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Reversible nilpotent centers with cubic nonlinearities

Leandro Bacelar[1] · Jaume Llibre[2]

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Abstract

A real analytic differential system having a center at the origin of coordinates after a linear change of variables and a rescaling of the time can be written in one of the following three forms:

$$\dot{x} = -y + X_2(x, y), \ \ \dot{y} = x + Y_2(x, y),$$

called a *linear type center*,

$$\dot{x} = y + X_2(x, y), \ \ \dot{y} = Y_2(x, y)$$

called a *nilpotent center*, and

$$\dot{x} = X_2(x, y), \ \ \dot{y} = Y_2(x, y)$$

called a *degenerate center*, where $X_2(x, y)$ and $Y_2(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin.

While there are many papers dedicated to study phase portraits of different classes of linear type centers, few papers studied the phase portraits of the nilpotent and degenerate centers. Here we classify the global phase portraits in the Poincaré disc of reversible nilpotent centers with cubic nonlinearities.

Keywords Nilpotent centers · Reversible centers · Cubic polynomials differential systems

Mathematics Subject Classification Primary: 34C05 · 34A05 · 37C10

1 Introduction and statement of the main results

These last years a big interest has appeared for understand better the nilpotent centers, see for instance [5–7, 9, 11–13, 15, 16]. In the present paper we study a new class of nilpotent centers.

✉ Jaume Llibre
jaume.llibre@uab.cat

Leandro Bacelar
leandrobacelar@usp.br

1 Instituto de Ciências Matemática e Computaçãço, Universidade de São Paulo, São Carlos, São Paulo 13566-590, Brazil

2 Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia, Spain

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The main goal of this study is to classify the phase portraits in the Poincaré disc of the differential systems

$$\dot{x} = y + a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3, \qquad \dot{y} = b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3, \quad (1)$$

with $a_i, b_i \in \mathbb{R}$, having at the origin a symmetric nilpotent center with respect to the y-axis.

From Theorem 3.5 of [10] in order that system (1) has a center or a focus we must have $b_0 < 0$.

The nilpotent center at the origin of coordinates of system (1) is symmetric with respect to the y-axis if and only if system (1) is invariant under the change $(x, y, t) \longrightarrow (-x, y, -t)$. Thus the differential system (1) is reduced to the differential system

$$\dot{x} = y + a_1x^2y + a_3y^3, \quad \dot{y} = b_0x^3 + b_2xy^2.$$

When $b_0 < 0$ these polynomial differential systems have a nilpotent center at the origin due to the mentioned symmetry.

In summary the objective of this paper is to characterize the phase portraits in the Poincaré disc of the differential systems

$$\dot{x} = y + ax^2y + by^3, \qquad \dot{y} = cx^3 + dxy^2, \quad (2)$$

with $a, b, c, d \in \mathbb{R}$ and $c < 0$.

Roughly speaking the Poincaré disc $\mathbb{D}^2$ is the closed unit disc in the plane $\mathbb{R}^2$, where its interior has been identified with the whole plane $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$, has been identified with the infinity of $\mathbb{R}^2$. Note that in the plane $\mathbb{R}^2$ we can go or come from the infinity in as many directions as points has the circle $\mathbb{S}^1$. A polynomial differential system defined in $\mathbb{R}^2$ can be extended analytically to the Poincaré disc $\mathbb{D}^2$. In this way we can study the dynamics of the polynomial differential systems in a neighborhood of the infinity. In the Appendix we summarize how to work in the Poincaré disc.

In the next proposition we reduce the four parameters of the differential system (2) to at most two parameters.

Proposition 1 *Differential system (2), with $c < 0$, after doing the following rescaling of the variables $(x, y, t) = (\alpha x, \beta y, \gamma t)$ becomes one of the following five systems:*

(a) If $d > 0$:

$$\dot{x} = y + a'x^2y + by^3, \qquad \dot{y} = -x^3 + xy^2, \quad (3)$$

where $a', b \in \mathbb{R}$, by choosing $\alpha = -d^{-1/2}$, $\beta = -(-c)^{1/2}/d$ and $\gamma = (-d/c)^{1/2}$.

