\varphi_{n_-}(x), \end{aligned} \quad (44) $$

where

$$ \begin{aligned} {}^{-}\varphi_{n_-}(t,x) &= -\theta(\pi_-)\,{}_{-}\varphi_{n_-}(t,x)\tilde{g}({}_{-}|^{-}), \\ {}_{+}\varphi_{n_-}(t,x) &= -\theta(-\pi_-)\,{}^{+}\varphi_{n_-}(t,x)\tilde{g}({}^{+}|_{+}). \end{aligned} \quad (45) $$

Applying an integral transformation of type (38) to solutions (45) we obtain:

$$ (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty}\tilde{M}^*(p_0,p_-)\,{}^{-}\varphi_{n_-}(t,x)dp_- = {}^{-}\varphi_{n_0}(t,x), $$

$$ (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty}\tilde{M}^*(p_0,p_-)\,{}_{+}\varphi_{n_-}(t,x)dp_- = {}_{+}\varphi_{n_0}(t,x). $$

Then

$$ \begin{aligned} {}_{+}^{-}\psi_{n_0}(X) &= (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty}\tilde{M}^*(p_0,p_-)\,{}_{+}^{-}\tilde{\psi}_{n_-}(X)dp_-, \\ {}_{+}^{-}\tilde{\psi}_{n_-}(X) &= (2\pi eE)^{-1/2}\int_{-\infty}^{+\infty}\tilde{M}(p_0,p_-)\,{}_{+}^{-}\psi_{n_0}(X)dp_0, \end{aligned} \quad (46) $$

where ${}_{+}^{-}\psi_{n_0}(X)$ are given by Eq. (24) with the functions $\varphi_{n_0}(x)$ denoted as ${}_{+}^{-}\varphi_{n_0}(X)$ and ${}_{+}^{-}\tilde{\psi}_{n_-}(X)$ are given by Eq. (44).

Using the second transformation from Eq. (46) and relations (26), we see that the following orthogonality relations hold:

$$ ({}_{+}\tilde{\psi}_{n_-}, {}_{-}\tilde{\psi}_{n'_-})_x = 0, \quad ({}^{-}\tilde{\psi}_{n_-}, {}^{+}\tilde{\psi}_{n'_-})_x = 0. $$

One can see that

$$ {}_{+}\tilde{\psi}_{n_-}(X) = 0 \quad \text{if} \quad \pi_- > 0, $$
$$ {}^{-}\tilde{\psi}_{n_-}(X) = 0 \quad \text{if} \quad \pi_- < 0. $$

Using Eqs. (42) and (46), and taking into account that coefficients $\tilde{g}'$ do not depend on p_0, we find from relations (27) that

$$ {}_{+}\tilde{\psi}_{n_-}(X = \tilde{g}({}_{+}|{}^{-})^{-1}[{}_{-}\tilde{\psi}_{n_-}(X)\tilde{g}({}_{-}|{}^{-}) + {}^{-}\tilde{\psi}_{n_-}(X)] = 0, \quad \text{if} \quad \pi_- > 0, $$

$$ {}^{-}\tilde{\psi}_{n_-}(X = \tilde{g}({}^{-}|{}_{+})^{-1}[{}^{+}\tilde{\psi}_{n_-}(X)\tilde{g}({}^{+}|{}_{+}) + {}_{+}\tilde{\psi}_{n_-}(X)] = 0, \quad \text{if} \quad \pi_- < 0. $$

Thus Eqs. (43) hold true with any $\tilde{g}({}_{-}|{}^{-})$ and $\tilde{g}({}_{+}|{}^{+})$ satisfying relations (28).

Note that similar sets of solutions of the Klein-Gordon equation for scalar particles confined between two capacitor plates were obtained in [26]. These solutions are related by an integral transformations similar to (42) and (46).

