LONG-TIME BEHAVIOR FOR SEMILINEAR EQUATION WITH TIME-DEPENDENT AND ALMOST SECTORIAL LINEAR OPERATOR

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Abstract. In this paper we study the solvability and asymptotic dynamics of a nonautonomous semilinear reaction-diffusion equation in a domain with a handle $\Omega_0 = \Omega \cup R_0$, formed by an open subset $\Omega \subset \mathbb{R}^N$ connected to a line segment R_0 at the ending points of the segment. We also assume that the linear part of this equation (the diffusion term) is time-dependent and the growth condition on the nonlinearity F is more general than linear growth. To obtain existence of local solution, the uniformly almost sectoriality of the family of linear operator associated to the evolution equation is explored. An abstract result on existence of mild solution for semilinear problems of the form

$$u_t + A(t)u = F(u),\ t > \tau; \quad u(\tau) = u_0,$$

where $A(t)$ is uniformly almost sectorial, is proved and we analyze its application to the equation in Ω_0. Through an iterative procedure we obtain estimates of the solution in the spaces L^{2^k}, for any $k \in \mathbb{N}$, resulting in global well-posedness of the solution and existence of pullback attractor. We also explore how the line segment R_0 impacts in the pullback attractor obtained. Those results are obtained without requiring monotonicity or asymptotic assumption on $A(t)$ as $t \to \infty$.

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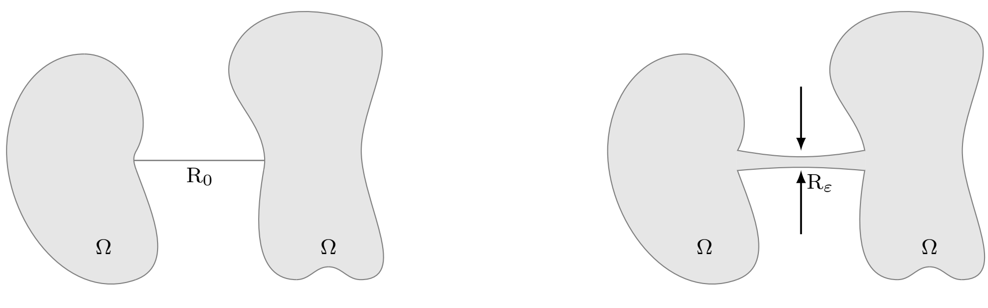
1. **Introduction**

In this work we study the asymptotic dynamics of a semilinear nonautonomous reaction-diffusion equation in a domain with a handle and with time-dependent linear operator. To be precise, let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain formed by two disjoint components: $\Omega = \Omega_L \cup \Omega_R$, $\overline{\Omega_L} \cap \overline{\Omega_R} = \emptyset$. Attached to this Ω, consider the line segment R_0 given by $R_0 = \{(r, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}; r \in (0, 1)\}$. We assume that Ω and R_0 are connected by the points $(0, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$ and $(1, 0) \in \mathbb{R} \times \mathbb{R}^{N-1}$, and that there exists a cylinder centered in the line segment R_0 that only intersects Ω in its bases (see Figure 1 - (A) below).



(A) Domain with handle Ω_0 (B) Dumbbell domain

Figure 1. Domain with a handle

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Consider in $\Omega_0 = \Omega \cup R_0$ the following reaction-diffusion equation:

$$\begin{cases} w_t - div(a(t,x)\nabla w) + w = f(w), & x \in \Omega,\ t > \tau, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \\ v_t - \partial_r(a(t,r)\partial_r v) + v = f(v), & r \in R_0,\ t > \tau, \\ v(p_0) = w(p_0) \text{ and } v(p_1) = w(p_1), \end{cases} \tag{1.1}$$

where $p_0 = (0, 0, ..., 0) \in \mathbb{R}^N$ and $p_1 = (1, 0, ..., 0) \in \mathbb{R}^N$ are the junction points between the sets Ω and R_0. We will refer to equations where the linear part depends on time as *singularly nonautonomous.* This terminology, adopted for instance in [9, 14], is not unanimous and, in the case we are considering here, does not refer to discontinuity or blow-up in time. We adopt it in order to easily distinguish between the case studied here to the well established case where there is no time-dependence on the linear operators.

Our goal is to analyze the long-time dynamics of this problem, obtaining global well-posedness, estimates for the solution (w, v) and existence of pullback attractor. We shall see that this problem can be solved with initial conditions in a L^p-like kind of space, producing solutions with more regularity, as a consequence of the parabolic structure of the problem. There are some particularities of this problem that make interesting this analysis. We enumerate them in the sequel.

Limit of reaction-diffusion equations on thin domains: Equation (1.1) is obtained as a limit equation of a sequence of semilinear reaction-diffusion equations in a domain with a thin channel as in Figure 1 - (B), called *dumbbell domains.* Arrieta et al. in [4, 5, 6] considered an autonomous version of (1.1) and developed a functional setting suited to study the dynamics of those equations in dumbbell domains. The limit case, when ε tends to zero, is the one we focus in this paper. The channel R_ε in the limit becomes the line segment R_0, to which we refer as a *handle.*

The linear operator is almost sectorial: The abstract initial value problem obtained from (1.1) is of parabolic type, but its linear associated operator has a deficiency in its resolvent estimate. This deficiency comes from the condition at the junction points p_0 and p_1, as proved in [5]. The linear operator in this case belongs to a class called *almost sectorial.* Operators in this class generate a special type of integrated semigroups, the *semigroups of growth* (introduced by Da Prato [16] and studied by [22, 25]). In Section 2 of this work we briefly introduce the notion of almost sectorial operator and we also present a slightly modified version of an abstract result given in [12] proving the existence of local solution for singularly nonautonomous equations in which features almost sectorial operators, extending the result presented in the aforementioned paper to incorporate other types of nonlinearities to which we can solve the semilinear problem.

The system is coupled only in one direction: Time-dependence of the diffusion term as well as the fact that the linear operator is not self-adjoint prevent us to construct an energy function for the equation. To study global well-posedness and the existence of attracting sets for the problem, we explore the equations in Ω and R_0 separately. The first equation (given in terms of w) is independent of the second equation (in terms of v), whereas the second one depends of the values that w assumes at the junction points. We say that those equations are coupled in only one direction. By knowing that a local solution exists, we can separate those equations and treat them independently. When they are separated, the problems are sectorial and we can use the regularizing effect well-known for equations with sectorial operators.

The equation in Ω: For w in Ω we have the following reaction-diffusion equation

$$\begin{cases} w_t - div(a(t,x)\nabla w) + w = f(w), & x \in \Omega,\ t > \tau, \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

which is also singularly nonautonomous equation, but now the linear operator is sectorial. Due to the time-dependence in the diffusion term, the approach to treat scalar reaction-diffusion equations via comparison results (as done in [7, 8]) is not available. Instead, we must use the analysis done in [9] for this same equation (1.2), where the authors obtained estimates for the solution through an iterative procedure: by knowing L^{2^k}−estimates, they were able to obtain $L^{2^{k+1}}-$ estimates. With those bounds and classical embedding theorems, global well-posedness and existence of attracting sets (for the component in Ω) follow.

The equation in R_0: On the line segment we have a reaction-diffusion equation with nonautonomous and nonlinear boundary condition,

$$\begin{cases} v_t - \partial_r(a(t,r)\partial_r v) + v = f(v), & r \in R_0,\ t > \tau, \\ v(t,p_0) = w(t,p_0) \text{ and } v(t,p_1) = w(t,p_1). \end{cases} \tag{1.3}$$

We perform a change of variables in a manner that the conditions on the boundary are incorporated to the equation. A dependence on w and on w_t will appear at this new equation and it will be necessary to provide ways to control the values that w and w_t assume at the points p_0 and p_1 in order for the equation in R_0, after the change of variables, to be properly defined. Once w and w_t are controlled and dissipation for the new problem is ensured, the long-time dynamics of $v(t)$ in R_0 is studied and we obtain the existence of pullback attracting sets for the equation in R_0.

Coupling the equations: With information on the asymptotic dynamics for w in Ω and v in R_0, we couple those two evolutions and obtain the existence of pullback attractors for the system in $\Omega_0 = \Omega \cup R_0$. This coupling provides several interesting insights on the pullback attractor, especially informations on how the dynamics in R_0 contributes to form the pullback attractor.

To sum up the ideas, in order to obtain local well-posedness we will treat the system in its integrity and use the theory of almost sectorial operators to construct the solution. Once existence of local solution is demonstrated, we separate the equations in Ω and R_0 and we use the fact that the linear operators associated to those are sectorial. We then apply the regularizing effect that differential equations with sectorial operator posses in order to obtain estimates and attracting sets for the solutions acting in Ω and R_0.

The iterative procedure developed to obtain estimates in L^{2^k} for the solutions is quite general and can be applied to other parabolic problems in which the linear operator is a second order regular elliptic boundary value problem (see [15, Section 1.2.4]). Moreover, no additional condition concerning monotonicity, decay or asymptotic behavior for the function $a(t,x)$ was necessary, which differs from some studies existent in the literature [11, 17, 19, 20, 29], where global well-posedness for the singularly nonautonomous case was only obtained after assuming one of those conditions.

To attend the proposed agenda, this paper is organized as follows. In Section 2 we present the abstract theory on singularly nonautonomous equations with almost sectorial operators and we prove a result on local well-posedness, Theorem 2.7. It differs from the result provided in [12], since it incorporates a wider class of Banach spaces and nonlinearities for which we can solve the problem. We also provide in this section a briefly review on sectorial operators and pullback attractors. Section 3 establishes the functional setting in which we will treat the evolution equation and the main results on the existence of local solution (Proposition 3.8), global solutions and pullback attractors (Theorem 3.10) for the problem. In Sections 4, 5 and 6 we prove the results enunciated in Section 3. To be precise, Section 4 deals with the equation in Ω, (1.2), Section 5 with the equation in R_0, (1.3), and in Section 6 we couple those informations.

2. **Abstract setting**

We separated this section into three parts: the first one deals with singularly nonautonomous semilinear problems with almost sectorial operators; in the second we treat the case where the linear operator is sectorial and we study regularization and smoothing effect of differential equations in which features sectorial operators; the last part is a brief review of pullback attractors.

2.1. **Local well-posedness for abstract semilinear problems with almost sectorial operators.** Consider the abstract semilinear evolution problem

$$\begin{cases} u_t + A(t)u = F(u), \quad t > \tau, \\ u(\tau) = u_0 \in Y \hookrightarrow X, \end{cases}$$

where X, Y are Banach spaces, Y is continuously embedded in X, which we denote by $Y \hookrightarrow X$, $A(t) : D(A(t)) \subset Y \to Y$, $t \in \mathbb{R}$, is a family of linear operators and $F : Y \to X$ a nonlinearity. We assume the following properties on the linear operators $A(t)$:

(P.1) $A(t) : D(A(t)) \subset Y \to Y$ is a closed densely defined linear operator, the domain $D = D(A(t))$ is fixed in time, and there exist constants $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, $C > 0$ and $\alpha \in (0, 1]$, independent of $t \in \mathbb{R}$, such that

$$\Sigma_\varphi \cup \{0\} \subset \rho(-A(t)),$$

where $\Sigma_\varphi := \{\lambda \in \mathbb{C}; |arg\lambda| \leq \varphi\}$, and

$$\left\|(\lambda + A(t))^{-1}\right\|_{\mathcal{L}(Y)} \leq \frac{C}{1 + |\lambda|^\alpha}, \quad \forall \lambda \in \Sigma_\varphi \cup \{0\}. \tag{2.1}$$

The family $\{A(t)\}_{t\in\mathbb{R}}$ is called $\alpha-$*uniformly almost sectorial* and α is the *constant of almost sectoriality.*

(P.2) There exists a less regular Banach space X such that $Y \hookrightarrow X$ and the extension of $A(t)$ in X is also $\alpha-$uniformly almost sectorial. Moreover, the resolvent of $A(t)$ has the following regularizing property: it takes elements of X into the Banach space Y and there exists $\beta \in (0, 1]$ such that

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(X,Y)} \leq \frac{C}{1 + |\lambda|^\beta}, \quad \text{for all } \lambda \in \Sigma_\varphi \cup \{0\}. \tag{2.2}$$

(P.3) There are constants $C > 0$ and $\delta \in (0, 1]$ such that, for any $t, \tau, s \in \mathbb{R}$,

$$\|[A(t) - A(\tau)]A^{-1}(s)\|_{\mathcal{L}(Y)} \leq C|t - \tau|^\delta. \tag{2.3}$$

We say that the function $\mathbb{R} \ni t \mapsto A(t)A^{-1}(s) \in \mathcal{L}(Y)$ is $\delta-$*uniformly Hölder continuous.*

Conditions (P.1) - (P.2) state that each operator $A(t)$ is almost sectorial and there is a uniformity in this almost sectoriality. As far as condition (P.3), note that if $\tau = s$ in (2.3) and (t, s) lies in a compact set of $\mathbb{R}^2$, then

$$\left\|A(t)A(s)^{-1}\right\|_{\mathcal{L}(Y)} \leq C.$$

2.1.1. *The autonomous and homogeneous linear problem.* For a fixed $\tau \in \mathbb{R}$, the linear operator $A(\tau)$ enjoys the properties (P.1) and (P.2) stated above. If $\alpha \in (0, 1)$ in (2.1) then $-A(\tau)$ does not generate a C_0-semigroup, nevertheless this *almost sectorial* operator generates a special type of semigroup, called *semigroup of growth* $1 - \alpha$. It was proved in [12] that $-A(\tau)$ generates a family of linear operators $T_{-A(\tau)}(t)$ given by

$$T_{-A(\tau)}(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t}(\lambda + A(\tau))^{-1} d\lambda, \quad \text{for all } t > 0, \tag{2.4}$$

where Γ is the contour of Σ_φ, that is, $\Gamma = \{re^{-i\varphi} : r > 0\} \cup \{re^{i\varphi} : r > 0\}$ and it is orientated with increasing imaginary part. Some properties of $T_{-A(\tau)}(\cdot)$ are listed below. They follow from the estimates for the linear operator $A(t)$ and the expression (2.4) for $T_{-A(\tau)}(\cdot)$. Its proof can be found in [5, Section 2].

Proposition 2.1. *For a fixed $\tau \in \mathbb{R}$, if $T_{-A(\tau)}(t)$, $t > 0$, is the family defined in* (2.4)*, then:*

(1) $T_{-A(\tau)}(\cdot)$ is a semigroup and satisfies $T_{-A(\tau)}(t)T_{-A(\tau)}(s) = T_{-A(\tau)}(t + s)$, for all $t, s > 0$.
(2) $T_{-A(\tau)}(t)$ has its images in $D(A(\tau)) \subset Y$, that is, the semigroup has a regularizing effect for the solution.
(3) $T_{-A(\tau)}(t)$ is a bounded linear operator and, for all $t > 0$,

$$\left\|T_{-A(\tau)}(t)\right\|_{\mathcal{L}(Y)} \leq Ct^{\alpha-1} \quad \text{and} \quad \left\|T_{-A(\tau)}(t)\right\|_{\mathcal{L}(X,Y)} \leq Ct^{\beta-1}.$$

For semigroups of growth, continuity of $[0,\infty) \ni t \mapsto T_{-A(\tau)}(t)y$ for each $y \in Y$ does not necessarily hold for $t = 0$. The estimates obtained for the family $T_{-A(\tau)}(t)$ in item (3) of Proposition 2.1 reflects the fact that there might exist $y \in Y$ such that $T_{-A(\tau)}(t)y \nrightarrow y$ as $t \to 0^+$. Those estimates justify the name *semigroup of growth* adopted for them. For further results on semigroups of growth, we recommend [16, 25]. The importance of the family given by (2.4) relies on the fact that it is closely related to the solution of the autonomous associated problem. For instance, consider the problem

$$\begin{cases} u_t + A(\tau)u = 0, \ t > 0; \\ u(0) = u_0 \in Y, \end{cases} \tag{2.5}$$

where $\tau \in \mathbb{R}$ is fixed. The next result states that $u(t) = T_{-A(\tau)}(t)u_0$ is a classical solution for (2.5).

