SLOW-FAST NORMAL FORMS ARISING FROM PIECEWISE SMOOTH VECTOR FIELDS

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Dedicated to the memory of Jorge Sotomayor Tello.

Abstract. We study planar piecewise smooth differential systems of the form

$$\dot{z} = Z(z) = \frac{1+\mathrm{sgn}(F)}{2}X(z) + \frac{1-\mathrm{sgn}(F)}{2}Y(z),$$

where $F:\mathbb{R}^2\to\mathbb{R}$ is a smooth map having 0 as a regular value. We consider linear regularizations Z_ε^φ of Z by replacing $\mathrm{sgn}(F)$ by $\varphi(F/\varepsilon)$ in the last equation, with $\varepsilon>0$ small and φ being a transition function (not necessarily monotonic). Nonlinear regularizations of the vector field Z whose transition function is monotonic are considered too. It is a well-known fact that the regularized system is a slow–fast system. In this paper we study typical singularities of slow-fast systems that arise from (linear or nonlinear) regularizations, namely fold, transcritical and pitchfork singularities. Furthermore, the dependence of the slow-fast system on the graphical properties of the transition function is investigated.

1. Introduction

In real life there are phenomena whose mathematical models are expressed by piecewise smooth vector fields, which have been studied at least since 1937. These systems are used in many branches of applied sciences, for example, Physics, Control Theory, Economics, Cell Mitosis, etc. For more details see, for instance, [3, 5].

A piecewise smooth vector field (or PSVF for short) is defined as follows: let Σ be a closed subset with empty interior of the ambient space (for example, a manifold embedded in $\mathbb{R}^n$). Such subset is called *discontinuity locus* and it divides the ambient space in finitely many open subsets $\{U_i\}_{i=1}^k$. In each open subset U_i is defined a smooth vector

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field. This paper deals with the case where a smooth curve divides a neighbourhood of $0 \in \mathbb{R}^2$ in two open regions. See Section 2 for a precise definition.

One of the most important question concerning PSVF's is: how to define the dynamics in Σ? In other words, how to define the transition between the dynamics defined in two different open sets?

Filippov [5] gave an answer defining the dynamics in Σ as the convex combination of two vector fields. This defines the so called *Sliding vector field*. We say that this vector field defined according to Filippov's ideas follows the *Filippov's convention*.

However, for some models, the Filippov's convention is not sufficient to describe the dynamics. For example, in [15] a model involving friction between an object and a flat surface was studied. The author gave an example that Filippov's convention takes into account only kinetic friction, while it is possible to consider static friction as well.

Another way to define the dynamics in the discontinuity locus Σ is combining two powerful tools: Regularizations of PSVF's and Blow-ups. A regularization process that is compatible with the Filippov's convention is the *Sotomayor-Teixeira regularization* [22], which consists in obtaining a one-parameter family of smooth vector fields Z_ε converging to Z when $\varepsilon \to 0$ (see Subsection 2.2). By using blow-up techniques, the regularized system $\dot{z} = Z_\varepsilon(z)$ becomes a slow-fast system, and therefore we are able to apply classical results on geometric singular perturbation theory (see Subsection 2.4) in the study of PSVF's. Such a link between Regularization Processes and geometric singular perturbation theory is a recent approach in mathematics and we refer to [1, 16, 17, 18, 19, 20] for further details. A similar approach can also be seen in [10].

Different regularization processes lead to different slow-fast systems, which gives rise to different sliding or sewing regions (see [19, 20, 21]). In this paper, we consider *linear* regularizations and *nonlinear* regularizations. See subsections 2.2 and 2.3 for precise definitions.

The dynamics of the *linearly* regularized system depends on the so called *transition function* φ, which can be monotonic or non monotonic. In this paper we highlighted the relation between the properties of the graph of φ, the properties of the slow-fast system, and the sliding regions of the PSVF's. See Theorem A below.

The main goal of this paper is to study typical singularities of slow-fast systems that arise from (linear or nonlinear) regularizations. For both linear and nonlinear regularizations are presented examples of

PSVF's such that, after (linear or nonlinear) regularization and directional blow-up, the slow-fast system presents normally hyperbolic, fold, transcritical or pitchfork singularities.

At some point, the reader may think that, after *linear* regularization and blow-up, it is possible to generate any slow-fast singularity, since it is just a matter of a suitable choice of the transition function. In general, this is not true. Indeed, we show that it does not exist a transition function that generate a pitchfork singularity. However, if we consider *nonlinear regularizations* it is possible to generate such a singularity (see Example 13). This shows that nonlinear regularizations are more general than the linear ones (see also [18, 20]).

Our main results, Theorems A, B and C are stated and proved in Section 3. In what follows, we briefly describe them.

Firstly, consider linear regularizations. Suppose that we drop the monotonicity condition of the transition function φ. In this context, we will prove that the critical points of φ give rise to non normally hyperbolic points of the critical set C_0 of $\dot{z} = Z_\varepsilon(z)$. For more details see Item (a) of Theorem A.

In addition, item (b) of Theorem A assures that we extend the classical Filippov sliding region when the transition function satisfy $|\varphi(x_0)| > 1$ for some x_0 in the open interval $(-1, 1)$. According to item (c) of the same Theorem, the dynamics in this extended sliding region is naturally defined using the classical Filippov sliding vector field. It is important to emphasize that item (c) of Theorem A was already proved in [21, Theorem 3]. For completness sake, we incorporated it in the statement of Theorem A and proved it as well.

Finally, item (d) of Theorem A says that there are cases in which it is not possible to apply geometric singular perturbation theory in order to define the sliding dynamics in some points of Σ. See Figure 1.

Slow-fast normal forms are well known in the literature (see Subsection 2.5 and the references therein). In Theorem B we state conditions that both PSVF and transition function must satisfy in order to generate classical slow-fast normal forms, such as fold and transcritical singularities. Moreover, we prove that there are slow-fast normal forms that can not be generated by *linear* regularization processes. This is the case of the pitchfork singularity. See Figure 2.

In order to generate pitchfork singularities, we must consider nonlinear regularization. Theorem C gives the conditions that must satisfy both monotonic transition function and vector field associated with the nonlinearly regularized system to generate this type of singularity.

Fold, transcritical and pitchfork singularities have very interesting dynamical properties. For example, M. Krupa, P. Szmolyan in [11,

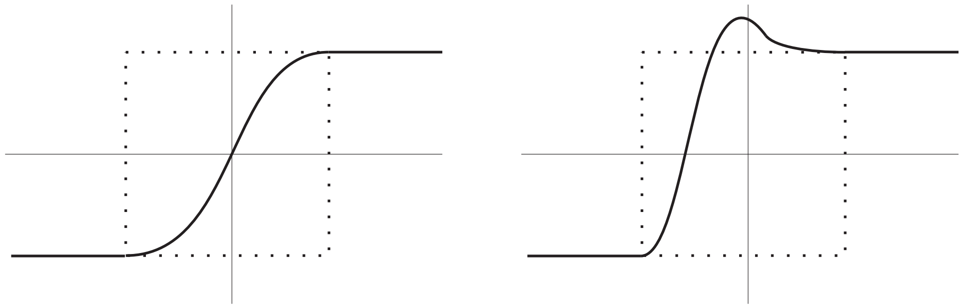

FIGURE 1. Monotonic transition function (left) and non monotonic transition function (right). The monotonic one generates only normally hyperbolic critical sets, and the sliding region coincides with the one proposed by Filippov. The non monotonic one has a critical point, which generates a non normally hyperbolic point of the critical manifold. Moreover, in this example, such a transition function extends the classical notion of sliding region.

