



There is no largest proper operator ideal

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Abstract

An operator ideal is proper if the only invertible operators it contains have finite rank. We answer a problem posed by Pietsch (Operator ideals, North-Holland, Amsterdam, 1980) by proving (i) that the ideal of inessential operators is not maximal among proper operator ideals, and (ii) that there is no largest proper operator ideal. Our proof is based on an extension of the construction by Aiena and González (Math Z 233:471–479, 2000), of an improjective but essential operator on Gowers–Maurey’s shift space X_S (Math Ann 307:543–568, 1997), through a new analysis of the algebra of operators on powers of X_S . We also prove that certain properties hold for general $\mathbb{C}$ -linear operators if and only if they hold for these operators seen as real: for example this holds for operators belonging to the ideals of strictly singular, strictly cosingular, or inessential operators, answering a question of González and Herrera (Stud Math 183(1):1–14, 2007). This gives us a frame to extend the negative answer to the problem of Pietsch to the real setting.

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1 Introduction

In this paper we consider operator ideals (or more generally, families of operators) in the sense of Pietsch [29]. Unless specified otherwise by space we mean infinite dimensional Banach space and by subspace we mean closed infinite dimensional subspace. An operator will be a bounded linear operator between Banach spaces, and $L(X, Y)$ denotes the space of operators between the spaces X and Y . If U is an operator ideal, then $U(X, Y)$ is the subset of operators of $L(X, Y)$ belonging to U . For all other unexplained notation see what follows.

In his book Pietsch considers a family of spaces associated to an ideal U , see [29] 2.1: the space ideal $\text{Space}(U)$ defined by

$$X \in \text{Space}(U) \Leftrightarrow \text{Id}_X \in U.$$

Of course this definition makes sense even when U is a family of operators which is not an ideal. Note that it is immediate that the space ideal of U coincides with the space ideal of its closure U^{clos} , [29] Proposition 4.2.8, so in this context one does not need to pay attention to whether the ideals considered are closed. In [29] 2.3.3 an ideal U is called *proper* if $\text{Space}(U) = F$, the class of finite dimensional spaces; or equivalently, if $U(X)$ is a proper ideal of $L(X)$ whenever X is infinite dimensional. Among proper ideals one can mention the ideals of finite rank, compact, strictly singular, strictly cosingular, or inessential operators, see definitions below. Problem [29] 2.3.6 asks whether there is a largest proper operator ideal. It is actually conjectured by Pietsch that such an ideal exists and is equal to the ideal In of inessential operators (a specific case of [29] Conjecture 4.3.7, see [29] 4.3.1).

Problem 1 (Pietsch, 1980) Is the ideal of inessential operators the largest proper operator ideal?

Problem 2 (Pietsch, 1980) More generally, does there exist a largest proper operator ideal?

This can also be seen as a special case of [29] Problem 2.2.8, where Pietsch asks whether, given a space ideal A (see [29] Definition 2.1.1), there exists a largest operator ideal U with $A = \text{Space}(U)$. Problems 1 and 2 correspond to the space ideal F of finite dimensional spaces for which $F = \text{Space}(\text{In})$.

Recall that an operator $R \in L(X, Y)$ is said to be *inessential*, $R \in \text{In}(X, Y)$, if $\text{Id}_X - TR$ is Fredholm for any $T \in L(Y, X)$ (equivalently $\text{Id}_Y - RT$ is Fredholm for any such T); otherwise we shall say that it is *essential*. Two spaces X and Y are *essentially incomparable* if $L(X, Y) = \text{In}(X, Y)$; equivalently, $L(Y, X) = \text{In}(Y, X)$.

There is a natural direction in which to investigate whether In is the largest proper operator ideal, which was suggested to the author by Manuel González. This would be to study the question in the setting of complex spaces as well as real spaces and obtain strong structural differences between the complex and the real cases. Indeed if some $\mathbb{C}$ -linear operator is essential as real but inessential as complex, then this might mean that one gets a larger proper ideal than the real ideal of inessential operators.

More generally it is a natural question, related to the study of complex structures on real Banach spaces, to understand the differences between real and complex versions of some classical operator ideals, and this is a first aim of this paper. More precisely we ask whether a $\mathbb{C}$ -linear operator belongs to a certain ideal as $\mathbb{C}$ -linear if and only if it does as an $\mathbb{R}$ -linear operator. It is obvious for example that an operator is compact as $\mathbb{C}$ -linear if and only if it is compact as $\mathbb{R}$ -linear. The question for strictly singular appears in [16] as Remark 2.7, and for inessential was personally asked by M. González.

While an $\mathbb{R}$ -strictly singular (resp. $\mathbb{R}$ -inessential), $\mathbb{C}$ -linear operator is clearly always $\mathbb{C}$ -strictly singular (resp. $\mathbb{C}$ -inessential), the converse is not immediate, since there are more real subspaces (resp. operators) than complex subspaces (resp. operators) in a complex space. However we shall show that the answer is actually positive, and holds also for many other classical ideals. The result depends on a characterization based on the notion of *self-conjugacy* of a complex ideal, see Proposition 5.

Theorem 1 *A $\mathbb{C}$ -linear operator is inessential as a complex operator if and only if it is inessential as a real operator. The same holds for the ideals of*

- *strictly singular operators,*
- *strictly cosingular operators,*
- *A -factorable operators if A is a complex and self-conjugate space ideal.*

Going back to Pietsch's problem, in particular the direction suggested above does not work. In a second part of the paper we use another approach to Problems 1 and 2, which we shall actually solve negatively.

An operator is *improjective*, $T \in \text{Imp}(X, Y)$, if the restriction of T to a complemented subspace of X is never an isomorphism onto a complemented subspace of Y , see Tarafdar [31]. When $L(X, Y) = \text{Imp}(X, Y)$ (equivalently $L(Y, X) = \text{Imp}(Y, X)$), then X and Y are said to be *projectively incomparable*. It is straightforward that all inessential operators are improjective, and that Id_X is never improjective for X infinite dimensional.

In 2000, Aiena and González proved that there exist operators which are improjective but not inessential, [1] Theorem 3.6. Actually they obtain two projectively incomparable spaces and an operator between them which is essential, [1] Proposition

3.7. This suggests a direction to find a proper ideal larger than In , providing a negative answer to Problem 1: since $\text{Id}_X \in \text{Imp}$ only when X is finite dimensional, we would be done if Imp were an operator ideal. However in the same paper Aiena and González prove that the improjective operators do not form an ideal, [1] Theorem 3.6.

The example of [1] relies on the theory of spaces with few operators (or exotic spaces) of Gowers-Maurey, see [27]. As commented in the Aiena-Gonzalez paper, while hereditarily indecomposable spaces (first defined by Gowers-Maurey [17]) have the property that all operators are either Fredholm or inessential, on the other hand, in indecomposable spaces operators are either Fredholm or improjective; so it is natural to consider an indecomposable space which is not HI. Their example is therefore based on the “shift space” X_S of Gowers-Maurey [18] which has these properties, see also Maurey’s surveys [26] and [27] for a more thorough description. Considering the complex version of X_S , they find an infinite codimensional subspace Y of X_S which is projectively incomparable with X_S ; however there is an operator $T \in L(X_S, Y)$ which is not inessential.

If X is a Banach space, $\text{Op}(X)$ denotes the family of X -factorable operators. This is an ideal if, e.g., X is isomorphic to its square. It is easy to see that two spaces X and X' are projectively incomparable if and only if $\text{Op}(X) \cap \text{Op}(X')$ is proper. So in particular $\text{Op}(X_S) \cap \text{Op}(Y)$ is proper and contains an operator which is not inessential. A negative answer to Problem 1 would follow if $\text{Op}(X_S) \cap \text{Op}(Y)$ were an ideal; but since X_S is not isomorphic to its square this has no reason to hold.

In this paper we show how to enhance Aiena-González’s result so that the associated Op -class is an ideal: we define $\text{Op}^{<\omega}(X)$ the class of operators which are X^n -factorable for some $n \in \mathbb{N}$ and observe that it is an ideal. The crucial point is then to go back to the construction of [18] to prove that all powers of the spaces X_S and Y (or possibly some technical variation of them) are projectively incomparable, which means that $U := \text{Op}^{<\omega}(X_S) \cap \text{Op}^{<\omega}(Y)$ is a proper ideal. Since the essential operator T defined in [1] belongs to U , the ideal of inessential operators is not the largest among proper ideals. This answers Problem 1 of Pietsch. Actually we prove slightly more:

Theorem 2 *The ideal of inessential operators is not maximal among proper operator ideals, i.e. there exists a proper operator ideal V with $\text{In} \subsetneq V$.*

Based on the observation of Aiena-González that their construction actually provides an example of two improjective operators whose sum is not improjective, we find two versions of the above ideal and two operators belonging to each of them but whose sum is invertible on X_S . As a corollary there actually cannot exist a largest proper ideal. So we have a stronger result, namely the answer to Problem 2 of Pietsch is also negative.

Theorem 3 *There is no largest proper operator ideal.*

These examples hold both in the real and complex setting. We actually use some ideas of the first part of the paper to extend our negative answers from the complex to the real setting. To be able to treat both the complex and real cases in a unified way, we shall replace the complex version (call it $X_S(\mathbb{C})$) of X_S used in the above

description, by the complexification $X := (X_S(\mathbb{R}))_{\mathbb{C}}$ of the real version $X_S(\mathbb{R})$ of X_S . While these two spaces are certainly not isomorphic, their algebras of operators have very similar properties, sufficiently for our purposes, and so all of the above applies to X . But additionally X is much easier to relate to a real space (through complexification), and this will provide us with a real solution based on operators on $X_S(\mathbb{R})$.

1.1 Background and definitions

In what follows I_X , or sometimes Id_X , denotes the identity map on X . We use the notation $X \simeq Y$ to mean that the spaces X and Y are linearly isomorphic.

We recall a few basic results about certain operator ideals and Fredholm theory. For more details we refer to [25] or to the survey of B. Maurey [26].

An operator $S \in L(X, Y)$ is strictly singular, $S \in SS(X, Y)$, when $S|_Z$ is never an isomorphism onto its range, for Z an (infinite dimensional) subspace of X ; it is strictly cosingular, $S \in CS(X, Y)$, when QS is never surjective for Q the quotient map onto a quotient of Y by some infinite codimensional subspace of Y . Both SS and CS are closed operator ideals.