3.2. Proper Time Representations

Spinor singular functions in SFQED with x-steps are defined by Eqs. (1)–(3). In the case of L-constant electric field, they can be found as sums over the above constructed solutions (see [4]). We note that for $L \to \infty$ it is sufficient to consider these sums over the Klein zone only. In this zone, solutions (30) and (31) satisfy the following orthonormality relations on the $t =$ const hyperplane:

$$ ({}_{\zeta}\psi_{n_0}, {}_{\zeta}\psi_{n_0'}) = ({}^{\zeta}\psi_{n_0}, {}^{\zeta}\psi_{n_0'}) = \delta_{\sigma,\sigma'}\delta(p_0 - p_0')\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)\mathcal{M}_{n_0}, $$
$$ ({}_{\zeta}\psi_{n_0}, {}^{\zeta}\psi_{n_0'}) = 0, $$
$$ (\psi, \psi') = \int \psi^\dagger(X)\psi'(X)d\mathbf{r}, $$
$$ \mathcal{M}_{n_0} = |\tilde{g}({}_{+}|{}^{-})|^2 = e^{\pi\lambda}. $$

Taking all that into account, the singular functions can be represented as:

$$ S^c(X, X') = \theta(t - t')S^-(X, X') - \theta(t' - t)S^+(X, X'), $$

$$ S^-(X, X') = i\sum_{n_0}\mathcal{M}_{n_0}^{-1} \times {}^{+}\psi_{n_0}(X)\tilde{g}({}^{+}|{}_{-})\tilde{g}({}^{-}|{}_{-})^{-1}\,{}^{-}\bar{\psi}_{n_0}(X'), \tag{47} $$

$$ S^+(X, X') = i\sum_{n_0}\mathcal{M}_{n_0}^{-1} \times {}_{-}\psi_{n_0}(X)\tilde{g}({}_{-}|{}^{+})\tilde{g}({}_{+}|{}^{+})^{-1}\,{}_{+}\bar{\psi}_{n_0}(X'); $$

$$ S(X, X') = S^-(X, X') + S^+(X, X'); \tag{48} $$

$$ S^c_{\text{in/out}}(X, X') = \theta(t - t')S^-_{\text{in/out}}(X, X') - \theta(t' - t)S^+_{\text{in/out}}(X, X'), $$

$$ S^-_{\text{in/out}}(X, X') = i\sum_{n_0}\mathcal{M}_{n_0}^{-1\mp}\psi_{n_0}(X)^{\mp}\bar{\psi}_{n_0}(X'), \tag{49} $$

$$ S^+_{\text{in/out}}(X, X') = i\sum_{n_0}\mathcal{M}_{n_0\mp}^{-1}\psi_{n_0}(X)_{\mp}\bar{\psi}_{n_0}(X'), $$

$$ \bar{\psi} = \psi^\dagger\gamma^0. $$

Using relations (27), we represent the singular functions $S^p(X, X')$ and $S^{\bar{p}}(X, X')$ given by Eq. (2) as follows:

$$ \begin{aligned} S^p(X, X') &= i\sum_{n_0}\mathcal{M}_{n_0}^{-1}\,{}_{-}\psi_{n_0}(X)\tilde{g}({}^{-}|{}_{-})^{-1}\,{}^{-}\bar{\psi}_{n_0}(X'), \\ S^{\bar{p}}(X, X') &= -i\sum_{n_0}\mathcal{M}_{n_0}^{-1}\,{}^{+}\psi_{n_0}(X)\tilde{g}({}_{+}|{}^{+})^{-1}\,{}_{+}\bar{\psi}_{n_0}(X'). \end{aligned} \tag{50} $$

We stress that both functions vanish in the absence of the vacuum instability.

Using Eqs. (42), (46), and (43), we obtain the following integral representations:

$$ \begin{aligned} S^-(X, X') &= -i\sum_\sigma\int dp_- d\mathbf{p}_\perp \\ &\theta(+\pi'_-)e^{-\pi\lambda/2}\,{}^{+}\tilde{\psi}_{n_-}(X)\,{}_{-}\bar{\tilde{\psi}}_{n_-}(X'), \\ S^+(X, X') &= -i\sum_\sigma\int dp_- d\mathbf{p}_\perp \\ &\theta(-\pi'_-)e^{-\pi\lambda/2}\,{}_{-}\tilde{\psi}_{n_-}(X)\,{}^{+}\bar{\tilde{\psi}}_{n_-}(X'), \end{aligned} \tag{51} $$

