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TEACHING ELEMENTARY MATHEMATICS TO ELEMENTARY TEACHERS IN BRAZIL

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Abstract

This work is part of a four-year research project that began in January 2023. The project was motivated by the recognition of the extremely limited mathematical training of early-year basic education teachers (working with children aged 6 to 10) and sought to develop a continuing training program based on interaction and collaboration with school teachers. The aim was to assess the impact of this training on teachers' Pedagogical Content Knowledge, understand the role of school collaboration in this potential contribution, and, based on empirical evidence, refine the training program. This project, which follows a Research-Based Design approach, is now at its midpoint, with several research and product developments underway. Here, we present part of this process. We begin with a brief overview of Brazil's public education system and the theoretical foundations of the continuing teacher training proposal. Next, we explain the partnership between the university research team and the school's teaching staff. After these preliminary considerations, we describe the first phase of the project, including the collaborative work with the school and the course design process currently under development and study. Finally, we outline the project's research and development progress and discuss next steps, including the training of multipliers.

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Keywords: arithmetics, teacher training, pedagogical content knowledge

In this paper, we present a research project in Brazil: the development and assessment of an in-service education course in arithmetic for elementary school teachers. The conceptual aims of this course are similar to those of the South African *Wits Maths Connect Primary* project, as stated by Venkat and Graven (2017, p. 169): " [...] shifting the nature of primary teachers' ways of working with mathematics, aiming to build connections and coherence in teachers' work with examples, representations and explanations."

The project is ongoing, and research results are not yet available. Our goal in describing and presenting this project is to receive valuable suggestions, contributions, and criticism from other researchers.

Overview of the Basic Education System in Brazil

Brazil's education system serves over 200 million people. Basic education consists of three stages: Early Childhood Education, Elementary and Middle School, and High School. Although the system follows the Federal Constitution and national regulations, it is administered by state governments and municipalities.

Elementary school (grades 1–9) is compulsory and divided into Early Years (ages 6–10) and Final Years (ages 11–14). In the Early Years (EY), a single teacher covers all subjects (except physical education and arts). Early Years teachers must complete a four-year Pedagogy degree. According to Edda Curi (2006), these programs dedicate at most 4% of total coursework to mathematics content or teaching methodology (Curi, 2006; Costa et al., 2016). This insufficient training in mathematics, its pedagogy, and didactics has a significant impact on student achievement and development.

A 2021 study conducted by the State of São Paulo's Secretary of Education (SEDUC-SP, 2021) (responsible for a school system with 3.5 million students) assessed the pandemic's impact on student learning. It found that by the end of 5th grade, students' mathematics knowledge aligned with expectations for students in the third quarter of 1st grade, a 4.2-year gap. These findings are consistent with broader trends, including Brazilian students' results in PISA assessment (OECD, 2023).

This context of inadequate teacher preparation in mathematics, leading to poor student outcomes, motivated a research project aimed at developing an in-service teacher training program in arithmetic.

In-service teacher training: theoretical premises

The mathematical content of the program focuses on the most fundamental topics: arithmetic, including numbers, the positional decimal system, and the four basic operations. Our premise regarding the knowledge teachers should develop aligns with Ma's (1999) concept of a *Profound Understanding of Fundamental Mathematics* (PUFM). This goes beyond a mere conceptual grasp of elementary mathematics; "it is the awareness of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics and the ability to provide a foundation for that conceptual structure and instill those basic attitudes in students" (Ma, 1999, p. 106).

From a structural perspective, we followed the recommendations of (Moriconi, 2017), who reviewed international literature on effective continuing teacher education. The study identified five key characteristics that increase the likelihood of positively impacting teachers' practice:

1. Focus on content and pedagogical content knowledge
2. Active learning
3. Considerable duration
4. Collective participation of teachers from the same school or grade
5. Coherence with school practices

Below, we briefly explain how we incorporated each characteristic.