(b) If $d < 0$:

$$\dot{x} = y + a'x^2y + by^3, \qquad \dot{y} = -x^3 - xy^2, \quad (4)$$

where $a', b \in \mathbb{R}$, by choosing $\alpha = (-d)^{-1/2}$, $\beta = (-c)^{1/2}/d$ and $\gamma = -(d/c)^{1/2}$;

(c) If $d = 0$ and $a > 0$:

$$\dot{x} = y + x^2y + a'y^3, \qquad \dot{y} = -x^3, \quad (5)$$

where $a' \in \mathbb{R}$, by choosing $\alpha = -a^{-1/2}$, $\beta = -(-c)^{-1/2}/a$ and $\gamma = (-a/c)^{1/2}$;

(d) For $d = 0$ and $a < 0$:

$$\dot{x} = y - x^2y + a'y^3, \qquad \dot{y} = -x^3, \quad (6)$$

where $a' \in \mathbb{R}$ by choosing $\alpha = (-a)^{-1/2}$, $\beta = (-c)^{1/2}/a$ and $\gamma = -(a/c)^{1/2}$;

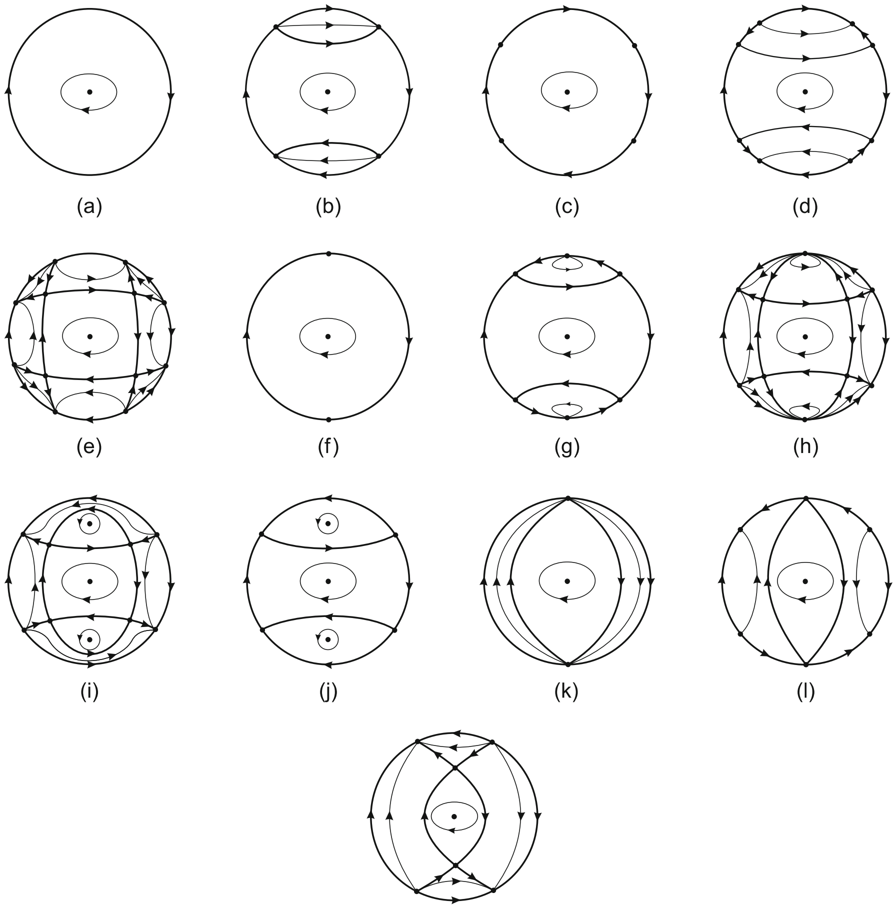

Fig. 1 Phase portraits of the differential system in the Poincaré disc

(e) *If* $d = a = 0$*:*

$$\dot{x} = y + a'y^3, \qquad \dot{y} = -x^3, \tag{7}$$

where $a' \in \mathbb{R}$*, by choosing* $\alpha = 1$ *and* $\beta = (-c)^{1/2}$*, and* $\gamma = (-c)^{-1/2}$*.*

In what follows we shall write a instead of a' in the differential systems of Proposition 1. The proof of Proposition 1 is trivial and we do not write it.

Theorem 2 *The phase portrait of the differential system (2) in the Poincaré disc is topologically equivalent to one of the phase portraits of Fig. 1. More precisely, the phase portrait of Fig. 1*

(a) *is realized by system (3) when either* $b > (1-a)^2/4$*, or* $b = (1-a)^2/4 > 0$ *and* $a > 1$*, or* $0 < b < (1-a)^2/4$ *and* $a > 1$*, or by system (4) with either* $0 < b < (1+a)^2/4$ *and* $a - b \geq 0$*, or* $0 < b = (1+a)^2/4$*, or by system (5) with* $a > 0$*, or by system (6) with* $a > 1/4$*, or by system (7) with* $a > 0$*;*