$$ \begin{aligned} S^p(X, X') &= -i\sum_\sigma\int dp_- d\mathbf{p}_\perp \\ &\theta(+\pi'_-)e^{-\pi\lambda}\,{}_{-}\tilde{\psi}_{n_-}(X)\,{}_{-}\bar{\tilde{\psi}}_{n_-}(X'), \\ S^{\bar{p}}(X, X') &= i\sum_\sigma\int dp_- d\mathbf{p}_\perp \\ &\theta(-\pi'_-)e^{-\pi\lambda}\,{}^{+}\tilde{\psi}_{n_-}(X)\,{}^{+}\bar{\tilde{\psi}}_{n_-}(X'). \end{aligned} \tag{52} $$

Using Eqs. (37) and (43), and taking into account the following sums over spin polarizations

$$ \sum_\sigma v_{1,\sigma}v_{1,\sigma}^\dagger = \sum_\sigma (v_{1,\sigma} \otimes v_{1,\sigma}^\dagger) = \Xi_+ = \frac{1}{2}(1 + \gamma^0\gamma^1), $$

we can rewrite representation (51) as follows:

$$ S^\pm(X, X') = \int \theta(\mp\pi'_-)Y^{(\pm)}(X, X'; p_-)dp_-, $$

$$ Y^{(\pm)}(X, X'; p_-) = \frac{1}{4\pi}\left[\gamma^0 + \frac{(m - \gamma_\perp\hat{\mathbf{p}}_\perp)}{\pi_-}\right]\Xi_+ \times \left[\gamma^0 + \frac{(m + \gamma_\perp\hat{\mathbf{p}}'^{*}_\perp)}{\pi'_-}\right]\gamma^0\,{}^{-}_{+}F, $$

$$
{}^{-}_{+}F = \frac{i}{(2\pi)^{d-2}} I_1 \exp\left\{\frac{i}{2}\left[eE\left(\frac{x_-^2 - x_-'^2}{2} - t^2 + t'^2\right)\right.\right.
$$
$$
\left.\left. - p_-(x_+ - x_+')\right] - im^2 {}^{-}_{+}a\right\},
$$

$$
I_1 = \int \exp[-i{}^{-}_{+}ap_\perp^2 + i(\mathbf{r}_\perp - \mathbf{r}_\perp')\mathbf{p}_\perp]d\mathbf{p}_\perp,
$$

$$
{}^{-}_{+}a = \frac{1}{2eE}\{\ln(\mp i\tilde{\pi}_-) - [\ln(\pm i\tilde{\pi}_-')]^*\},
$$

$$
\tilde{\pi}_- = \pi_-/\sqrt{eE}, \quad \tilde{\pi}_-' = \pi_-'/\sqrt{eE}.
$$

For the complex variable ${}^{-}_{+}a$, we chose the main branch of the logarithm, i.e.,

$$
\mathrm{Re}({}^{-}_{+}a) = \frac{1}{2eE}\ln\left|\frac{\pi_-}{\pi_-'}\right|,
$$

$$
\mathrm{Im}({}^{-}_{+}a) = \mp\frac{\pi}{2eE}\mathrm{sgn}(\pi_-)\theta(-\pi,\pi_-').
$$

Calculating the Gaussian integral I_1, we obtain:

$$
{}^{-}_{+}F(X,X';p_-) = i\left(\frac{-i}{4\pi{}^{-}_{+}a}\right)^{\frac{d-2}{2}} \exp\left\{-i\frac{\pi_- + \pi_-'}{4}y_+\right.
$$
$$
\left. + ie\Lambda - i{}^{-}_{+}am^2 + \frac{i}{4{}^{-}_{+}a}|\mathbf{r}_\perp - \mathbf{r}_\perp'|^2\right\}, \tag{53}
$$
$$
\Lambda = -Ey_0(x + x')/2, \quad y_\pm = x_\pm - x_\pm'.
$$