Lemma 2.2. *(*[5, Lemma 2.1 and Lemma 2.4]*) Let $T_{-A(\tau)}(t)$ be the linear operator defined in* (2.4)*. The mapping $(0,\infty) \ni t \mapsto T_{-A(\tau)}(t) \in \mathcal{L}(Y)$ is differentiable and*

$$\frac{d}{dt}T_{-A(\tau)}(t) = -A(\tau)T_{-A(\tau)}(t).$$

In particular, for $u_0 \in Y$, $u(t) = T_{-A(\tau)}(t)u_0$ is a classical solution of (2.5) *for $t > 0$.*

2.1.2. *The nonautonomous linear associated problem.* Consider the singularly nonautonomous problem

$$\begin{cases} u_t + A(t)u = 0, \quad t > \tau, \ \tau \in \mathbb{R}, \\ u(\tau) = u_0 \in Y. \end{cases} \tag{2.6}$$

We search for a two parameter family of linear operator $U(t,\tau)$ that, in some sense, is connected to the solution of the evolution equation and plays a similar role that the semigroup $T_{-A(\tau)}(t)$ does for the autonomous case (2.5). Ideally, if $u(t,\tau,u_0)$ is the local solution for (2.6), we would be searching for a family $U(t,\tau)$ such that $u(t,\tau,u_0) = U(t,\tau)u_0$. This problem was studied simultaneously by Sobolevskii [24] and Tanabe [26, 27, 28] for the sectorial case. The almost sectorial case was then considered in [12]. We briefly motivate the formal computation that inspires the definition of $U(t,\tau)$.

Suppose $U(t,\tau)$ is a family satisfying $\partial_t U(t,\tau) = -A(t)U(t,\tau)$ (that is, a solution of (2.6)). Also, assume that there exists another family $\Phi(t,\tau) \in \mathcal{L}(Y)$ such that $U(t,\tau)$ is obtained trough the integral equation below

$$U(t,\tau) = T_{-A(\tau)}(t-\tau) + \int_\tau^t T_{-A(s)}(t-s)\Phi(s,\tau)ds. \tag{2.7}$$

Differentiating (2.7) in t, adding $A(t)U(t,\tau)$ on both sides and using $\partial_t U(t,\tau) + A(t)U(t,\tau) = 0$, we deduce

$$0 = \Phi(t,\tau) - [A(\tau) - A(t)]T_{-A(\tau)}(t-\tau) - \int_\tau^t [A(s) - A(t)]T_{-A(s)}(t-s)\Phi(s,\tau)ds.$$

If we denoted

$$\varphi_1(t,\tau) = [A(\tau) - A(t)]T_{-A(\tau)}(t-\tau), \tag{2.8}$$

then $\Phi(t,\tau)$ would have to satisfy

$$\Phi(t,\tau) = \varphi_1(t,\tau) + \int_\tau^t \varphi_1(t,s)\Phi(s,\tau)ds. \tag{2.9}$$

If we had a family $\Phi(t,\tau)$ satisfying (2.9), then we could proceed in the reverse way to obtain $U(t,\tau)$. This is actually what the authors in [12, Section 2] did and we enunciate in the sequel.

Proposition 2.3. [12, Section 2] *Let $A(t), t \in \mathbb{R}$, be a family of linear operators satisfying (P.1) - (P.3), where $\alpha \in (0,1]$ is the constant of almost sectoriality and $\delta \in (0,1]$ the constant of Hölder continuity.*

(1) The family $\{\varphi_1(t,\tau) \in \mathcal{L}(Y);\ t > \tau\}$ given by (2.8) *is continuous in the uniform operator topology, that is, $\{(t,\tau) \in \mathbb{R}^2; t > \tau\} \ni (t,\tau) \mapsto \varphi_1(t,\tau) \in \mathcal{L}(Y)$ is continuous and its norm can be estimated by*

$$\|\varphi_1(t,\tau)\|_{\mathcal{L}(Y)} \leq C(t-\tau)^{\alpha+\delta-2}, \quad \textit{for } t > \tau.$$

(2) If $\alpha+\delta>1$, then there exists a unique family $\{\Phi(t,\tau)\in\mathcal{L}(Y);\ t>\tau\}$ that satisfies (2.9) *and this family is continuous, that is, $\{(t,\tau)\in\mathbb{R}^2; t>\tau\}\ni(t,\tau)\mapsto\Phi(t,\tau)\in\mathcal{L}(Y)$ is continuous. Moreover, for any $T>0$, there exists $C=C(T)>0$ such that*

$$\|\Phi(t,\tau)\|_{\mathcal{L}(Y)}\le C(t-\tau)^{\alpha+\delta-2}\quad \text{for all } 0<t-\tau\le T.$$

(3) If $\alpha+\delta>1$, then there exists a unique two parameter family $\{U(t,\tau)\in\mathcal{L}(Y);\ t>\tau\}$ given by (2.7) *such that $\{(t,\tau)\in\mathbb{R}^2; t>\tau\}\ni(t,\tau)\mapsto U(t,\tau)\in\mathcal{L}(Y)$ is continuous and, for each $T>0$, there exists constant $C=C(T)>0$ such that*

$$\|U(t,\tau)\|_{\mathcal{L}(Y)}\le C(t-\tau)^{\alpha-1}\quad \text{for all } 0<t-\tau\le T.$$

We refer to the family $U(t,\tau)$ as linear process of growth $1-\alpha$ associated to $A(t),\ t\in\mathbb{R}$.

Note that the existence of the family $U(t,\tau)$ depends on the condition $\alpha+\delta>1$, which is trivially satisfied in the sectorial case ($\alpha=1$). In the sequel we present some additional properties of $U(t,\tau)$ that can be found in [10].

Proposition 2.4. *Let $A(t)$, $t\in\mathbb{R}$, be a family of linear operators satisfying (P.1) - (P.3) and $U(t,\tau)$ the linear process associated to $A(t),\ t\in\mathbb{R}$. Then*

(1) $U(t,s)U(s,\tau)=U(t,\tau)$, for all $\tau<s<t$.
(2) $(\tau,\infty)\ni t\mapsto U(t,\tau)\in\mathcal{L}(Y)$ is strongly differentiable and $\partial_t U(t,\tau)=-A(t)U(t,\tau)$.

As it happens for the semigroup $T_{-A(\tau)}(\cdot)$, the linear process $U(t,\tau)$ also has a regularizing property, taking elements from X into Y (see [10, Theorem 2.13]). In this case, $U(t,\tau)\in\mathcal{L}(X,Y)$ and we obtain the following estimate for the linear process.

Lemma 2.5. *Assume that (P.1) - (P.3) hold and let $\beta\in(0,1]$ be the constant in* (2.2)*. If $\alpha+\delta>1$ then for any $T>0$ there exists $C=C(T)>0$ such that*

$$\|U(t,\tau)\|_{\mathcal{L}(X,Y)}\le C(t-\tau)^{\beta-1},\quad \text{for all } 0<t-\tau\le T.$$

Proof. We first note that as a consequence of condition (P.2), we also obtain $\|\Phi(t,\tau)\|_{\mathcal{L}(X)}\le C(T)(t-\tau)^{\alpha+\delta-1}$. Therefore, expression (2.7) for the linear process and estimates obtained in Proposition 2.1 imply that

$$\begin{aligned}\|U(t,\tau)\|_{\mathcal{L}(X,Y)}&\le\|T_{-A(\tau)}(t-\tau)\|_{\mathcal{L}(X,Y)}+\int_\tau^t\|T_{-A(s)}(t-s)\|_{\mathcal{L}(X,Y)}\|\Phi(s,\tau)\|_{\mathcal{L}(X)}ds\\&\le C(t-\tau)^{\beta-1}+C(T)\int_\tau^t(t-s)^{\beta-1}(s-\tau)^{\alpha+\delta-2}ds\\&\le C(t-\tau)^{\beta-1}+C(T)(t-\tau)^{\alpha+\beta+\delta-2}\mathcal{B}(\beta,\alpha+\delta-1)\\&\le C(t-\tau)^{\beta-1}\left[1+C(T)(t-\tau)^{\alpha+\delta-1}\mathcal{B}(\beta,\alpha+\delta-1)\right]\\&\le C(T)(t-\tau)^{\beta-1}.\end{aligned}$$

□

Earlier we mentioned that the linear process $U(t,\tau)$ indeed recover the solution of the nonautonomous homogeneous problem (2.6) by considering $u(t)=U(t,\tau)u_0$. Not only $U(t,\tau)$ solves the nonautonomous homogeneous problem (2.6), but it is also an important tool to solve the semilinear problem, as we discuss next.

2.1.3. *Existence of local solution for the semilinear problem.* Consider the semilinear problem

$$\begin{cases}u_t+A(t)u=F(u),\ t>\tau;\\ u(\tau)=u_0\in Y\hookrightarrow X,\end{cases}\tag{2.10}$$

where $A(t),t\in\mathbb{R}$, satisfies (P.1) - (P.3). We assume the following condition for F:

(NL) The nonlinearity $F : Y \to X$ has a polynomial growth of order ρ, that is, there exist constants $C > 0$ and $\rho \geq 1$, such that for every $u, v \in Y$,

$$\|F(u) - F(v)\|_X \leq C \|u - v\|_Y \left(1 + \|u\|_Y^{\rho-1} + \|v\|_Y^{\rho-1}\right),$$
$$\|F(u)\|_X \leq C(1 + \|u\|_Y^{\rho}).$$

As it happens for the semigroup $T_{-A(\tau)}(t)$, the family $U(t,\tau)$ might be discontinuous at the initial time $t = \tau$ [10, Section 2]. Consequently, we shall search for a solution for the semilinear problem that has the same type of discontinuity of $U(t,\tau)$.

Definition 2.6. *A function $u : (\tau, \tau + t_0] \to Y$ is a* mild solution *in $(\tau, \tau + t_0]$ for* (2.10) *if*

(1) $u(\cdot) \in \mathcal{C}((\tau, \tau + t_0], Y)$ and $\lim_{t \to \tau^+} \|u(t) - U(t,\tau)u_0\|_Y = 0$,

(2) $u(\cdot)$ is given by the variation of constants formula

$$u(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F(u(s))ds, \qquad \textit{for all } t \in (\tau, \tau + t_0].$$

The following theorem proves the existence of mild solution for the problem studied. It slightly differs from Theorem 3.1 in [12], where the authors approached the problem using fractional power spaces. The way we enunciate this theorem allows more general situations, as the nonautonomous equation (1.1) that we wish to study, where F is not sub-linear and presents a certain growth. We shall focus now on the case where the problem is not sectorial, that is, $\alpha \in (0,1)$. The sectorial case ($\alpha = 1$) will be treated separately on the next subsection.

Theorem 2.7. *Assume that conditions (P.1) - (P.3) hold. Let $\alpha \in (0,1)$ be the constant of uniformly almost sectoriality, $\delta \in (0,1]$ be the constant of uniformly Hölder continuity and $\beta \in (0,1]$ the constant given in (P.2). Additionally, assume that $\alpha + \delta > 1$ and $F : Y \to X$ satisfies(NL) with*

$$1 \leq \rho < \frac{\beta}{1 - \alpha}. \tag{2.11}$$

Then, for every $u_0 \in Y$, there exists $t_0 > 0$ such that the initial value problem (2.10) *has a unique mild solution defined in $(\tau, \tau + t_0]$ and t_0 depends on u_0, but can be chosen uniformly for u_0 in bounded subsets of Y.*

Proof. Given $\tau \in \mathbb{R}$, $u_0 \in Y$ and $\mu > 0$, we search for solutions in the space

$$K(t_0, u_0) = \left\{ v \in \mathcal{C}((\tau, \tau + t_0], Y);\ \sup_{t \in (\tau, \tau + t_0]} \|v(t) - U(t,\tau)u_0\|_Y \leq \mu \right\},$$

where $t_0 > 0$ will be suitably chosen later. If we set $\|\xi\|_K = \sup_{t \in (\tau, \tau + t_0]} \|\xi(t) - U(t,\tau)u_0\|_Y$, then $K(t_0, u_0)$ is a Banach space with this norm. Note that

$$(t-\tau)^{1-\alpha} \|v(t)\|_Y \leq (t-\tau)^{1-\alpha} \|v(t) - U(t,\tau)u_0\|_Y + (t-\tau)^{1-\alpha} \|U(t,\tau)u_0\|_Y \leq t_0^{1-\alpha}\mu + C\|u_0\|_Y \leq k,$$

and k can be chosen uniformly for u_0 in bounded subsets of Y. The condition $\alpha + \delta > 1$ guarantees the existence of the family $U(t,\tau)$ (see Proposition 2.3 and Lemma 2.5). We consider the operator

$$(\Psi v)(t) := U(t,\tau)u_0 + \int_\tau^t U(t,s)F(v(s))ds, \quad t \in (\tau, \tau + t_0],$$

defined in $K(t_0, u_0)$. From the continuity of $(t,\tau) \mapsto U(t,\tau)u_0$ it follows that $\Psi v \in C((\tau, \tau + t_0], Y)$. We prove that, for small values of t_0, Ψ is a contraction in $K(t_0, u_0)$. The Banach Fixed Point Theorem will ensure then the existence of a unique fixed point for Ψ in $K(t_0, u_0)$, which is the solution of (2.10). Given $v \in K(t_0, u_0)$ and using the estimates available for the linear process $U(t,\tau)$, we have

$$\begin{aligned}
\|(\Psi v)(t) - U(t,\tau)u_0\|_Y &\leq \int_\tau^t \|U(t,s)F(v(s))\|_Y \, ds \leq \int_\tau^t \|U(t,s)\|_{\mathcal{L}(X,Y)} \|F(v(s))\|_X \, ds \\
&\leq C \int_\tau^t (t-s)^{\beta-1}(1 + k^\rho (s-\tau)^{(\alpha-1)\rho}) ds \\
&\leq C(t-\tau)^\beta + Ck^\rho (t-\tau)^{\beta-(1-\alpha)\rho} \mathcal{B}(\beta, 1 - (1-\alpha)\rho).
\end{aligned}$$

Condition (2.11) implies $\beta - (1-\alpha)\rho > 0$ and also $1 - (1-\alpha)\rho > 0$. Therefore,

$$\|\Psi v(t) - U(t,\tau)u_0\|_Y \leq C[t_0^\beta + t_0^{\beta-(1-\alpha)\rho}] \leq \mu,$$

for t_0 small enough. Moreover, $\|\Psi v(t) - U(t,\tau)v_0\|_Y \xrightarrow{t\to\tau^+} 0$. Also,

$$\begin{aligned}\|\Psi v(t) - \Psi w(t)\|_Y &\leq \int_\tau^t \|U(t,s)\|_{\mathcal{L}(X,Y)} \|F(v(s)) - F(w(s))\|_X \, ds \\ &\leq \int_\tau^t C(t-s)^{\beta-1} \|v(s) - w(s)\|_Y (1 + \|v(s)\|_Y^{\rho-1} + \|w(s)\|_Y^{\rho-1}) ds \\ &\leq C \int_\tau^t (t-s)^{\beta-1} \left(1 + 2k^{\rho-1}(s-\tau)^{-(1-\alpha)(\rho-1)}\right) ds \, \|v - w\|_K \\ &\leq C \left[(t-\tau)^\beta + 2k^{\rho-1}(t-\tau)^{\beta-(1-\alpha)(\rho-1)}\mathcal{B}(\beta, 1-(1-\alpha)(\rho-1))\right] \|v-w\|_K\end{aligned}$$

and from condition (2.11), $\beta - (1-\alpha)(\rho-1) > 0$ and, consequently, $1 - (1-\alpha)(\rho-1) > 0$. Therefore, for t_0 sufficiently small and any $t \in (\tau, \tau + t_0)$,

$$\|\Psi v(t) - \Psi w(t)\|_Y \leq C \left[t_0^\beta + t_0^{\beta-(1-\alpha)(\rho-1)}\right] \|v - w\|_K \leq \frac{1}{2} \|v - w\|_K .$$

□

2.2. **Semilinear problems with sectorial operators: regularization and smoothing effect.** If $\alpha = 1$ in (P.1), then instead of an almost sectorial operator $A(t)$, $t \in \mathbb{R}$, we have a sectorial operator for which the usual theory on semigroup and linear process generation holds [23]. A family $A(t)$, $t \in \mathbb{R}$, satisfying (P.1) with $\alpha = 1$ is called *uniformly sectorial.*

In this case there is no deficiency in the resolvent estimate, each operator $-A(\tau)$ generates an analytic C_0-semigroup such that $[0,\infty) \ni t \mapsto T_{-A(\tau)}(t)y$ is continuous (including at $t = 0$), for any $y \in Y$. If (P.3) holds, we can also ensure existence of a linear process $U(t,\tau) \in \mathcal{L}(Y)$ associated to $A(t)$, $t \in \mathbb{R}$, such that $[\tau,\infty) \ni t \to U(t,\tau)y \in Y$ is continuous (including at $t = \tau$). The estimates for those linear families are:

$$\|T_{-A(\tau)}(t)\|_{\mathcal{L}(Y)} \leq C \quad \text{and} \quad \|U(t,\tau)\|_{\mathcal{L}(Y)} \leq C.$$

Moreover, parabolic problems in which features sectorial operators have well known regularizing properties [9, 24] which is useful in the analysis of the long time dynamics of the solution. To take advantage of this well established regularizing property, we introduce the scale of fractional powers associated to the operators $A(t)$.