(b) *is realized by system (3) when* $0 < b < (1-a)^2/4$, $-1 < a < 1$ *and* $a + b \geq 0$, *or by system (4) with* $0 < b < (1+a)^2/4$ *and* $a - b < 0$, *or by system (5) with* $a = 0$, *or by system (6) with* $a = 1/4$;

(c) *is realized by system (3) when either* $b = (1-a)^2/4 > 0$ *and* $a < 1$, *or* $b = 0$ *and* $a \geq 1$, *or by system (4) with* $b = 0$ *and* $a \geq 0$, *or by system (7) with* $a = 0$;

(d) *is realized by system (3) when either* $0 < b < (1-a)^2/4$, $a < -1$ *and* $a + b \geq 0$, *or* $0 < b < (1-a)^2/4$, $-1 < a < 1$ *and* $a + b = 0$, *or* $0 < b < (1-a)^2/4$, $a < -1$ *and* $a + b \geq 0$, *or by system (6) with* $0 < a < 1/4$;

(e) *is realized by system (3) when either* $0 < b < (1-a)^2/4$, $-1 < a < 1$ *and* $a + b < 0$, *or* $0 < b < (1-a)^2/4$ *and* $a = -1$, *or* $0 < b < (1-a)^2/4$, $a < -1$ *and* $a + b < 0$;

(f) *is realized by system (3) when* $b = 0$ *and* $a \geq 1$;

(g) *is realized by system (3) when* $b = 0$ *and* $0 \leq a < 1$;

(h) *is realized by system (3) when* $b < 0$ *and* $a < 0$;

(i) *is realized by system (3) when* $b < 0$ *and* $a \leq -1$, *or* $b < 0$, $a > -1$ *and* $a + b < 0$;

(j) *is realized by system (3) when* $b < 0$, $a > -1$ *and* $a + b \geq 0$;

(k) *is realized by system (4) with* $b = 0$ *and* $-1 \leq a < 0$;

(l) *is realized by system (4) with* $b = 0$ *and* $a < -1$, *or by system (6) with* $a = 0$;

(m) *is realized by system (4) with* $b < 0$, *or by system (5) with* $a < 0$, *or by system (6) with* $a < 0$, *or by system (7) with* $a < 0$.

Theorem 2 is proved along the Sect. 2, 3, 4, 5, 6. The five differential systems of Proposition 1 are integrable, their first integrals are given at beginning of Sect. 2, 3, 4, 5, 6, respectively.