12] studied the dynamics of the slow-fast system around this type of singularities for $\varepsilon > 0$ and built a map of transition between transversal sections. By applying these results together with Theorems B and C one can determine the local dynamics of the system regularized around these singularities and thus it is possible to make a global study of the dynamics of these systems.

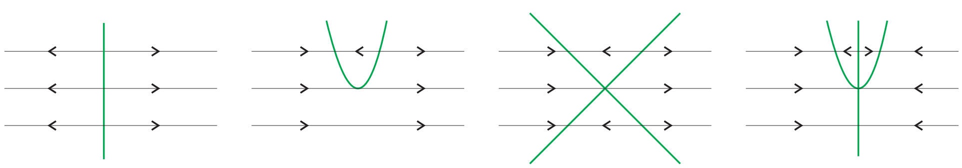

FIGURE 2. From the left to the right: normally hyperbolic, fold, transcritical and pitchfork points of a slow-fast system. It is not possible to generate the last one with linear regularizations, for any transition function. However, it is possible to generate it with nonlinear regularizations. The critical set is highlighted in green.

The paper is organized as follows. In Section 2 we present some introductory notions on PSVF, regularization processes, geometric singular perturbation theory and slow-fast normal forms. In Section 3 we state and prove Theorems A, B, and C.

2. Preliminaries on piecewise smooth vector fields and geometric singular perturbation theory

This section is devoted to establishing some basic results and notation that will be used throughout the paper.

2.1. **Piecewise smooth vector fields.**

Let $F : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a sufficientlly smooth function and consider C^r vector fields $X, Y : U \subset \mathbb{R}^2 \to \mathbb{R}^2$. A C^r *piecewise smooth vector field* $Z : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ (or PSVF for short) is given by

$$(1) \quad Z(\mathbf{x}) = \frac{1}{2}\bigg(\Big(1 + \operatorname{sgn}\big(F(\mathbf{x})\big)\Big)X(\mathbf{x}) + \Big(1 - \operatorname{sgn}\big(F(\mathbf{x})\big)\Big)Y(\mathbf{x})\bigg)$$

where $\mathbf{x} \in U$ and we assume that Z is multi-valued in the set

$$\Sigma = \{\mathbf{x} \in U; F(\mathbf{x}) = 0\},$$

which is called *discontinuity locus* or *discontinuity set*. The set of all C^r piecewise smooth vector fields is denoted by Ω^r. A PSVF is also denoted by $Z = (X, Y)$ in order to emphasize the dependency on the smooth vector fields X and Y.

The *Lie derivative* of F with respect to the vector field X is given by $XF = \langle X, \nabla F\rangle$ and $X^iF = \langle X, \nabla X^{i-1}F\rangle$ for all integer $i \geq 2$. This allows us to define the following regions in Σ:

(1) *Filippov sewing region*:
$$\Sigma^w = \big\{\mathbf{x} \in \Sigma\ ;\ XF(\mathbf{x}) \cdot YF(\mathbf{x}) > 0\big\};$$
(2) *Filippov sliding region*:
$$\Sigma^s = \big\{\mathbf{x} \in \Sigma\ ;\ XF(\mathbf{x}) \cdot YF(\mathbf{x}) < 0\big\}.$$

We emphasize that in the literature these sets are simply called sewing region and sliding region, respectively. Nevertheless, in [19] the authors presented a new definition of such regions, which depends on the type of regularization adopted (see Definitions 7 and 8). Due to this fact, we will call these regions as *Filippov regions* in order to stress that we are talking about the classical definition of sewing and sliding. See Figure 3.

A point $\mathbf{x}_0 \in \Sigma$ is a *PS-tangency point* if $XF(\mathbf{x}_0) = 0$ or $YF(\mathbf{x}_0) = 0$. We say that $\mathbf{x}_0$ is a *PS-fold point* of X if $XF(\mathbf{x}_0) = 0$ and $X^2F(\mathbf{x}_0) \neq 0$. If $X^2F(\mathbf{x}_0) > 0$, $\mathbf{x}_0$ is a *PS-visible fold* of X and if $X^2F(\mathbf{x}_0) < 0$ we say that $\mathbf{x}_0$ is an *PS-invisible fold* of X. Analogously we define PS-tangency points and PS-fold points of Y. Note that if $Y^2F(\mathbf{x}_0) < 0$, $\mathbf{x}_0$ is a PS-visible fold of Y and if $Y^2F(\mathbf{x}_0) > 0$ the point $\mathbf{x}_0$ is a PS-invisible fold of Y. If $\mathbf{x}_0$ is a PS-fold of both X and Y, we say that $\mathbf{x}_0$ is a *PS-fold-fold*. Finally, we say that $\mathbf{x}_0 \in \Sigma$ is a *PS-cusp point* if $XF(\mathbf{x}_0) = X^2F(\mathbf{x}_0) = 0$ and $X^3F(\mathbf{x}_0) \neq 0$.

Singularities of slow-fast systems will be discussed later. Throughout this paper, a singularity of a PSVF will be called *PS-singularity*, and a singularity of a slow-fast system when $\varepsilon = 0$ will be called *SF-singularity*.

Following Filippov's convention [5], one can define a vector field in $\Sigma^s \subset \Sigma$. The *Filippov sliding vector field* associated to $Z \in \Omega^r$ is the vector field $Z^\Sigma : \Sigma \to T\Sigma$ given by

$$Z^\Sigma(\mathbf{x}) = \frac{1}{YF - XF}\Big(X \cdot YF - Y \cdot XF\Big), \tag{2}$$

which is the convex combination between X and Y.

A *regularization of a PSVF*, Z, is a 1-parameter family of smooth vector fields Z_ε, $\varepsilon > 0$, satisfying that Z_ε converges pointwise to Z on $\mathbb{R}^2 \backslash \Sigma$, when $\varepsilon \to 0$ (see [20]). In this paper is considered two types of regularizations: the *linear* and *nonlinear* ones.

2.2. Linear regularization of piecewise smooth vector fields.

The regularization process proposed by Sotomayor and Teixeira in [22] is a powerfull tool in the study of piecewise smooth vector fields. With this technique, it is possible to construct a family of smooth vector fields $\{Z_\varepsilon\}_\varepsilon$ such that $Z_\varepsilon \to Z_0 = Z$ when $\varepsilon \to 0$.

We say that $\varphi : \mathbb{R} \to \mathbb{R}$ is a *transition function* if the following conditions are satisfied:

(1) φ is sufficiently smooth;
(2) $\varphi(t) = -1$ if $t \leq -1$ and $\varphi(t) = 1$ if $t \geq 1$;
(3) $\varphi'(t) > 0$ if $s \in (-1, 1)$. This condition is called *monotonicity*.