Each operator $A(t)$, $t \in \mathbb{R}$, is positive and its fractional powers are well defined (in the sense of [3]). For every $\omega > 0$, $[A(t)]^\omega : D([A(t)]^\omega) \subset Y \to Y$ is a linear operator and it generates a scale of Banach spaces given by the fractional power spaces $\{[Y(t)]^\omega\}_{\omega \geq 0}$, where

$$[Y(t)]^\omega = D([A(t)]^\omega) \quad \text{and} \quad \|\cdot\|_{[Y(t)]^\omega} = \|\cdot\|_{D([A(t)]^\omega)} = \|[A(t)]^\omega \cdot\|_Y.$$

From (P.3) and the fact that $\|A(t)A^{-1}(\tau)\|_{\mathcal{L}(Y)} \leq C$, we conclude that the norms

$$\|\cdot\|_{[Y(t)]^\omega} = \|[A(t)]^\omega \cdot\|_Y \quad \text{and} \quad \|\cdot\|_{[Y(\tau)]^\omega} = \|[A(\tau)]^\omega \cdot\|_Y$$

are equivalent. Therefore, we can fix a time t and refer to Y^ω as domain of any operator $[A(t)]^\omega$. Henceforth, we obtain a scale of fractional powers $\{Y^\omega\}_{\omega\geq 0}$ that does not depend on $t \in \mathbb{R}$ and we can extend it for negative powers by defining, for $\omega \in (0,1)$, $Y^{-\omega} = [Y^\omega]^*$, where $[Y^\omega]^*$ stands for the dual of Y^ω (see [3] and [30] for the negative fractional power spaces). For this scale we have the following estimates.

Proposition 2.8. [14, Theorem 2.2] *For $0 \leq \gamma \leq \omega < 1 + \delta$, there exists a constant $C(\omega,\gamma) > 0$ such that*

$$\|A(t)^\omega U(t,\tau)A(\tau)^{-\gamma}\|_{\mathcal{L}(Y)} \leq C(\omega,\gamma)(t-\tau)^{\gamma-\omega}, \quad \textit{for all } t > \tau, \ \tau \in \mathbb{R}.$$

The scale of fractional power space allows us to obtain solutions in more regular spaces than Y itself, that is, on fractional powers of Y. Consider the abstract semilinear problem:

$$\begin{cases} u_t + A(t)u = F(t,u), \ t > \tau; \\ u(\tau) = u_0 \in Y \hookrightarrow Y^{-\theta}, \end{cases} \tag{2.12}$$

where F is a nonlinearity such that $F : \mathbb{R} \times Y \to Y^{-\theta}$, $0 \leq \theta < 1$. In [24, Theorem 7] local well-posedness of (2.12) was proved, which we state next:

Theorem 2.9. [24, Theorem 7] *Let $A(t)$, $t \in \mathbb{R}$, be a family of linear operators satisfying (P.1) with $\alpha = 1$ and (P.3) with constant $\delta \in (0, 1]$ of Hölder continuity. Moreover, assume that $F : \mathbb{R} \times Y \to Y^{-\theta}$ is a locally Hölder continuous function in the first variable and locally Lipschitz in the second variable, $0 \leq \theta < 1$. If $U(t, \tau)$ denotes the linear process associated to $A(t)$, then, given any $u_0 \in Y$,*

$$u(t) = U(t, \tau)u_0 + \int_\tau^t U(t, s)F(s, u(s))ds$$

is a strong solution for (2.12)*, that is,*

(1) $u(\cdot) \in \mathcal{C}([\tau, T), Y) \cap \mathcal{C}^1((\tau, T), Y)$ and $u(t)$ satisfies the equation in (2.12) *for $t > \tau$.*
(2) $u(t) \in Y^\xi$, for any $0 \leq \xi < 1 - \theta$ and $t > \tau$.
(3) If $\|u(t)\|_Y$ is bounded in any bounded set $[\tau, t^*]$, then $u(\cdot)$ is globally defined in time.
(4) $u_t(t) \in Y^\xi$, for any $0 \leq \xi < \delta - \theta$ and $t > \tau$.

Remark 2.10. *Usually theorem above is stated with the initial condition in Y^γ, $\leq 0 \leq \gamma < 1$, and $F : \mathbb{R} \times Y^\gamma \to Y$. As a consequence, $u(t) \in Y^\xi$, $0 \leq \xi < 1$, and $u_t(t) \in Y^\omega$, $0 \leq \omega < \delta$. The case presented above is more suitable for the application we will consider and it is obtained from the classical presentation considering a realization of $A(t)$ in the space $Z := Y^{-\theta}$.*

Note that this type of result is similar to Theorem 2.7 that we proved earlier. However, the sectoriality of $A(t)$, $t \in \mathbb{R}$ allows us to obtain more regularity for the solution $u(t)$ and its derivative $u_t(t)$. In particular, u is a strong solution for the problem. We call this property *regularization.*

2.3. **Pullback attractor.** In this last part of the abstract theory we provide a brief review in the theory of pullback attractors. For more details, we recommend [13]. Let Y be a Banach space and $\{S(t, \tau) : Y \to Y; t \geq \tau\}$ a family of operators satisfying:

(1) $S(t, t) = I_Y$, for all $t \in \mathbb{R}$, and $S(t, \tau) = S(t, s)S(s, \tau)$, for all $t \geq s \geq \tau$, $\tau \in \mathbb{R}$.
(2) $(\tau, \infty) \ni t \mapsto S(t, \tau)u_0$ is continuous for all $u_0 \in Y$.

Such family is called a *process in* Y and we also denote it by $S(\cdot, \cdot)$. We will usually call it *nonlinear process* to distinguish from the family $U(t, \tau)$ associated to $A(t)$. To compare the distance between two sets in the phase space Y, we use the *Hausdorff semidistance*: $dist(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$, for $A, B \subset Y$, where d is the metric in Y.

Definition 2.11. *The* pullback attractor *of $S(\cdot, \cdot)$ is a family $\mathcal{A}(\cdot) = \{\mathcal{A}(t) \subset Y;\ t \in \mathbb{R}\}$ satisfying:*

(1) $\mathcal{A}(t)$ is compact for all $t \in \mathbb{R}$.
(2) $\mathcal{A}(\cdot)$ is invariant *by $S(\cdot, \cdot)$, that is, $S(t, s)\mathcal{A}(s) = \mathcal{A}(t)$, for all $t \geq s$, $s \in \mathbb{R}$.*
(3) $\mathcal{A}(\cdot)$ pullback attracts bounded sets of Y, that is, for $B \subset Y$ bounded and $t \in \mathbb{R}$, $dist(S(t, s)B, \mathcal{A}(t)) \overset{s \to -\infty}{\longrightarrow} 0$.
(4) $\mathcal{A}(\cdot)$ is the minimal closed family that satisfies $(1) - (3)$.

The existence of such object in the phase space is guaranteed whenever we find a family of compact pullback attracting sets.

Theorem 2.12. [13, Theorem 2.12] *A process $S(\cdot, \cdot)$ has a pullback attractor $\mathcal{A}(\cdot)$ if, and only if, there exists a family of compact sets $K(\cdot)$ that pullback attracts bounded sets of Y.*

Corollary 2.13. *If there exists a fixed compact set $K \subset Y$ such that, for any bounded set $B \subset Y$,*

$$dist(S(t, s)B, K) \to 0 \text{ when } s \to -\infty,$$

then $S(\cdot, \cdot)$ has a pullback attractor $\mathcal{A}(\cdot)$ such that $\bigcup_{t \in \mathbb{R}} \mathcal{A}(t) \subset K$.

The description of the pullback attractor can be given in terms of global bounded solutions, which we define in the sequel.

Definition 2.14. *A continuous function $\xi : \mathbb{R} \to Y$ is a* global solution *of $S(\cdot,\cdot)$ if for any $t, s \in \mathbb{R}$, $t \geq s$, $S(t,s)\xi(s) = \xi(t)$. Moreover, we say that a global solution $\xi(\cdot) : \mathbb{R} \to Y$ of $S(\cdot,\cdot)$ is* bounded in the past *if there exists $\tau \in \mathbb{R}$ such that $\{\xi(t) : \ t \leq \tau\}$ is bounded in Y.*

Proposition 2.15. [13, Theorem 1.17] *Suppose that the pullback attractor $\mathcal{A}(\cdot)$ for the process $S(\cdot,\cdot)$ is bounded in the past. Then*

$$\mathcal{A}(t) = \{\xi(t) : \ \xi(\cdot) : \mathbb{R} \to Y \ \textit{is a global solution bounded in the past}\}.$$

3. Singularly nonautonomous reaction-diffusion equation in Ω_0

In this section we present a functional setting in which we treat the equation (1.1) and the main results concerning local solvability, global well-posedness and existence of pullback attractor. We assume that the following conditions for (1.1) hold:

(A.1). The function $a : \mathbb{R} \times \overline{\Omega_0} \to \mathbb{R}^+$ is continuously differentiable in each of the sets that form Ω_0, that is, $a \in \mathcal{C}^1(\mathbb{R} \times \Omega, \mathbb{R}^+)$ and $a \in \mathcal{C}^1(\mathbb{R} \times R_0, \mathbb{R}^+)$. The differentiability on R_0 means that if we consider the function $h : (0,1) \mapsto \mathbb{R}^+$ given by $h(r) = a(t,(r,0))$, then $h \in \mathcal{C}^1(0,1)$.

(A.2). The function $a(t,x)$ has its image in a closed interval $[a_0, a_1] \subset (0,\infty)$ and if we denote by $a'(t,x) = \frac{d}{dt}a(t,x)$, $b(t,x) := \nabla_x a(t,x)$ the gradient function (in x) of $a(t,x)$, then $a'(t,x)$ and $b(t,x)$ are both bounded, that is, $a'(t,x) \in L^\infty(\Omega_0)$ and $b(t,x) \in [L^\infty(\Omega_0)]^N$.

(A.3). The functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are $\delta-$*Hölder continuous in the first variable* with $\delta \in (0,1]$:

$$|a(t,x) - a(s,x)| \leq C|t-s|^\delta, \quad |b(t,x) - b(s,x)| \leq C|t-s|^\delta.$$

(A.4). The nonlinearity $f \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ and satisfies a *polynomial growth condition*, that is,

$$|f'(\xi)| \leq C(1 + |\xi|^{\rho-1}), \text{ for some } \rho \geq 1. \tag{3.1}$$

Remark 3.1. *We use x for the variable that takes values in Ω, r for the variable that takes values in R_0 and $t, s, \tau \in \mathbb{R}$ for variables representing time. Note that r has the form $(z,0) \in \mathbb{R} \times \mathbb{R}^{N-1}$, with $z \in [0,1]$. We consider r as an element in the interval $[0,1]$ and treat $v_r(t,r)$ as the derivative of v in the real variable $r \in [0,1]$.*

The phase space in which we will solve the equation is the Banach space

$$U_p^0 = L^p(\Omega) \times L^p(0,1) \quad \text{ with norm } \|(w,v)\|_{U_p^0} = \|w\|_{L^p(\Omega)} + \|v\|_{L^p(0,1)},$$

and (1.1) originates the following abstract singular semilinear evolution equation:

$$\begin{cases} (w,v)_t + A(t)(w,v) = F(w,v), & t > \tau; \\ (w,v)(\tau) = (w_0,v_0) \in U_p^0, \end{cases}$$

where $A(t) : D(A(t)) \subset U_p^0 \to U_p^0$ is the linear operator with time-independent domain $D = D(A(t))$ given by

$$D = \left\{(w,v) \in W^{2,p}(\Omega) \times W^{2,p}(0,1) : \partial_n w = 0 \text{ in } \partial\Omega \text{ and } v(p_i) = w(p_i), i = 0,1\right\}, \tag{3.2}$$

$$A(t)(w,v) = (-div(a(t,x)\nabla w) + w, -\partial_r(a(t,r)\partial_r v) + v), \quad \text{for } (w,v) \in D, \tag{3.3}$$

and the nonlinearity F is given by

$$F(w,v)(x) = \begin{cases} f(w(x)), & x \in \Omega, \\ f(v(r)), & r \in R_0. \end{cases} \tag{3.4}$$

Remark 3.2. *Conditions at p_0 and p_1 in* (3.2) *only makes sense if $w \in C(\overline{\Omega})$. Therefore, $p > \frac{N}{2}$ must be required in order to ensure that $W^{2,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ (see* [1, Theorem 5.4]*).*

Under those conditions, we have the following properties for the family $A(t), t \in \mathbb{R}$.

Lemma 3.3. *Let $A(t) : D \subset U_p^0 \to U_p^0$ be defined as in* (3.3)*, then there exists $C > 0$ such that*

$$\|[A(t) - A(s)]A(\tau)^{-1}\|_{\mathcal{L}(U_p^0)} \leq C|t-s|^\delta, \quad \textit{for all } \tau, s, t \in \mathbb{R}.$$

Proof. For $(w,v)\in D$, we have

$$A(t)(w,v)-A(s)(w,v)=\left(-div\left([a(t,x)-a(s,x)]\nabla w\right),-\partial_r\left([a(t,r)-a(t,s)]\partial_r v\right)\right)$$

and

$$\begin{aligned}\int_\Omega |div([a(t,x)-a(s,x)]\nabla w(x))|^p\,dx &= \int_\Omega |\nabla_x([a(t,x)-a(s,x)])\nabla w(x)+[a(t,x)-a(s,x)]\Delta w|^p\,dx\\ &\le C|t-s|^{\delta p}\int_\Omega\left\{\frac{|\nabla_x a(t,x)-\nabla_x a(s,x)|}{|t-s|^\delta}\right\}^p|\nabla w(x)|^p dx+C|t-s|^{\delta p}\int_\Omega\left\{\frac{|a(t,x)-a(s,x)|}{|t-s|^\delta}\right\}^p|\Delta w(x)|^p dx\\ &\le C|t-s|^{\delta p}\|w\|^p_{W^{2,p}(\Omega)}.\end{aligned}$$

The same calculation on the line segment R_0 provides

$$\int_0^1 |\partial_r\left([a(t,r)-a(s,r)]\partial_r v(r)\right)|^p\,dx\le C|t-s|^{\delta p}\|v\|^p_{W^{2,p}(0,1)}.$$

Therefore, $\|[A(t)-A(s)](w,v)\|^p_{U^0_p}\le C|t-s|^{p\delta}\|(w,v)\|^p_D$, for all $(w,v)\in D$. Taking the $p-th$ roots on both sides and replacing (w,v) by $A(\tau)^{-1}(\tilde w,\tilde v)$, we deduce

$$\|[A(t)-A(s)]A(\tau)^{-1}(\tilde w,\tilde v)\|_{U^0_p}\le C|t-s|^\delta\|(\tilde w,\tilde v)\|_{U^0_p},\quad \forall(\tilde w,\tilde v)\in U^0_p.$$

□

In [12, Proposition 4.1] and [5, Proposition 3.1] several properties of the family $A(t),t\in\mathbb{R}$, are presented, including its almost sectoriality. We enunciate them in the sequel and in the last statement we provide information about the spectrum of $A(t)$ that can be found in [5, Section 3.2].

Proposition 3.4. *The family of linear operators $A(t),t\in\mathbb{R}$, satisfies:*

(1) $A(t)$ is a closed linear operator and it has a fixed dense domain D.

(2) $A(t)$ has compact resolvent and the semigroup $T_{-A(t)}(s)$ is compact.

(3) There exists $\varphi\in\left(\frac{\pi}{2},\pi\right)$ and $C>0$ (independent of t) such that $\Sigma_\varphi\subset\rho(-A(t))$, for all $t\in\mathbb{R}$, and, for $\frac{N}{2}<q\le p$, $\lambda\in\Sigma_\varphi\cup\{0\}$, we have

$$\|(\lambda+A(t))^{-1}\|_{\mathcal{L}(U^0_q,U^0_p)}\le\frac{C}{|\lambda|^\beta+1},$$

for each $0<\beta<1-\frac{N}{2q}-\frac12\left(\frac1q-\frac1p\right)$. In particular, the case $q=p$ yields

$$\|(\lambda+A(t))^{-1}\|_{\mathcal{L}(U^0_p)}\le\frac{C}{|\lambda|^\alpha+1},$$

for $0<\alpha<1-\frac{N}{2p}<1$.