2 Analysis of system (3)

In this section for the differential system (3), first we provide its first integrals, after we study the local phase portraits at the finite and infinite singular points, and finally we show the global phase portraits in the Poincaré disc.

2.1 First integrals

In what follows we use log and coth to represent, respectively, the logarithmic and hyperbolic cotangent functions.

Proposition 3 *The first integrals* $H = H(x, y) \in \mathbb{C}[x, y]$ *of the differential systems (3) are:*

(a) *If* $(a-1)^2 - 4b \neq 0$, *then*

$$H(x, y) = (2(1+a)\coth^{-1}\left(\frac{\sqrt{(a-1)^2-4b}(1+(a+b)y^2)}{(1+a+2ax^2+2bx^2+(a-1)(a+b)y^2)}\right) + \sqrt{(a-1)^2-4b}\cdot\log(1+(a+b)x^4+(a+2b-1)y^2+b(a+b)y^4+ x^2(1+a+(a-1)(a+b)y^2)).$$

(b) *If* $(a-1)^2 - 4b = 0$ *and* $a \neq 1$, *then*

$$H(x, y) = \frac{(4+(1+a)^2y^2)}{(4+2(1+a)x^2+(a^2-1)y^2)} + \log(4+2(1+a)x^2+(a^2-1)y^2).$$

(*c*) *If* $(a-1)^2-4b=0$ *and* $a=1$, *then*

$$H(x,y)=\frac{1+y^2}{1+x^2}+\log(4(1+x^2)).$$

Proof Let $(x(t), y(t))$ be an arbitrary solution of the differential Eq. (3). Since for all the functions H of the different statements of the proposition under the corresponding assumptions satisfy

$$\frac{dH(x(t),y(t))}{dt}=\frac{\partial H(x,y)}{\partial x}\frac{dx}{dt}+\frac{\partial H(x,y)}{\partial y}\frac{dy}{dt}=0,$$

the functions H are first integrals. □

2.2 The finite singular points

See Subsect. 7.1 of the Appendix for the definitions of hyperbolic, semihyperbolic and nilponent singular points.

Proposition 4 *The finite singular points of the differential system (3) are:*

(*a*) *The nilpotent center at the origin;*
(*b*) *if* $b<0$ *the points* $P_\pm=\left(0,\pm\sqrt{\frac{1}{-b}}\right)$ *are centers; and*
(*c*) *if* $a+b<0$ *the four additional points* $Q_{1\pm}=\left(\sqrt{\frac{-1}{a+b}},\pm\sqrt{\frac{-1}{a+b}}\right)$ *and* $Q_{2\pm}=\left(-\sqrt{\frac{-1}{a+b}},\pm\sqrt{\frac{-1}{a+b}}\right)$ *are hyperbolic saddles.*

Proof It is immediate to check that the origin, $P_\pm$ and the points $Q_{1\pm}$ and $Q_{2\pm}$ are the possible finite singular points. Furthermore, $P_\pm$ are real if and only if $b<0$ and the points $Q_{1\pm}$ and $Q_{2\pm}$ are real if and only if $a+b<0$, so statement (a) is proved. For $b<0$, for the Jacobian matrix of the differential system (3) at $P_\pm$, the eigenvalues are $\pm(-2/b)^{1/2}i$, i.e. purely imaginary, so these singular points must be centers or foci, and since these points lies in the y-axis and the system is invariant under the change $(x,y,t)\longrightarrow(-x,y,-t)$, we get that $P_\pm$ are centers. For $a+b<0$, the determinant of the Jacobian matrix of the differential system (3) at the points $Q_{1\pm}$ and $Q_{2\pm}$ is $4/(a+b)$, therefore by Theorem 2.15 of [10] these points are hyperbolic saddles and the proposition is proved. □