Throughout this paper it will be clear that, by dropping the monotonicity condition, it is possible to obtain different critical manifolds of the slow-fast system associated to the regularization. Moreover, non monotonic transition functions can expand the Filippov sliding region in Σ (see [19] and Theorem A below).

Definition 1. *Let φ be a transition function. A φ-linear regularization of a piecewise smooth vector field $Z = (X, Y)$ is an one-parameter family Z_ε^φ of smooth vector fields given by*

$$Z_\varepsilon^\varphi(\mathbf{x}) = \left(\frac{1}{2} + \frac{\varphi_\varepsilon\big(F(\mathbf{x})\big)}{2}\right)X(\mathbf{x}) + \left(\frac{1}{2} - \frac{\varphi_\varepsilon\big(F(\mathbf{x})\big)}{2}\right)Y(\mathbf{x}); \tag{3}$$

with $\varphi_\varepsilon(s) = \varphi\left(\dfrac{s}{\varepsilon}\right)$ for $\varepsilon > 0$. When φ is monotonic, we say that (3) is the ST-regularization (Sotomayor–Teixeira Regularization) of Z.

Intuitively, regularizing piecewise smooth vector field means to replace the discontinuity set Σ by a stripe (a tubular neighbourhood of Σ) of width 2ε. Outside this stripe, the vector fields Z_ε^φ and Z coincide, and inside the stripe the vector field Z_ε^φ can be seen as the "average" between X and Y.

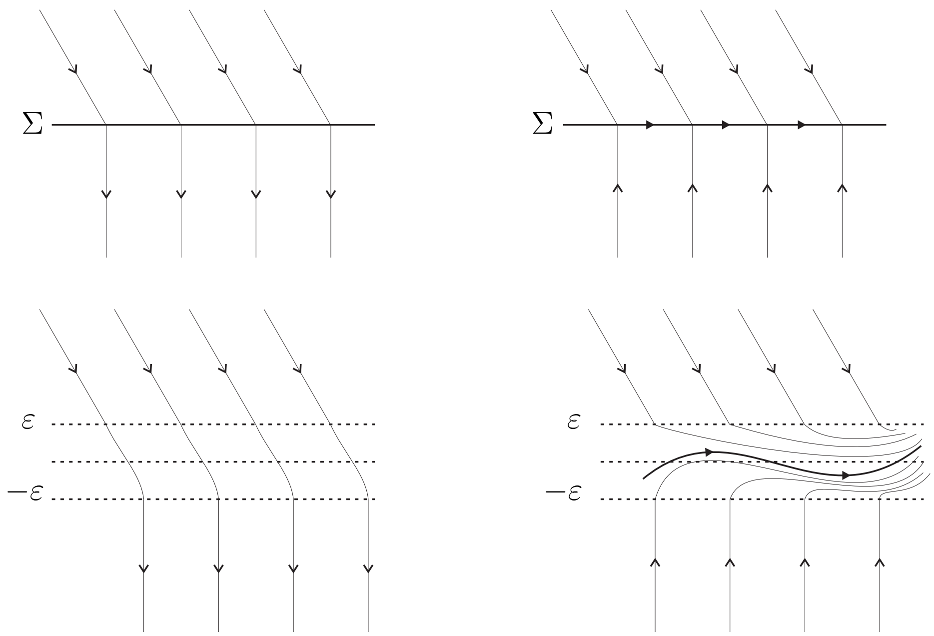

FIGURE 3. ST-Regularization of a Filippov sewing region (left) and a Filippov sliding region (right).

2.3. **Nonlinear regularization of piecewise smooth vector fields.** In [18, 20] the authors considered another way to generalize the notions of sliding region and sliding vector field by means of *nonlinear regularizations*.

Definition 2. *A continuous combination of X and Y is a 1-parameter family of smooth vector fields $\widetilde{Z}(\lambda, .)$, with $\lambda \in [-1,1]$, such that $\widetilde{Z}(1,p) = X(p)$, $\widetilde{Z}(-1,p) = Y(p)$, $\forall p \in U$.*

Now we define $\varphi-$nonlinear regularization of $Z = (X,Y)$.

Definition 3. *Let $\widetilde{Z}(\lambda,p)$ be a continuous combination of X and Y. A $\varphi-$nonlinear regularization of $Z = (X,Y)$ is the 1-parameter family given by $\widetilde{Z}(\varphi(\frac{F}{\varepsilon}),p)$.*

Recall that if $F > \varepsilon$, then $\varphi(\frac{F}{\varepsilon}) = 1$ and $\widetilde{Z}(\varphi(\frac{F}{\varepsilon}),p) = X(p)$; and if $F < -\varepsilon$, then $\varphi(\frac{F}{\varepsilon}) = -1$ and $\widetilde{Z}(\varphi(\frac{F}{\varepsilon}),p) = Y(p)$ (see Figure 4).

In [18, Theorem 1], it was shown the following result: *Let φ be a monotonic transition function and ψ a non-monotonic transition function. If Z_ε^ψ is a ψ-linear regularization, then there exists an unique*

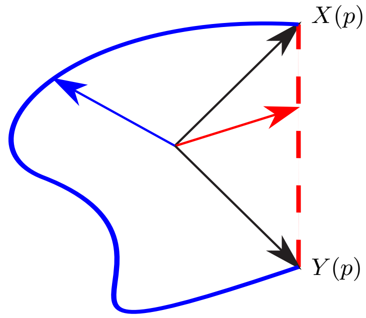

FIGURE 4. Linear (red) and nonlinear (blue) regularizations.

continuous combination $\widetilde{Z}(\lambda, p)$ *such that* $Z_\varepsilon^\psi(p) = \widetilde{Z}(\varphi(\frac{F}{\varepsilon}), p)$. However, in general the converse is not true (see Theorems B and C).

2.4. **Geometric singular perturbation theory.**

In the 1970s, Neil Fenichel wrote several papers on invariant manifold theory, which allowed a rigorous study of slow-fast systems (i.e., systems of differential equations with multiple time scales). We refer to [8, 9, 23] for a careful introduction on slow-fast systems, as well as details of the proof given in Fenichel's original paper [4]. The book [13] contains introductory notions, applications and more sophisticated concepts on this subject. For applications in Biology, see [7] and the references therein. Finally, see [2] for results concerning geometric singular perturbation theory for systems with many time scales.

A system of the form

$$\varepsilon \dot{x} = f(x, y, \varepsilon); \quad \dot{y} = g(x, y, \varepsilon); \tag{4}$$

is called *slow-fast system*, where $(x, y) \in \mathbb{R}^2$, $0 < \varepsilon \ll 1$ and $f, g : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ are sufficiently smooth. The dot $\cdot$ represents the derivative of the functions $x(\tau)$ and $y(\tau)$ with respect to the variable τ.

If we write $t = \frac{\tau}{\varepsilon}$, then system (4) becomes

$$x' = f(x, y, \varepsilon); \quad y' = \varepsilon g(x, y, \varepsilon); \tag{5}$$

in which the apostrophe ' denotes the derivative of the functions $x(t)$ and $y(t)$ with respect to the variable t. Observe that the parameter $\varepsilon = \frac{\tau}{t}$ represents the ratio of the time scales.