- Mathematical Content and Pedagogical Content Knowledge

  We recognized the need to strengthen mathematical knowledge before addressing pedagogical content knowledge (PCK). We intentionally intertwined these elements to keep classroom application central while valuing our academic mathematical expertise alongside teachers' practical knowledge. For PCK, we examined several analytical models (Ball et al., 2008, Carrillo et al., 2017) and adopted the framework proposed by the Coactive Initiative (Baumert et al., 2010).

- Active Learning

  Meetings were designed as dialogues. We used slides with minimal text to guide discussions, and much of the structured content emerged as summaries of these conversations. Additionally, sessions incorporated interactive activities to promote reflection.

- Considerable Duration

  The course was designed for 60–180 hours. This flexibility accommodates the heterogeneity of public school systems.

- Collective School Participation

  The first phase involved one school with significant participation from primary teachers. For broader implementation, researchers can only make recommendations, as decisions rest with educational administrators.

- Coherence with School Practices

  Our primary goal is strengthening teachers' knowledge, independent of specific methodologies. We introduced diverse pedagogical resources while encouraging autonomous teaching decisions, acknowledging the importance of school-level coherence.

The following section details the program's development process, with emphasis on these five characteristics.

Design methodology: a partnership between school and academia

This project received funding from the São Paulo State Research Agency (Fapesp) and the State Department of Education (SEDUC-SP). The research team consists of university researchers, professors, and students from Unicamp (referred to as the university team), along with teachers and pedagogical advisors from a partner school—the Edson Luis Lima Souto Municipal Elementary School (referred to as the school team). Fapesp provided monthly scholarships to compensate the school team for their participation.

From 2023 to 2024, the university team visited the partner school weekly. Mornings were devoted to observing and participating in mathematics classes (grades 1–5), while afternoons were reserved for meetings with school staff. These weekly sessions were supplemented by three intensive meetings (lasting 2–4 days each) during school breaks. All university-school team interactions were recorded.

In the project's second year, we expanded documentation to include classroom footage, accumulating over 80 hours of raw material. This is being edited into 40 short episodes.

This partnership aimed to develop a continuing education program grounded in classroom realities—one that public school teachers would find relatable and applicable to their students.

Weekly meetings prioritized discussions of mathematical content and associated pedagogical knowledge, aiming to foster growth in teachers' Pedagogical Content Knowledge (PCK). We adopted the Coactive Initiative framework (Baumert et al., 2010), which evaluates PCK across three dimensions: knowledge of students and their backgrounds (*PCK-Students*); multiple representations and explanations of concepts (*PCK-Instruction*); and diverse approaches to solving mathematical problems (*PCK-Task*). In parentheses, we used the same abbreviated terms employed by the authors in their appendix.

Given practical constraints on measuring student learning outcomes, we assessed the course's impact through teacher PCK—the strongest predictor of student learning gains (Baumert et al., 2010). We developed parallel PCK tests (Versions A/B), each with 11 questions including 3 shared anchor items. To control for test difficulty differences, half of participants received Version A first, while the other half began with Version B. These instruments were co-developed with researchers Stefan Krauss and Nicole Steib (University of Regensburg).

The partnership culminated in a three-day intensive workshop where teams co-planned six mathematics lessons. Teachers then implemented these in their classrooms, with recordings analyzed collectively. This condensed *Lesson Study* process revealed marked improvements in teaching practices and conceptual clarity—outcomes that validated our transition to developing a broader arithmetic course for teachers.

The structure of the Experimental Course

The primary goal of this course is to strengthen teachers' mathematical knowledge and pedagogical understanding, focusing on number concepts, the decimal system, and the four arithmetic operations: addition, subtraction, multiplication, and division. The course does not promote any specific methodology or technology; rather, it aims to expand teachers' instructional repertoire.