2.3 The infinite singular points

Proposition 5 *The infinite singular points in the local chart* U_1 *of system (3) are shown in Table 1, where*

$$p_\pm=\left(\pm\frac{1}{\sqrt{1-a}},0\right),\qquad s_\pm=\left(\pm\sqrt{\frac{1-a}{2b}},0\right),$$

$$r_\pm=\left(\pm\sqrt{\frac{1-a-\sqrt{(1-a)^2-4b}}{2b}},0\right),\qquad m_\pm=\left(\pm\sqrt{\frac{1-a+\sqrt{(1-a)^2-4b}}{2b}},0\right).$$

Proof From Subsect. 7.3 the Poincaré compactification of system (3) in the local chart U_1 is

$$\dot u=-1+u^2-au^2-bu^4-u^2v^2,\quad \dot v=-uv(a+bu^2+v^2). \tag{8}$$

Table 1 The infinite equilibria in the local chart U_1.

Values of a and b			Infinite equilibria in U_1
$b > (1-a)^2/4$			No equilibria
$b = (1-a)^2/4 > 0$	$a > 1$		No equilibria
	$a < 1$		$p_\pm$ are formed by two hyperbolic sectors
$0 < b < (1-a)^2/4$	$a > 1$		No equilibria
	$-1 < a < 1$	$a+b > 0$	$m_\pm$ are hyperbolic nodes, m_+^s, m_-^u $r_\pm$ are hyperbolic saddles
		$a+b = 0$	$m_\pm$ are hyperbolic nodes, m_+^s, m_-^u $r_\pm$ are semihyperbolic saddles
		$a+b < 0$	$m_\pm$ are hyperbolic nodes, m_+^s, m_-^u $r_\pm$ are hyperbolic nodes, r_+^u, r_-^s
	$a = -1$		$m_\pm$ are hyperbolic nodes, m_+^s, m_-^u $r_\pm$ are hyperbolic nodes, r_+^u, r_-^s
	$a < -1$	$a+b > 0$	$m_\pm$ are hyperbolic saddles $r_\pm$ are hyperbolic nodes, r_+^u, r_-^s
		$a+b = 0$	$m_\pm$ are semihyperbolic saddles $r_\pm$ are hyperbolic nodes, r_+^u, r_-^s
		$a+b < 0$	$m_\pm$ are hyperbolic nodes, m_+^s, m_-^u $r_\pm$ are hyperbolic nodes, r_+^u, r_-^s
$b = 0$	$a \geq 1$		No equilibria
	$0 < a < 1$		$p_\pm$ are hyperbolic saddles
	$a = 0$		$p_\pm$ are semihyperbolic saddles
	$a < 0$		$p_\pm$ are hyperbolic nodes, p_+^u, p_-^s
$b < 0$	$a \leq -1$		$r_\pm$ are hyperbolic nodes, r_+^u, r_-^s
	$a > -1$	$a+b > 0$	$r_\pm$ are hyperbolic saddles
		$a+b = 0$	$r_\pm$ are semihyperbolic saddles
		$a+b < 0$	$r_\pm$ are hyperbolic nodes, r_+^u, r_-^s

Assume that $b > (1-a)^2/4$. Then the possible infinite singular points $r_\pm$ and $m_\pm$ are complex.

Assume that $b = (1-a)^2/4 > 0$ and $a > 1$. Then the possible infinite singular points $p_\pm$ are complex.

Assume that $b = (1-a)^2/4 > 0$ and $a < 1$. Then system (3) has the two infinite singular points $p_\pm$. The Jacobian matrix of system (3) evaluated at $p_\pm$ is identically zero. Doing blow ups as in the proof of the next Proposition 6 we obtain that the local phase portrait of the infinite singular points $p_\pm$ is formed by two hyperbolic sectors, and consequently the two separatrices are on the line of infinity.

Assume that $0 < b < (1-a)^2/4$ and $a > 1$. Then the possible infinite singular points $r_\pm$ and $m_\pm$ are complex.