Consider equation (4) and set $\varepsilon = 0$. We obtain the so called *slow system* given by

$$0 = f(x, y, 0); \quad \dot{y} = g(x, y, 0). \tag{6}$$

This equation is also known in the literature as *reduced problem* or *slow vector field*. Note that (6) is not an ODE, but it is an *algebraic differential equation* (ADE).

Solutions of (6) are contained in the set

$$C_0 = \Big\{(x, y) \in \mathbb{R}^2 \ ; \ f(x, y, 0) = 0\Big\}.$$

Definition 4. *The set C_0 is called critical set. In the case where C_0 is a manifold, C_0 is called critical manifold.*

On the other hand, setting $\varepsilon = 0$ in equation (5) we obtain the so called *fast system*

$$x' = f(x, y, 0); \quad y' = 0. \tag{7}$$

System (7) is also known in the literature as *layer problem*, *layer equation* or *fast vector field*. Moreover, the system (7) can be seen as a system of ordinary differential equations, where $y \in \mathbb{R}$ is a parameter and the critical set C_0 is a set of equilibrium points of (7).

The main goal of geometric singular perturbation theory is to study systems (6) and (7) in order to obtain information of the full system (4). Observe that the systems (4) and (5) are equivalent when $\varepsilon > 0$, since they only differ by time scale.

Definition 5. *We say that $\mathbf{x}_0 \in C_0$ is normally hyperbolic if $f_x(\mathbf{x}_0) \neq 0$. The set of all normally hyperbolic points of C_0 will be denoted by $\mathcal{NH}(C_0)$.*

Recall that the nomenclature *PS-singularity* and *SF-singularity* is adopted in order to emphasize when p is a singularity of the piecewise smooth vector field (1) or a singularity of the slow system (6).

2.5. Normal forms of slow-fast systems.

In what follows we briefly recall some normal forms of slow-fast systems. An overview on this subject can be found in Chapter 4 of [13], and the reader can see the references therein for further details of the proofs. The normal forms of planar SF-generic transcritical and SF-generic pitchfork singularities were given in [12].

We say that the critical manifold $C_0 = \{f(x, y, 0) = 0\}$ has a *planar SF-generic fold* (or SF-fold for short) at the origin if

$$\begin{aligned} f_x(0,0,0) = 0; \quad f_{xx}(0,0,0) \neq 0; \\ f_y(0,0,0) \neq 0 \ \text{ and } \ g(0,0,0) \neq 0. \end{aligned} \tag{8}$$

In order to obtain a *SF-generic transcritical singularity* at the origin, the planar slow-fast system (4) must satisfy the following conditions:

$$\begin{aligned} f(0,0,0) = f_x(0,0,0) = f_y(0,0,0) = 0; \\ \det \operatorname{Hes}(f) < 0; \quad f_{xx}(0,0,0) \neq 0 \neq g(0,0,0); \end{aligned} \tag{9}$$

where $\operatorname{Hes}(f)$ denotes the Hessian matrix of f.

On the other hand, in order to obtain a *SF-generic pitchfork singularity* at the origin we must require the following conditions:

$$\begin{aligned} f(0,0,0) = f_x(0,0,0) = f_{xx}(0,0,0) = f_y(0,0,0) = 0; \\ f_{xxx}(0,0,0) \neq 0, \quad f_{xy}(0,0,0) \neq 0, \quad g(0,0,0) \neq 0. \end{aligned} \tag{10}$$

The normal forms of planar SF-generic fold, SF-generic transcritical and SF-generic pitchfork singularities were given in [11, 12]. The normal form of a normally hyperbolic point can be found in [13].

Theorem 6 gathers the results mentioned in the previous paragraph. The notation $\mathcal{O}$ denotes the higher order terms of a function. Moreover, in each case, λ denotes a constant that depends on the conditions of non-degeneracy of each SF-singularity (see [11, 12, 13] for details).

Theorem 6. *There exists a smooth change of coordinates such that for (x, y) sufficiently small the System (5) can be written as*

(a): *If the slow-fast system (5) satisfies the non-degeneracy conditions (8) of a planar SF-generic fold:*

$$x' = y + x^2 + \mathcal{O}(x^3, xy, y^2, \varepsilon); \quad y' = \varepsilon\Big(\pm 1 + \mathcal{O}(x, y, \varepsilon)\Big); \tag{11}$$

(b): *If the slow-fast system (5) satisfies the non-degeneracy conditions (9) of a SF-generic transcritical singularity:*

$$x' = x^2 - y^2 + \lambda\varepsilon + \mathcal{O}(x^3, x^2y, xy^2, y^3, \varepsilon x, \varepsilon y, \varepsilon^2); \quad y' = \varepsilon\Big(1 + \mathcal{O}(x, y, \varepsilon)\Big); \tag{12}$$

(c): *If the slow-fast system (5) satisfies the non-degeneracy conditions (10) of a SF-pitchfork singularity:*

$$x' = x(y - x^2) + \lambda\varepsilon + \mathcal{O}(x^2y, xy^2, y^3, \varepsilon x, \varepsilon y, \varepsilon^2); \quad y' = \varepsilon\Big(\pm 1 + \mathcal{O}(x, y, \varepsilon)\Big). \tag{13}$$

(d): *If $0 \in C_0$ is a normally hyperbolic point:*

$$\left\{ \begin{array}{rcl} x' & = & \Lambda(x, y, \varepsilon)x; \\ y' & = & \varepsilon\Big(h(y, \varepsilon) + H(x, y, \varepsilon)(x)\Big); \end{array} \right. \tag{14}$$

where x is sufficiently small, Λ, h and H are C^{r-1} in all arguments. Moreover, $\Lambda(x, y, \varepsilon)$ is non-zero, and $H(x, y, \varepsilon)$ is linear when applied to x.

3. Regularizations and typical SF-singularities

The relation between (*linear*) regularization of piecewise smooth vector fields and slow-fast systems had led mathematicians in a new direction in the research in qualitative theory of ordinary differential equations. By applying a directional blow-up, it is possible to transform a (*linearly*) regularized vector field into a slow-fast system. This approach was used for the first time in [1] in the context of planar piecewise smooth vector fields, and lately by [16] in the 3-dimensional case. The n-dimensional case was discussed in [17].

This study starts considering a planar piecewise smooth vector field whose discontinuity set is a smooth curve and *linear* regularizations. Without loss of generality, we adopt a coordinate system such that $Z = (X, Y)$ is written as

$$\dot{z} = Z(z) = \frac{1+\operatorname{sgn}(x)}{2}X(z) + \frac{1-\operatorname{sgn}(x)}{2}Y(z), \quad z = (x, y) \tag{15}$$

that is, the discontinuity set is a straight line. A linear regularization of (15) is the family

$$\dot{z} = Z_\varepsilon^\varphi(z) = \frac{1+\varphi(x/\varepsilon)}{2}X(z) + \frac{1-\varphi(x/\varepsilon)}{2}Y(z), \tag{16}$$

where $X = (f_1, f_2)$, $Y = (g_1, g_2)$ are applied in $z = (x, y)$. We emphasize that in this study the transition function φ is not necessarily monotonic.