(4) The spectrum of $A(t)$ consists entirely of isolated eigenvalues, all of them positive and real

$$\sigma(A(t))=\{\lambda_i(t):1=\lambda_1(t)\le\lambda_2(t)\le...\le\lambda_n(t)\le...\}.$$

Remark 3.5. *The operator $A(t)$, $t\in\mathbb{R}$, given in* (3.3) *differs from the operators considered in* [5, 12]. *However, the proof of each statement above is exactly the same as the one presented in* [5], *since it only depends on the sectoriality of the operators $-div(a(t,x)\nabla w)+w$ in Ω and $-\partial_r(a(t,r)\partial_r v)+v$ (with Dirichlet boundary condition) in R_0, and on Sobolev embeddings.*

Lemma 3.3 and Proposition 3.4 imply that $A(t),t\in\mathbb{R}$, satisfies conditions (P.1) - (P.3) if we identify $X=U^0_q$ and $Y=U^0_p$. Since $\Omega=\Omega_0\cup R_0$ is bounded, we have $U^0_p\hookrightarrow U^0_q$.

We turn our attention to the nonlinearity f. The growth condition (3.1) and the Mean Value Theorem imply the existence of a constant $C>0$ such that

$$\begin{aligned}|f(\xi)-f(\psi)|&\le C|\xi-\psi|(1+|\xi|^{\rho-1}+|\psi|^{\rho-1}),\\ |f(\xi)|&\le C(1+|\xi|^\rho).\end{aligned}$$

Lemma 3.6. *Let F be the nonlinearity defined in* (3.4) *and suppose that* (3.1) *is satisfied. Then F takes elements of U^0_p to elements in U^0_q, that is, $F : U^0_p \to U^0_q$, where $q = \frac{p}{\rho}$. Furthermore, for each $(w, v), (\tilde{w}, \tilde{v}) \in U^0_p$, we have*

$$\|F(w,v) - F(\tilde{w},\tilde{v})\|_{U^0_q} \leq C\|(w,v) - (\tilde{w},\tilde{v})\|_{U^0_p}(1 + \|(w,v)\|^{\rho-1}_{U^0_p} + \|(\tilde{w},\tilde{v})\|^{\rho-1}_{U^0_p}),$$
$$\|F(t,(w,v))\|_{U^0_q} \leq C(1 + \|(w,v)\|^{\rho}_{U^0_p}).$$

Proof. We only verify the first inequality. The second follows in a similar way. Note that

$$\|F(w,v) - F(\tilde{w},\tilde{v})\|_{U^0_q} = \left[\int_\Omega |f(w(x)) - f(\tilde{w}(x))|^q dx\right]^{\frac{1}{q}} + \left[\int_0^1 |f(v(s)) - f(\tilde{v}(s))|^q ds\right]^{\frac{1}{q}}.$$

We consider the integrals separately. It follows from Hölder's inequality that

$$\begin{aligned}\int_\Omega |f(w(x)) - f(\tilde{w}(x))|^q dx &\leq \int_\Omega C^q|w(x) - \tilde{w}(x)|^q(1 + |w(x)|^{q(\rho-1)} + |\tilde{w}(x)|^{q(\rho-1)})dx \\ &\leq C\,\|w - \tilde{w}\|^q_{L^p(\Omega)}\left(1 + \int_\Omega |w|^{q(\rho-1)\frac{p}{p-q}} + |\tilde{w}|^{q(\rho-1)\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\end{aligned}$$

and we used that $q(\rho-1)\frac{p}{p-q} = p$. Therefore,

$$\begin{aligned}\left[\int_\Omega |f(w(x)) - f(\tilde{w}(x))|^q dx\right]^{\frac{1}{q}} &\leq C\,\|w - \tilde{w}\|_{L^p(\Omega)}\left(1 + \|w\|^p_{L^p(\Omega)} + \|\tilde{w}\|^p_{L^p(\Omega)}\right)^{\frac{p-q}{pq}} \\ &\leq C\,\|w - \tilde{w}\|_{L^p(\Omega)}\left(1 + \|w\|^{\rho-1}_{L^p(\Omega)} + \|\tilde{w}\|^{\rho-1}_{L^p(\Omega)}\right).\end{aligned}$$

With the same reasoning, we deduce that

$$\left[\int_0^1 |f(v(s)) - f(\tilde{v}(s))|^q ds\right]^{\frac{1}{q}} \leq C\,\|v - \tilde{v}\|_{L^p(0,1)}\,(1 + \|v\|^{\rho-1}_{L^p(0,1)} + \|\tilde{v}\|^{\rho-1}_{L^p(0,1)}).$$

Using the above inequalities, we obtain the desired estimate. □

3.1. **Local well-posedness and maximal growth ρ.** Consider the abstract problem

$$\begin{cases}(w,v)_t + A(t)(w,v) = F(w,v), \ t > \tau; \\ (w,v)(\tau) = (w_0,v_0) \in U^0_p \hookrightarrow U^0_q.\end{cases} \tag{3.5}$$

By considering $Y = U^0_p$ and $X = U^0_q$, the family $A(t), t \in \mathbb{R}$, satisfies conditions (P.1),(P.2) and (P.3) and the nonlinearity F satisfies (NL). In this case, the constant of almost sectoriality α and constant $\beta \in (0,1)$ are any real numbers in the intervals

$$0 < \alpha < 1 - \frac{N}{2p} =: \alpha^+ \quad \text{and} \quad 0 < \beta < 1 - \frac{N}{2q} - \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right) =: \beta^+.$$

In order to ensure local well-posedness, we must verify the two remaining inequalities that appear as a restriction in Theorem 2.7: $\alpha + \delta > 1$ and $1 \leq \rho < \frac{\beta}{1-\alpha}$.

Lemma 3.7. *Let $\rho = \frac{p}{q}$ and $\delta \in (0,1]$. There exist $0 < \beta < \beta^+$ and $0 < \alpha < \alpha^+$ such that:*

(1) $\alpha + \delta > 1$ holds if, and only if

$$p > \frac{N}{2\delta}. \tag{3.6}$$

.

(2) $1 \leq \rho < \frac{\beta}{1-\alpha}$ holds if and only if,

$$p > N \quad \text{and} \quad \frac{p(2N+1)}{2p+1} < q \leq p. \tag{3.7}$$

Proof. Note that $\alpha^+ + \delta > 1$ if and only if $p > \frac{N}{2\delta}$ and $\frac{p}{q} = \rho < \frac{\beta^+}{1-\alpha^+}$, that is, $\frac{p}{q} < \frac{1-\frac{N}{2q}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}{1-\left(1-\frac{N}{2p}\right)}$ if and only if $q > \frac{p(2N+1)}{2p+1}$. Also, the condition $\frac{p(2N+1)}{2p+1} < q \leq p$ only makes sense if $p > N$.

□

Note that any lower bound for q is actually an upper-bound for $\rho = \frac{p}{q}$ and inequality (3.7) can be restated as an upper-bound ρ_c for the growth of F given by

$$\rho_c = \frac{p}{\frac{p(2N+1)}{2p+1}} = \frac{2p+1}{2N+1}.$$

The local well-posedness in U_p^0 follows from Theorem 2.7.

Proposition 3.8. *Assume that conditions (A.1) - (A.4) are satisfied,* $p > \max\left\{N, \frac{N}{2\delta}\right\}$ *and* $1 \le \rho < \frac{2p+1}{2N+1}$ *. If* $U(t,\tau)$ *is the linear process of growth* $1-\alpha$ *associated to* $A(t)$*,* $t \in \mathbb{R}$*, then* (3.5) *has a local mild solution in* U_p^0*,* $(w,v)(\cdot) : (\tau, \tau+T) \to U_p^0$*, given by*

$$(w,v)(t) = U(t,\tau)u_0 + \int_\tau^t U(t,s)F((w,v)(s))ds, \qquad \text{for all } t \in (\tau, T).$$

3.2. **Global well-posedness and existence of pullback attractor.** In order to obtain global well-posedness, we require a dissipativeness assumption on f given in terms of the first eigenvalue of $A(t)$, $t \in \mathbb{R}$:

(D). The nonlinearity f satisfies

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < 1.$$

From the definition of lim sup, we can restate (D) in a different manner that shall be useful in the following.

Lemma 3.9. *Assume that condition (D) holds. There exists* $0 < \gamma_1 < 1$ *such that, for each* $\gamma \in (0, \gamma_1)$*, one can find a constant* $M > 0$ *for which the following inequality holds for all* $s \in \mathbb{R}$*:*

$$f(s)s \le (1-\gamma)s^2 + M. \tag{3.8}$$

We enunciate in the sequel the theorem on global well-posedness and existence of pullback attractor for problem (3.5) in the space U_p^0. The proof of the following theorem is postponed to Section 6 and, as we shall see then, another restriction on the growth of f will appear as a consequence of the fact that we must ensure $w_t \in \mathcal{C}(\overline{\Omega})$ in order to make sense of $w_t(p_0)$ and $w_t(p_1)$.

Theorem 3.10. *Assume that (A.1) - (A.4) and (D) are satisfied,* $p > \max\{N, \frac{N}{2\delta}\}$ *and* $1 \le \rho < \min\left\{\frac{2\delta p}{N}, \frac{2p+1}{2N+1}\right\}$*. Then the solution* $(w,v)(t)$ *for the problem* (3.5) *defines a nonlinear process*

$$S(t,\tau) = (w,v)(t,\tau,(w_0,v_0)),$$

in $U_p^0 = L^p(\Omega) \times L^p(0,1)$ *which has a pullback attractor* $\mathcal{A}(t)$ *in* U_p^0*. Moreover,* $\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)$ *lies in a compact subset of* $\mathcal{C}^{1,\eta}(\Omega) \times \mathcal{C}^{1,\eta}(0,1)$*, for some* $\eta > 0$*, and pullback attracts bounded sets of* U_p^0 *in the topology of* $\mathcal{C}^{1,\eta}(\Omega) \times \mathcal{C}^{1,\eta}(0,1)$*.*

Note that it is only required to know N and δ in order to establish values of p and ρ for which the problem can be locally and globally solved. For instance, in figure below the shadowed region represents, for $N = 3$ and two different values for δ, all possible values for p and ρ such that global well-posedness is guaranteed.

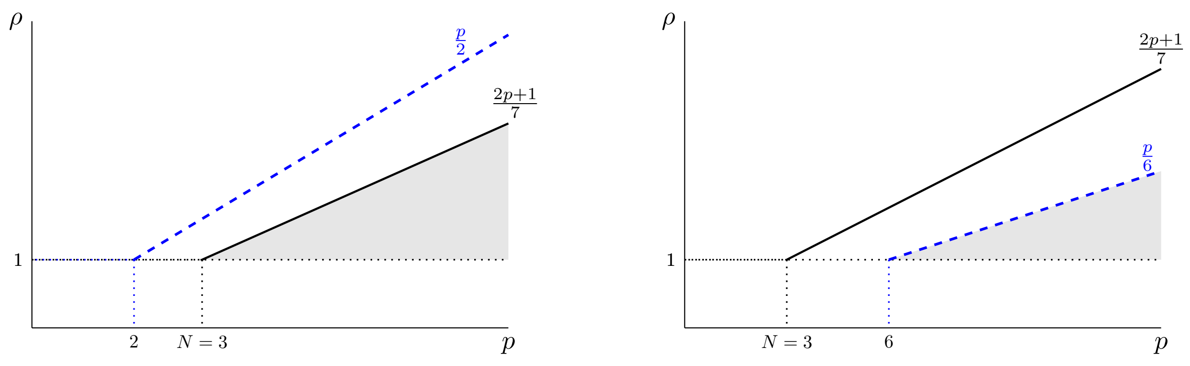


FIGURE 2. Maximal growth $N = 3$ and $\delta = \frac{3}{4}$ (left-side) or $\delta = \frac{1}{4}$ (right-side)

4. The equation in Ω

We already have a local flow in $U_p^0 = L^p(\Omega) \times L^p(0,1)$ due to Proposition 3.8, but we shall consider the solutions $w = w(t)$ and $v = v(t)$ separately, which allows us to embed them into more regular spaces, as a consequence of the sectoriality that appears once we separate the problems. We begin with the semilinear equation in Ω given by

$$\begin{cases} w_t - div(a(t,x)\nabla w) + w = f(w), & x \in \Omega,\ t > \tau, \\ \partial_n w = 0, & x \in \partial\Omega, \end{cases}$$

and we assume that conditions (A.1) to (A.4) and (D) hold, $\delta \in (0,1]$ is the Hölder constant, $p > \max\{N, \frac{N}{2\delta}\}$ and $1 \leq \rho < \min\left\{\frac{2\delta p}{N}, \frac{2p+1}{2N+1}\right\}$ is the growth of f. Let $A_\Omega(t) : D(A_\Omega(t)) \subset L^p(\Omega) \to L^p(\Omega)$, $t \in \mathbb{R}$, be the family of linear operators given by

$$A_\Omega(t)w = -div(a(t,x)\nabla w) + w,\ x \in \Omega, \quad \text{with domain} \quad D(A_\Omega(t)) = W := \{W^{2,p}(\Omega); \partial_n w = 0\}, \tag{4.1}$$

We present some properties of this family that follow from [23, Theorem 7.3.5] and from Lemma 3.3.

Lemma 4.1. *Let $A_\Omega(t)$, $t \in \mathbb{R}$ be the family defined in (4.1). Then $A_\Omega(t)$, $t \in \mathbb{R}$, is uniformly sectorial and uniformly $\delta-$Hölder continuous, that is, there exists constant $C > 0$ such that, for all $\lambda \in \Sigma_\varphi \cup \{0\}$ and $\tau, s, t \in \mathbb{R}$,*

$$\|(\lambda + A_\Omega(t))^{-1}\|_{\mathcal{L}(L^p(\Omega))} \leq \frac{C}{1+|\lambda|} \quad \text{and} \quad \|[A_\Omega(t) - A_\Omega(s)]A_\Omega(\tau)^{-1}\|_{\mathcal{L}(L^p(\Omega))} \leq C|t-s|^\delta.$$

Moreover, the spectrum of $A_\Omega(t)$ consists entirely of isolated eigenvalues, all of them positive, real and satisfying $\sigma(A_\Omega(t)) = \{\mu_i(t) : 1 = \mu_1(t) \leq \mu_2(t) \leq ... \leq \mu_n(t) \leq ...\}$.

Therefore, $A_\Omega(t)$ is a positive sectorial operator and its fractional powers $[A_\Omega(t)]^\omega$, $\omega \in \mathbb{R}$, are well-defined. As we discussed in Subsection 2.2, we can obtain a scale of fractional power spaces $W^\omega = D[(A_\Omega(t)]^\omega)$, for a fixed $t \in \mathbb{R}$ and $\omega > 0$. This scale is extended to negative fractional powers by considering $W^{-\omega} = [W^\omega]^*$. In this case, $W^0 = L^p(\Omega)$ and $W^1 = W = D(A_\Omega(t))$.