Assume that $0 < b < (1-a)^2/4$, $-1 < a < 1$ and $a + b > 0$. Then system (3) has the infinite singular points $r_\pm$ and $m_\pm$. Using Theorem 2.15 of [10] $m_\pm$ are hyperbolic nodes, m_- is unstable (we denote it as m_-^u), and m_+ is stable (we denote it as m_+^s), and $r_\pm$ are hyperbolic saddles.

Assume that $0 < b < (1-a)^2/4$, $-1 < a < 1$ and $a + b = 0$. Then system (3) has the infinite singular points $r_\pm$ and $m_\pm$. Using Theorem 2.15 of [10] $m_\pm$ are hyperbolic nodes, m_- is unstable (we denote it as m_-^u), and m_+ is stable (we denote it as m_+^s), and $r_\pm$ usign Theorem 2.19 of [10] are semihyperbolic saddles.

Assume that $0 < b < (1-a)^2/4$, $-1 < a < 1$ and $a + b < 0$. Then system (3) has the infinite singular points $r_\pm$ and $m_\pm$, all of them are hyperbolic nodes by Theorem 2.15 of [10], m_- and r_+ are unstable, and m_+ and r_- are stable.

Assume that $0 < b < (1-a)^2/4$ and $a = -1$. Then system (3) has the infinite singular points $r_\pm$ and $m_\pm$, all of them are hyperbolic nodes by Theorem 2.15 of [10], m_- and r_+ are unstable, and m_+ and r_- are stable.

Assume that $0 < b < (1-a)^2/4$, $a < -1$ and $a+b > 0$. Then system (3) has the infinite singular points $r_\pm$ and $m_\pm$, all of them are hyperbolic and by Theorem 2.15 of [10], $m_\pm$ are saddles, and r_+ is an unstable node and r_- is a stable node.

Assume that $0 < b < (1-a)^2/4$, $a < -1$ and $a+b = 0$. Then system (3) has the infinite singular points $r_\pm$ and $m_\pm$. By Theorem 2.19 $m_\pm$ are semihyperbolic saddles. By Theorem 2.15 of [10], $r_\pm$ are hyperbolic nodes, r_+ is unstable and r_- is stable.

Assume that $0 < b < (1-a)^2/4$, $a < -1$ and $a+b < 0$. Then system (3) has the infinite singular points $r_\pm$ and $m_\pm$, all of them are hyperbolic nodes by Theorem 2.15 of [10], m_- and r_+ are unstable, and m_+ and r_- are stable.

Assume that $b = 0$ and $a \geq 1$. Then the possible infinite singular points $p_\pm$ are complex.

Assume that $b = 0$ and $0 < a < 1$. Then system (3) has the two infinite singular points $p_\pm$. The Jacobian matrix of system (3) evaluated at $p_\pm$ has eigenvalues $\pm 2\sqrt{1-a}$ and $\mp a/\sqrt{1-a}$. Using Theorem 2.15 of [10] the infinite singular points $p_\pm$ are hyperbolic saddles.

Assume that $b = 0$ and $a = 0$. Then system (3) has the two infinite singular points $p_\pm$. The Jacobian matrix of system (3) evaluated at $p_\pm$ has eigenvalues $\pm 2\sqrt{1-a}$ and 0. Using Theorem 2.19 of [10] the infinite singular points $p_\pm$ are semihyperbolic saddles.

Assume that $b = 0$ and $a < 0$. Then system (3) has the two infinite singular points $p_\pm$. The Jacobian matrix of system (3) evaluated at $p_\pm$ has eigenvalues $\pm 2\sqrt{1-a}$ and $\mp a/\sqrt{1-a}$. Using Theorem 2.15 of [10] the infinite singular points $p_\pm$ are hyperbolic nodes, p_+ unstable and p_- stable.