An operator $T : X \rightarrow Y$ is Fredholm if it has closed image and finite dimensional kernel and cokernel. It is finitely singular if there exists a finite codimensional (closed) subspace Y of X such that the restriction $T|_Y$ is an isomorphism onto its range TY - this terminology appears in [18]; such operators are more classically called upper semi-Fredholm, as in [1]. It is infinitely singular otherwise, which is equivalent to saying that for any $\varepsilon > 0$ there exists an infinite dimensional subspace Z of X such that $\|T|_Z\|$ is at most ε ([26] Proposition 3.2). From this last characterization it is also useful to note (i) that the class of infinitely singular operators is preserved by strictly singular perturbations, and (ii) that an operator is infinitely singular as soon as its restriction to some infinite dimensional subspace is infinitely singular.

Recall that K denotes the closed ideal of compact operators. We have the following classical inclusions:

$$K \subseteq SS \cap CS \subseteq SS + CS \subseteq \text{In} \subseteq \text{Imp}$$

The ideal of inessential operators is closely related to Fredholm theory; in particular an inessential perturbation of a Fredholm operator is Fredholm (and so this holds as well for compact or strictly singular perturbations).

A Banach space is decomposable if it is the (topological) direct sum of two infinite dimensional closed subspaces, indecomposable otherwise, and hereditarily indecomposable (HI) if it contains no decomposable subspace. The first example of an HI space was due to Gowers-Maurey [17] and since then a great number of other indecomposable or HI examples with various additional properties have been obtained (some of which may be found in [27]).

2 Complex ideals versus real ideals

In this section we recall and develop tools to compare $\mathbb{R}$ -linear and $\mathbb{C}$ -linear behaviours of operators, with Theorem 1 as our objective.

2.1 Complex structures

The theory of complex structures on Banach spaces was born after the example by Bourgain (1986) of two spaces which are linearly isometric as real spaces but not isomorphic as complex spaces [5]. Actually the two spaces used by Bourgain are complex conjugate and so the real linear isometry is just the identity map between them.

A complex structure on a real space X is the space X equipped with a $\mathbb{C}$ -linear structure whose underlying real structure coincides with the original one. Allowing renormings, this is in correspondence with $\mathbb{R}$ -linear operators J on X of square equal to $-I_X$, which define the multiplication $x \mapsto i.x$. The number of complex structures on a space is understood up to ($\mathbb{C}$ -linear) isomorphism and has been studied in several papers. For example a real space is said to have *unique complex structure* if it admits complex structures and all of them are mutually isomorphic. Examples of spaces with unique complex structure are: (a) the Hilbert space (folklore or the next list of examples), (b) the spaces ℓ_p , $L_p(0, 1)$, c_0 , $C([0, 1])$ and more generally real spaces admitting a complex structure and whose complexification is primary (Kalton, Theorem 28 in [14]), (c) an hereditarily indecomposable example [12], (d) a non-classical example with a subsymmetric basis [9], and (e) others. Examples of spaces without complex structure are James space [10], a uniformly convex space of Szarek [30], the original Gowers-Maurey space [17], as well as many other spaces with small spaces of operators. “Extremely non-complex” real spaces are considered in [23]. Complex versus quaternionic structures on some exotic real spaces are studied in [28].

In [12] are also provided spaces with exactly n complex structures, whenever $n \geq 2$. This also gives examples of spaces with a complex structure which is not unique but still is isomorphic to its conjugate. An example with exactly $\aleph_0$ complex structures is due to Cuellar [8], and one with $2^{\aleph_0}$ and additional properties is due to Anisca [2] (it is not hard to check that the original example of Bourgain also admits $2^{\aleph_0}$ such structures). See also [3] for considerations on the number of complex structures in the setting of complexity of equivalence relations on Polish spaces.

In [20] Kalton, using a variation of Kalton-Peck space Z_2 from [22], defined a much simpler example of complex space $Z_2(\alpha)$ (α a non-zero real parameter) not isomorphic to its conjugate $\overline{Z_2(\alpha)}$ (which here identifies with $Z_2(-\alpha)$). According to the proof of [20] Theorem 2, see [7], it actually holds that $Z_2(\alpha)$ does not even embed into $\overline{Z_2(\alpha)}$. Regarding Z_2 it seems to be an interesting open question whether it admits a unique complex structure. Finally the most extreme example seems to appear in [12], with a space admitting exactly two complex structures, which are conjugate (and therefore $\mathbb{R}$ -linearly isometric) but totally incomparable as complex spaces (meaning that no $\mathbb{C}$ -linear subspace of one is $\mathbb{C}$ -isomorphic to a $\mathbb{C}$ -linear subspace of the other).

These examples show that there can be quite a variety of complex structures on a given real space, and therefore it is a natural and non trivial question not only to relate properties of operators seen as $\mathbb{R}$ -linear or seen as $\mathbb{C}$ -linear, but also seen as $\mathbb{C}$ -linear with respect to different complex structures on the same real space.

We refer to Pietsch [29] for background on operator ideals. In this paper we shall use the word *class* to define a family of normed spaces which is stable under isomorphisms. A *class* of operators which does not necessarily define an ideal is also defined in the sense of Pietsch, i.e. with varying domain and codomain.

The concept of complexification of real spaces, and of linear operators on them, is well-known, and recalled below. It is for example extremely useful in order to use spectral theory in the context of real spaces. There is a less well-known and almost trivial process, which we shall call here *realification*, and which is simply the one obtained by “forgetting” the multiplication by i on a space and “only remembering” the $\mathbb{R}$ -linear structure.

We list the definitions of complexification and realification in various situations below. Before that, let us fix an important notation. Since we shall always go back and forth between real and complex ideals or classes, to avoid confusion and when relevant we shall reserve lower case letters ($u, ss, cs, in, \dots$) for classes of $\mathbb{R}$ -linear operators and upper case letters ($U, SS, CS, IN, \dots$) for classes of $\mathbb{C}$ -linear operators. The same will hold for classes of spaces ($a, \dots$ for classes of real spaces, $A, \dots$ for classes of complex spaces).

2.2 Normed spaces

The complexification $X_{\mathbb{C}}$ of a real space X is the space $X \oplus X$ equipped with the complex structure associated to $J(x, y) = (-y, x)$. Elements of $X_{\mathbb{C}}$ are often denoted $x + iy$, $x, y \in X$, although we shall usually prefer the notation (x, y) to avoid confusion. Regarding the realification:

Definition 1 Let X be a complex space. The realification $X_{\mathbb{R}}$ of X is the space X equipped with the real structure underlying its complex structure.

As is usual we denote by $\overline{X}$ the conjugate of the complex space X , i.e. the space X equipped with the law $\lambda.x := \overline{\lambda}x$. It is clear that the realifications of X and $\overline{X}$ coincide. Note also that if T is $\mathbb{C}$ -linear from X to Y , then it also acts as a $\mathbb{C}$ -linear operator, denoted $\overline{T}$, from $\overline{X}$ to $\overline{Y}$.

Remark 1 The following hold:

1. if X is a real space then $(X_{\mathbb{C}})_{\mathbb{R}} = X \oplus X$.
2. if X is a complex space then $(X_{\mathbb{R}})_{\mathbb{C}} \simeq X \oplus \overline{X}$; specifically, the formula

$$R_X(x, y) = (x + y, iy - ix)$$

defines a $\mathbb{C}$ -linear isomorphism R_X from $X \oplus \overline{X}$ onto $(X_{\mathbb{R}})_{\mathbb{C}}$.

Proof 1. is obvious. For 2., the use of the map R_X is essentially an observation of N.J. Kalton which appears in a first form in [14] Lemma 27 and then more clearly

in a paper of W. Cuellar Carrera [9] Lemma 2.1. It is clear that R_X defines an $\mathbb{R}$ -linear isomorphism, and the fact that it is a $\mathbb{C}$ -linear isomorphism follows from the computation:

$$R_X(i(x, y)) = R_X(ix, -iy) = (i(x - y), y + x) = i(y + x, i(y - x)) = iR_X(x, y).$$

□

2.3 Classes of spaces

It is then natural to define complexification and realification of classes of spaces, where we recall that the classes are understood to be invariant by isomorphism.

Definition 2 If a is a class of real spaces, we define the class $a_{\mathbb{C}}$ of complex spaces by

$$X \in a_{\mathbb{C}} \Leftrightarrow X_{\mathbb{R}} \in a.$$

If A is a class of complex spaces, we define the class $A_{\mathbb{R}}$ of real spaces by

$$X \in A_{\mathbb{R}} \Leftrightarrow X_{\mathbb{C}} \in A.$$

Remark 2 The following hold:

1. If X is a real space and a a class of real spaces, then $X \in (a_{\mathbb{C}})_{\mathbb{R}}$ iff $X^2 \in a$.
2. If X is a complex space and A a class of complex spaces, then $X \in (A_{\mathbb{R}})_{\mathbb{C}}$ iff $X \oplus \overline{X} \in A$.

2.4 Linear operators

Similar concepts are defined for bounded linear operators.

Definition 3 If T is $\mathbb{R}$ -linear from X to Y then its *complexification* $T_{\mathbb{C}}$ from $X_{\mathbb{C}}$ to $Y_{\mathbb{C}}$ is well-known, and defined as

$$T_{\mathbb{C}}(x, y) = (Tx, Ty).$$

Conversely for T $\mathbb{C}$ -linear between complex spaces X and Y , its *realification* $T_{\mathbb{R}}$ will be T seen as $\mathbb{R}$ -linear between $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$.

Note that $T \mapsto T_{\mathbb{C}}$ is an algebra homomorphism from the space $\mathcal{L}(X, Y)$ to $\mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$, and that $T \mapsto T_{\mathbb{R}}$ is an algebra homomorphism from $\mathcal{L}(X, Y)$ to $\mathcal{L}(X_{\mathbb{R}}, Y_{\mathbb{R}})$. As a consequence:

Remark 3 The following hold:

1. if T is $\mathbb{R}$ -linear then the realification of the complexification of T is $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ acting from X^2 to Y^2 .

2. if T is $\mathbb{C}$ -linear then the complexification of the realification of T may be seen as $\begin{pmatrix} T & 0 \\ 0 & \overline{T} \end{pmatrix}$ acting from $X \oplus \overline{X}$ to $Y \oplus \overline{Y}$, in the sense that

$$(T_{\mathbb{R}})_{\mathbb{C}} = R_Y \circ \begin{pmatrix} T & 0 \\ 0 & \overline{T} \end{pmatrix} \circ R_X^{-1},$$

where R_Y (resp. R_X) is the identification between $Y \oplus \overline{Y}$ and $(Y_{\mathbb{R}})_{\mathbb{C}}$ (resp. between $X \oplus \overline{X}$ and $(X_{\mathbb{R}})_{\mathbb{C}}$) defined in Remark 1.