Lemma 4.2. [21, Theorem 1.6.1] *Let $\{W^\omega\}_{-1\leq\omega\leq 1}$ be the scale of fractional powers defined above. Then, W^ω is compactly embedded in W^γ, whenever $\omega < \gamma$ and the following embeddings hold:*

$$\begin{array}{ll} W^\omega \hookrightarrow \mathcal{C}^{1,\eta}(\Omega) & \text{for some } \eta > 0, \text{ if } \omega > \frac{1}{2} + \frac{N}{2p}, \\ W^\omega \hookrightarrow \mathcal{C}^{\nu}(\Omega) & \text{for some } \nu > 0, \text{ if } \omega > \frac{N}{2p}, \\ W^\omega \hookrightarrow L^r(\Omega) & \text{for some } r > p, \text{ if } \omega > \frac{N}{2p} - \frac{N}{2r}, \\ L^r(\Omega) \hookrightarrow W^{-\omega} & \text{for some } 1 \leq r < p, \text{ if } \omega > \frac{N}{2r} - \frac{N}{2p}. \end{array}$$

We shall connect in the sequel those informations of fractional powers to the nonlinearity. Let F_Ω be the Nemitskiĭ operator associated to f in Ω,

$$F_\Omega(w)(x) = f(w(x)), \quad \text{for all } x \in \Omega.$$

Lemma 4.3. *Let ω be any real number such that $\omega > \frac{N\rho}{2p} - \frac{N}{2p}$. Then we have $F_\Omega : L^p(\Omega) \to W^{-\omega}$, F_Ω is locally Lipschitz and, for $w, \tilde{w} \in L^p(\Omega)$,*

$$\|F_\Omega(w) - F_\Omega(\tilde{w})\|_{W^{-\omega}} \leq C\|w - \tilde{w}\|_{L^p(\Omega)}(1 + \|w\|^{\rho-1}_{L^p(\Omega)} + \|\tilde{w}\|^{\rho-1}_{L^p(\Omega)}),$$
$$\|F_\Omega(w)\|_{W^{-\omega}} \leq C(1 + \|w\|^\rho_{L^p(\Omega)}).$$

Proof. In the proof of Lemma 3.6, we saw that $F_\Omega : L^p(\Omega) \to L^{\frac{p}{\rho}}(\Omega)$ and satisfies $\|F_\Omega(w)\|_{L^{\frac{p}{\rho}}(\Omega)} \leq C(1+\|w\|^\rho_{L^p(\Omega)})$. Using Lemma 4.2 we deduce that $L^{\frac{p}{\rho}}(\Omega) \hookrightarrow W^{-\omega}$ provided $\omega > \frac{N\rho}{2p} - \frac{N}{2p}$ and we obtain

$$\|F_\Omega(w)\|_{W^{-\omega}} \leq C(1 + \|w\|^\rho_{L^p(\Omega)}).$$

The other inequality follows analogously. ∎

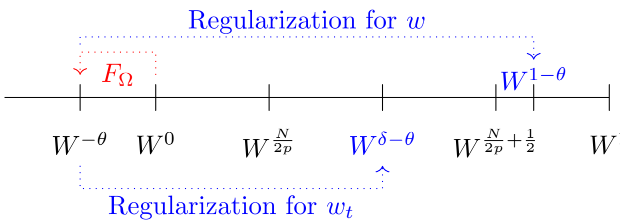
4.1. **Regularization in Ω.** Fix a $\theta > \frac{N\rho}{2p} - \frac{N}{2p}$ such that $F_\Omega : L^p(\Omega) \to W^{-\theta}$. The linear family $A_\Omega(t), t \in \mathbb{R}$, and the nonlinearity F_Ω satisfy the conditions of Theorem 2.9 and the evolution problem in Ω:

$$\begin{cases} w_t + A_\Omega(t)w = F_\Omega(w), \quad t > \tau, \\ w(\tau) = w_0 \in L^p(\Omega) \hookrightarrow W^{-\theta}, \end{cases} \tag{4.2}$$

can be locally solved in $L^p(\Omega)$. Denote by $U_\Omega(t,\tau) \in \mathcal{L}(L^p(\Omega))$ the linear process associated to the family $A_\Omega(t), t \in \mathbb{R}$, and, from the variation of constants formula, w can be given as

$$w(t,\tau,w_0) = U_\Omega(t,\tau)w_0 + \int_\tau^t U_\Omega(t,s)F_\Omega(w(s))ds. \tag{4.3}$$

The fact that this semilinear problem (4.2) has a parabolic structure allows us to gain more regularity for $w(t)$ and $w_t(t)$. As a matter of fact, Theorem (2.9) states that $w(t) \in W^{1-\theta}$ and $w_t(t) \in W^{\delta-\theta}$, for any $t > \tau$. In the next auxiliary lemma, we will see that there exists θ such that $w(t) \in W^{1-\theta} \hookrightarrow \mathcal{C}^{1,\eta}(\Omega)$, for some $\eta > 0$, and $w_t(t) \in W^{\delta-\theta} \hookrightarrow \mathcal{C}^{\nu}(\Omega)$, for some $\nu > 0$. Those embeddings are necessary to the change of variables we will perform in next section, but in order to ensure $w_t \in \mathcal{C}^\nu(\Omega)$ for some ν, restriction $\rho < \frac{2\delta p}{N}$ appears, as we see next.



Lemma 4.4. *Let $p > \max\{N, \frac{N}{2\delta}\}$ and $1 \le \rho < \min\{\frac{2\delta p}{N}, \frac{2p+1}{2N+1}\}$. There exists $\theta > 0$ such that $F_\Omega : L^p(\Omega) \to W^{-\theta}$, $W^{1-\theta} \hookrightarrow \mathcal{C}^{1,\eta}(\Omega)$ and $W^{\delta-\theta} \hookrightarrow \mathcal{C}^\nu(\Omega)$, for some $\eta > 0$ and $\nu > 0$. To be precise, θ can be any number in the interval*

$$\mathcal{I} = \left(\frac{N\rho}{2p} - \frac{N}{2p}, \min\left\{\delta, \tfrac{1}{2}\right\} - \frac{N}{2p}\right). \tag{4.4}$$

Proof. Lemma 4.3 already provides a lower bound for θ, that is $\theta > \frac{N\rho}{2p} - \frac{N}{2p}$, and from embeddings on Lemma 4.2, we will only have $W^{1-\theta} \hookrightarrow \mathcal{C}^{1,\eta}(\Omega)$ if $1 - \theta > \frac{1}{2} + \frac{N}{2p}$, which implies that $\theta < \frac{1}{2} - \frac{N}{2p}$ and $W^{\delta-\theta} \hookrightarrow \mathcal{C}^\nu(\Omega)$ if $\delta - \theta > \frac{N}{2p}$, that is, $\theta < \delta - \frac{N}{2p}$. Therefore, θ must be on both intervals

$$\theta \in \left(\frac{N\rho}{2p} - \frac{N}{2p}, \frac{1}{2} - \frac{N}{2p}\right) \cap \left(\frac{N\rho}{2p} - \frac{N}{2p}, \delta - \frac{N}{2p}\right).$$

Note that $\frac{1}{2} - \frac{N}{2p}$ is indeed greater than $\frac{N\rho}{2p} - \frac{N}{2p}$ as a consequence of $\rho < \frac{2p+1}{2N+1}$ and $p > N$. On the other hand, $\delta - \frac{N}{2p}$ is greater than $\frac{N\rho}{2p} - \frac{N}{2p}$ provided $\rho < \frac{2\delta p}{N}$. $\square$

From now on, we shall choose θ in (4.2) such that it satisfies the conditions of Lemma 4.4. In this case, the local solution $w = w(t,\tau,w_0)$ of (4.2) is in $\mathcal{C}^{1,\eta}(\Omega)$ and $w_t(t,\tau,w_0)$ is in $\mathcal{C}^\nu(\Omega)$, for any $t > \tau$ and some $\eta, \nu > 0$, even though the initial condition was in $L^p(\Omega)$.

4.2. **Estimates for w, w_t and global existence.** The evolution problem (4.2) has the particularity of having a time-dependent linear operator, which causes some difficulty when studying the long-time dynamics. In [9] the authors used a method inspired in the Moser-Alikakos technique [2, 18] to obtain global well-posedness, existence of pullback attractors and estimates for w and w_t. We summarize the results obtained there that will be necessary in the sequel. In order to simplify the presentation, we gather all the necessary assumptions that we will require on those results in the sequel:

(S) Assume that (A.1) to (A.4) and (D) hold, $M, \gamma > 0$ are the constants obtained in (3.8), $p > \max\{N, \frac{N}{2\delta}\}$ and $1 \le \rho < \min\{\frac{2\delta p}{N}, \frac{2p+1}{2N+1}\}$. Moreover, let θ be any fixed real number on the interval $\mathcal{I}$ given in (4.4), implicating that $W^{1-\theta} \hookrightarrow \mathcal{C}^{1,\eta}(\Omega)$ and $W^{\delta-\theta} \hookrightarrow \mathcal{C}^\nu(\Omega)$, for some $\eta, \nu > 0$.

The estimates for w and w_t now follow.

Proposition 4.5. [9, Proposition 6.12] *Assume that (S) holds. There exist positive constants E_1, E_2, F_1 and F_2 depending only on θ such that, for any $w_0 \in L^\infty(\Omega)$ and $\tau < t-1 < t$, the solution $w = w(t,\tau,w_0)$ of* (4.2) *satisfies, as long as the solution exists,*

$$\|w(t,\tau,w_0)\|_{W^{1-\theta}} \leq E_1 e^{-\gamma(t-\tau)} \left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + E_2,$$
$$\|w_t(t,\tau,w_0)\|_{W^{\delta-\theta}} \leq F_1 e^{-\gamma(t-\tau)} \left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + F_2.$$

Note that the assumption of $w_0 \in L^\infty(\Omega)$ in Proposition 4.5 does not represent any restriction when proving global well-posedness of the problem. Indeed, given $w_0 \in L^p(\Omega)$, then any small evolution $w(\tau+\varepsilon) = w(\tau+\varepsilon,\tau,w_0)$ starting at this point will be an element of $W^{1-\theta} \hookrightarrow L^\infty(\Omega)$ and $w(t,\tau,w_0) = w(t,\tau+\varepsilon, w(\tau+\varepsilon))$, $t > \tau+\varepsilon$, due to the uniqueness of solution. Therefore, the $\|w(t,\tau,w_0)\|_{W^{1-\theta}}$ remains bounded for t large. In particular, $\|w(t)\|_{L^p(\Omega)}$ also remains bounded since $W^{1-\theta} \hookrightarrow L^p(\Omega)$, and $w(t,\tau,w_0)$ is globally defined in time, generating a nonlinear process $S_\Omega(t,\tau) : L^p(\Omega) \to L^p(\Omega)$ given by

$$S_\Omega(t,\tau)w_0 = U_\Omega(t,\tau)w_0 + \int_\tau^t U_\Omega(t,s)F_\Omega(w(s))ds, \quad \text{for all } t > \tau.$$

Moreover, the restriction $\tau < t-1 < t$ only appears to emphasize that the estimate holds if we consider any small evolution of the system. The value $t-1$ could be replaced by $t-\varepsilon$ for any ε arbitrarily small. Finally, we note that the estimate on the derivative implies that $t \mapsto |w_t(t,x)| \in \mathbb{R}$ is bounded for each $x \in \overline{\Omega}$, since $W^{\delta-\theta} \hookrightarrow \mathcal{C}^\nu(\Omega)$.

4.3. **Existence of pullback attractor for $S_\Omega(\cdot,\cdot)$.** From the $W^{1-\theta}$–estimate for $w(t,\tau,w_0)$ the authors in [9] were able to construct a pullback attracting set, as stated in next proposition.

Proposition 4.6. [9, Theorem 6.13] *Assume that (S) holds and let $E_2 > 0$ be the constant in Proposition 4.5. The closed ball in $W^{1-\theta}$ centered in zero and with radius E_2, $B_{W^{1-\theta}}[0,E_2]$, is a pullback attracting set for the process $S_\Omega(t,\tau)$ in the topology of $W^{1-\theta}$.*

Since $W^{1-\theta} \overset{c}{\hookrightarrow} L^p(\Omega)$, this closed pullback attracting ball $B_{W^{1-\theta}}[0,E_2]$ is a compact pullback attracting set in $L^p(\Omega)$. The existence of pullback attractor for $S_\Omega(t,\tau)$ now follows.

Theorem 4.7. [9, Theorem 6.14] *Assume that (S) holds. The solution $w(t)$ for* (4.2) *defines a nonlinear process $S_\Omega(t,\tau)$ in $L^p(\Omega)$ which has a pullback attractor $\mathcal{A}_\Omega(t)$ in $L^p(\Omega)$. Moreover, $\bigcup_{t\in\mathbb{R}} \mathcal{A}_\Omega(t)$ lies in a compact subset of $\mathcal{C}^{1,\eta}(\Omega)$, for some $\eta > 0$, and pullback attracts bounded sets of $L^p(\Omega)$ in the topology of $\mathcal{C}^{1,\eta}(\Omega)$.*

5. The equation on the line segment R_0

We turn our attention to the reaction-diffusion equation that takes place at the line segment R_0. The evolution in R_0 is subordinated to the values w assume at the points, p_0 and p_1, at time t. This works as a time-dependent nonlinear boundary condition for the equation, that is,

$$\begin{cases} v_t(t,r) - \partial_r[a(t,r)\partial_r v(t,r)] + v(t,r) = f(v), & t > \tau, r \in (0,1), \\ v(t,0) = w(t,p_0) \text{ and } v(t,1) = w(t,p_1), & t > \tau. \end{cases} \tag{5.1}$$

We assume that all conditions presented in (S) hold. Consequently, $v(t,i) = w(t,p_i)$, $i = 0,1$, makes sense since $w(t,\cdot) \in \mathcal{C}^{1,\eta}(\Omega)$. We shall denote $w(t)$ at the junction points p_i by $w(t,p_i)$ or $w(t,\tau,w_0)(p_i)$, $i = 0,1$, if we wish to emphasize both the initial condition and the value at p_i. Each initial condition $w_0 \in L^p(\Omega)$ determines a different evolution equation (5.1) in R_0. In this section we study global existence and asymptotic dynamics for the problem (5.1) when a given function $w(t,\tau,w_0)$ is the solution for the problem in Ω.

So far problem (5.1) has nonautonomous and nonlinear boundary conditions, which prevent us to write it as an abstract semilinear differential equation. To get over this problem, we shall make a change of variable that incorporates this boundary condition into the equation. For that, consider the problem:

$$\begin{cases} \partial_r(a(t,r)\partial_r\xi) = 0, \quad r \in (0,1), \\ \xi(t,0) = w(t,\tau,w_0)(p_0);\ \xi(t,1) = w(t,\tau,w_0)(p_1). \end{cases} \tag{5.2}$$

The solution $\xi(t,r)$ of (5.2) will be central in the change of variable, which we describe next. Integrating two times in r and using the boundary condition, we obtain the solution for this problem.

Lemma 5.1. *Given $\tau \in \mathbb{R}$ and $w_0 \in L^p(\Omega)$,*

$$\xi(t,r) = w(t,\tau,w_0)(p_0)\mathcal{X}_0(t,r) + w(t,\tau,w_0)(p_1)\mathcal{X}_1(t,r), \quad t \geq \tau,\ r \in [0,1], \tag{5.3}$$

is the solution of the equation (5.2) associated to $(\tau,w_0) \in \mathbb{R} \times L^p(\Omega)$, where

$$\mathcal{X}_0(t,r) = \left[\frac{\int_r^1 \frac{1}{a(t,\theta)}d\theta}{\int_0^1 \frac{1}{a(t,\theta)}d\theta}\right], \quad \mathcal{X}_1(t,r) = \left[\frac{\int_0^r \frac{1}{a(t,\theta)}d\theta}{\int_0^1 \frac{1}{a(t,\theta)}d\theta}\right]. \tag{5.4}$$

We will use the notation $\xi(t,r;(\tau,w_0))$ whenever wish to emphasize the dependence on τ and w_0. Some properties of the functions $\mathcal{X}_i$, $i = 0,1$, are given below and they follow directly from expressions (5.4) and from (A.3).

Lemma 5.2. *The functions $\mathcal{X}_i(t,r)$, $i = 0,1$, are differentiable in t, and twice differentiable in r. Moreover, $\mathcal{X}_i(t,r)$, $\partial_t\mathcal{X}_i(t,r)$, $\partial_r\mathcal{X}_i(t,r)$ and $\partial_r^2\mathcal{X}_i(t,r)$ are bounded functions.*

As we did in the previous section, we will assume that $w_0 \in L^\infty(\Omega)$. There is no loss of generality in doing so, since any small evolution $w(t,\tau,w_0)$, $t > \tau$, belongs to $\mathcal{C}^{1,\eta}(\Omega)$.