Assume that $b < 0$ and $a \leq -1$. Then system (3) has the two infinite singular points $r_\pm$. Using Theorem 2.15 of [10] the infinite singular points $r_\pm$ are hyperbolic nodes, r_+ unstble, r_- stable.

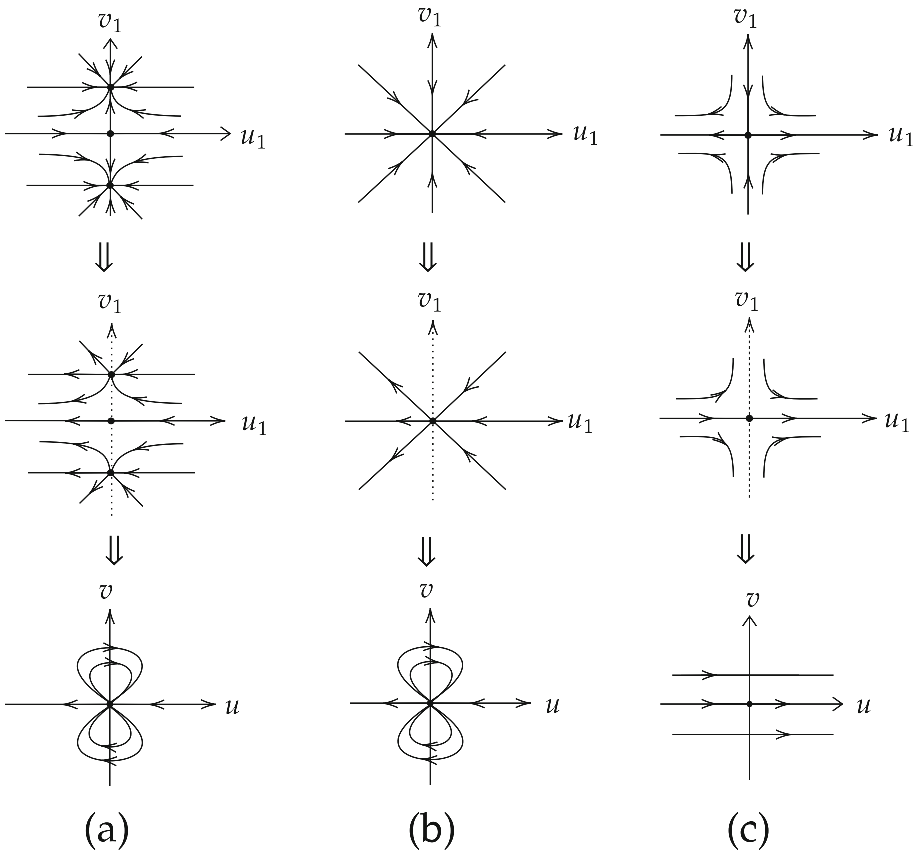

Fig. 2 The blow up of the infinite singular point at the origin of the local chart U_2 of system (9): **a** for $a < 0$, **b** for $0 \leq a < 1$, and **c** for $a \geq 1$. The three pictures of the first row of this figure correspond to differential system (11). The three pictures of the second row of this figure correspond to differential system (10). Finally the three pictures of the third row of this figure correspond to differential system (9), i.e. to the local phase portrait at the origin (0, 0) of the chart U_2

Assume that $b < 0$, $a > -1$ and $a + b > 0$. Then system (3) has the two infinite singular points $r_{\pm}$. Using Theorem 2.15 of [10] the infinite singular points $r_{\pm}$ are hyperbolic saddles.

Assume that $b < 0$, $a > -1$ and $a + b > 0$. Then system (3) has the two infinite singular points $r_{\pm}$. Using Theorem 2.19 of [10] the infinite singular points $r_{\pm}$ are semihyperbolic saddles.

Assume that $b < 0$, $a > -1$ and $a + b < 0$. Then system (3) has the two infinite singular points $r_{\pm}$. Using Theorem 2.15 of [10] the infinite singular points $r_{\pm}$ are hyperbolic nodes, r_+ unstble, r_- stable.