After a directional blow-up of the form $x = \varepsilon\widetilde{x}$, one obtains the slow-fast system (dropping the tilde in order to simplify the notation)

$$\varepsilon\dot{x} = \frac{f_1+g_1}{2} + \varphi(x)\left(\frac{f_1-g_1}{2}\right); \quad \dot{y} = \frac{f_2+g_2}{2} + \varphi(x)\left(\frac{f_2-g_2}{2}\right); \tag{17}$$

where f_1, f_2, g_1, g_2 are applied in $(\varepsilon x, y)$. Denote the critical set of (17) by C_0, which is given by

$$C_0 = \left\{(x, y)\ ;\ \frac{f_1(0,y)+g_1(0,y)}{2} + \varphi(x)\left(\frac{f_1(0,y)-g_1(0,y)}{2}\right) = 0\right\}.$$

Now, we recall the definitions of sliding and sewing points presented in [19]. Observe that such notions are local.

Definition 7. *A point $p \in \Sigma$ is a sliding point if there is an open set $U \ni p$ and a family of smooth manifolds $S_\varepsilon \subset U$ such that*

1. *For each ε, S_ε is invariant by the regularized system (16);*
2. *For each compact subset $K \subset U$, the sequence $S_\varepsilon \cap K$ converges to $\Sigma \cap K$ as $\varepsilon \to 0$ according to Hausdorff distance.*

Definition 8. *We say that* $p \in \Sigma$ *is a sewing point if* $XF(p) \cdot YF(p) \neq 0$ *and there is an open set* $U \ni p$ *and local coordinates defined in* U *such that*

(1) $\Sigma = \{x = 0\}$;
(2) *For each* $\varepsilon > 0$, *the* horizontal vector field $v(x, y) = (1, 0)$ *is a generator of the regularized system* (16) *in* U.

Intuitively, a point p is a sewing point if the flow of (16) around p is transversal to Σ.

Concerning linear regularizations, if the transition function is monotonic and the discontinuity set is smooth, the dynamics of the sliding vector field according to Filippov's convention is equivalent to the dynamics of the slow system associated (see [16, Theorem 1.1]). However, if we do not consider monotonic transition functions, one can obtain different dynamics of the (*linearly*) regularized vector field and consequently different singular perturbation problems, which can lead us to different definitions of sliding or sewing regions. See [19, 20] and Theorem A below. *Nonlinear* regularizations also lead us to different notions of sewing and sliding. See [18, 20].

Before we state Theorem A, let us introduce some notation. In what follows, $\Pi : \mathbb{R}^2 \to \Sigma$ is the canonical projection $\Pi(x, y) = (0, y)$.

From the definitions discussed previously, it is clear that different (linear or nonlinear) regularizations lead to different slow-fast systems, which gives rise to different sliding or sewing regions. In order to emphasize the dependency of the regularization adopted, we will call these sets as the *r-Sliding* and *r-Sewing regions*, and we will denote them as Σ_r^s and Σ_r^w respectively. It can be shown that $\Sigma_r^s \cap \Sigma_r^w = \emptyset$ (see [19, Remark 5]).

Consider the Filippov sliding vector field Z^Σ associated to the PSVF (15). Although in the literature it is only considered the dynamics of Z^Σ in the Filippov sliding or escaping regions, the domain $D\Big(Z^\Sigma\Big) \subset \Sigma$ of Z^Σ may be greater than Σ^s. In this sense, for our purposes, the domain $D\Big(Z^\Sigma\Big)$ of Z^Σ is the subset of Σ in which Z^Σ is well defined, and not only the Filippov sliding region Σ^s.

Theorem A. *Consider the PSVF* (15) *and denote its Filippov sliding vector field by* Z^Σ, *which domain is the set* $D\Big(Z^\Sigma\Big) \subset \Sigma$. *Consider linear regularization of* Z *and let* φ *be a transition function, not necessarily monotonic. Let* $\Pi : \mathbb{R}^2 \to \Sigma$ *be the canonical projection and* $x_0 \in (-1, 1)$. *Then the following hold:*

(a): *If $\varphi'(x_0)=0$, then the set of points (x_0,y) such that*

$$f_1(x_0,y)+g_1(x_0,y)+\varphi(x_0)\Big(f_1(x_0,y)-g_1(x_0,y)\Big)=0$$

is contained in $C_0\backslash\mathcal{NH}\Big(C_0\Big)$. In other words, critical points of φ gives rise to non normally hyperbolic points of the critical set C_0 of (17).

(b): *Suppose that $(x_0,y_0)\in\mathcal{NH}(C_0)$ and $|\varphi(x_0)|>1$. Then $\Pi\Big(\mathcal{NH}(C_0)\Big)\cap\Sigma^w\neq\emptyset$. Moreover, $\Sigma^s\subsetneq\Sigma^s_r$. In other words, if $|\varphi(x_0)|>1$ then the r-sliding region is greater than the classical Filippov sliding region.*

(c): *If $(0,y_0)\in\Sigma^s_r$ satisfies $f_1(0,y_0)\neq g_1(0,y_0)$, near such a point the dynamics in the r-sliding region Σ^s_r is given by the classical Filippov sliding vector field Z^Σ. In other words, even if we extend Σ^s to Σ^s_r, the dynamics in such a set is given by Z^Σ. In particular, (x_0,y_0) is an SF-equilibrium point of (17) if, and only if, $(0,y_0)$ is an equilibrium point of Z^Σ.*

(d): *If*

$$\Pi\Big(C_0\Big)\cap\left(\Sigma\backslash D\Big(Z^\Sigma\Big)\right)\neq\emptyset,$$

then $(0,y_0)\in\Pi\Big(C_0\Big)\cap\left(\Sigma\backslash D\Big(Z^\Sigma\Big)\right)$ is a tangency point for both vector fields X and Y simultaneously, and the line $\{y=y_0\}$ is a component of C_0. See Figure 5.

Proof. **(a):** Without loss of generality, we suppose that $x_0=0$. Expanding the first equation of (17) in Taylor series, one obtains

$$x'=\frac{1}{2}\Big((f_1+g_1)+\varphi(0)(f_1-g_1)\Big)+\frac{1}{2}\Big(\varphi'(0)(f_1-g_1)\Big)x+\ \dots$$

A point of the form $(0,y,0)$ is normally hyperbolic if, and only if, the following conditions are satisfied:

$$(f_1+g_1)+\varphi(0)(f_1-g_1)=0,\quad \varphi'(0)(f_1-g_1)\neq 0. \tag{18}$$

Therefore, if $\varphi'(0)=0$ (that is, 0 is a critical point of the transition function), then $(0,y,0)$ is not normally hyperbolic.

(b): We already know that $\Sigma^s\subset\Sigma^s_r$ (see [21, Theorem 3]). Now, we prove that Σ^s_r contains points that do not belong to Σ^s. Since $(0,y_0)\in\mathcal{NH}(C_0)$ and $|\varphi(0)|>1$, define the constant a as

$$a=\frac{\varphi(0)+1}{\varphi(0)-1}\ \Leftrightarrow\ \varphi(0)=\frac{a+1}{a-1}.$$

Then the conditions (18) can be rewritten as

$$g_1 = af_1, \quad \varphi'(0) \neq 0, \tag{19}$$

where f_1 and g_1 are applied in $(0, y)$ and $a \neq 1$. Note that the condition $a \neq 1$ is naturally satisfied with the assumptions above. Observe that $a < 0$ if, and only if, $|\varphi(0)| < 1$. Analogously, it can be checked that $a > 0$ if, and only if, $|\varphi(0)| > 1$.