Taking into account this result, we can write the functions $S^\pm(X, X')$ in the following form:

$$
S^\pm(X,X') = \mp(\gamma P + m)\Delta^\pm(X,X'),
$$

$$
\Delta^\pm(X,X') = \int dp_-\theta(\mp\pi_-')^{-}_{+}f(X,X';p_-),
$$

$$
{}^{-}_{+}f(X,X';p_-) = \exp(-eE\gamma^0\gamma^1{}^{-}_{+}a)^{-}_{+}f^{(0)},
$$

$$
{}^{-}_{+}f^{(0)} = -\left(\frac{-i}{4\pi}\right)^{d/2}({}^{-}_{+}a)^{\frac{2-d}{2}}\exp\left\{-i{}^{-}_{+}am^2\right.
$$
$$
+ \frac{i}{4{}^{-}_{+}a}|\mathbf{r}_\perp - \mathbf{r}_\perp'|^2 - i\frac{\pi_- + \pi_\perp'}{4}y_+ + ie\Lambda \tag{54}
$$
$$
\left. -\frac{1}{2}\{\ln(\mp i\pi_-) + [\ln(\mp i\pi_-')]^*\}\right\}.
$$

Assuming $y_- \neq 0$, we perform the change of the variable $s = {}^-a$ in the integral $\Delta^+(X, X')$ and the change of the variable $s = {}_+a$ in the integral $\Delta^-(X, X')$:

$$
\Delta^+(X,X') = \int_{-\infty}^{+\infty}\theta(-\pi_-')^- f(X,X';p_-)dp_-
$$
$$
= \int_{\Gamma_c}\tilde{f}(X,X';s)\frac{eEds}{\sinh(eEs)}
$$
$$
- \theta(+y_-)\int_{\Gamma_c - \Gamma_2 - \Gamma_1}\tilde{f}(X,X';s)\frac{eEds}{\sinh(eEs)},
$$

$$
\Delta^-(X,X') = \int_{-\infty}^{+\infty}\theta(+\pi_-')_+ f(X,X';p_-)dp_-
$$
$$
= \int_{\Gamma_c}\tilde{f}(X,X';s)\frac{eEds}{\sinh(eEs)}
$$
$$
- \theta(-y_-)\int_{\Gamma_c - \Gamma_2 - \Gamma_1}\tilde{f}(X,X';s)\frac{eEds}{\sinh(eEs)},
$$

$$
\tilde{f}(X,X';s)
$$
$$
= -\left(\frac{-i}{4\pi s}\right)^{d/2}\frac{1}{s}\exp\left[-eE\gamma^0\gamma^1 s + ie\Lambda - ism^2\right.
$$
$$
\left. + \frac{i}{4s}|\mathbf{r}_\perp - \mathbf{r}_\perp'|^2 - \frac{i}{4}eE\coth(eEs)(y_0^2 - y_1^2)\right].
$$

All the integration contours are shown in Fig. 1. Closing the integration contour $\Gamma_c - \Gamma_2 - \Gamma_1$ as $\mathrm{Re}s \to \pm\infty$, one can transform it into the contour Γ (see Fig. 2).

As a result, we obtain:

$$
S^\pm(X,X') = (\gamma P + m)\Delta^\pm(X,X'),
$$
$$
\mp\Delta^\pm(X,X') = \int_{\Gamma_c} f(X,X';s)ds \tag{55}
$$
$$
- \theta(\pm y_-)\int_\Gamma f(X,X';s)ds.
$$

The Fock-Schwinger kernel $f(X, X'; s)$ is given by Eq. (18) with the gauge dependent term Λ given by Eq. (53). Note that this term can be represented as an integral along a line (19), where in this case $A_\mu(X)$ is the potential given by Eq. (23).