Proof 1. is clear. For 2. we compute

$$((T_{\mathbb{R}})_{\mathbb{C}} \circ R_X)(x, x') = (T_{\mathbb{R}})_{\mathbb{C}}(x + x', ix' - ix) = (T(x + x'), T(ix' - ix)),$$

and

$$(R_Y \circ \begin{pmatrix} T & 0 \\ 0 & \overline{T} \end{pmatrix})(x, x') = R_Y(Tx, Tx') = (Tx + Tx', i(Tx' - Tx)).$$

Then we observe that the two expressions coincide by $\mathbb{C}$ -linearity of T . $\square$

2.5 Classes of operators and/or ideals

Finally we define complexification and realification for classes of operators. We shall see that these definitions behave well with operator ideals in the sense of Pietsch.

Definition 4 1. Let u be a class of $\mathbb{R}$ -linear operators. We define the complexification $u_{\mathbb{C}}$ of u by

$$T \in u_{\mathbb{C}} \Leftrightarrow T_{\mathbb{R}} \in u$$

2. Let U be a class of $\mathbb{C}$ -linear operators. We define the realification $U_{\mathbb{R}}$ of U by

$$T \in U_{\mathbb{R}} \Leftrightarrow T_{\mathbb{C}} \in U$$

Lemma 1 If u is a real (closed) ideal of operators then $u_{\mathbb{C}}$ is a complex (closed) ideal. If U is a complex (closed) ideal of operators then $U_{\mathbb{R}}$ is a real (closed) ideal.

For $u_{\mathbb{C}}$ note that this relies on the fact that if $T \in u_{\mathbb{C}}$ then $iT \in u_{\mathbb{C}}$, because $(iT)_{\mathbb{R}} = iT_{\mathbb{R}} \in u$ since i is an $\mathbb{R}$ -linear operator and u is a real ideal.

The following natural notion will prove extremely important.

2.6 Conjugate classes and/or ideals

Definition 5 For U a complex class of operators let us denote by $\overline{U}$ the conjugate class, i.e.

$$T \in \overline{U} \Leftrightarrow \overline{T} \in U.$$

Definition 6 A complex class U of operators is self-conjugate if $\overline{U} = U$.

The class $\overline{U}$ is not to be mistaken with the closure of U , which is denoted U^{clos} . The proof of the next proposition is left as an exercise.

Proposition 1 *The ideals of compact, strictly singular, strictly cosingular, inessential operators, and the class of improjective operators are self-conjugate.*

Proposition 2 *If u is a real class of operators, then $u_{\mathbb{C}}$ is self-conjugate.*

Proof For a $\mathbb{C}$ -linear operator T the $\mathbb{R}$ -linear operators $T_{\mathbb{R}}$ and $(\overline{T})_{\mathbb{R}}$ coincide. $\square$

To develop examples of ideals which are not self-conjugate, we consider $\text{Op}(X)$, the class of X -factorable operators, i.e. operators which factor through the Banach space X .

Definition 7 If X is a Banach space, then $\text{Op}(X)$ denotes the class of X -factorable operators, i.e. for $T \in L(Y, Z)$, $T \in \text{Op}(X)$ iff $T = UV$ for some $V \in L(Y, X)$ and $U \in L(X, Z)$.

Let us note the useful observation that $\overline{\text{Op}(X)} = \text{Op}(\overline{X})$ whenever X is a complex space. We recall the well-known fact:

Proposition 3 *If X is a Banach space which contains a complemented subspace isomorphic to X^2 , then $\text{Op}(X)$ is an operator ideal.*

Note that $\text{Op}(X)$ has no reason to be closed in general.

Proposition 4 *Let X be a complex space which is not isomorphic to a complemented subspace of $\overline{X}$. Then $\text{Op}(X)^{\text{clos}}$ is not self conjugate. In particular $\text{Op}(X)$ is not self-conjugate.*

Proof We shall prove that I_X does not belong to $\overline{\text{Op}(X)}^{\text{clos}} = \text{Op}(\overline{X})^{\text{clos}}$.

Indeed assume there exist $A : \overline{X} \rightarrow X$ and $B : X \rightarrow \overline{X}$ such that $T := I_X - AB$ has norm $\|T\| < \varepsilon$. Then for ε small enough $AB = I - T$ would be an isomorphism on X and therefore B would be an isomorphic embedding of X into $\overline{X}$. Finally the image BX would be complemented in $\overline{X}$ by $B(I - T)^{-1}A$. This is a contradiction. $\square$

Of course spaces not isomorphic to a complemented subspace of their conjugate and at the same time isomorphic to their squares (so that $\text{Op}(X)$ is an ideal) must be rather exotic. We present two examples of such spaces and therefore of ideals which are not self-conjugate.

Example 1 If F is the complex HI space totally incomparable with its conjugate from [12], then the ideal $\text{Op}(\ell_2(F))^{\text{clos}}$ is not self conjugate.

Proof The space F is complemented in $\ell_2(F)$ but does not embed in $\overline{\ell_2(F)} = \ell_2(\overline{F})$. Indeed, see for example [6], a space which embeds into $\ell_2(\overline{F})$ either contains a copy of ℓ_2 (which cannot hold in the case of the HI space F) or embeds into $\overline{F}^n$ for some n , which contradicts the total incomparability of F with $\overline{F}$. $\square$

A less exotic example, even “elementary” in the words of Kalton, is provided by him in [20].

Example 2 If $Z_2(\alpha)$ is the version of Kalton-Peck complex space defined by Kalton [20], then $\text{Op}(Z_2(\alpha))^{\text{clos}}$ is an ideal which is not self conjugate, for $\alpha \neq 0$.

Proof The space $Z_2(\alpha)$ does not embed into its conjugate, if $\alpha \neq 0$, see [20] Proof of Theorem 2 and [7]. On the other hand, it admits a canonical 2-dimensional “symmetric decomposition” in the same way as Z_2 does and in particular is isomorphic to its square. $\square$

3 Applications to real and complex versions of ideals

3.1 Real and complex versions of classical ideals

We use the analysis of the previous section to relate a certain correspondence between real and complex versions of ideals to the self-conjugacy property.

Proposition 5

1. Let u be a real ideal. Then $(u_{\mathbb{C}})_{\mathbb{R}} = u$.
2. Let U be a complex ideal. Then $(U_{\mathbb{R}})_{\mathbb{C}} = U \cap \overline{U}$.
3. A complex ideal U is self-conjugate if and only if $(U_{\mathbb{R}})_{\mathbb{C}} = U$.

Proof 1. Indeed $T \in (u_{\mathbb{C}})_{\mathbb{R}}$ if and only if $T_{\mathbb{C}} \in u_{\mathbb{C}}$ if and only if $(T_{\mathbb{C}})_{\mathbb{R}} \in u$, which means that $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ belongs to u and is equivalent to $T \in u$ by the ideal properties.

2. $T \in (U_{\mathbb{R}})_{\mathbb{C}}$ if and only if $T_{\mathbb{R}} \in U_{\mathbb{R}}$ if and only if $(T_{\mathbb{R}})_{\mathbb{C}} \in U$, which by Remark 32. means that $\begin{pmatrix} T & 0 \\ 0 & \overline{T} \end{pmatrix}$ acting on $X \oplus \overline{X}$ belongs to U ; this is equivalent to $T, \overline{T} \in U$ by the ideal properties.

3. follows from 2. and from Proposition 2. $\square$

We shall consider the real and complex versions of the ideals of strictly singular, strictly cosingular, inessential operators, and of the class of improjective operators. We denote ss, cs, in, imp the real versions and SS, CS, IN, IMP the complex versions of these.

Let us first note that a $\mathbb{C}$ -linear operator is $\mathbb{C}$ -strictly singular as soon as it is $\mathbb{R}$ -strictly singular. In our language

$$ss_{\mathbb{C}} \subset SS.$$

It is an easy exercise that the property

$$u_{\mathbb{C}} \subset U$$

also holds if $u = cs, in, imp$ and $U = CS, IN, IMP$, respectively. Actually we have

Proposition 6 *An $\mathbb{R}$ -linear map T is strictly singular (resp. strictly cosingular, inessential, improper) if and only if $T_{\mathbb{C}}$ is strictly singular (resp. strictly cosingular, inessential, improper). In other words,*

$$U_{\mathbb{R}} = u$$

holds if $u = ss, cs, in, imp$ and $U = SS, CS, IN, IMP$, respectively.

Proof We use Proposition 5. Since $ss_{\mathbb{C}} \subset SS$ then $ss = (ss_{\mathbb{C}})_{\mathbb{R}} \subset SS_{\mathbb{R}}$. Conversely if $T : X \rightarrow Y$ is not strictly singular, let $Z \subset X$ be a subspace such that $TI_{Z,X}$ is an isomorphism into Y . Then $T_{\mathbb{C}}I_{Z_{\mathbb{C}},X_{\mathbb{C}}}$ is a $\mathbb{C}$ -linear isomorphism from $Z_{\mathbb{C}}$ into $Y_{\mathbb{C}}$ and since $Z_{\mathbb{C}}$ is a $\mathbb{C}$ -linear subspace of $X_{\mathbb{C}}$, $T_{\mathbb{C}}$ is not strictly singular. Summing up $T \notin ss \Rightarrow T \notin SS_{\mathbb{R}}$.

Since $cs_{\mathbb{C}} \subset CS$, the inclusion $cs \subset CS_{\mathbb{R}}$ holds. Conversely if $T : X \rightarrow Y$ is not strictly cosingular, then let Q be the quotient map onto the quotient Z of Y by some infinite codimensional subspace of Y for which QT is surjective. Then $Q_{\mathbb{C}}$ is the quotient map from $Y_{\mathbb{C}}$ onto the quotient $Z_{\mathbb{C}}$ of $Y_{\mathbb{C}}$, and $Q_{\mathbb{C}}T_{\mathbb{C}}$ is surjective, therefore $T_{\mathbb{C}}$ is not strictly cosingular.

Since $in_{\mathbb{C}} \subset IN$, the inclusion $in \subset IN_{\mathbb{R}}$ holds. Conversely if $T : X \rightarrow Y$ is not inessential, let $U : Y \rightarrow X$ be such that $\text{Id} - UT$ is not Fredholm. Then $(\text{Id} - UT)_{\mathbb{C}} = \text{Id} - U_{\mathbb{C}}T_{\mathbb{C}}$ is not Fredholm, and therefore $T_{\mathbb{C}}$ is not inessential.