Lemma 5.3. *Assume that (S) hold and let $(\tau,w_0) \in \mathbb{R} \times L^\infty(\Omega)$. There exist constants G_1, G_2 independent of (τ,w_0), such that, for $\tau < t-1 < t$,*

$$|\xi(t,r;(\tau,w_0))| + |\xi_t(t,r;(\tau,w_0))| \leq G_1 e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + G_2.$$

Proof. From expression (5.3) for $\xi(t,r)$ and Lemma 5.2, we obtain

$$|\xi(t,r)| \leq C(|w(t,\tau,w_0)(p_0)| + |w(t,\tau,w_0)(p_1)|)$$

and

$$\begin{aligned} |\xi_t(t,r)| &\leq \left| w_t(t,p_0)\mathcal{X}_0(t,r) + w(t,p_0)\frac{d}{dt}\mathcal{X}_0(t,r) + w_t(t,p_1)\mathcal{X}_1(t,r) + w(t,p_1)\frac{d}{dt}\mathcal{X}_1(t,r)\right| \\ &\leq C(|w_t(t,p_0)| + |w(t,p_0)| + |w_t(t,p_1)| + |w(t,p_1)|). \end{aligned}$$

Therefore, from the estimates obtained in Proposition 4.5 for w and w_t, from the fact that $w(t) \in \mathcal{C}^{1,\eta}(\Omega)$ for some $\eta > 0$ and $w_t(t) \in \mathcal{C}^\nu(\Omega)$ for some $\nu > 0$, we deduce after relabeling the constants that

$$\begin{aligned} |\xi(t,r)| + |\xi_t(t,r)| &\leq C(|w(t,p_0)| + |w(t,p_1)| + |w_t(t,p_0)| + |w_t(t,p_1)|) \\ &\leq C(2E_1 + 2F_1)e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + C(2E_2 + 2F_2) \\ &\leq G_1 e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + G_2. \end{aligned}$$

□

5.1. **Change of variables and associated problem with Dirichlet boundary conditions.** If we consider $z(t,r) = v(t,r) - \xi(t,r)$, then the initial value problem (5.2) becomes

$$\begin{cases} z_t - \partial_r(a(t,r)\partial_r z) + z = \psi(t,z), & t > \tau, r \in (0,1), \\ z(t,0) = 0 \text{ and } z(t,1) = 0, & t > \tau, \\ z(\tau,r) = v_0(r) - \xi(\tau,r) =: z_0 \in L^p(0,1), \end{cases} \tag{5.5}$$

where

$$\psi(t,z) = -\xi - \xi_t + f(z+\xi)$$

is a nonlinearity depending on ξ and ξ_t. Therefore, $\psi(t,z)$ also depends on $(\tau, w_0) \in \mathbb{R} \times L^p(\Omega)$ and we can emphasize this dependence by denoting

$$\psi(t,z;(\tau,w_0)) = -\xi(t,r;(\tau,w_0)) - \xi_t(t,r;(\tau,w_0)) + f\left(\xi(t,r;(\tau,w_0)) + z\right).$$

We refer to (5.5) as the *associated problem with Dirichlet boundary condition* and we define the linear operator $A_{R_0}(t) : D(A_{R_0}(t)) \subset L^p(0,1) \to L^p(0,1)$ given by

$$A_{R_0}(t)z = -\partial_r(a(t,r)\partial_r z) + z, \quad \text{and} \quad D(A_{R_0}(t)) = V = W^{2,p}(0,1) \cap W_0^{1,p}(0,1).$$

Note that this linear operator $A_{R_0}(t)$ is essentially the same as $A_\Omega(t)$ defined in (4.1) for functions in the lower dimensional space $R_0 = (0,1)$. Therefore, $A_{R_0}(t)$ is uniformly sectorial, $\delta-$Hölder continuous, positive and has a spectrum given by $\sigma(A_{R_0}(t)) = \{\nu_i(t) : 1 = \nu_1(t) \leq \nu_2(t) \leq ... \leq \nu_n(t) \leq ...\}$. Its fractional powers are well-defined and we obtain a scale of Banach spaces

$$V^\omega = D([A_{R_0}(t)]^\omega), \quad 0 \leq \omega \leq 1,$$

where $V^0 = L^p(0,1)$, $V^1 = V = D(A_{R_0}(t))$ and $V^{-\omega} = [V^\omega]^*$. Moreover, V^ω satisfies the same embeddings that W^ω satisfies in Lemma (4.2), with $N = 1$ and $\Omega = (0,1)$ instead. Let Ψ be the Nemitskiĭ operator associated to ψ, that is,

$$\Psi(t,z)(r) = \psi(t, z(r)).$$

We shall prove that Ψ satisfies the conditions necessary to ensure local well posedness for the equation in R_0.

5.2. **Properties for the nonlinearity.** We start by proving that $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies an appropriate dissipative condition for problem (5.5) and we study how this dissipation depends on the choice of (τ, w_0).

Proposition 5.4. *Assume that (S) holds. Given any $(\tau, w_0) \in \mathbb{R} \times L^\infty(\Omega)$ there exist a constant H independent of t, τ or w_0, such that, for all $\tau < t-1 < t$,*

$$z\psi(t,z) \leq (1-\gamma)z^2 + e^{-\gamma(t-\tau)} m(\|w_0\|_{L^\infty}) + H, \tag{5.6}$$

where $m(\|w_0\|_{L^\infty})$ is a constant that depends on $\|w_0\|_{L^\infty(\Omega)}$, given by

$$m(\|w_0\|_{L^\infty}) = C\left[\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)} + \|w_0\|^{\rho^2}_{L^\infty(\Omega)}\right],$$

for a constant $C > 0$. In particular, $m(\|w_0\|_{L^\infty})$ can be uniformly chosen for w_0 in bounded subsets of $L^\infty(\Omega)$.

Proof. From dissipativeness condition (D) on f and the fact that both functions ξ and ξ_t are bounded, we obtain $\limsup_{|z|\to\infty} \dfrac{\psi(t,z)}{z} < 1$. Therefore $\inf_{r>0} \sup_{|z|>r} \dfrac{\psi(t,z)}{z} < 1$ and there exists $r_1 > 0$ such that $S := \sup_{|z|>r_1} \dfrac{\psi(t,z)}{z} < 1$. We denote $\gamma_1 := 1 - S$ and we can assume $0 < \gamma_1 < 1$.

Hence, for $|z| > r_1$ and $\gamma \in (0, \gamma_1)$, $\frac{\psi(t,z)}{z} \leq S = 1 - \gamma_1 < 1 - \gamma$, which implies that

$$z\psi(t,z) < (1-\gamma)z^2, \quad |z| > r_1.$$

For $|z| < r_1$, in order to obtain an estimate for Ψ, we first note that

$$|f(\xi(t)+z)| \leq C(1 + |\xi(t)+z|^\rho) \leq C2^\rho(1 + r_1^\rho + |\xi(t)|^\rho) \leq C + C|\xi(t)|^\rho.$$

Since $\Psi(t,z) = -\xi(t) - \xi_t(t) + f(\xi(t)+z)$, we deduce, for $z \in [-r_1, r_1]$, that

$$\begin{aligned}
|z\psi(t,z)| &\leq r_1\left[|f(\xi(t)+z)| + |\xi(t)| + |\xi_t(t)|\right] \leq Cr_1 + r_1|\xi(t)|^\rho + r_1|\xi(t)| + r_1|\xi_t(t)| \\
&\leq Cr_1 + Cr_1 G_1^\rho e^{-\rho\gamma(t-\tau)}\left(\|w_0\|^\rho_{L^\infty(\Omega)} + \|w_0\|^{\rho^2}_{L^\infty(\Omega)}\right) + Cr_1 G_2^\rho + G_1 e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + r_1 G_2 \\
&\leq e^{-\gamma(t-\tau)} C\left[\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)} + \|w_0\|^{\rho^2}_{L^\infty(\Omega)}\right] + H,
\end{aligned}$$

where we used the fact that $e^{-\rho\gamma(t-\tau)} \leq e^{-\gamma(t-\tau)}$ and grouped every constant term inside C and H. Those two estimates together allow us to conclude that

$$z\psi(t,z) \leq (1-\gamma)z^2 + e^{-\gamma(t-\tau)} C\left[\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)} + \|w_0\|^{\rho^2}_{L^\infty(\Omega)}\right] + H.$$

□

The Nemitskiĭ operator $\Psi(t,z)(x)=\psi(t,z(x))$ has the following properties:

Lemma 5.5. *Assume that (S) holds and let $(\tau,w_0)\in\mathbb{R}\times L^\infty(\Omega)$. Let ω be any real number such that $\omega>\frac{\rho}{2p}-\frac{1}{2p}$. Then $\Psi:\mathbb{R}\times L^p(0,1)\to V^{-\omega}$ is locally Höder continuous in the first variable and locally Lipschitz in the second variable. Moreover, there exists a constant $C>0$ such that, for $\tau<t-1<t$, we have*

$$\|\Psi(t,z)\|_{V^{-\omega}}\leq C\left[e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+1+\|z\|^\rho_{L^p(0,1)}\right], \tag{5.7}$$

where $m(\|w_0\|_{L^\infty})$ is the constant obtained in Proposition 5.4.

Proof. First we note that $\psi(t,z)=f(z+\xi)-\xi(t)-\xi_t(t)$ and we immediately deduce that ψ is locally Lipschitz on z since f is locally Lipschitz. As far as ψ being locally Hölder continuous in t, it reduces to proving this property to $t\mapsto\xi(t)$ and $t\mapsto\xi_t(t)$. But those functions are given in terms of $w(t)$, $w_t(t)$, $\mathcal{X}_i(t,r)$ and $\partial_t\mathcal{X}_i(t,r)$, which are all locally Hölder continuous in t. Indeed, $w(t)$ is continuously differentiable in t and, by differentiating in t the variation of constant formula (4.3), we obtain w_t is also Hölder continuous in t. As far as $\mathcal{X}_i(t,r)$ and $\partial_t\mathcal{X}_i(t,r)$ local Hölder continuity in t follows from the properties required for $a(t,x)$.

Finally, we observe that $\Psi:\mathbb{R}\times L^p(0,1)\to L^{\frac{p}{\rho}}(0,1)$ since it has the same growth in z that $F:L^p(0,1)\to L^{\frac{p}{\rho}}(0,1)$ does. The desired estimate follows from noticing that $\xi,\xi_t\in L^\infty(0,1)$ and satisfies the estimate obtained in Lemma 5.3. We obtain, adjusting the constants whenever necessary,

$$\begin{aligned}
&\|\Psi(t,z)\|_{L^{\frac{p}{\rho}}(0,1)}\\
&\quad=\|-\xi(t)-\xi_t(t)+F(\xi(t)+z)\|_{L^{\frac{p}{\rho}}(0,1)}\leq\|\xi(t)+\xi_t(t)\|_{L^{\frac{p}{\rho}}(0,1)}+\|F(\xi(t)+z)\|_{L^{\frac{p}{\rho}}(0,1)}\\
&\quad\leq\|\xi(t)+\xi_t(t)\|_{L^{\frac{p}{\rho}}(0,1)}+C\left(1+\|\xi(t)+z\|^\rho_{L^p(0,1)}\right)\leq\|\xi(t)+\xi_t(t)\|_{L^\infty(0,1)}+C+C\|\xi(t)\|^\rho_{L^\infty(0,1)}+C\|z\|^\rho_{L^p(0,1)}\\
&\quad\leq G_1e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)}+\|w_0\|^\rho_{L^\infty(\Omega)}\right)+G_2+CG_1^\rho e^{-\rho\gamma(t-\tau)}\left(\|w_0\|^\rho_{L^\infty(\Omega)}+\|w_0\|^{\rho^2}_{L^\infty(\Omega)}\right)+G_2^\rho+C\|z\|^\rho_{L^p(0,1)}\\
&\quad\leq C\left[e^{-\gamma(t-\tau)}\left(\|w_0\|_{L^\infty(\Omega)}+\|w_0\|^\rho_{L^\infty(\Omega)}+\|w_0\|^{\rho^2}_{L^\infty(\Omega)}\right)+1+\|z\|^\rho_{L^p(0,1)}\right]\\
&\quad\leq C\left[e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty(\Omega)})+1+\|z\|^\rho_{L^p(0,1)}\right].
\end{aligned}$$

Moreover, $L^{\frac{p}{\rho}}(0,1)\hookrightarrow V^{-\omega}$ as long as $\omega>\frac{\rho}{2p}-\frac{1}{2p}$, which proves inequality (5.7). ☐

5.3. **Local well-posedness in R_0.** Let $\theta\in\mathcal{I}$ be the fixed real number established in Assumption (S). Since $\theta>\frac{N\rho}{2p}-\frac{N}{2p}$, then $\theta>\frac{\rho}{2p}-\frac{1}{2p}$ and $\Psi:L^p(0,1)\to V^{-\theta}$. We can restate problem (5.5) in the abstract form

$$\begin{cases}z_t+A_{R_0}(t)z=\Psi(t,z), & t>\tau,\\ z(0)=z_0=v_0-\xi(\tau,w_0)\in L^p(0,1)\hookrightarrow V^{-\theta},\end{cases} \tag{5.8}$$

and in this case, $A_{R_0}(t)$ and $\Psi:L^p(0,1)\to V^{-\theta}$ satisfy all the conditions of Theorem 2.9 to ensure local well-posedness. Moreover, the solution $z(t,\tau,z_0)$ belongs to the space $V^{1-\theta}$, for any $t>\tau$, and since $\theta\in\mathcal{I}$, we obtain $1-\theta>\frac{1}{2}+\frac{N}{2p}>\frac{1}{2}+\frac{1}{2p}$, implying that

$$z(t,\tau,z_0)\in V^{1-\theta}\hookrightarrow\mathcal{C}^{1,\eta}(0,1),$$

for some $\eta>0$. We denote by $U_{R_0}(t,\tau):L^p(0,1)\to L^p(0,1)$ the linear process associated to $A_{R_0}(t)$, $t\in\mathbb{R}$, and z can be given as

$$z(t,\tau,z_0)=U_{R_0}(t,\tau)z_0+\int_\tau^t U_{R_0}(t,s)\Psi(t,z(s))ds,\qquad\text{for } t\in(\tau,\tau+T). \tag{5.9}$$

This discussion proves the following proposition.

Proposition 5.6. *Assume that condition (S) holds and let $(\tau,w_0)\in\mathbb{R}\times L^\infty(\Omega)$ and $z_0\in L^p(0,1)$. Then problem (5.8) is locally well-posed and its solution $z=z(t)\in V^{1-\theta}\hookrightarrow\mathcal{C}^{1,\eta}(0,1)$, for some $\eta>0$.*

To obtain global well-posedness of the solution, we must ensure that $\|z(t,\tau,z_0)\|_{L^p(0,1)}$ remains bounded, which we do in the sequel.

5.4. **Estimates in the channel** R_0**.** We will obtain estimates for $z(t)$ in the spaces $L^2(0,1)$ and $L^{2^k}(0,1)$. For some k sufficiently large, $L^{2^k}(0,1)$ will be embed in $L^p(0,1)$, implying that $\|\cdot\|_{L^p(0,1)} \leq \|\cdot\|_{L^{2^k}(0,1)}$, which allows us to obtain estimate for the solution on the phase space that we are solving the equation, ensuring global existence of the solution. Due to the regularizing property that the parabolic equation (5.8) has, those estimates can be improved to the more regular space $V^{1-\theta} \hookrightarrow \mathcal{C}^{1,\eta}(0,1)$. Those ideas are similar to the ones obtained in [9] for w in Ω, but some extra care must be taken when using the estimate (5.6) for ψ due to its dependence on the elapsed time $t-\tau$.

Moreover, in the results that follow we shall assume that $z_0 \in L^\infty(0,1)$. This does not cause any loss of generality, since if $z_0 \in L^p(0,1)$, then after any arbitrarily small evolution $t > \tau$ in time, $z(t,\tau,z_0)$ will be an element of $V^{1-\theta} \hookrightarrow \mathcal{C}^{1,\eta}(0,1) \hookrightarrow L^\infty(0,1)$. Note that $z_0 \in L^\infty(0,1)$ if and only if $(w_0,v_0) \in L^\infty(\Omega) \times L^\infty(0,1)$, since $z_0 = v_0 - \xi(\tau, w_0)$.

5.4.1. L^{2^k}*-Estimates for the solution.* We start with the case $k=1$, that is, we obtain L^2- estimate for the solution $z = z(t,\tau,z_0)$.

Proposition 5.7. *Suppose that condition (S) holds and let* $(w_0,v_0) \in L^\infty(\Omega) \times L^\infty(0,1)$ *and* $z_0 = v_0 - \xi(\tau,w_0) \in L^\infty(0,1)$*. Then the solution* $z(\cdot,\tau,z_0)$ *to* (5.5) *satisfies, for* $\tau < t-1 < t$ *and as long as it exists,*

$$\|z(t,\tau,z_0)\|_{L^2(0,1)} \leq 2^{\frac{1}{2}} \left[e^{-\gamma(t-\tau)} \|z_0\|_{L^2(0,1)} + \left[\frac{2}{\gamma} e^{-\gamma(t-\tau)} m(\|w_0\|_{L^\infty}) + \frac{2}{\gamma} H \right]^{\frac{1}{2}} \right],$$

where H *and* $m(\|w_0\|_{L^\infty})$ *are given in Proposition 5.4.*

Proof. We take the inner product in $L^2(0,1)$ of the equation (5.5) with z,

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2(0,1)} = -\int_0^1 a(t,r)|\partial_r z|^2 dr - \|z\|^2_{L^2(0,1)} + \int_0^1 \psi(t,z) z dr$$

and from inequality (5.6), we obtain

$$\|z\|^2_{L^2(0,1)} + \frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2(0,1)} \leq (1-\gamma)\|z\|^2_{L^2(0,1)} + e^{-\gamma(t-\tau)} m(\|w_0\|_{L^\infty}) + H,$$
$$2\gamma\|z\|^2_{L^2(0,1)} + \frac{d}{dt}\|z\|^2_{L^2(0,1)} \leq e^{-\gamma(t-\tau)} \left[2m(w_0) + 2e^{\gamma(t-\tau)} H \right].$$

We multiply by $e^{2\gamma(t-\tau)}$ the above inequality, resulting in

$$\frac{d}{dt}\left[e^{2\gamma(t-\tau)} \|z\|^2_{L^2(0,1)} \right] \leq e^{\gamma(t-\tau)} \left[2m(w_0) + e^{\gamma(t-\tau)} 2H \right].$$

Integrating from τ to t, and using that $m_2(w_0) + e^{\gamma(t-\tau)} H$ is increasing in t, we obtain

$$e^{2\gamma(t-\tau)} \|z(t)\|^2_{L^2(0,1)} - \|z_0\|^2_{L^2(0,1)} \leq e^{\gamma(t-\tau)} \frac{1}{\gamma} \left[2m(\|w_0\|_{L^\infty}) + e^{\gamma(t-\tau)} 2H \right].$$

Therefore,

$$\|z(t)\|^2_{L^2(0,1)} \leq e^{-2\gamma(t-\tau)} \|z_0\|^2_{L^2(0,1)} + \left[\frac{2}{\gamma} e^{-\gamma(t-\tau)} m(\|w_0\|_{L^\infty}) + \frac{2}{\gamma} H \right],$$

and the disered inequality follows by taking the square root on both sides. □

From the L^2-estimates of the solution, we shall obtain estimates on the $L^{2^k}-$norm of the solution through an iterated process. As we will see, we first construct a recursive formula for $\|z(t)\|_{L^{2^k}}$ in terms of $\|z(t)\|_{L^{2^{k-1}}}$.