It was demonstrated in [18] that

$$
\int_\Gamma F(X,X';s)ds = 0, \quad y_\mu y^\mu < 0,
$$

which implies that integrals $\Delta^\pm(X, X')$ in Eq. (55) can be written as:

$$
\mp\Delta^\pm(X,X') = \int_{\Gamma_c} f(X,X';s)ds
$$
$$
- \theta(\pm y_0)\int_\Gamma f(X,X';s)ds. \tag{56}
$$

Thus, we have derived the Schwinger integral representation (16) for the causal propagator (47) in the case under consideration and have demonstrated that the commutation function (48) has the universal structure (17):

$$S^c(X,X') = (\gamma P + m)\Delta^c(X,X'),$$
$$\Delta^c(X,X') = \int_{\Gamma_c} f(X,X';s)ds,$$
$$S(X,X') = (\gamma P + m)\Delta(X,X'),$$
$$\Delta(X,X') = \operatorname{sgn}(t-t')\int_{\Gamma} f(X,X';s)ds.$$

In fact the above result represents an indirect prove that sets of solutions constructed above are complete on the t-const hyperplane. Representation (56) holds true for arbitrary X and X' in spite of the fact that the change of variables in integral (54) was performed under the condition $y_- \neq 0$. All this justifies that representation (56) is equivalent to (47).

In the same manner, we represent the singular functions $S^p(X,X')$ and $S^{\bar{p}}(X,X')$ given by Eq. (52) as:

$$S^p(X,X') = \int dp_-\theta(+\pi'_-)\tilde{Y}^{(-)}(X,X';p_-),$$
$$S^{\bar{p}}(X,X') = \int dp_-\theta(-\pi'_-)\tilde{Y}^{(+)}(X,X';p_-),$$
$$\tilde{Y}^{(\pm)}(X,X';p_-) = \frac{1}{4\pi}\left[\gamma^0 + \frac{(m-\gamma_\perp\hat{\mathbf{p}}_\perp)}{\pi_-}\right]\Xi_+ \times\left[\gamma^0 + \frac{(m+\gamma_\perp\hat{\mathbf{p}}'^*_\perp)}{\pi'_-}\right]\gamma^0\tilde{F}^{(\pm)},$$
$$\tilde{F}^{(\pm)} = \frac{\pm iI_2}{(2\pi)^{d-2}}\exp\left\{\frac{i}{2}\left[eE\left(\frac{x_-^2 - x'^2_-}{2} - t^2 + t'^2\right) - p_-(x_+ - x'_+)\right] - i\left(b_\pm - \frac{i\pi}{2eE}\right)m^2\right\},$$
$$I_2 = \int \exp\left[-i\left(b_\pm - \frac{i\pi}{2eE}\right)p_\perp^2 + i(\mathbf{r}_\perp - \mathbf{r}'_\perp)\mathbf{p}_\perp\right]d\mathbf{p}_\perp,$$
$$b_\pm = (\ln(\pm i\tilde{\pi}_-) - [\ln(\pm i\tilde{\pi}'_-)]^*)/(2eE).$$

Assuming $y_- \neq 0$, we perform the change of variables $s = b_- - i\pi/(2eE)$,

$$\operatorname{Re}(s) = \frac{1}{2eE}\log\left|\frac{\pi_-}{\pi'_-}\right|,$$
$$\operatorname{Im}(s) = -\frac{\pi}{2eE}[\theta(+\pi_-) + \theta(-\pi_-)],$$

in the integral $S^p(X,X')$ and $s = b_+ - i\pi/(2eE)$ in the integral $S^{\bar{p}}(X,X')$,

$$\operatorname{Re}(s) = \frac{1}{2eE}\log\left|\frac{\pi_-}{\pi'_-}\right|,$$
$$\operatorname{Im}(s) = -\frac{\pi}{2eE}[\theta(-\pi_-) + \theta(-\pi'_-)].$$

Then

$$\begin{aligned} S^p(X,X') &= -\int_{\Gamma_p} f(X,X';s)ds - \theta(-y_-)\int_{\Gamma_2+\Gamma_3-\Gamma_p} f(X,X';s)ds, \\ S^{\bar{p}}(X,X') &= -\int_{\Gamma_p} f(X,X';s)ds - \theta(+y_-)\int_{\Gamma_2+\Gamma_3-\Gamma_p} f(X,X';s)ds. \end{aligned} \tag{57}$$