Since $|\varphi(0)| > 1$, then $a > 0$ and points of $\mathcal{NH}(C_0)$ of the form $(0, y)$ such that $g_1(0, y) = af_1(0, y)$ are projected in the Filippov sewing region Σ^w by Π.

By [19, Theorem 4.2], we have the inclusion $\Pi\big(\mathcal{NH}(C_0)\big) \subset \Sigma^s_r$. This means that $(0, y) \notin \Sigma^s$ is a sliding point, which implies that $\Sigma^s \subsetneq \Sigma^s_r$.

(c): Setting $\varepsilon = 0$ in the first equation of (17), we have

$$\varphi(x) = \frac{g_1(0, y) + f_1(0, y)}{g_1(0, y) - f_1(0, y)}.$$

Combining this expression with the second equation of (17), we obtain

$$\begin{aligned}\dot{y} &= \frac{f_2 + g_2}{2} + \left(\frac{g_1 + f_1}{g_1 - f_1}\right)\left(\frac{f_2 - g_2}{2}\right)\\ &= \frac{(g_1 - f_1)(f_2 + g_2) + (g_1 + f_1)(f_2 - g_2)}{2(g_1 - f_1)}\\ &= \frac{g_1(0, y)f_2(0, y) - f_1(0, y)g_2(0, y)}{g_1(0, y) - f_1(0, y)}\end{aligned}$$

which is exactly the expression of Z^Σ. Therefore, the dynamics in the r-sliding region Σ^s_r is given by the classical Filippov sliding vector field Z^Σ.

(d): The domain of Z^Σ is precisely the set

$$D\Big(Z^\Sigma\Big) = \{(0, y) \in \Sigma \ ; \ g_1(0, y) \neq f_1(0, y)\}.$$

If $(0, y_0) \notin D\Big(Z^\Sigma\Big)$ and $(0, y_0) \in \Pi\Big(C_0\Big)$, then $g_1(0, y_0) = f_1(0, y_0)$. From the expression of C_0, $(0, y_0)$ must be a tangency point for both X and Y. Moreover, the equation $f_1(0, y_0) = 0$ assures that the horizontal line $\{y = y_0\}$ is a component of the critical manifold C_0. See Figure 5.

□

Item (a) of Theorem A assures that, in order to generate SF–singularities with linear regularizations, we may drop the monotonicity of the

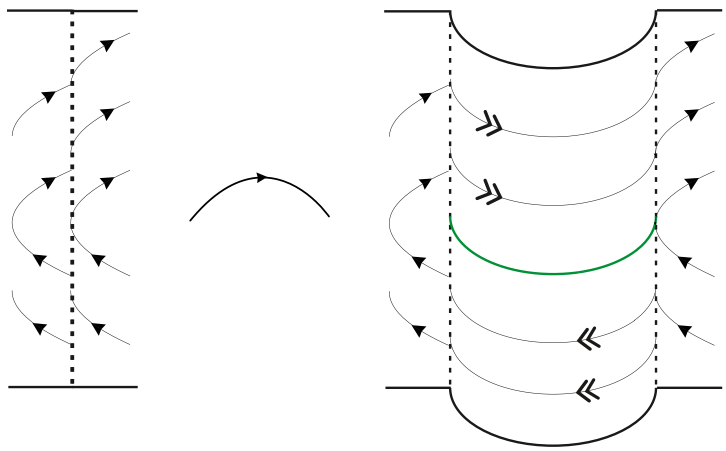

FIGURE 5. Level $\varepsilon = 0$ of the regularized vector field. The semi-cylinder represents the blowing up locus and the flows with simple arrow and with double arrow represent the slow and the fast system, respectively. Statement (d) says that, if the projection $\Pi(C_0)$ on Σ contains a point $p = (0, y_0)$ that do not belong to the domain $D\Big(Z^\Sigma\Big)$ of the Filippov sliding vector field Z^Σ, then p is a tangency point for both X and Y, that is, $g_1(p) = f_1(p) = 0$. Moreover, the critical manifold C_0 (highlighted in green) contains a line $y = y_0$. It is not possible to define dynamics in Σ through p using geometric singular perturbation theory.

transition function φ (see also Theorem B). Moreover, $\varphi(0) = 1$ implies $f_1(0, y) = 0$, that is, there is a PS-tangency point between X and Σ. Analogously, $\varphi(0) = -1$ implies $g_1(0, y) = 0$, that is, there is a PS-tangency point between Y and Σ.

Following our notation, [21, Theorem 3] assures that $\Sigma^s \subset \Sigma^s_r$. However, in our statement we give a condition such that $\Sigma^s \subsetneq \Sigma^s_r$. In other words, if $(x_0, y_0) \in \mathcal{NH}(C_0)$ and $|\varphi(x_0)| > 1$ for $x_0 \in (-1, 1)$, then there exists a point $(0, y_0) \in \Sigma^s_r$ that does not belong to Σ^s.

According to item (c), the dynamics in r-sliding region Σ^s_r is naturally extended using the classical Filippov sliding vector field. Finally, item (d) says that $\Pi\big(C_0\big)$ is entirely contained in $D\Big(Z^\Sigma\Big)$, unless C_0 contains horizontal lines. This means that we can not define a sliding dynamics in $\Sigma\backslash\Big(D\big(Z^\Sigma\big)\Big)$ using geometric singular perturbation theory.

Now, we are concerned in establishing conditions that both piecewise smooth vector field and transition function must satisfy in order to generate SF-singularities.

Theorem B. *Consider the PSVF (15) and let φ be a transition function, not necessarily monotonic. After linear regularization and directional blow-up, it is possible to generate normally hyperbolic points, SF-fold singularities and SF-transcritical singularities. However, it is not possible to generate SF-pitchfork singularities.*

Proof. Let φ be a transition function (not necessarily monotonic) and $Z = (X, Y)$ be a PSVF, in which $X = (f_1, f_2)$ and $Y = (g_1, g_2)$.