In general, the course content is consistent with the main conclusions of the 23rd ICMI Study devoted to whole number arithmetic: "increased emphasis on problem solving (as opposed to simply focusing on teaching arithmetical fluency) and, second, the importance of working with concrete and pictorial representations," as presented in the final discussion of the chapter on professional development (Venkat et al., 2018).

Moreover, the course aims to increase teachers' understanding and repertoire by presenting multiple representations and exploring their different advantages and strengths. For example, the course explores the number line when discussing computation strategies, and base-ten blocks when presenting intermediary and final algorithms. We note that these two representations are discussed in Venkat et al. (2018) from different perspectives, including those of European and Macao participants.

As for the dynamic of the course, it is structured around slide presentations with minimal text, showcasing mathematical and classroom situations. These materials are used in an expository-dialogical manner. Our objective is to foster the principle expressed in the title of Reinhart's (2000) article: Never Say Anything a Kid Can Say! As an example, we presented and discussed the Gattegno Chart with the teachers (which they had never seen before), and afterward showed a video of a teacher leading a very similar discussion with 4th-grade students.

Below, we outline key aspects of the course, which are interwoven throughout the content. In parentheses, we indicate the PCK dimensions (as quoted in the previous section) that are related in each aspect.

Topic structure* (*PCK-Instruction and Task*): *meanings and representations, strategies and mental calculation, algorithms

Each mathematical topic is introduced by exploring its various meanings and representations. For example, in subtraction, we not only address the commonly known interpretations, such as take-away, difference, and part-whole, but also emphasize mathematical modeling of these situations (e.g., 7−3=x, x+3=7, and 3+x=7). Before delving into problem-solving methods and computation, we ensure that these foundational concepts are well understood.

Regarding calculations, we distinguish between mental or approximate calculation strategies and algorithms. When discussing algorithms, we place particular emphasis on intermediate or semi-written algorithms, which, while less concise or efficient, significantly enhance mathematical intelligibility, as explored in the next section.

***Intelligibility of mathematics* (*PCK-Instruction and Task*)**

As previously mentioned, Early Years teachers often receive minimal mathematics training. Their perspective on mathematics tends to be procedural—focused on *how* to perform calculations—while algorithms are often perceived as "black boxes" that simply yield correct answers. A crucial aspect of the course is to demystify mathematics and create "aha moments," fostering a shift in teachers' attitudes towards the subject (see Liljedahl, 2005).

We define *intermediate algorithms* as a sequence of suboptimal methods that, while not computationally efficient, provide clear insight into their underlying logic.