Closing the integration contour $\Gamma_2 + \Gamma_3 - \Gamma_p$ as $\operatorname{Re}s \to \pm\infty$ one can transform it into the contour Γ_p^1 (see Fig. 2) with its radius tending to zero. As a result, we arrive at the equation:

$$\begin{aligned} S^p(X,X') &= (\gamma P + m)\Delta^p(X,X'), \\ \Delta^p(X,X') &= -\int_{\Gamma_p} f(X,X';s)ds - \theta(-y_-)\int_{\Gamma_p^1} f(X,X';s)ds; \\ S^{\bar{p}}(X,X') &= (\gamma P + m)\Delta^{\bar{p}}(X,X'), \\ \Delta^{\bar{p}}(X,X') &= -\int_{\Gamma_p} f(X,X';s)ds - \theta(+y_-)\int_{\Gamma_p^1} f(X,X';s)ds. \end{aligned} \tag{58}$$

Note that in the limit $s \to \pm 0 - i\pi/eE$ one has that

$$\begin{aligned} \lim_{s\to\pm 0 - i\pi/eE} f(X,X';s) &= \pm f_\perp(X,X')\delta(y^0)\delta(y^1), \\ f_\perp(X,X') &= -i\left(\frac{eE}{4\pi^2}\right)^{\frac{d-2}{2}} \times\exp\left(i\pi\gamma^0\gamma^1 - \frac{\pi m^2}{eE} - \frac{eE}{4\pi}|\mathbf{r}_\perp - \mathbf{r}'_\perp|^2\right). \end{aligned} \tag{59}$$

Taking Eq. (59) into account, one can localize singularities in the integral in Eq. (58):

$$\int_{\Gamma_p^1} f(X,X';s)ds = \theta(y_1^2 - y_0^2)\Delta_R^0(X,X'),$$

$$\Delta_R^0(X,X') = \int_{\Gamma_R^p} f(X,X';s)ds.$$

Here Γ_R^p is a clockwise circle around the point $s = -i\pi/eE$ with a small enough radius R, inside of which the function $f(x, x', s)$ does not have any other singularities.

The final forms for singular functions $S^{p/\bar{p}}$ are:

$$S^{p/\bar{p}}(X,X') = (\gamma P + m)\Delta^{p/\bar{p}}(X,X'),$$

$$-\Delta^p(X,X') = \int_{\Gamma_p} f(X,X';s)ds + \theta(y^1)\int_{\Gamma_p^1} f(X,X';s)ds, \quad (60)$$

$$-\Delta^{\bar{p}}(X,X') = \int_{\Gamma_p} f(X,X';s)ds + \theta(-y^1)\int_{\Gamma_p^1} f(X,X';s)ds.$$

Note that the contour Γ_p^1 can be transformed into the contour $\Gamma_2 + \Gamma_3 - \Gamma_p$. The step function $\theta(\pm y^1)$ can be represented as a function $\theta(\mathbf{yE}/E)$ of the projection $\mathbf{yE}/E$ on the field direction of the displacement vector y.

Using Eq. (60), we obtain proper-time representations for singular functions $S^c_{\text{in/out}}(X, X')$ and $S^{\mp}_{\text{in/out}}(X, X')$,

$$S^c_{\text{in/out}}(X,X') = S^{p/\bar{p}}(X,X') + S^c(X,X'),$$
$$S^{\mp}_{\text{in/out}}(X,X') = \mp S^{p/\bar{p}}(X,X') + S^{\pm}(X,X'). \quad (61)$$

Note that the change of variables in integral (57) was performed under the condition $y_- \neq 0$. However, representations (52) hold true for any y_-.

One can verify that representations (60) and therefore Eq. (61) are valid for arbitrary X and X'. To this end, one needs to check that (60) satisfy the same Dirac equation as (50) for any X and X'. First we verify that integrals (60) satisfy the Dirac equation for any X and X'. Then, we verify that Cauchy conditions for distributions (50) coincide with (60) at $t = t'$.

We note that the corresponding scalar singular functions can be derived from representations for the spinor functions $\Delta^{\pm}(X, X')$, $\Delta^c(X, X')$, $\Delta(x, x')$, and $\Delta^{p/\bar{p}}(X, X')$ was putting formally all the γ-matrices to zero.