Lemma 5.8. *Suppose that condition (S) holds and let* $(w_0,v_0) \in L^\infty(\Omega) \times L^\infty(0,1)$, $z_0 = v_0 - \xi(\tau,w_0) \in L^\infty(0,1)$*. If* $z(\cdot,\tau,z_0)$ *is the solution of* (5.5) *then, given* $k \in \mathbb{N}$*, there exists constant* $c > 0$ *independent of* k *such that, for any* $\tau < t-1 < t$,

$$\begin{aligned} \|z(t)\|^{2^k}_{L^{2^k}(0,1)} \leq{}& e^{-2^k(t-\tau)} \|z_0\|^{2^k}_{L^{2^k}(0,1)} + c(2^k)^{\frac{3}{2}} e^{-2^k(t-\tau)} \int_\tau^t e^{2^k(s-\tau)} \|z(s)\|^{2^k}_{L^{2^{k-1}}(0,1)} ds \\ &+ \left[\tfrac{2}{\gamma} e^{-\gamma(t-\tau)} m(\|w_0\|_{L^\infty}) + \tfrac{2}{\gamma} H \right], \end{aligned} \tag{5.10}$$

as long as the solution exists. The constants H *and* $m(\|w_0\|_{L^\infty})$ *are given in Proposition 5.4.*

Proof. Multiplying the equation in (5.5) by z^{2^k-1} and integrating in $(0,1)$, we obtain

$$\int_0^1 z_t z^{2^k-1}dr = \int_0^1 \partial_r(a(t,r)\partial_r z)z^{2^k-1}dr - \int_0^1 z^{2^k}dr + \int_0^{-1}\psi(t,z)zz^{2^k-2}dr.$$

Note that the term on the left side is equal $\frac{1}{2^k}\frac{d}{dt}\int_0^1 z^{2^k}dr$ and let $M(t,\tau,w_0)=e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+H$. From estimate (5.6) for $\psi(t,z)z$, we obtain

$$\int_0^1 \psi(t,z)zz^{2^k-2}dr \le \int_0^1(1-\gamma)z^{2^k}dr + M(t,\tau,w_0)\int_0^1 z^{2^k-2}dr \le \int_0^1(1-\gamma)z^{2^k}dr + M(t,\tau,w_0)\left[\int_0^1[z^{2^k}+1]dr\right]$$

and in the last inequality we used the fact that $a^{2^k-2}<a^{2^k}+1$ for any positive a. Thus,

$$\frac{1}{2^k}\frac{d}{dt}\int_0^1 z^{2^k}dr \le \int_0^1 \partial(a(t,r)\partial_r z)z^{2^k-1}dr + [M(t,\tau,w_0)-\gamma]\int_0^1 z^{2^k}dr + M(t,\tau,w_0).$$

Since $M(t,\tau,w_0)$ is independent of $r\in(0,1)$, we can proceed exactly as in the proof of Lemma 6.10 of [9] and we shall obtain the following differential inequality for the L^{2^k}-norm of $z(t)$:

$$\frac{d}{dt}\left[e^{2^k(t-\tau)}\|z(t)\|^{2^k}_{L^{2^k}(0,1)}\right] \le e^{2^k(t-\tau)}c(2^k)^{\frac{3}{2}}\|z(t)\|^{2^k}_{L^{2^{k-1}}(0,1)} + e^{2^k(t-\tau)}2^k\left[e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+H\right]. \tag{5.11}$$

From (5.11) we derive (5.10) by integrating the above expression from τ to t and by noticing that

$$\begin{aligned}\int_\tau^t 2^k e^{(2^k-\gamma)(s-\tau)}[m(\|w_0\|_{L^\infty})+e^{\gamma(s-\tau)}H]ds &\le \frac{2^k}{2^k-\gamma}e^{(2^k-\gamma)(t-\tau)}[m(\|w_0\|_{L^\infty})+e^{\gamma(t-\tau)}H]\\ &\le \frac{2^k}{2^k-\gamma}e^{2^k(t-\tau)}\left[e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+H\right] \le \frac{2^{k-1}}{2^k-\gamma}e^{2^k(t-\tau)}\left[\frac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+\frac{2}{\gamma}H\right]\\ &\le e^{2^k(t-\tau)}\left[\frac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+\frac{2}{\gamma}H\right],\end{aligned}$$

since $\gamma\in(0,1)$ and $\frac{2^{k-1}}{2^k-\gamma}<1$. □

From the recurrence formula obtained in Lemma 5.8, we can derive $L^{2^k}-$estimates for the solution $z=z(t,\tau,z_0)$.

Proposition 5.9. *Assume condition (S) holds and let $(w_0,v_0)\in L^\infty(\Omega)\times L^\infty(0,1)$, $z_0=v_0-\xi(\tau,w_0)\in L^\infty(0,1)$. If $z(\cdot,\tau,z_0)$ is the solution of* (5.5) *then, given any $k\in\mathbb{N}$, there exists a constant $D(k,\gamma)$ that depends on k and γ such that, for $\tau<t-1<t$,*

$$\|z(t)\|_{L^{2^k}(0,1)} \le D(k,\gamma)\left\{e^{-\gamma(t-\tau)}\|z_0\|_{L^{2^k}(0,1)} + \left[\tfrac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+\tfrac{2}{\gamma}H\right]^{\frac{1}{2}}\right\},$$

as long as the solution exists. The constants H and $m(\|w_0\|_{L^\infty})$ are given in Proposition 5.4.

Proof. The result follows from induction. For $k=1$ the statement is already proved in Proposition 5.7. Assume that the statement holds for an integer $k-1$ greater than 1, that is, there exists $D(k-1,\gamma)$ such that

$$\|z(t)\|_{L^{2^{k-1}}(0,1)} \le D(k-1,\gamma)\left\{e^{-\gamma(t-\tau)}\|z_0\|_{L^{2^{k-1}}(0,1)} + \left[\tfrac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+\tfrac{2}{\gamma}H\right]^{\frac{1}{2}}\right\}.$$

We prove that it also holds for k. To simplify the notation, we shall incorporate terms that depend only on k and γ inside one constant that will always be denoted by $D(k,\gamma)$. From the recurrence formula (5.10) we can estimate L^{2^k} from the $L^{2^{k-1}}-$bound. First, consider the integral that appears in (5.10):

$$\begin{aligned}&\int_\tau^t e^{2^k(s-\tau)}\|z(s)\|^{2^k}_{L^{2^{k-1}}(0,1)}ds\\ &\le \int_\tau^t e^{2^k(s-\tau)}D(k-1,\gamma)^{2^k}\left\{e^{-\gamma(s-\tau)}\|z_0\|_{L^{2^{k-1}}(0,1)} + \left[\tfrac{2}{\gamma}e^{-\gamma(s-\tau)}m(\|w_0\|_{L^\infty})+\tfrac{2}{\gamma}H\right]^{\frac{1}{2}}\right\}^{2^k}ds\\ &\le \int_\tau^t e^{2^k(s-\tau)}D(k-1,\gamma)(2)^{2^k}\left\{e^{-2^k\gamma(s-\tau)}\|z_0\|^{2^k}_{L^{2^{k-1}}(0,1)}\right\}ds\end{aligned}$$

$$+\int_\tau^t e^{2^k(s-\tau)}D(k-1,\gamma)(2)^{2^k}e^{-2^{(k-1)}\gamma(s-\tau)}\left[\tfrac{2}{\gamma}m(\|w_0\|_{L^\infty})+\tfrac{2}{\gamma}He^{\gamma(s-\tau)}\right]^{\frac{2^k}{2}}ds$$
$$\leq D(k,\gamma)\frac{e^{2^k(1-\gamma)(t-\tau)}}{2^k(1-\gamma)}\|z_0\|^{2^k}_{L^{2^{k-1}}(0,1)}+D(k,\gamma)\frac{e^{2^{(k-1)}(2-\gamma)(t-\tau)}}{2^{(k-1)}(2-\gamma)}\left[\tfrac{2}{\gamma}m(\|w_0\|_{L^\infty})+\tfrac{2}{\gamma}e^{\gamma(t-\tau)}H\right]^{\frac{2^k}{2}}$$
$$\leq D(k,\gamma)e^{2^k(1-\gamma)(t-\tau)}\|z_0\|^{2^k}_{L^{2^{k-1}}(0,1)}+D(k,\gamma)e^{2^k(t-\tau)}\left[\tfrac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+\tfrac{2}{\gamma}H\right]^{\frac{2^k}{2}}.$$

Therefore,

$$c(2^k)^{\frac{3}{2}}e^{-2^k(t-\tau)}\int_\tau^t e^{2^k(s-\tau)}\|z(s)\|^{2^k}_{L^{2^{k-1}}(0,1)}ds$$
$$\leq D(k,\gamma)e^{-2^k\gamma(t-\tau)}\|z_0\|^{2^k}_{L^{2^{k-1}}(0,1)}+D(k,\gamma)\left[\tfrac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+\tfrac{2}{\gamma}H\right]^{\frac{2^k}{2}}.$$

We shall replace the above inequality in (5.10), but first note that $\|\cdot\|_{L^{2^{k-1}}(0,1)}\leq\|\cdot\|_{L^{2^k}(0,1)}$ since the embedding constant of $L^{2^k}(0,1)\hookrightarrow L^{2^{k-1}}(0,1)$ is less than 1. Moreover, we note that $e^{-2^k(t-\tau)}\leq e^{-2^k\gamma(t-\tau)}$, since $\gamma\in(0,1)$, and we can assume that the constants $D(k,\gamma)$ and $\frac{2}{\gamma}H$ are greater than one (we enlarge them if that is not the case). Under those considerations and recalling that we incorporate new constants inside the same $D(k,\gamma)$ whenever necessary, we obtain

$$\|z(t)\|^{2^k}_{L^{2^k}(0,1)}\leq e^{-2^k(t-\tau)}\|z_0\|^{2^k}_{L^{2^k}(0,1)}+D(k,\gamma)e^{-2^k\gamma(t-\tau)}\|z_0\|^{2^k}_{L^{2^k}(0,1)}+D(k,\gamma)\left[\tfrac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+\tfrac{2}{\gamma}H\right]^{\frac{2^k}{2}}$$
$$+\left[\tfrac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+\tfrac{2}{\gamma}H\right]$$
$$\leq D(k,\gamma)\left\{e^{-2^k\gamma(t-\tau)}\|z_0\|^{2^k}_{L^{2^k}(0,1)}+\left[\tfrac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+\tfrac{2}{\gamma}H\right]^{\frac{2^k}{2}}\right\}.$$

Extracting the 2^k-th root on both sides and making adjustments on the constant $D(k,\gamma)$, we obtain the desired estimate. ☐

5.5. **Global well-posedness.** For any $k>\log_2 p$, we have $L^{2^k}(0,1)\hookrightarrow L^p(0,1)$. Therefore,

$$\|z(t)\|_{L^p(0,1)}\leq\|z(t)\|_{L^{2^k}(0,1)}$$

and there exists a constant $C>0$ such that, for $w_0\in L^\infty(\Omega)$, $z_0\in L^\infty(0,1)$ and $\tau<t-1<t$, we obtain

$$\|z(t)\|_{L^p(0,1)}\leq C\left\{e^{-\gamma(t-\tau)}\|z_0\|_{L^\infty(0,1)}+\left[\tfrac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty})+\tfrac{2}{\gamma}H\right]^{\frac{1}{2}}\right\}. \tag{5.12}$$

Therefore, $z(t,\tau,z_0)$ is globally defined in time, generating a nonlinear process $S_{R_0}(t,\tau):L^p(0,1)\to L^p(0,1)$:

$$S_{R_0}(t,\tau)z_0=U_{R_0}(t,\tau)z_0+\int_\tau^t U_{R_0}(t,s)\Psi(s,z(s))ds,\quad \text{for all } t>\tau.$$

5.6. **Regularization and estimates in $V^{1-\theta}$.** The estimates of the solution obtained so far allow us to derive estimates in better norms, due to the regularizing effect that the parabolic equation (5.8) has on the solution. To be precise, we can obtain estimates in $V^{1-\theta}$, which is embedded in $\mathcal{C}^{1,\eta}(0,1)$ for some $\eta>0$.

Proposition 5.10. *Suppose that condition (S) holds and let $(w_0,v_0)\in L^\infty(\Omega)\times L^\infty(0,1)$ and $z_0=v_0-\xi(\tau,w_0)\in L^\infty(0,1)$. If $z(\cdot,\tau,z_0)$ is the solution of (5.8), then there exist constants $K_1,K_2>0$ independent of τ, w_0 or z_0, such that, for any $\tau<t-1<t$,*

$$\|z(t)\|_{V^{1-\theta}}\leq K_1e^{-\frac{\gamma}{2}(t-\tau)}n\left[\|w_0\|_{L^\infty},\|z_0\|_{L^\infty}\right]+K_2, \tag{5.13}$$

where $n\left[\|w_0\|_{L^\infty},\|z_0\|_{L^\infty}\right]$ is a constant that depends on $\|w_0\|_{L^\infty(\Omega)}$ and $\|z_0\|_{L^\infty(0,1)}$ and can be uniformly chosen for w_0 in bounded subsets of $L^\infty(\Omega)$ and z_0 in bounded subsets of $L^\infty(0,1)$.