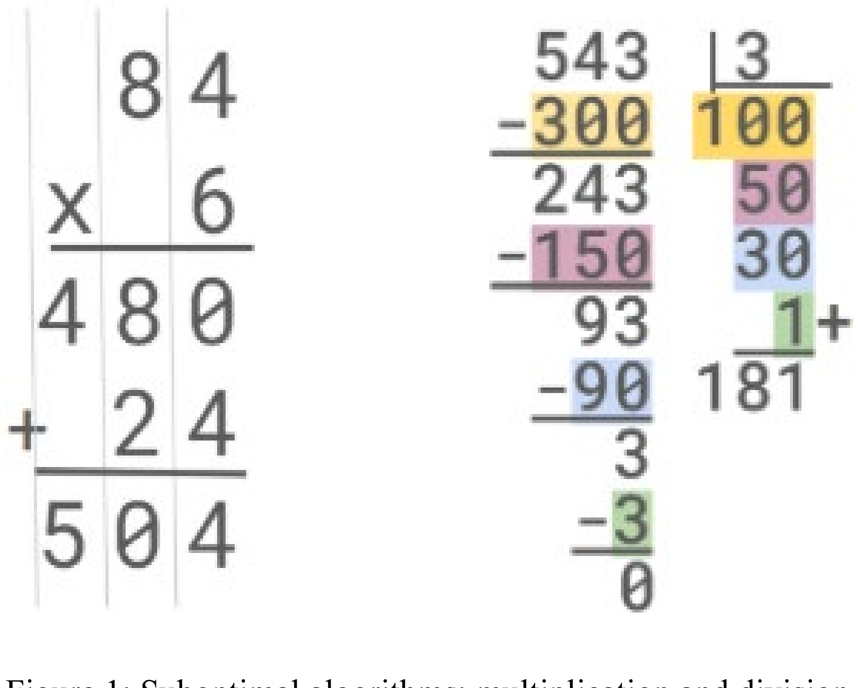

Figure 1: Suboptimal algorithms: multiplication and division

Types of situation and course dynamics

The course incorporates a diverse range of learning dynamics, systematically integrating the following activities:

a) (*PCK-Student*) *Analysis of student errors and written work*, fostering Pedagogical Content Knowledge (PCK) and *noticing* skills, as explored by Mason (2002).

b) (*PCK-Student*) *Analysis of classroom videos* recorded at the partner school. These are short excerpts capturing different scenarios, from individual silent work to whole-class mathematical discussions. Selected clips are used to spark reflection and discussion (see Santagata, 2021).

c) (*PCK-Instruction*) *Exploration of teaching materials*, including resources from the National Center for Excellence in the Teaching of Mathematics (NCETM) and the Freinet Pedagogy Self-Corrective Notebooks.[1]

d) (*PCK-Student, Task and Instruction*) *Reflection on real classroom situations*, adapted from the Subject Knowledge Audit Questions developed by NCETM. Beyond their formative role, we realize that the regular use of these activities suggests that we can try to use the answers to these tasks also as a way of assessing PCK development as a process.[2]

Recording and organizing content

While slide presentations with minimal text facilitate interaction and discussion, they do not serve as a lasting reference. In response to feedback from teachers at the partner school, two measures were adopted:

1. *Pre-distributed slide thumbnails* – Teachers receive printed thumbnails of the slides before sessions, allowing them to take notes during discussions.
2. *Concise summary texts* – To consolidate key ideas and concepts, short written summaries are provided. These texts, mindful of teachers' demanding schedules, are authored by Leonardo Barichello (University of

[1] The main material produced by the NCETM used in the course are the Primary Mastery Professional Development materials, available at https://shorturl.at/YUNGJ, the AutoCorrect notebooks exist only in print and have very little literature. For more information, see (Monthuber, 2002).

[2] We intend to systematize this evaluation by developing a coding book from the feedback material offered to teachers by the NCETM.

São Paulo), a team researcher who does not participate in the training sessions.

Experimental Course: first trial

In the first semester of 2025, we launched the first structured course utilizing the material developed over the previous two years. Given the context, we have two classes, each comprising 40 teachers. In every case, two teachers from the same school participate: one working in the classroom and the other serving as a coordinator.

Each class will attend six sessions, each lasting six hours. These sessions will take place at the university on Tuesdays. The researchers, who are also the authors of this text, lead the sessions together, both are present for the entire meeting.

This inaugural edition is part of the research project and will be rigorously monitored and evaluated using a variety of instruments: (a) PCK pre- and post-tests, as described in the Design Section; (b) Analysis of responses to the Subject Knowledge Audit Questions; (c) Systematic analysis of video use to facilitate teacher discussions, following the methodological steps outlined in Borko et al. (2008); (d) Full recordings of all sessions, enabling further study to enhance both researcher understanding and course development; (e) Three experienced basic education teachers with strong academic backgrounds attending as critical observers.

At the time of writing, only the first session had been conducted with both classes. While it is too early for in-depth analysis, one particularly promising aspect stands out: from the very first session, teachers engaged intensely in discussions. It was a fantastic start!