Since $imp_{\mathbb{C}} \subset IMP$, the inclusion $imp \subset IMP_{\mathbb{R}}$ holds. Conversely if $T : X \rightarrow Y$ is not improper, let W be complemented in X and Z in Y such that T restricts to an isomorphism between W and Z . Then $T_{\mathbb{C}}$ restricts to an isomorphism between the complemented subspaces $W_{\mathbb{C}}$ and $Z_{\mathbb{C}}$ of $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ respectively, so is not improper. $\square$

Corollary 7 *A $\mathbb{C}$ -linear operator is strictly singular (resp. strictly cosingular, inessential) if and only if it is strictly singular (resp. strictly cosingular, inessential) as $\mathbb{R}$ -linear. In other words*

$$U = u_{\mathbb{C}}$$

holds if $u = ss, cs, in, imp$ and $U = SS, CS, IN, IMP$, respectively.

Proof Since $ss = SS_{\mathbb{R}}$, it follows that $ss_{\mathbb{C}} = (SS_{\mathbb{R}})_{\mathbb{C}}$ and this is equal to SS by Proposition 5, since SS is self-conjugate. The same reasoning holds for strictly cosingular and inessential operators. $\square$

We formalize these ideas as follows:

Proposition 8 *Let U be a complex ideal, and let $u = U_{\mathbb{R}}$, i.e., $T \in u \Leftrightarrow T_{\mathbb{C}} \in U$. Then the following are equivalent:*

1. *for any $\mathbb{C}$ -linear operator T between two complex spaces, $T \in U$ if and only if T seen as $\mathbb{R}$ -linear is in u ,*
2. $u_{\mathbb{C}} = U$,
3. U is self-conjugate.

Definition 8 When $u = U_{\mathbb{R}}$ and 1-2-3 of Proposition 8 hold, we say that (u, U) is a *regular pair of ideals*.

Corollary 9 *The pairs (ss, SS) , (cs, CS) , and (in, IN) are regular.*

In terms of complex structures on a real Banach space, this also means:

Corollary 10 *If (u, U) is a regular pair of ideals, then an operator belonging to U with respect to a complex structure on the real space X , also belongs to U with respect to any other complex structure on X for which it is $\mathbb{C}$ -linear.*

Another very relevant family of operator ideals are the ideals $\text{Op}(A)$, generalizing Definition 7 of $\text{Op}(X)$. According to [29] Definition 2.1.1 a *space ideal* A is a class of spaces containing the finite dimensional ones and stable under taking direct sums and complemented subspaces. The ideal $\text{Op}(A)$ is defined in [29] 2.2.1:

Definition 9 If A is a space ideal, then $T \in \text{Op}(A)$ if and only if T is X -factorable for some $X \in A$.

If A is a complex space ideal we define in an obvious way the conjugate space ideal $\overline{A}$ by

$$X \in \overline{A} \Leftrightarrow \overline{X} \in A,$$

and say that A is self-conjugate if $A = \overline{A}$. If A is complex, we also denote by $\text{op}(A)$ the ideal of $\mathbb{R}$ -linear operators which factor (by $\mathbb{R}$ -linear operators) through $X_{\mathbb{R}}$ for some $X \in A$.

Proposition 11 *Let A be a complex and self-conjugate space ideal. Then the pair $(\text{op}(A), \text{Op}(A))$ is a regular pair of ideals.*

Proof We claim that $\text{op}(A) = \text{Op}(A)_{\mathbb{R}}$. Indeed assume T is an $\mathbb{R}$ -linear operator factoring through $X_{\mathbb{R}}$ for some $X \in A$. Then $T_{\mathbb{C}}$ factors through $(X_{\mathbb{R}})_{\mathbb{C}}$. Since $(X_{\mathbb{R}})_{\mathbb{C}}$ is isomorphic to $X \oplus \overline{X}$ by Remark 12., and since A is a self-conjugate space ideal, it also belongs to A . So $T_{\mathbb{C}}$ belongs to $\text{Op}(A)$, which means by definition that T belongs to $\text{Op}(A)_{\mathbb{R}}$. Conversely if $T_{\mathbb{C}}$ belongs to $\text{Op}(A)$, then the matrix $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ belongs to $\text{op}(A)$ from which it follows easily that T itself belongs to $\text{op}(A)$. Since the claim holds, the result follows from the fact that $\text{Op}(A)$ is obviously self conjugate and from Proposition 8. $\square$

The above extends obviously to ideals of operators $T \in L(Y, Z)$ which factorize through A as operators of $L(Y, Z^{**})$. As an easy application we also obtain the regular pair of ideals: (real ℓ_p -factorable operators, complex ℓ_p -factorable operators), (real σ -integral operators, complex σ -integral operators),..., see [29] 19.3 and 23 for details. We also leave as an exercise to the reader to find examples of regular pair of ideals related to the ideal U_T of operators factorizing through a given operator T (under the necessary restrictions).

3.2 Improjective operators and examples of non-regular pairs

Since improjective operators do not form an ideal, according to [1], Proposition 8 does not apply to them. What is true is the following slightly more restrictive statement:

Proposition 12 *Let X, Y be two complex Banach spaces such that the space $\text{Imp}(X \oplus \bar{X}, Y \oplus \bar{Y})$ is a linear subspace of $L(X \oplus \bar{X}, Y \oplus \bar{Y})$. Then a $\mathbb{C}$ -linear operator T between X and Y is improjective if and only if it is improjective as $\mathbb{R}$ -linear.*

Proof We already observed that $\mathbb{R}$ -improjective implies $\mathbb{C}$ -improjective. Assume now T is not improjective as $\mathbb{R}$ -linear. Then $(T_{\mathbb{R}})_{\mathbb{C}}$ is not improjective between $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$, which by Remark 32. is equivalent to saying that $\begin{pmatrix} T & 0 \\ 0 & \bar{T} \end{pmatrix}$ is not improjective from $X \oplus \bar{X}$ to $Y \oplus \bar{Y}$. Since $\begin{pmatrix} T & 0 \\ 0 & \bar{T} \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \bar{T} \end{pmatrix}$, the hypothesis implies that $\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$ is not improjective or that $\begin{pmatrix} 0 & 0 \\ 0 & \bar{T} \end{pmatrix}$ is not improjective. In the first case, T not improjective from X to Y , and in the second case, $\bar{T}$ is not improjective from $\bar{X}$ to $\bar{Y}$, or equivalently, since Imp is self-conjugate, again $T : X \rightarrow Y$ is not improjective. $\square$

We can use the examples of non-self-conjugate ideals from Sect. 2 to give immediate examples of pairs which are not regular, showing that the hypotheses of Proposition 11 are necessary. Let X be $Z_2(\alpha)$ or $\ell_2(F)$ and consider the complex and real ideals $\text{Op}(X)$, the ideal of X -factorable $\mathbb{C}$ -linear operators (factorizing with $\mathbb{C}$ -linear maps), and $\text{op}(X)$, the ideal of X -factorable $\mathbb{R}$ -linear operators (factorizing with $\mathbb{R}$ -linear maps). Then:

Example 3 The pair $(\text{op}(X), \text{Op}(X))$ is not a regular pair.

Question 1 Find other natural examples of regular or non-regular pairs of ideals.

4 A solution to the problem of Pietsch

Recall that an operator ideal (or class) U is proper if $I_X \in U$ implies X finite dimensional, and that $\text{Op}(X)$ is the class of X -factorable operators. We first list a few useful facts.

Proposition 13 *Let U be an operator ideal, and X, Y, Z be infinite dimensional Banach spaces. Then*

1. U is proper if and only if $U \subseteq \text{Imp}$,
2. if U is proper, then so is the operator ideal $\text{In} + U$,
3. $\text{Id}_Z \in \text{Op}(X)$ if and only if Z embeds complementably in X ,
4. $\text{Op}(X) \cap \text{Op}(Y)$ is proper if and only if X and Y are projectively incomparable.

Proof 1. The class Imp is proper and being proper is hereditary; this proves the “if” part. For the “only if” part, assume $U \not\subseteq \text{Imp}$ for some ideal U , let $T \in L(Y_0, Z_0)$ be

an element of U which is not improjective, and let $t \in L(Y, Z)$ be an isomorphism between infinite dimensional complemented subspaces Y of Y_0 and Z of Z_0 which witnesses that T is not improjective. The ideal properties of U imply that t and t^{-1} belong to U . It follows that also $\text{Id}_Y = t^{-1} \circ t$ belongs to U , implying that U is not proper.

2. Assume Id_Z belongs to $\text{In} + U$, and let S be some inessential operator on Z such that $\text{Id}_Z + S$ belongs to U . Since $\text{Id}_Z + S$ is Fredholm with index 0 and by the ideal properties of U , we deduce that some automorphism on Z belongs to U , and therefore that Id_Z itself belongs to U . In particular, U is non proper as soon as $\text{In} + U$ is not.

3. If $\text{Id}_Z = TV$ is a factorization witnessing that $\text{Id}_Z \in \text{Op}(X)$, then VT is a projection onto the isomorphic copy VZ of Z . Conversely if T is an embedding of Z into X whose range TZ is complemented by a projection P of X onto Z , then $\text{Id}_Z = PT \in \text{Op}(X)$.

4. the class $\text{Op}(X) \cap \text{Op}(Y)$ is not proper if and only if there exists an infinite dimensional space Z such that $\text{Id}_Z \in \text{Op}(X) \cap \text{Op}(Y)$, i.e. by 3. Z embeds complementably in both X and Y . $\square$

Item 4. in Proposition 13 suggest a way of constructing new proper ideals. However the problem is that $\text{Op}(X)$ is not in general an ideal, unless for example X is isomorphic to its square; but this last property is unlikely to happen for Gowers-Maurey spaces. To remedy this obstruction we extend $\text{Op}(X)$ as follows:

Definition 10 Let X be a Banach space. We denote by $\text{Op}^{<\omega}(X)$ the ideal of operators which are X^n -factorable for some $n \in \mathbb{N}$.

It is clear that $\text{Op}^{<\omega}(X)$ is an ideal: if $R, T \in L(Y, Z)$ are X^m and X^n -factorable respectively, then $R+T$ is X^{m+n} -factorable. See for example the proof of [29] Theorem 2.2.2. From Proposition 134. we deduce:

Remark 4 Let X, Y be infinite dimensional Banach spaces. Then the ideal $\text{Op}^{<\omega}(X) \cap \text{Op}^{<\omega}(Y)$ is proper if and only X^m and Y^n are projectively incomparable for all $m, n \in \mathbb{N}$.