The proof is given by direct computations. The idea is to compare the coefficients of the Taylor expansion at the origin of the function that defines the critical set C_0 of (17) with the expressions of the normal forms given in Subsection 2.5. With this procedure, we obtain that such coefficients must satisfy the following conditions in order to generate SF-singularities:

(a): *Fenichel normal form (normally hyperbolic point):*

$$f_1(0,0) - g_1(0,0) \neq 0, \quad \varphi'(0) \neq 0; \tag{20}$$

(b): *SF-generic Fold:*

$$\begin{aligned} & f_1(0,0) - g_1(0,0) \neq 0, \quad \varphi'(0) = 0, \quad \varphi''(0) \neq 0; \\ & \varphi(0) = \frac{g_1(0,0) + f_1(0,0)}{g_1(0,0) - f_1(0,0)}; \\ & \big(f_{1,y}(0,0) + g_{1,y}(0,0)\big) + \varphi(0)\big(f_{1,y}(0,0) - g_{1,y}(0,0)\big) \neq 0. \end{aligned} \tag{21}$$

(c): *SF-Transcritical singularity:*

$$\begin{aligned} & f_1(0,0) - g_1(0,0) \neq 0, \quad \varphi'(0) = 0, \quad \varphi''(0) \neq 0; \\ & \varphi(0) = \frac{g_1(0,0) + f_1(0,0)}{g_1(0,0) - f_1(0,0)}; \\ & \big(f_{1,y}(0,0) + g_{1,y}(0,0)\big) + \varphi(0)\big(f_{1,y}(0,0) - g_{1,y}(0,0)\big) = 0; \\ & \begin{vmatrix} \frac{1}{4}\Big((f_1 - g_1)\varphi''(0)\Big) & 0 \\ 0 & \frac{1}{4}\Big((1+\varphi(0))f_{1,yy} + (1-\varphi(0))g_{1,yy}\Big) \end{vmatrix} < 0; \end{aligned} \tag{22}$$

where $f_1, g_1, f_{1,yy}$ and $g_{1,yy}$ are computed at $(0,0)$.

(d): *SF-Pitchfork singularity:* it is not possible to generate this kind of SF-singularity, for any transition function φ. Indeed, such a SF-singularity would lead us to require

$$\begin{aligned} & f_1(0,0) - g_1(0,0) \neq 0, \quad \varphi'(0) = 0, \quad \varphi''(0) = 0, \quad \varphi'''(0) \neq 0; \\ & \big(f_{1,y}(0,0) + g_{1,y}(0,0)\big) + \varphi(0)\big(f_{1,y}(0,0) - g_{1,y}(0,0)\big) = 0; \\ & \varphi(0) = \frac{g_1(0,0) + f_1(0,0)}{g_1(0,0) - f_1(0,0)}, \quad \varphi'(0)\big(f_{1,y}(0,0) - g_{1,y}(0,0)\big) \neq 0; \end{aligned} \tag{23}$$

and therefore the transition function would satisfy $\varphi'(0) = 0$ and $\varphi'(0) \neq 0$ simultaneously, which is a contradiction.

□

Remark 9. *Notice that the SF-fold, SF-transcritical, and SF-pitchfork singularities are non normally hyperbolic points.*

Due to Theorem B, one can start to search for examples of regularized systems that possess normally hyperbolic points, SF-fold singularities and SF-transcritical singularities. In the following example we present a regularized system that has a SF-transcritical singularity at the origin.

Example 10. *Consider the normal form of a PS-cusp singularity*

$$Z(x,y) = \begin{cases} X(x,y) = \Big(-y^2, 1\Big), & \text{if } x > 0; \\ Y(x,y) = \Big(1, 1\Big), & \text{if } x < 0. \end{cases} \tag{24}$$

Recall that the origin is a PS-cusp singularity and $\Sigma^s = \Sigma \backslash \{0\}$. Now, consider the transition function φ given by

$$\varphi(t) = \begin{cases} -1, & \text{if } t \leq -1; \\ -\frac{3t^5}{2} + t^4 + \frac{5t^3}{2} - 2t^2 + 1, & \text{if } -1 \leq t \leq 1; \\ 1, & \text{if } t \geq 1; \end{cases} \tag{25}$$

in which $t_0 = 0$ and $t_1 = \frac{8}{15}$ are local maximum and minimum, respectively. See Figure 6.

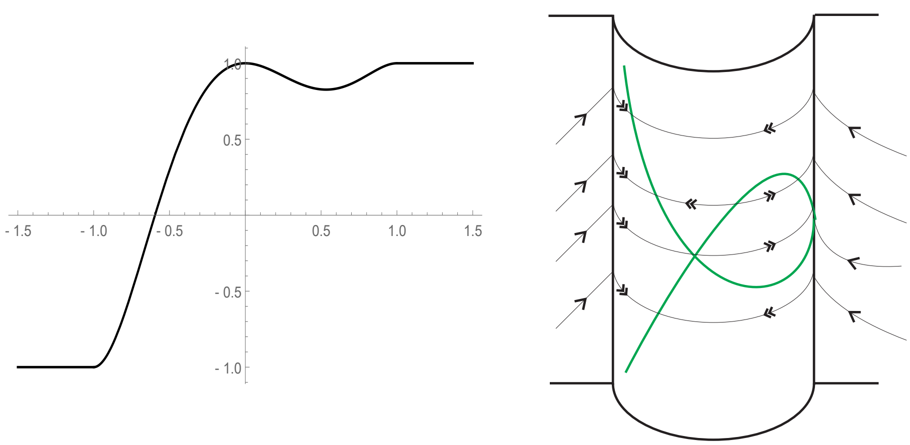

FIGURE 6. Graphic of the monotone transition function φ (left) and the linear regularized system (26) (right). The critical manifold is highlighted in green.

After regularization and blow-up, one obtains the slow-fast system

$$\text{(26)} \qquad \begin{cases} \varepsilon\dot{x} &= \frac{1}{4}\left(x^2(x-1)^2(3x+4)\left(y^2+1\right)-4y^2\right); \\ \dot{y} &= 1. \end{cases}$$

Observe that for $x=0$ and $x=\frac{8}{15}$, the critical manifold presents non normally hyperbolic points. In particular, the origin is a transcritical singularity.

It is important to note that this example can be generalized as follows.

Corollary 11. *Suppose that the origin is a PS-cusp singularity of the PSVF (15) and let φ be a non-monotonic transition function such that $\varphi(0)=1$, $\varphi'(0)=0$, and $\varphi''(0)\neq 0$. If $g_1(0,0)\varphi''(0)f_{1,yy}(0,0)>0$, then the regularized system associated with Z has a SF-transcritical singularity at origin.*

Proof. Suppose that the origin is a regular-cusp singularity of the PSVF (15), that is,

- $XF(0,0)=f_1(0,0)=0$;
- $X^2F(0,0)=f_{1,y}(0,0)f_2(0,0)=0$, thus $f_{1,y}(0,0)=0$;
- $X^3F(0,0)=f_{1,yy}(0,0)(f_2(0,0))^2\neq 0$, hence $f_{1,yy}(0,0)\neq 0$;
- $g_1(0,0)\neq 0$;

where $F(x,y)=x$ and XF is the Lie derivative of F with respect to the vector field X. Then, we get that

- $(f_1-g_1)(0,0)=-g_1(0,0)\neq 0$;
- $\varphi(0)=1$;
- $\begin{vmatrix} \frac{1}{4}\Big((f_1-g_1)\varphi''(0)\Big) & 0 \\ 0 & \frac{1}{4}\Big((1+\varphi(0))f_{1,yy}+(1-\varphi(0))g_{1,yy}\Big) \end{vmatrix} = -\frac{g_1\varphi''(0)f_{1,yy}}{8}.$

Since $\varphi'(0)=0$ and $g_1(0,0)\varphi''(0)f_{1,yy}(0,0)>0$, then the conditions obtained in the proof of Theorem B imply that the origin is a SF-transcritical singularity. □

Using the definition of a PS-fold singularity of the PSVF (15) and Theorem B we obtain the following result.