Spin-offs

The project has led to a series of developments and by-products that are noteworthy in their own right. The collection of videos showcasing classroom situations and the PCK tests described earlier were part of the initial research plan. The videos have been well received by teachers, promoting rich discussions about teaching and learning (see Borko et al. (2008) for analysis categories). The collection is currently being organized and will be made available for pre-service and in-service teacher education. The reliability of the pre- and post-intervention PCK tests is being studied, and if validated, they will be made available for use by researchers. Notably, to the best of our knowledge, the PCK test is the first tool designed to evaluate an intervention in two stages: pre- and post-test.

Throughout the project, small artifacts were developed for classroom use, including a set of magnetic pieces representing base-ten (Dienes) blocks.

Beyond these specific contributions, the university team's engagement with the school fostered collaboration in structuring the school team's proposals, which includes the development of the following:

Arithmetic diagnosis: At the beginning of the year, a decision was made to dedicate a few hours per week to regrouping 4th-grade students into more homogeneous groups to address learning gaps. Since the teachers were unfamiliar with their new classes, a diagnostic tool was needed to support the grouping process. The university team developed a set of 150 questions—30 for each grade level—to be used adaptively. This material has already been applied to many students, revised, and reformulated, ultimately becoming a tool that enables teachers to conduct their own assessments while receiving feedback on the main challenges faced by both the class and individual students.

Error analysis: The students' responses to the diagnostic questions provided far more information than was required for grouping purposes. Since the questions were open-ended, we took on the challenge of deciphering the students' reasoning. The richness of this data and the insights it provided led to a research sub-project: a comparative study of student errors in Brazil and Germany. Two tools were designed, each comprising 11 questions, targeted at 1st- and 3rd-grade students. Data collection has already been completed in Brazil and is scheduled to take place in Germany in June 2025.

Preliminary insights

The systematic analysis of the data produced in this project was planned to begin in July 2025, following the conclusion of the first experimental trial course. However, some preliminary analyses and observations suggest that we can expect significant changes in the knowledge and practices of the teachers.

With some teachers from the partner school, we observed qualitative improvements when comparing the pre- and post-PCK tests. Moreover, for two years, we attended these teachers' math classes weekly. We observed significant changes in their practices: they now engage in meaningful math talks with their students; children's errors and mistakes are treated as learning opportunities for the whole class; the learning goals are clearer, and the choice of activities is more coherent with those goals.

Finally, in the undergoing trial of the Experimental Course, we are having a strong engagement of the teachers and extraordinary feedback.

Next steps

In the first semester of 2025, the first experimental course is underway, with two classes of 40 teachers each. As previously mentioned, its impact will be assessed using multiple tools to evaluate PCK development. Each session is carefully planned and discussed by the project's research team in collaboration with partner teachers.

At the same time, the team of partner teachers has begun developing materials to support multiplier training – our next challenge. We are open and eager for comments and suggestions.

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ASSESSMENT OF PRIMARY SCHOOL STUDENT TEACHERS' BASIC ARITHMETIC COMPETENCIES – DEVELOPING A DIAGNOSTIC TEST AND A COMPETENCY-LEVEL MODEL

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Abstract

An important aspect of teacher knowledge, as a relevant factor of good teaching, is mathematical content knowledge. Due to the relevance of mathematical knowledge in the school context, especially basic competencies - also known as elementary mathematics -, that become relevant as a solid foundation for learning new mathematical content, we have addressed this important knowledge with our research. There is a lack of diagnostic instruments for assessing basic competencies at the beginning of university. In the following paper, we present the process of conceptualizing a diagnostic test and its fit using the first pilot data and the development of a competence level model for diagnosing arithmetic competencies.

Keywords: competence level model, elementary arithmetic

Introduction

Arithmetic competencies are of great relevance for everyday life and are the basis for opening up many further mathematical areas such as algebra. Therefore, many of those competencies that are addressed in the early arithmetic classroom are considered fundamental or 'basic' for further learning. 'Basic' refers to the competencies of elementary arithmetic at primary and secondary school levels.

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