We now consider X_S , the “shift-space” defined by Gowers and Maurey in [18], see also [27] and more details in [26] (see also [19] for considerations on equivalence of projections on X_S). The space X_S is an indecomposable, non hereditarily indecomposable space, admitting a Schauder basis for which the shift operator S is an isometric embedding, implying that X is isomorphic to its hyperplanes. Actually the complex version of X_S has the very strong following rigidity property:

Proposition 14 (Gowers–Maurey) *The following are equivalent for a subspace Y of X_S :*

1. Y is isomorphic to X_S
2. Y is complemented in X_S
3. Y is finite codimensional in X_S

We shall use the next crucial proposition, whose proof is postponed until the next section and is of a more technical nature. The proof involves multidimensional versions

of the machinery used by Gowers and Maurey in [18], and therefore requires some familiarity with the use of K -theory for algebras of operators on Banach spaces and in particular properties of Fredholm operators, as quite well explained in [26]. It also requires certain facts of the K -theory of the Wiener algebra $A(\mathbb{T})$, as well as some conditions to apply complexification and obtain the real case. For these reasons we keep those details for the next section.

Proposition 15 *Let X_S be the real or complex shift space of Gowers-Maurey. Assume $m, n \in \mathbb{N}$. Let Y be an infinite codimensional subspace of X_S . Then there is no isomorphism between an infinite dimensional complemented subspace of X_S^m and a subspace of Y^n .*

Let us note here that we shall actually prove that a complemented subspace of X_S^m must be isomorphic to X_S^q for some $q \leq m$, and therefore Proposition 15 will follow from the fact that X_S does not embed into Y^n . Note also that the case $m = n = 1$ in the complex case is immediate from Proposition 14 and this is the idea that was used by Aiena and González in [1].

Let us first mimic the construction of [1] inside X_S . The first observation is that the spectral properties of the shift operator S are similar to the usual properties of the shift on ℓ_2 , as follows.

Given $t \in \mathbb{T}$ (resp. $\{-1, 1\}$ in the real case), the operator $\text{Id} - tS$ is injective. We claim that its image is not closed; indeed otherwise $\text{Id} - tS$ would be an isomorphism onto its image, and this is false, by considering for any $N \in \mathbb{N}$, the vector

$$x_N = \sum_{n=1}^N t^n e_n,$$

which has norm at least $N / \log_2(N + 1)$ by [18] Theorem 5, while

$$(\text{Id} - tS)(x_N) = te_1 - t^{N+1}e_{N+1}$$

has norm at most 2. This implies that for any $t \in \mathbb{T}$ (resp. $\{-1, 1\}$ in the real case) and for some compact operator K_t on X_S , the operator

$$T_t := \text{Id} - tS + K_t$$

has image of infinite codimension (see for example [24] Theorem 5.4). Denote $Y_t = \overline{\text{Im}(T_t)}$ and consider T_t as an operator into Y_t .

Proposition 16 *Given $t \in \mathbb{T}$ (resp. $\{-1, 1\}$ in the real case), the ideal*

$$U_t := \text{Op}^{<\omega}(X_S) \cap \text{Op}^{<\omega}(Y_t)$$

is a proper ideal which is not contained in the ideal of inessential operators.

Proof Since Y_t is infinite-codimensional, by Proposition 15, all powers of X_S and of Y_t are projectively incomparable, or equivalently, $\text{Op}^{<\omega}(X_S) \cap \text{Op}^{<\omega}(Y_t)$ is a proper ideal. Denote by i_{Y_t, X_S} the canonical inclusion of Y_t inside X_S . The operator $T_t : X_S \rightarrow Y_t$ belongs to U_t , and it is essential, since $\text{Id} - \frac{1}{2}i_{Y_t, X_S}T_t = \text{Id} - \frac{1}{2}(\text{Id} - tS + K_t) = \frac{1}{2}(\text{Id} + tS - K_t)$ is not Fredholm. $\square$

Theorem 4

1. In is not maximal among proper ideals: there exists a proper operator ideal V with $\text{In} \subsetneq V \subsetneq \text{Imp}$;
2. there is no largest proper ideal.

Proof 1. Pick $V = V_t := \text{In} + U_t$ which is a proper operator ideal by Proposition 132., contained in Imp by Proposition 131., and not equal to In by Proposition 16. 2. An ideal U containing all proper ideals must contain U_1 and U_{-1} . Therefore the operators $T_1 = \text{Id} - S + K_1$ and $T_{-1} = \text{Id} + S + K_{-1}$ belong to U .

Then the Fredholm operator $i_{Y_1, X_S} \circ T_1 + i_{Y_{-1}, X_S} \circ T_{-1} = 2\text{Id} + K_1 + K_{-1}$ belongs to U , and therefore $\text{Id} = \text{Id}_{X_S}$ belongs to U . Since X_S is infinite dimensional, U cannot be proper. $\square$

5 The proof of projective incomparability

This section is devoted to the proof of Proposition 15.

5.1 Complex version versus complexification of the shift space

We recall a few facts from [18]. If $X_S(\mathbb{K})$ is the version of the shift space defined on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then there exists an algebra homomorphism and projection map Φ from $L(X_S(\mathbb{K}))$ to some algebra of operators denoted $\mathcal{A}$. S denotes the right shift and L the left shift on the canonical basis of X_S . Elements of $\mathcal{A}$ are those of the form $\Phi(T) = \sum_{k \geq 0} a_k S^k + \sum_{k \geq 1} a_{-k} L^k$ for some sequence $(a_k)_k \in \ell_1(\mathbb{Z}, \mathbb{K})$, which we shall denote $(a_k(T))_k$, and we have that $\|\Phi(T)\| = \sum_{k \in \mathbb{Z}} |a_k(T)|$. For simplification we shall denote $\Phi(T) = \sum_{k \in \mathbb{Z}} a_k S^k$ in the situation above, even if S is not formally invertible. The map Φ has the property that $T - \Phi(T)$ is strictly singular for any $T \in L(X_S(\mathbb{K}))$, which allows to reduce most of the study of operators on $X_S(\mathbb{K})$ to operators in $\mathcal{A}$.

From this the authors of [18] concentrate on the complex case, in which case $\ell_1(\mathbb{Z})$ identifies with the Wiener algebra $A(\mathbb{T})$ of complex valued functions in $C(\mathbb{T})$ whose Fourier series have absolutely summable coefficients.

We may use the complex version $X_S(\mathbb{C})$ of X_S to give a negative answer to the problem of Pietsch in the complex case. In order to be able to treat the real case as well we shall see that it is enough to replace $X_S(\mathbb{C})$ by the complexification of the real version of X_S , denoted $(X_S(\mathbb{R}))_{\mathbb{C}}$. A few comments are in order. Both $X_S(\mathbb{C})$ and $(X_S(\mathbb{R}))_{\mathbb{C}}$ have natural Schauder bases and contain two canonical isometric real subspaces W and iW , where W is the space generated by real linear combinations of elements of the basis. While in the complexification $(X_S(\mathbb{R}))_{\mathbb{C}}$ these two form a direct

sum, this is probably not the case inside $X_S(\mathbb{C})$. Indeed the “no shift” version of the norm of this space is the norm on Gowers-Maurey’s HI space, which is known to be HI as a real space (see the comments on p475 of [12]) and therefore indecomposable as a real space, and it is probable that similarly W and iW do not form a direct sum in $X_S(\mathbb{C})$. This makes it more difficult to study real subspaces of $X_S(\mathbb{C})$ and suggests the use of $(X_S(\mathbb{R}))_{\mathbb{C}}$ instead.

Consider the complexification $(X_S(\mathbb{R}))_{\mathbb{C}}$. Note that it is equipped with the complexification of the shift operator on $X_S(\mathbb{R})$, which is just the shift operator on $(X_S(\mathbb{R}))_{\mathbb{C}}$ with its natural basis, and which we denote also S ; therefore S is a power bounded, isomorphic embedding on the space, inducing an isomorphism with its hyperplanes. Likewise the complexification of the left shift is power bounded. By classical results about complexifications, operators on the space are of the form $T = A + iB$, where A, B are $\mathbb{R}$ -linear operators (meaning that the formula $(A + iB)(x + iy) = Ax - By + i(Bx + Ay)$ holds); it follows that

$$\begin{aligned} T(x + iy) &= \sum_{k \in \mathbb{Z}} (a_k + ib_k) S^k(x + iy) + (V + iW)(x + iy) \\ &= \sum_{k \in \mathbb{Z}} \lambda_k S^k(x + iy) + (V + iW)(x + iy), \end{aligned}$$

where the series λ_k is absolutely summable in $\mathbb{C}$, the action of S on the complex space $(X_S(\mathbb{R}))_{\mathbb{C}}$ is identified with the shift operator S there, and where V, W are strictly singular. By the results of Sect. 3 this is the same as saying that $T - \sum_k \lambda_k S^k$ is strictly singular as a complex operator. Therefore we may also define an algebra homomorphism and projection map (again called Φ) from $L((X_S(\mathbb{R}))_{\mathbb{C}})$ to the algebra (again denoted $\mathcal{A}$) of operators of the form $\Phi(T) = \sum a_k S^k$ for $(a_k)_k \in \ell_1(\mathbb{Z}, \mathbb{C})$ denoted $(a_k(T))_k$.

Summing up, in what follows, X will denote either the complex version $X_S(\mathbb{C})$ of the shift space, or the complexification $(X_S(\mathbb{R}))_{\mathbb{C}}$ of the real version of the shift space, and $\mathcal{A}$ and ϕ the corresponding algebra and map.

As in [18], Ψ is the map defined from $L(X)$ to the Wiener algebra $A(\mathbb{T})$ by

$$\Phi(T) = \sum_{k \in \mathbb{Z}} a_k S^k \Rightarrow \Psi(T)(e^{i\theta}) = \sum_{k \in \mathbb{Z}} a_k e^{ki\theta}.$$

While in the case of $X = X_S(\mathbb{C})$, Ψ induces an isometric isomorphism between $\mathcal{A}$ and $A(\mathbb{T})$ ([18] Lemma 11), in the case of $X = (X_S(\mathbb{R}))_{\mathbb{C}}$ the map $\Psi|_{\mathcal{A}}$ is just an isomorphism, whose norm and norm of the inverse depend on the equivalent norm chosen on $(X_S(\mathbb{R}))_{\mathbb{C}}$ (by [18] Lemma 11 in the real case). This does not affect the rest of our computations.

We shall also denote by Φ the induced projection from $L(X^n, X^m) = M_{m,n}(L(X))$ onto $M_{m,n}(\mathcal{A})$, i.e. if $T = (T_{ij})_{i,j} \in M_{m,n}(L(X))$ then we define

$$\Phi((T_{ij})_{i,j}) = (\Phi(T_{ij}))_{i,j}$$

and we note that

$$\Phi(T) = \sum_k A_k S^k,$$

where $A_k = A_k(T) \in M_{m,n}(\mathbb{C})$ is the matrix $(a_k(T_{ij}))_{i,j}$, which is a less cumbersome notation than the more detailed $(\sum_k a_k^{i,j} S^k)_{i,j}$, with $a_k^{i,j} = a_k(T_{ij})$.