Corollary 12. *Suppose that the origin is a PS-fold singularity of the PSVF (15) and let φ be a non-monotonic transition function such that $\varphi(0)=1$, $\varphi'(0)=0$, and $\varphi''(0)\neq 0$. Then the regularized system associated with Z has a SF-fold singularity at origin.*

At some point, the reader may think that, after non monotonic linear regularization and blow-up, it is possible to generate any SF-singularity, since it is just a matter of a suitable choice of the transition function.

In general, this is not true. Indeed, Theorem B assures that it does not exist a transition function that generates a SF-pitchfork singularity. This leads us to consider nonlinear regularizations.

3.1. **Nonlinear regularization and SF-singularities.**

In what follows, we present a version of Theorem B for nonlinear regularization.

Theorem C. *Consider the PSVF (15) and let φ be a monotonic transition function. After φ-nonlinear regularization $\widetilde{Z}(\varphi(\frac{x}{\varepsilon}), x, y)$ and directional blow-up, it is possible to generate normally hyperbolic points, SF-fold singularities, SF-transcritical singularities and SF-pitchfork singularities.*

Proof. Let φ be a monotonic transition function and $Z = (X, Y)$ be a PSVF. Consider the φ-nonlinear regularization $\widetilde{Z}(\varphi(\frac{x}{\varepsilon}), x, y)$ of Z, where $\widetilde{Z} = (\widetilde{Z}^1, \widetilde{Z}^2)$. The proof is given by direct computations. The idea is to compare the coefficients of the Taylor expansion of the function $\widetilde{Z}^1(\varphi(\tilde{x}), \varepsilon\tilde{x}, y)$ near $(0, 0, 0)$ with the expressions of the normal forms given in Subsection 2.5 and use that $\varphi'(t) \neq 0$ for all $t \in (-1, 1)$. With this procedure, we obtain that such coefficients must satisfy the following conditions in order to generate SF-singularities:

(a): *Fenichel normal form (normally hyperbolic point):*

$$\widetilde{Z}^1_\lambda(\varphi(0), 0, 0) \neq 0; \tag{27}$$

(b): *SF-generic Fold:*

$$\begin{aligned} &\widetilde{Z}^1(\varphi(0), 0, 0) = 0; \quad \widetilde{Z}^1_\lambda(\varphi(0), 0, 0) = 0; \quad \widetilde{Z}^1_{\lambda\lambda}(\varphi(0), 0, 0) \neq 0; \\ &\widetilde{Z}^1_y(\varphi(0), 0, 0) \neq 0; \quad \widetilde{Z}^2(\varphi(0), 0, 0) \neq 0. \end{aligned} \tag{28}$$

(c): *SF-Transcritical singularity:*

$$\begin{aligned} &\widetilde{Z}^1(\varphi(0), 0, 0) = 0; \quad \widetilde{Z}^1_\lambda(\varphi(0), 0, 0) = 0; \quad \widetilde{Z}^1_{\lambda\lambda}(\varphi(0), 0, 0) \neq 0; \\ &\widetilde{Z}^1_y(\varphi(0), 0, 0) \neq 0; \quad \widetilde{Z}^2(\varphi(0), 0, 0) \neq 0; \\ &\left(\widetilde{Z}^1_{\lambda y}(\varphi(0), 0, 0)\right)^2 - \widetilde{Z}^1_{\lambda\lambda}(\varphi(0), 0, 0)\widetilde{Z}^1_{yy}(\varphi(0), 0, 0) > 0. \end{aligned} \tag{29}$$

(d): *SF-Pitchfork singularity:*

$$\begin{aligned} &\widetilde{Z}^1(\varphi(0), 0, 0) = 0; \quad \widetilde{Z}^1_\lambda(\varphi(0), 0, 0) = 0; \quad \widetilde{Z}^1_{\lambda\lambda}(\varphi(0), 0, 0) = 0; \\ &\widetilde{Z}^1_y(\varphi(0), 0, 0) = 0; \quad \widetilde{Z}^1_{\lambda\lambda\lambda}(\varphi(0), 0, 0) \neq 0; \quad \widetilde{Z}^1_{\lambda y}(\varphi(0), 0, 0) \neq 0; \\ &\widetilde{Z}^2(\varphi(0), 0, 0) \neq 0. \end{aligned} \tag{30}$$

□

To end this section, we present an example of a nonlinear regularized system with SF-pitchfork singularity.

Example 13. *Let $Z = (X, Y)$ be a PSVF defined on $\mathbb{R}^2$ with $F(x, y) = x$, $X(x, y) = ((x+1)y+1, -1)$, $Y(x, y) = ((x-1)y-1, -1)$. Consider the continuous combination of X and Y given by*

$$\widetilde{Z}(\lambda, x, y) = \Big((x+\lambda)y + \lambda^3, -1\Big).$$

Assume that the monotonic transition function φ satisfies $\varphi(0) = 0$ and $\varphi'(0) \neq 0$ (for example, $\varphi(t) = -\frac{t^5}{2} + \frac{t^3}{2} + t$, for all $t \in (-1, 1)$). Thus, after nonlinear regularization and directional blow-up we obtain

$$\varepsilon\dot{\hat{x}} = (\varepsilon\hat{x} + \varphi(\hat{x}))y + \varphi(\hat{x})^3; \quad \dot{y} = -1; \tag{31}$$

where $\hat{x} = \frac{x}{\varepsilon}$. Notice that (31) satisfies conditions (30) and therefore the origin is a SF-pitchfork singularity. See Figure 7.

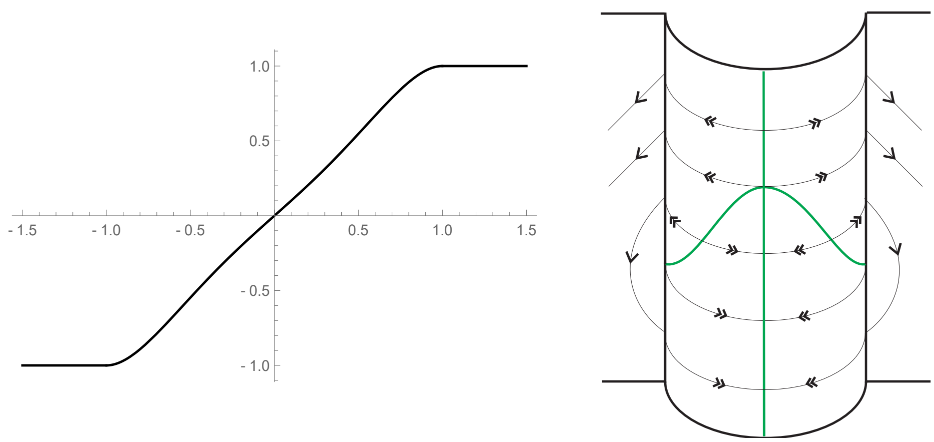

Figure 7. Graphic of the monotone transition function φ (left) and the φ-nonlinear regularization (31) of f and g (right). The critical manifold is highlighted in green.

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5. Data availability

All data generated or analyzed during this study are included in this article.

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