Proof. Since $\theta \in \mathcal{I}$ defined in (4.4), we have $\theta > \frac{\rho}{2p} - \frac{1}{2p}$. Therefore, there exists $\varepsilon > 0$ sufficiently small such that $\theta - \varepsilon > \frac{\rho}{2p} - \frac{1}{2p}$ and Lemma 5.5 implies that $\Psi : L^p(0,1) \to V^{-\theta+\varepsilon}$. We use the Variation of Constants Formula (5.9), Proposition 2.8 applied to the process $U_{R_0}(t,\tau)$ and estimate (5.7) for $\|\Psi(t,\cdot)\|_{V^{-\theta+\varepsilon}}$ to obtain

$$\begin{aligned}\|z(t,\tau,z_0)\|_{V^{1-\theta}} &= \|z(t,t-1,z(t-1,\tau,z_0))\|_{V^{1-\theta}}\\ &\le \|U_{R_0}(t,t-1)z(t-1,\tau,z_0)\|_{V^{1-\theta}} + \int_{t-1}^t \|U_{R_0}(t,s)\|_{\mathcal{L}(V^{-\theta+\varepsilon},V^{1-\theta})}\|\Psi(s,z(s))\|_{V^{-\theta+\varepsilon}}ds\\ &\le C\|z(t-1,\tau,z_0)\|_{L^p(0,1)} + C\int_{t-1}^t (t-s)^{-1+\varepsilon}\|\psi(s,z(s))\|_{L^p(0,1)}ds.\\ &\le C\|z(t-1,\tau,z_0)\|_{L^p(0,1)} + C\int_{t-1}^t (t-s)^{-1+\varepsilon}\left[e^{-\gamma(s-\tau)}m(\|w_0\|_{L^\infty}) + 1 + \|z(s)\|^\rho_{L^p(0,1)}\right]ds.\end{aligned}$$

Using estimate (5.12) for $\|z(t)\|_{L^p(0,1)}$ and (5.7) for $\|\Psi(t,z)\|_{V^{-\theta}}$, we deduce

$$\begin{aligned}&\|z(t,\tau,z_0)\|_{V^{1-\theta}}\\ &\le C\left\{e^{-\gamma(t-1-\tau)}\|z_0\|_{L^\infty(0,1)} + \left[\tfrac{2}{\gamma}e^{-\gamma(t-1-\tau)}m(\|w_0\|_{L^\infty}) + \tfrac{2}{\gamma}H\right]^{\frac12}\right\}\\ &\quad + C\int_{t-1}^t (t-s)^{-1}\left[e^{-\gamma(s-\tau)}m(\|w_0\|_{L^\infty}) + 1 + \|z(s)\|^\rho_{L^p(0,1)}\right]ds\\ &\le Ce^{\gamma}e^{-\gamma(t-\tau)}\|z_0\|_{L^\infty(0,1)} + Ce^{\frac{\gamma}{2}}\left[\tfrac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty}) + \tfrac{2}{\gamma}e^{\gamma}H\right]^{\frac12} + C\left[e^{-\gamma(t-1-\tau)}m(\|w_0\|_{L^\infty}) + 1\right]\tfrac{1}{\varepsilon}\\ &\quad + C\int_{t-1}^t (t-s)^{-1+\varepsilon}\left\{e^{-\rho\gamma(s-\tau)}\|z_0\|^\rho_{L^\infty(0,1)} + \left[\tfrac{2}{\gamma}e^{-\gamma(s-\tau)}m(\|w_0\|_{L^\infty}) + \tfrac{2}{\gamma}H\right]^{\frac{\rho}{2}}\right\}ds\\ &\le Ce^{\gamma}e^{-\gamma(t-\tau)}\|z_0\|_{L^\infty(0,1)} + Ce^{\frac{\gamma}{2}}\left[\tfrac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty}) + \tfrac{2}{\gamma}e^{\gamma}H\right]^{\frac12} + Ce^{\gamma}\left[e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty}) + e^{\gamma}\right]\\ &\quad + Ce^{\rho\gamma}e^{-\rho\gamma(t-\tau)}\|z_0\|^\rho_{L^\infty(0,1)} + Ce^{\frac{\rho\gamma}{2}}\left[\tfrac{2}{\gamma}e^{-\gamma(t-\tau)}m(\|w_0\|_{L^\infty}) + \tfrac{2}{\gamma}H\right]^{\frac{\rho}{2}}.\end{aligned}$$

Using the fact that $e^{\rho\gamma(t-\tau)} \le e^{\gamma(t-\tau)} \le e^{-\frac{\gamma}{2}(t-\tau)}$, we can regroup all the constants together and we can restate the above estimate as

$$\|z(t,\tau,z_0)\|_{V^{1-\theta}} \le K_1e^{-\frac{\gamma}{2}}\left[\|z_0\|_{L^\infty(0,1)} + \|z_0\|^\rho_{L^\infty(0,1)} + m(\|w_0\|_{L^\infty}) + m(\|w_0\|_{L^\infty})^{\frac12} + m(\|w_0\|_{L^\infty})^{\frac{\rho}{2}}\right] + K_2,$$

where K_1, K_2 are positive constants, independent of t, τ, w_0 and z_0. By denoting

$$n\left[||w_0\|_{L^\infty(\Omega)}, \|z_0\|_{L^\infty(0,1)}\right] = \|z_0\|_{L^\infty(0,1)} + \|z_0\|^\rho_{L^\infty(0,1)} + m(\|w_0\|_{L^\infty}) + m(\|w_0\|_{L^\infty})^{\frac12} + m(\|w_0\|_{L^\infty})^{\frac{\rho}{2}},$$

we obtain the desired inequality. □

5.7. **Existence of pullback attracting set for** $z(t)$**.** Note that given any $\tau \in \mathbb{R}$ and $w_0 \in L^\infty(\Omega)$, inequality (5.13) states that the closed ball centered in zero and with radius K_2 in $V^{1-\theta}$, $B_{V^{1-\theta}}[0,K_2]$, is a pullback attracting set for the solutions of (5.8). Moreover, this ball does not depend on the choice of $w_0 \in L^\infty(\Omega)$. As a matter of fact, for any $w_0 \in L^p(\Omega)$, due to the immediate regularization effect of the equation in Ω, we have $B_{V^{1-\theta}}[0,K_2]$ is a pullback attracting set for problem (5.8) associated to $(\tau, w_0) \in \mathbb{R}\times L^p(0,1)$. This discussion implicates in the following proposition.

Proposition 5.11. *Assume that (S) holds and let $K_2 > 0$ be the constant in Proposition 5.10. For any $(\tau,w_0) \in \mathbb{R}\times L^p(\Omega)$, the closed ball in $V^{1-\theta}$ centered in zero and with radius K_2, $B_{V^{1-\theta}}[0,K_2]$, is a pullback attracting set for the process $S_{R_0}(t,\tau)$ associated to (5.8) in the topology of $V^{1-\theta}$.*

Since $V^{1-\theta} \overset{c}{\hookrightarrow} L^p(0,1)$, this closed pullback attracting ball $B_{V^{1-\theta}}[0,K_2]$ is a compact pullback attracting set in $L^p(0,1)$. The existence of pullback attractor for $S_{R_0}(t,\tau)$ inside $B_{V^{1-\theta}}[0,K_2]$ now follows.

Theorem 5.12. *Assume that (S) holds and let $(\tau,w_0) \in \mathbb{R}\times L^p(\Omega)$. The solution $z(t,\tau,z_0)$ for (5.8) defines a nonlinear process $S_{R_0}(t,\tau)$ in $L^p(0,1)$ which has a pullback attractor $\mathcal{A}^{w_0}_{R_0}(t)$ in $L^p(0,1)$. Moreover, $\bigcup_{t\in\mathbb{R}}\mathcal{A}^{w_0}_{R_0}(t)$ lies in a compact subset of $\mathcal{C}^{1,\eta}(0,1)$, for some $\eta > 0$, and pullback attracts bounded sets of $L^p(0,1)$ in the topology of $\mathcal{C}^{1,\eta}(0,1)$.*

Recall that for each $w_0 \in L^p(\Omega)$, it corresponds a distinct equation in R_0, so we explicitly incorporate this dependence on the pullback attractor associated to $z = z(t)$, by denoting $\mathcal{A}_{R_0}^{w_0}(t)$. This index is useful to understand how the attractor contributes to form the attractor of the equation in $\Omega_0 = \Omega \cup R_0$.

5.8. **Returning to the original equation in R_0.** So far we have obtained global existence and estimates in $V^{1-\theta}$ for the solution $z(t)$ of (5.8) in $(0,1)$. However, $z(t)$ is the solution after the change of variable, while $v(t) = z(t) + \xi(t)$. We can prove the following result.

Proposition 5.13. *Suppose that condition (S) holds and let $(w_0, v_0) \in L^\infty(\Omega) \times L^\infty(0,1)$ and $w = w(t)$ be the solution of the equation* (4.2) *in Ω. If $v(\cdot,\tau,z_0)$ is the solution of the equation* (5.1) *in R_0, then there exist constants $L_1, L_2 > 0$ independent of τ, w_0 or v_0, such that, for any $\tau < t-1 < t$,*

$$\|v(t)\|_{V^{1-\theta}} \leq L_1 e^{-\frac{\gamma}{2}(t-\tau)} \kappa\left[\|w_0\|_{L^\infty}, \|v_0\|_{L^\infty}\right] + L_2,$$

where $\kappa\left[\|w_0\|_{L^\infty}, \|v_0\|_{L^\infty}\right]$ depends on $\|w_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{L^\infty(0,1)}$ and can be uniformly chosen for (w_0, v_0) in bounded subsets of $L^\infty(\Omega) \times L^\infty(0,1)$.

Proof. It follows from Lemma 5.2 that $\mathcal{X}_i(t,\cdot) \in W^{2,p}(0,1)$ and $\|\mathcal{X}_i\|_{W^{2,p}(0,1)} \leq C$. Since $W^{2,p}(0,1) \hookrightarrow V^{1-\theta}$, we have $\|\mathcal{X}_i\|_{V^{1-\theta}} \leq C$ and from the estimate for $w(t)$ obtained in Proposition 4.5 and the constant $m(\|w_0\|_{L^\infty})$ defined in Proposition 5.4, we deduce

$$\begin{aligned}
\|\xi(t,r;(\tau,w_0))\|_{V^{1-\theta}} &\leq |w(t,\tau,w_0)(p_0)| \|\mathcal{X}_0(t,r)\|_{V^{1-\theta}} + |w(t,\tau,w_0)(p_1)| \|\mathcal{X}_1(t,r)\|_{V^{1-\theta}} \\
&\leq C(|w(t,\tau,w_0)(p_0)| + |w(t,\tau,w_0)(p_1)|) \\
&\leq CE_1 e^{-\gamma(t-\tau)} \left(\|w_0\|_{L^\infty(\Omega)} + \|w_0\|^\rho_{L^\infty(\Omega)}\right) + CE_2 \\
&\leq CE_1 e^{-\gamma(t-\tau)} m(\|w_0\|_{L^\infty}) + CE_2.
\end{aligned}$$

Now it follows from Proposition 5.10 that

$$\begin{aligned}
\|v(t)\|_{V^{1-\theta}} &\leq \|z(t)\|_{V^{1-\theta}} + \|\xi(t)\|_{V^{1-\theta}} \\
&\leq K_1 e^{-\frac{\gamma}{2}(t-\tau)} n\left(\|w_0\|_{L^\infty}, \|z_0\|_{L^\infty}\right) + K_2 + CE_1 e^{-\gamma(t-\tau)} m(\|w_0\|_{L^\infty}) + CE_2 \\
&\leq L_1 e^{-\frac{\gamma}{2}(t-\tau)} \kappa\left(\|w_0\|_{L^\infty}, \|v_0\|_{L^\infty}\right) + L_2,
\end{aligned}$$

where we adjusted the constants and used the fact that $\|v_0\|_{L^\infty(0,1)} \leq \|z_0\|_{L^\infty(0,1)} + \|\xi(t)\|_{L^\infty(0,1)}$. □

6. Coupling the dynamics: Existence of pullback attractor

We finally return to the semilinear problem

$$\begin{cases} (w,v)_t(t) = -A(t)(w,v)(t) + F((w,v)(t)), \ t > \tau; \\ (w,v)(\tau) = (w_0, v_0) \in U_p^0. \end{cases} \tag{6.1}$$

Global well-posedness and existence of pullback attracting set for the problem in $\Omega_0 = \Omega \cup R_0$ now follows directly from Propositions 4.5 and 5.13.

Proposition 6.1. *Suppose that condition (S) holds and let $(w_0, v_0) \in U_p^0$ and $\theta \in \mathcal{I}$. The following statements hold:*

(1) The solution $(w(t), v(t)) = (w,v)(t,\tau,(w_0,v_0))$ of (6.1) *exists for all $t > \tau$ and defines a nonlinear process*

$$S(t,\tau)(w_0,v_0) = (w(t),v(t)) = U(t,\tau)(w_0,v_0) + \int_\tau^t U(t,s)F(t,s)ds.$$

(2) The set $B_{W^{1-\theta}}[0,E_2] \times B_{V^{1-\theta}}[0,L_2]$, where $E_2, L_2 > 0$ are the constants obtained in Propositions 4.5 and 5.13, respectively, is a pullback attracting set for the process $S(t,\tau)$. Moreover, $B_{W^{1-\theta}}[0,E_2] \times B_{V^{1-\theta}}[0,L_2]$ attracts bounded sets of U_p^0 the $W^{1-\theta} \times V^{1-\theta}-$topology.

Proof. The fact that $(w(t), v(t))$ generates a nonlinear process follows directly from the global existence of w and v, obtained in Theorem 4.7 and Theorem 5.12. Now, assume that $B \subset U_p^0$ is a bounded set. After any small evolution in time we obtain a new set $S(\tau+\varepsilon,\tau)B \subset U_p^0$ which is now bounded in $L^\infty(\Omega) \times L^p(0,1)$. This follows from the regularizing properties for $w(t)$ and $v(t) = z(t) + \xi(t)$ discussed in Sections 4 and 5. Therefore, Propositions 4.5 and 5.13 proves that $B_{W^{1-\theta}}[0,E_2] \times B_{V^{1-\theta}}[0,L_2]$ pullback attracts $S(\tau+\varepsilon,\tau)B$ (hence attracts B) in the stronger norm $W^{1-\theta} \times V^{1-\theta}$. □

We can finally prove Theorem 3.10.

6.1. **Proof of Theorem 3.10.** Under the conditions required in Theorem 3.10, all the statements present in (S) hold, implicating that $W^{1-\theta} \overset{c}{\hookrightarrow} \mathcal{C}^{1,\eta}(\Omega)$ and $V^{1-\theta} \overset{c}{\hookrightarrow} \mathcal{C}^{1,\eta}(0,1)$, for some $\eta > 0$. It follows from Proposition 6.1 that $B_{W^{1-\theta}}[0,E_2] \times B_{V^{1-\theta}}[0,L_2] \overset{c}{\hookrightarrow} U_p^0$ is a compact pullback attracting set for the process $S(t,\tau)$ in U_p^0. Therefore, Corollary 2.13 implies the existence of a pullback attractor

$$\mathcal{A}(t) \subset B_{W^{1-\theta}}[0,E_2] \times B_{V^{1-\theta}}[0,L_2], \quad \forall t \in \mathbb{R},$$

that attracts bounded sets of U_p^0 in the topology of $W^{1-\theta} \times V^{1-\theta}$. Moreover, since $B_{W^{1-\theta}}[0,E_2] \times B_{V^{1-\theta}}[0,L_2] \overset{c}{\hookrightarrow} \mathcal{C}^{1,\eta}(\Omega) \times \mathcal{C}^{1,\eta}(0,1)$, the last statement follows. □

6.2. **Further properties of the pullback attractor.** We can derive some conclusions and observations from the steps taken to obtain the existence of pullback attractor:

(1) Let $\mathcal{A}_\Omega(t)$ denote the pullback attractor for the equation in Ω (obtained in Theorem 4.7) and $\mathcal{A}(t)$ denote the pullback attractor obtained in Theorem 3.10 for the problem in Ω_0. Then $\mathcal{A}(t)$ has two components, one acting in Ω and the other in the channel R_0. Since the dynamics in Ω is independent of R_0, the pullback attractor in Ω, $\mathcal{A}_\Omega(t)$, and the part of $\mathcal{A}(t)$ in Ω must be the same. In other words, if $\Pi_1 : U_p^0 \to L^p(\Omega)$ is the projection in the first coordinate, then $\Pi_1(\mathcal{A}(t)) = \mathcal{A}_\Omega(t)$.

(2) Since

$$\bigcup_{t\in\mathbb{R}} \mathcal{A}_\Omega(t) \subset B_{W^{1-\theta}}[0,E_2] \quad \text{and} \quad \bigcup_{t\in\mathbb{R}} \mathcal{A}(t) \subset B_{W^{1-\theta}}[0,E_2] \times B_{V^{1-\theta}}[0,L_2]$$

both attractors are bounded in the past and they are given by the union of all bounded global solutions (Proposition 2.15), that is

$$\mathcal{A}_\Omega(t) = \{\phi(t);\ \phi : \mathbb{R} \to L^p(\Omega) \text{ is a bounded global solution}\}$$
$$\mathcal{A}(t) = \{(\phi,\zeta)(t);\ (\phi,\zeta) : \mathbb{R} \to L^p(\Omega) \times L^p(0,1) \text{ is a bounded global solution}\}.$$

(3) If $w_0 \in L^p(\Omega)$ is such that there exists a bounded global solution $\phi : \mathbb{R} \to L^p(0,1)$ with $\phi(\tau) = w_0$, then ϕ originates one problem in the channel R_0 (as in (5.5)). The solution $v(t)$ in the channel is given by $v(t,r) = \xi(t,r;(\tau,w_0)) + z(t,r)$ and in Theorem 5.12 we proved the existence of a pullback attractor $\mathcal{A}_{R_0}^{w_0}(t)$ in $L^p(0,1)$. In this case,

$$\xi(t) + \mathcal{A}_{R_0}^{w_0}(t)$$

pullback attracts the solution $v(t)$ in the channel and since $\xi(t,r;(\tau,w_0))$ is bounded, we have

$$\big(\phi(t), \xi(t) + \mathcal{A}_{R_0}^{w_0}(t)\big) \subset \mathcal{A}(t).$$

In other words, for a bounded global solution ϕ in Ω such that $\phi(\tau) = w_0$, the set $\big(\phi(t), \xi(t) + \mathcal{A}_{R_0}^{w_0}(t)\big)$ is a "piece" of the pullback attractor. This illustrates how the dynamics in the channel can collaborate to form the pullback attractor.

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