Likewise we define a map Ψ from $L(X^n, X^m)$ to $M_{m,n}(A(\mathbb{T}))$ by the formula

$$\Psi(T)(e^{i\theta}) = \sum_k A_k(T) e^{ik\theta}.$$

We shall make use of some notation and results of K -theory of Banach algebras. If A is a unital Banach algebra, then $M_\infty(A)$ denotes the set of (n, n) -matrices of elements of A of arbitrary size, i.e. $M_\infty(A) = \cup_n M_n(A)$ with the natural embeddings of $M_n(A)$ into $M_{n+1}(A)$. Idempotents of $M_\infty(A)$ coincide with idempotents in one of the $M_n(A)$. Among them I_n denotes the identity on $M_n(A)$ (seen inside $M_\infty(A)$). As usual $GL_n(A)$ denotes the set of invertibles in $M_n(A)$, and we also define $GL_\infty(A) = \cup_n GL_n(A)$ with the natural embedding of $GL_n(A)$ inside $GL_{n+1}(A)$ defined by adding ones along the diagonal. If $A \subseteq L(X)$ is an algebra of operators on a space X then I_n will also be denoted Id_{X^n} or I_{X^n} . Two idempotents P, Q of $M_\infty(A)$ are *similar* if there exists some $N \in \mathbb{N}$ and some M in $GL_N(A)$, such that, denoting the natural copy of M inside $GL_\infty(A)$ still by M , the relation

$$P = M^{-1}QM$$

holds. Note in particular that if P and Q are two similar idempotents in $M_\infty(L(X))$ for some X then the images PX and QX are isomorphic. Regarding the very basic results of K -theory we shall use, we refer to [4] for background and [26] for a survey in a language familiar to Banach space specialists.

5.2 Properties in the Wiener algebra $A(\mathbb{T})$

We recall classical or easy properties of the algebra $C(\mathbb{T})$ of continuous complex functions on the complex circle $\mathbb{T}$, the Wiener algebra $A(\mathbb{T})$ of functions in $C(\mathbb{T})$ with absolutely summable Fourier series, and their matrix algebras. They are certainly folklore but not always easy to find explicitly in the literature, so we sometimes prefer to give a short proof rather than a too abstract or too general argument. We recall Wiener's Lemma [32]: if an element of $A(\mathbb{T})$ is invertible in $C(\mathbb{T})$ (i.e. does not vanish anywhere on $\mathbb{T}$), then its inverse belongs to $A(\mathbb{T})$ as well. See [26], either Lemma 7.2 for a Banach space theoretic proof or the commentary after Proposition 2.2 for the classical proof.

Proposition 17 *The following hold*

1. *An element M of $M_n(C(\mathbb{T}))$ is invertible if and only if $\det(M)$ is invertible in $C(\mathbb{T})$*
2. *An element M of $M_n(A(\mathbb{T}))$ which is invertible in $M_n(C(\mathbb{T}))$ must be invertible in $M_n(A(\mathbb{T}))$*
3. *The set $GL_n(A(\mathbb{T}))$ of invertibles in $M_n(A(\mathbb{T}))$ is dense in $GL_n(C(\mathbb{T}))$*

Proof 1. follows from the cofactor formula in the abelian algebra $C(\mathbb{T})$. 2. follows from the cofactor formula and the fact that $\det(M)^{-1}$ belongs to $A(\mathbb{T})$ by Wiener's Lemma. 3. since $A(\mathbb{T})$ is dense in $C(\mathbb{T})$, $M_n(A(\mathbb{T}))$ is dense in $M_n(C(\mathbb{T}))$. If M is invertible in $M_n(C(\mathbb{T}))$ then an element of $M_n(A(\mathbb{T}))$ close enough to M will be invertible in $M_n(C(\mathbb{T}))$ and therefore in $M_n(A(\mathbb{T}))$. $\square$

Lemma 2 *Two idempotents of $M_\infty(A(\mathbb{T}))$ which are similar in $M_\infty(C(\mathbb{T}))$ are similar in $M_\infty(A(\mathbb{T}))$.*

Proof Let P and Q be such idempotents, and let M be invertible in some $GL_N(C(\mathbb{T}))$ such that $Q = MPM^{-1}$. By Proposition 17 3. we may find a perturbation M' of M belonging to $GL_N(A(\mathbb{T}))$. Then $Q' = M'PM'^{-1}$ is an idempotent of $M_N(A(\mathbb{T}))$ which is similar to P in $M_N(A(\mathbb{T}))$, but also to Q if M' was chosen close enough to M . Indeed it is a classical and immediate computation (valid in any Banach algebra) that Q and Q' are similar through the invertible $U = I - Q(Q' - Q) + (Q - Q')Q$ as soon as Q' is close enough to Q in $M_N(C(\mathbb{T}))$ (see e.g. [26] Lemma 9.2). Since Q and Q' belong to the algebra $M_N(A(\mathbb{T}))$, U is an invertible of $M_N(A(\mathbb{T}))$. $\square$

5.3 Complemented subspaces in powers of X

Recall that X is either $X_S(\mathbb{C})$ or $(X_S(\mathbb{R}))_{\mathbb{C}}$. We now prove several results indicating how the rigidity properties of X proved in [18] carry over to its powers X^n . As a first result and for clarity let us quickly repeat the ideas of [18] to show that $X_S(\mathbb{R})_{\mathbb{C}}$ also satisfies the equivalence of Proposition 14.

Proposition 18 *The following are equivalent for an infinite dimensional subspace Y of X*

1. *Y is isomorphic to X*
2. *Y is complemented in X*
3. *Y is finite codimensional in X*

Proof 3. $\Rightarrow$ 2. is trivial, and 3. $\Rightarrow$ 1. is due to the existence of the shift operator S . 1. $\Rightarrow$ 3.: if there is an embedding of X into X , it is not infinitely singular, and it follows that it must be Fredholm. This can be seen as a consequence of Corollary 20, whose proof follows below. 2. $\Rightarrow$ 3.: If P is a projection on X then $\Psi(P)$ is an idempotent in $A(\mathbb{T})$, therefore it is either constantly 0 or 1, meaning that $\Phi(P)$ is either I_X or 0. Then either P or $\text{Id} - P$ is a strictly singular projection and therefore has finite rank. So $Y = PX$ has finite codimension. $\square$

We now prove a technical lemma which is a multidimensional version of Lemma 14 from [18]. It can be thought of as carrying spectral properties of the shift map S

over to general operators on powers of X . This involves two aspects: passing from S to linear combinations of powers of S as in [18], and passing from the one dimensional to the multidimensional setting. The version presented here strongly benefited from suggestions of the referee.

Lemma 3 *Let $T \in L(X^n, X^m)$, for $m, n \in \mathbb{N}$. If some non-zero $(\alpha_i)_{i=1, \dots, n} \in \mathbb{C}^n$ belongs to $\text{Ker}(\Psi(T)(t))$ for some $t \in \mathbb{T}$, then the restriction of T to the subspace $\{(\alpha_1 x, \dots, \alpha_n x), x \in X\}$ of X^n is infinitely singular.*

Proof To give a reasonably detailed proof we first fix some useful notation. For $k = n$ or m , $\alpha = (\alpha_1, \dots, \alpha_k)$ in $\mathbb{C}^k$ and $x \in X$ we define $\alpha \odot x := (\alpha_1 x, \dots, \alpha_k x) \in X^k$. Equipping $\mathbb{C}^k$ and X^k with the respective ℓ_1 -sum norms we note and shall use the estimate $\|\alpha \odot x\| \leq \|\alpha\| \|x\|$. We also equip (m, n) scalar matrices with their corresponding norm as operators from $\mathbb{C}^n$ to $\mathbb{C}^m$ (equipped with their respective ℓ_1 -sum norms), and for such a matrix A and for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{C}^m$, we denote

$$\beta = A\alpha \Leftrightarrow \begin{pmatrix} \beta_1 \\ \dots \\ \beta_m \end{pmatrix} = A \begin{pmatrix} \alpha_1 \\ \dots \\ \alpha_n \end{pmatrix}.$$

Note the estimate $\|A\alpha\| \leq \|A\| \|\alpha\|$, that we shall use along the proof.

Recall that $\Psi(T)(t) = \sum_k t^k A_k$, where the A_k are (m, n) scalar matrices, and that $\alpha = (\alpha_1, \dots, \alpha_n)$ belonging to the kernel of the matrix $\Psi(T)(t)$ means that

$$0 = \sum_k t^k A_k \alpha = \sum_k t^k A_k \begin{pmatrix} \alpha_1 \\ \dots \\ \alpha_n \end{pmatrix}.$$

Recall also that $\Phi(T) = \sum_k A_k S^k$.

For $N \in L$ (the lacunary set of integers defined in [18]), consider the vector

$$x_N := x_N(t) = \frac{\log_2(1 + N^2)}{N^2} \sum_{j=N^2}^{2N^2} t^{-j} e_j \in X.$$

The vector x_N has norm 1 as stated in the proof of [18] Lemma 14 (a consequence of [18] Lemma 7). We claim that if $\alpha = (\alpha_1, \dots, \alpha_n)$ belongs to the kernel of $\Psi(T)(t)$, then

$$\Phi(T)(\alpha \odot x_N) = \Phi(T)(\alpha_1 x_N, \dots, \alpha_n x_N)$$

tends to 0 when N tends to ∞ . This will imply that the restriction of $\Phi(T)$ to the subspace $\{\alpha \odot x, x \in X\}$ is infinitely singular: indeed by choosing values of N sufficiently far apart, we can then define a subspace Y generated by a basic sequence of vectors of the form $\alpha \odot x_N$, $N \in L$, with successive supports on the natural basis of X^n , so that the restriction of $\Phi(T)$ to Y has norm at most some given $\varepsilon > 0$. Since $T - \Phi(T)$ is strictly singular, this implies, as commented in the Background section, that T itself is infinitely singular on the required subspace $\{\alpha \odot x, x \in X\}$.

To prove the claim let us fix a non-zero α in the kernel of $\Psi(T)(t)$, assuming without loss of generality that $\|\alpha\| = 1$. Taking N large enough so that $\|\Phi(T) - U_N\|$ is less than some given $\varepsilon > 0$, with $U_N = \sum_{k=-N}^N A_k S^k$, it is sufficient to prove that $\|U_N(\alpha \odot x_N)\| \leq \varepsilon$ for N large enough.