In this consideration for the L-constant electric field, the electric field is directed along the axis x, $E_x = E$. It is clear that choosing the opposite direction corresponds to the reflection $x \to -x$, $x' \to -x'$, and to the change $\gamma^1 \to -\gamma^1$. Therefore, it is sufficient to consider the case of the one direction of the electric field. We see that representations (60) and (61) coincide up to a field gauge with representations (20) and (21). They can be easily written in a covariant form by using the field strength tensor $F_{\mu\nu}$; see Eq. (22). We recall that the latter were found for a constant electric field given by the time-dependent potential in the framework of a general formulation of QED for such a case [3].

4. DISCUSSION

In this article we have constructed and studied singular functions in SFQED with T-constant electric field and in SFQED with L-constant electric field. For both cases we find in- and out-solutions of the Dirac equation in the special forms of light cone variables. With help of these solutions, we construct the Fock-Schwinger proper-time integral representations for all kinds of singular functions needed for calculation of probability amplitudes of processes and average values of physical quantities. The Fock-Schwinger proper-time integral representations for singular functions in SFQED with L-constant electric field are obtained for the first time. The representations for singular functions in SFQED with T-constant electric field are obtained for an arbitrary orientation of the external electric field, which non-trivially generalizes results of [18, 19].

After standard ultraviolet regularization and renormalization, all physical quantities that can be obtained using a causal propagator $S^c(X, X')$ are finite in the limit $L \to \infty$ and $T \to \infty$. For example, it can be seen for the vacuum matrix element of an energy-momentum tensor,

$$\langle T_{\mu\nu}\rangle^c = \langle 0,\text{out}|T_{\mu\nu}|0,\text{in}\rangle c_v^{-1},$$

$$T_{\mu\nu} = \frac{1}{2}(T^{\text{can}}_{\mu\nu} + T^{\text{can}}_{\nu\mu}),$$

$$T^{\text{can}}_{\mu\nu} = \frac{1}{4}\{[\hat{\Psi}^\dagger(X)\gamma^0, \gamma_\mu P_\nu \hat{\Psi}(X)] + [P^*_\nu \hat{\Psi}^\dagger(X)\gamma^0, \gamma_\mu \hat{\Psi}(X)]\}$$

that can be presented as

$$\langle T_{\mu\nu}\rangle^c = i\,\text{tr}[A_{\mu\nu}S^c(X,X')]|_{X=X'},$$

$$A_{\mu\nu} = \frac{1}{4}[\gamma_\mu(P_\nu + P'^*_\nu) + \gamma_\nu(P_\mu + P'^*_\mu)].$$

It is natural that the vacuum instability problem and its manifestation are completely different in the L-constant field and the T-constant field for finite intervals T and L. However, in the limits $T, L \to \infty$, the corresponding characteristics of the vacuum instability turn out to be the same. In particular, this fact can be interpreted as follows: both cases represent different regularizations of the vacuum instability in the idealized case of the constant uniform electric field.

However, this equivalence may be absent for the average values of physical quantities, which can be obtained using singular functions $S^c_{\mathrm{in/out}}(X, X')$. For example, vacuum mean values of an energy-momentum tensor,

$$\langle T_{\mu\nu}\rangle_{\mathrm{in/out}} = \langle 0, \mathrm{in/out}|T_{\mu\nu}|0, \mathrm{in/out}\rangle$$

can be presented as

$$\langle T_{\mu\nu}\rangle_{\mathrm{in/out}} = i\mathrm{tr}[A_{\mu\nu}S^c_{\mathrm{in/out}}(X, X')]|_{X=X'},$$

where contributions of $S^{p/\bar{p}}(X, X')$ are involved. Such contributions grow indefinitely in the limit of either $T \to \infty$ or $L \to \infty$ (see [5, 20]). This is due to unlimited growth of number density of created pairs of electrons and positrons. Hence it follows that in such problems there is an essential physical difference between T-constant and L-constant fields.

Nevertheless, in cases where the corresponding contributions to the Feynman diagrams are finite, one can use the obtained proper-time representations of $S^{p/\bar{p}}(X, X')$. In these cases a regularization for a constant uniform electric field by the L-constant field is equivalent to the regularization by the T-constant field if $T, L \to \infty$.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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