We note that

$$U_N(\alpha \odot x_N) = \sum_{k=-N}^N A_k S^k(\alpha \odot x_N) = \frac{\log_2(1+N^2)}{N^2} \sum_{k=-N}^N \sum_{j=N^2}^{2N^2} t^{-j} A_k S^k(\alpha \odot e_j),$$

where we used the linearity of the maps $x \mapsto (AR)(\alpha \odot x)$, for all A (m,n)-scalar matrices, R operators on X , and $\alpha \in \mathbb{C}^n$. Note also the key formula $(AR)(\alpha \odot x) = (A\alpha) \odot Rx$, from which

$$A_k S^k(\alpha \odot e_j) = (A_k \alpha) \odot e_{j+k}$$

holds. We deduce that

$$\frac{N^2}{\log_2(1+N^2)} U_N(\alpha \odot x_N) = \sum_{k=-N}^N \sum_{j=N^2}^{2N^2} t^{-j} (A_k \alpha) \odot e_{j+k},$$

which, by using $i = j + k$ and reorganizing, becomes

$$\frac{N^2}{\log_2(1+N^2)} U_N(\alpha \odot x_N) = \sum_{i=N^2-N}^{2N^2+N} t^{-i} \sum_{k=\max(-N, i-2N^2)}^{\min(N, i-N^2)} t^k (A_k \alpha) \odot e_i \quad (1)$$

Note that the inside sums

$$y_i(\alpha) := \sum_{k=\max(-N, i-2N^2)}^{\min(N, i-N^2)} t^k (A_k \alpha) \odot e_i$$

are uniformly bounded over i : indeed the e_i 's are normalized and therefore

$$\|y_i(\alpha)\| \leq \sum_{k=-\infty}^{+\infty} \|A_k \alpha\| \leq \sum_{k=-\infty}^{+\infty} \|A_k\| < +\infty.$$

In the sum over i in (1) we have $4N$ terms $t^{-i} y_i(\alpha)$ corresponding to $N^2 - N \leq i < N^2 + N$ and $2N^2 - N < i < 2N^2 + N$, and therefore the sum of these terms is dominated in norm by

$$4N \sum_{k=-\infty}^{+\infty} \|A_k\| \quad (2)$$

We note that for each of the remaining terms $t^{-i}y_i(\alpha)$ in (1), corresponding to $i \in I_N := [N^2 + N, 2N^2 - N]$, the sum defining $y_i(\alpha)$ runs exactly over k in $[-N, N]$. So the sum of those $t^{-i}y_i(\alpha)$ is

$$\sum_{i \in I_N} t^{-i}y_i(\alpha) = \sum_{i \in I_N} t^{-i} \sum_{k=-N}^N t^k(A_k\alpha) \odot e_i = \left(\sum_{k=-N}^N t^k(A_k\alpha) \right) \odot \left(\sum_{i \in I_N} t^{-i}e_i \right) \quad (3)$$

where we used the bilinearity of the maps $(\alpha, x) \mapsto \alpha \odot x$.

Using that α belongs to the kernel of $\sum_k t^k A_k$ and that $\|A_k\alpha\| \leq \|A_k\|$, we have that

$$\left\| \sum_{k=-N}^N t^k(A_k\alpha) \right\| = \left\| \sum_{|k|>N} t^k(A_k\alpha) \right\| \leq \sum_{|k|>N} \|A_k\| \quad (4)$$

On the other hand

$$\sum_{i \in I_N} t^{-i}e_i = \frac{N^2}{\log_2(1+N^2)}x_N - \sum_{i \in [N^2, N^2+N[\cup]2N^2-N, 2N^2]} t^{-i}e_i,$$

so

$$\left\| \sum_{i \in I_N} t^{-i}e_i \right\| \leq \frac{N^2}{\log_2(1+N^2)} + 2N \leq \frac{2N^2}{\log_2(1+N^2)} \quad (5)$$

for large enough $N \in L$. From (3),(4) and (5) we deduce

$$\left\| \sum_{i \in I_N} t^{-i}y_i(\alpha) \right\| \leq \frac{2N^2}{\log_2(1+N^2)} \sum_{|k|>N} \|A_k\| \quad (6)$$

Inserting in (1) the estimates (2) and (6) corresponding to i outside and inside of I_N respectively, we finally obtain

$$\|U_N(\alpha \odot x_N)\| \leq \frac{4 \log_2(1+N^2)}{N} \sum_{k \in \mathbb{N}} \|A_k\| + 2 \sum_{k > |N|} \|A_k\| \quad (7)$$

This tends to 0 when N tends to infinity, as claimed. $\square$

Following the terminology from [26], we shall say that a scalar valued map ϕ defined on $\mathbb{T}$ vanishes on $\mathbb{T}$ if $\phi(t) = 0$ for some $t \in \mathbb{T}$, and does not vanish on $\mathbb{T}$ if $\phi(t) \neq 0$ for all $t \in \mathbb{T}$.

Proposition 19 *The following hold for $n \in \mathbb{N}$:*

1. Let $T \in L(X, X^n)$, written in blocks as $T = \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix}$. If there exists some $t \in \mathbb{T}$ such that $\Psi(T_i)(t) = 0$ for all $i = 1, \dots, n$, then T is infinitely singular.
2. Let $T \in L(X^n)$. If $\det(\Psi(T))$ vanishes on $\mathbb{T}$ then T is infinitely singular.
3. Let $T \in L(X^n)$. If $\det(\Psi(T))$ does not vanish on $\mathbb{T}$ then T is Fredholm.

Proof Assertion 1. follows from setting $\alpha = 1$ in Lemma 3.

Likewise, if $t \in \mathbb{T}$ is such that $\det(\Psi(T)(t)) = 0$ then there exists a non-zero $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ belonging to $\text{Ker}(\Psi(T)(t))$, and therefore T is infinitely singular on some associated infinite dimensional subspace of X^n by Lemma 3. As commented in the Background section, this means that T itself is infinitely singular, and this proves 2.

3. If $\det(\Psi(T)(t)) \neq 0$ for all $t \in \mathbb{T}$ then $\Psi(T)$ is invertible in $M_n(A(\mathbb{T}))$ by Proposition 17 1. 2. Let U be an operator such that $\Psi(U) = \Psi(T)^{-1}$. From $\Psi(TU) = \text{Id}$ and $\Psi(UT) = \text{Id}$ we deduce $TU - \text{Id}$ and $UT - \text{Id}$ are strictly singular, and therefore T is Fredholm. $\square$

Corollary 20 *Operators on X^n are either Fredholm or infinitely singular. In particular the space X^n is not isomorphic to its subspaces of infinite codimension.*

In the next proposition we shall use the fundamental fact that the monoid of similarity classes of idempotents in $M_\infty(C(\mathbb{T}))$ is $\mathbb{N}$, or equivalently, that the rank (i.e. for $A \in M_\infty(C(\mathbb{T}))$ the common rank of all matrices $A(t)$ for $t \in \mathbb{T}$), is the only associated similarity invariant. This is a consequence of the essential fact that $K_1(\mathbb{C}) := K_0(C_1(\mathbb{T}))$ identifies with the set of homotopy classes of invertibles in $GL_n(\mathbb{C})$ and therefore is $\{0\}$ by contractibility of $GL_n(\mathbb{C})$ (here $C_1(\mathbb{T})$ denotes elements of $C(\mathbb{T})$ which vanish in 1). See for example [4] Theorem 8.2.2, which also reformulates as the K_0 -group of $C(\mathbb{T})$ being equal to $\mathbb{Z}$, see for example [4], Example 5.3.2 (c), or [26], Example 1 p49 or Examples 9.4.1.

Proposition 21 *Let $n \in \mathbb{N}$. A complemented subspace of X^n is isomorphic to X^m for some $m \leq n$.*

Proof Let P be a projection defined on X^n and note that $\Phi(P)$ is also a projection, which is a strictly singular perturbation of P . According to the Lemma on p49 of [26], the map P is therefore similar to a projection onto either some finite codimensional subspace of $\Phi(P)X^n$, or $\Phi(P)X^n \oplus E$ where E is finite dimensional. Therefore PX^n is a finite dimensional perturbation of $\Phi(P)X^n$ and since X^m is isomorphic to its finite dimensional perturbations, it is enough to prove the assertion for $\Phi(P)$. In other words we may assume that $P \in M_n(\mathcal{A})$.

The image of P through Ψ is an idempotent of $M_n(A(\mathbb{T}))$ and in particular of $M_n(C(\mathbb{T}))$. By the fact before the proposition, $\Psi(P)$ is similar inside $M_\infty(C(\mathbb{T}))$ to one of the canonical projections I_m (i.e. the identity of $M_m(C(\mathbb{T}))$). According to Lemma 2, it follows that $\Psi(P)$ is similar to I_m inside $M_\infty(A(\mathbb{T}))$, i.e.

$$\Psi(P) = MI_mM^{-1}$$

for some invertible M in $M_N(A(\mathbb{T}))$ of appropriate dimension, and therefore the relation lifts to

$$P = U \text{Id}_{X^m} U^{-1}$$

for some invertible U in $GL_N(\mathcal{A})$ (seeing also P and Id_{X^m} as operators on X^N in the canonical way). It follows that PX^n is isomorphic to X^m . Finally $m \leq n$ as a consequence of Corollary 20. $\square$

The proof of the above proposition implies the more technical result which follows:

Lemma 4 *If $P \in M_n(\mathcal{A})$ is a projection on X^n such that PX^n is isomorphic to X , then there exist operators $U_1, \dots, U_n, V_1, \dots, V_n$ in $\mathcal{A}$ such that*

$$PX^n = \{(U_1x, \dots, U_nx), x \in X\}$$

and such that

$$U_1V_1 + \dots + U_nV_n = \text{Id}_X.$$

Proof By the above $P = U \text{Id}_X U^{-1}$ for some $U \in GL_N(\mathcal{A})$ in the appropriate dimension N , but it is easily checked that this dimension may be assumed to be n and therefore $U \in GL_n(\mathcal{A})$. It follows that P admits the matrix representation

$$P = (U_i V_j)_{1 \leq i, j \leq n}$$

with $\sum_i V_i U_i = \text{Id}_X$, where $(U_1, \dots, U_n)$ is the first column of U and $(V_1, \dots, V_n)$ is the first row of U^{-1} and therefore these operators belong to $\mathcal{A}$. Note also that $\text{Id}_X = U_1V_1 + \dots + U_nV_n$ since $\mathcal{A}$ is abelian. We have the formula

$$P(x_1, \dots, x_n) = (U_1z, \dots, U_nz)$$

where $z = \sum_i V_i x_i$ and since $\sum_i V_i U_i = \text{Id}_X$, z takes all possible values in X . Therefore

$$PX^n = \{(U_1x, \dots, U_nx), x \in X\}.$$

$\square$

A 1-dimensional subspace of $\mathbb{C}^n$ generated by a vector a is complemented by the orthogonal projection $p(v) = \frac{\langle a, v \rangle}{\|a\|^2} a$. In the next lemma we show how a similar result holds in X^n for operators in $M_n(\mathcal{A})$. By $\text{diag}(M)$ we shall denote the diagonal block matrix operator on X^n with $M \in L(X)$ on the diagonal. For arbitrary $W \in \mathcal{A}$, we denote by $\overline{W}$ the operator in $\mathcal{A}$ such that $\Psi(\overline{W}) = \overline{\Psi(W)}$. That is, if $W = \sum_{n \in \mathbb{Z}} a_n S^n$, then $\overline{W} = \sum_{n \in \mathbb{Z}} \overline{a_{-n}} S^n$. We extend this definition in an obvious way to elements of $M_{m,n}(\mathcal{A})$. Finally if $T \in M_{n,1}(\mathcal{A})$, then $T^t \in M_{1,n}(\mathcal{A})$ denotes the transposed matrix of T .

Lemma 5 Assume $T \in M_{n,1}(\mathcal{A})$ is finitely singular from X to X^n . Then TX is complemented in X^n by the projection $P = B\text{diag}(A^{-1})$, where $A := \overline{T}^t \cdot T \in \mathcal{A}$ and $B = T \cdot \overline{T}^t \in M_n(\mathcal{A})$.

Proof Let us see T as a column

$$T = \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix},$$

where each T_i is an operator in $\mathcal{A}$. Since T is finitely singular, $\Psi(A) = \sum_i |\Psi(T_i)|^2$ does not vanish on $\mathbb{T}$, by Proposition 19 1.. So it is invertible (in $A(\mathbb{T})$ by Wiener's lemma), and so A is invertible in $\mathcal{A}$. The map P takes values in TX and we claim that $PT = T$, implying that P is a projection onto TX . The claim follows from the computation (using that $\mathcal{A}$ is abelian)

$$PT = T\overline{T}^t \text{diag}(A^{-1})T = T\overline{T}^t (A^{-1}T_i)_i = T \sum_i (\overline{T}_i A^{-1}T_i) = TAA^{-1} = T. \quad \square$$

We can now prove the main technical result of this section.

Proposition 22 Assume $m, n \in \mathbb{N}$. Let Y be an infinite codimensional subspace of X . Then there is no isomorphism between a complemented subspace of X^m and a subspace of Y^n .

Proof Assume there is such an isomorphism. By Proposition 21, it follows that there exists an isomorphic embedding R of X into Y^n . We denote $T = \Phi(R)$, and since $R - T$ is strictly singular, we note that T is finitely singular. So by Lemma 5, $P = T\overline{T}^t \text{diag}(A^{-1})$ is a projection onto TX , where $A := \overline{T}^t T$.

Let U_i, V_i be given for P by Lemma 4. Therefore, and letting $s := T - R$,

$$\begin{aligned} \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix} &= P \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix} = T\overline{T}^t \text{diag}(A^{-1}) \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix} \\ &= s\overline{T}^t \text{diag}(A^{-1}) \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix} + R\overline{T}^t \text{diag}(A^{-1}) \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix}. \end{aligned}$$

Since s is strictly singular, the operator $\begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix}$ is therefore the sum of a strictly singular

operator $\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$ and of an operator with values in Y^n , which implies that $U_i - s_i$

takes values in Y for $i = 1, \dots, n$. Then the operator $\sum_i (U_i - s_i) V_i$ takes values in Y , and on the other hand it is equal to $\sum_i U_i V_i - \sum_i s_i V_i = \text{Id}_X - \sum_i s_i V_i$. We would therefore obtain a strictly singular perturbation of the identity with values in an infinite codimensional subspace of X , a contradiction with the stability properties of the Fredholm class. $\square$

We finally arrive to the objective of this section, the *Proof of Proposition 15*: if Y is an infinite codimensional subspace of X_S (real or complex), then there is no isomorphism between a complemented subspace of X_S^m and a subspace of Y^n .

Proof In the complex case this is just Proposition 22 for $X = X_S(\mathbb{C})$. In the real case, if Z is a complemented subspace of X_S^m isomorphic to a subspace of Y^n , then $Z_{\mathbb{C}}$ is a complemented subspace of $(X_S)_{\mathbb{C}}^m$ isomorphic to a subspace of $Y_{\mathbb{C}}^n$, and therefore $Z_{\mathbb{C}}$ (and Z) must be finite dimensional by Proposition 22 in the case $X = (X_S(\mathbb{R}))_{\mathbb{C}}$. $\square$

6 Comments and problems

Our results leave open the following new version of [29] Problem 2.2.8.

Problem 3 For which space ideals A does there exist a largest operator ideal U with $A = \text{Space}(U)$?

Recall that a space ideal is a class of spaces containing the finite dimensional ones and stable under taking direct sums and complemented subspaces. For any ideal U , $\text{Space}(U)$ is a space ideal, [29] Theorem 2.1.3, and conversely a space ideal A always coincides with $\text{Space}(\text{Op}(A))$, [29] Theorem 2.2.5. And our main result is that the answer to Problem 3 is negative for the space ideal F of finite dimensional spaces.

Our techniques can actually be used to obtain a negative answer for for several additional classical space ideals, including the space ideal H of spaces isomorphic to a Hilbert space (finite or infinite dimensional) or the space ideal of superreflexive spaces. This is consequence of the next general proposition. Recall that two Banach spaces are said to be totally incomparable when no infinite dimensional subspace of one is isomorphic to a subspace of the other.

Proposition 23 *Let X_S be Gowers-Maurey's shift space. Let A be a space ideal with the property that all Banach spaces in A are totally incomparable with X_S . Then there is no largest ideal among the class of operator ideals U satisfying $A = \text{Space}(U)$.*

Proof Given spaces X, X' , denote by $\text{Op}_A^{<\omega}(X, X')$ the class of operators which factor both through $W \oplus X^n$, for some $n \in \mathbb{N}$ and $W \in A$, and through $W' \oplus X'^p$, for some $p \in \mathbb{N}$ and $W' \in A$. This is an ideal by the hypothesis that A is a space ideal. We make the following *Claim*: If Y is an infinite codimensional subspace of X_S , then $\text{Space}(\text{Op}_A^{<\omega}(X_S, Y)) = A$.

Assuming the *Claim* holds, apply it to Y_1 and Y_{-1} from Proposition 16. So the ideals $\text{Op}_A^{<\omega}(X_S, Y_1)$ and $\text{Op}_A^{<\omega}(X_S, Y_{-1})$ both have their space ideal equal to A . On the other hand for $i = -1, 1$ the operator T_i defined before Proposition 16 belongs to $\text{Op}_A^{<\omega}(X_S, Y_i)$ and therefore, since $i_{Y_1, X_S} \circ T_1 + i_{Y_{-1}, X_S} \circ T_{-1}$ is Fredholm on X_S , the

operator Id_{X_S} belongs to the sum of the two ideals. Since X_S does not belong to A , this implies that $\text{Space}(\text{Op}_A^{<\omega}(X_S, Y_1) + \text{Op}_A^{<\omega}(X_S, Y_{-1}))$ contains a space which is not in A . Therefore there is no largest ideal U among those with $\text{Space}(U) = A$.

To prove the *Claim*, we use the classical result of Edelstein and Wojtaszczyk [11] that total incomparability between two spaces Z_1, Z_2 implies that any complemented subspace Z of $Z_1 \oplus Z_2$ is isomorphic to the sum of two complemented subspaces of Z_1 and of Z_2 respectively. First note that if Z belongs to A then Id_Z obviously factorizes through $Z \oplus X_S$ and through $Z \oplus Y$, and therefore belongs to $\text{Op}_A^{<\omega}(X_S, Y)$. So we have the inclusion $A \subseteq \text{Space}(\text{Op}_A^{<\omega}(X_S, Y))$. Conversely assume Id_Z belongs to $\text{Op}_A^{<\omega}(X_S, Y)$. Then by Remark 4 and total incomparability, it follows from the Edelstein-Wojtaszczyk result that (i) Z must be isomorphic to $W \oplus X'$ for some $W \in A$ and X' complemented into some power X_S^m and (ii) Z must be isomorphic to $W' \oplus Y'$ for some $W' \in A$ and some complemented subspace Y' of some power Y^n . From (i) and (ii), X' embeds complementably into $W' \oplus Y'$. Total incomparability of W' with Y' , of X' with W' , and the Edelstein-Wojtaszczyk result imply that X' embeds complementably into $Y' \oplus F$ for some finite-dimensional F , and therefore into some power of Y . By Proposition 15 this implies that X' is finite dimensional and therefore that Z belongs to A . We have proved the reverse inclusion $\text{Space}(\text{Op}_A^{<\omega}(X_S, Y)) \subseteq A$. $\square$

Since X_S is separable reflexive, the following remain open:

- Question 2** 1. Let $REFL$ denote the space ideal of reflexive spaces. Does there exist a largest operator ideal U with $\text{Space}(U) = REFL$?
2. Same question for the space ideal of separable spaces.

It may be amusing to observe that it follows from Proposition 21 that the class A of spaces isomorphic to some power of X_S is a space ideal. Therefore $A = \text{Space}(\text{Op}^{<\omega}(X_S))$ by [29] Theorem 2.2.5.

Some natural comments and questions about examples from the first part of the paper are also included below.

Question 3 Are the spaces $Z_2(\alpha)$ and $\overline{Z_2(\alpha)}$ from [20] projectively incomparable for $\alpha \neq 0$? essentially incomparable?

Ferenczi-Galego [13] prove that if a space is essentially incomparable with its conjugate, then it does not contain a complemented subspace with an unconditional basis. For $Z_2(\alpha)$ (more generally, for twisted Hilbert spaces), by Kalton [21], a complemented subspace with an unconditional basis would have to be hilbertian. We do not know whether $Z_2(\alpha)$ contains a complemented Hilbertian copy (for Z_2 this is impossible, by [22] Corollary 6.7). It may be worth pointing out that the above result from [13] actually holds (with the same proof) for projective incomparability:

Proposition 24 *If a complex space X is projectively incomparable with its conjugate, then it does not contain a complemented subspace with an unconditional basis.*

Proof If Y is a subspace of X with an unconditional basis (e_n) , then $\overline{Y}$ is a subspace of $\overline{X}$ which is isomorphic to Y by the map $\sum_i \lambda_i e_i \mapsto \sum_i \overline{\lambda_i} e_i$. If Y is complemented in X then $\overline{Y}$ is complemented in $\overline{X}$. $\square$

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Declarations

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