This completes the proof of the proposition. □

Proposition 6 *The origin of the local chart U_2 of system (3) is an infinite singular point if and only if $b = 0$. Its local phase portrait is given in Fig. 2a if $a < 0$, 2b if $0 \leq a < 1$, and 2c if $a \geq 1$.*

Proof From Subsect. 7.3 the Poincaré compactification of system (3) in the local chart U_2

$$\dot{u} = b - (1-a)u^2 + v^2 + u^4, \quad \dot{v} = -uv + u^3 v \tag{9}$$

Therefore the origin of U_2 is an infinite singular point if and only if $b = 0$. In this case the Jacobian matrix of the differential system (9) at the origin is identically zero, and for the analysis of the local phase portrait at the origin, we will apply vertical blow ups, see Subsect. 7.2 of the Appendix.

We are able to do the vertical blow up $(u, v) = (u_1, u_1 v_1)$, since $u = 0$ is not a characteristic direction. Rewriting the system in the coordinates u_1 and v_1 we get

$$\dot{u_1} = u_1^2(a - 1 + u_1^2 + v_1^2), \quad \dot{v_1} = -u_1 v_1(a + v_1^2). \tag{10}$$

Eliminating the common factor u_1 in the expressions of $\dot{u_1}$ and $\dot{v_1}$ by rescaling the time variable we get the differential system

$$\dot{u_1} = u_1(a - 1 + u_1^2 + v_1^2), \quad \dot{v_1} = -v_1(a + v_1^2). \tag{11}$$

So along the axis $u_1 = 0$, the differential system (11) has three singular points $(0, 0)$ and $(0, \pm\sqrt{-a})$ if $a < 0$ and only the origin if $a \geq 0$.

By Theorems 2.15 and 2.19 of [10] we obtain that $(0, \pm\sqrt{-a})$ are hyperbolic stable nodes, and for $(0, 0)$ we have that it is a semihyperbolic saddle if $a = 1$, a hyperbolic stable node if $0 \leq a < 1$, a semihyperbolic stable node if $a = 0$ and a hyperbolic saddle if $a > 1$ or $a < 0$.

With these information going back through the blow ups we obtain Fig. 2, and we finish the proof of proposition. □

2.4 Phase portraits in the Poincaré disc

Combining the Propositions 4, 5 and 6 we obtain the following for the differential system (3).

Assume that $b > (1-a)^2/4$. Then the unique finite singular point is the nilpotent center at $(0, 0)$, and there are no infinite singular points. We claim that the center at $(0, 0)$ is global, i.e. $\mathbb{R}^2 \setminus \{(0, 0)\}$ is filled with periodic orbits. Indeed, if the center is not global let γ be the last periodic orbit surrounding the center. Consider the Poincaré map or the first return map defined on a small transversal segment Σ intersect the orbit γ. This is an analytic map of one variable that on the piece of the segment Σ contained in the bounded region limited by γ is the identity map. So this map is the identity on the whole segment Σ, in contradiction with the assumption that γ was the last orbit surrounding the center $(0, 0)$. Hence the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1a.

Assume that $b = (1-a)^2/4 > 0$ and $a > 1$. Again the unique finite singular point is the nilpotent center at $(0, 0)$, and there are no infinite singular points. Using the same arguments than in the case $b > (1-a)^2/4$, it follows that the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1a.

Assume that $b = (1-a)^2/4 > 0$ and $a < 1$. Then the unique finite singular point is the nilpotent center at $(0, 0)$, and the two infinite singular points $p_\pm$ in the local chart U_1 are formed by two hyperbolic sectors, of course we also have two infinite singular points in the local chart V_1 formed by two hyperbolic sectors. Hence using the arguments in the proof of the case $b > (1-a)^2/4$ the center $(0, 0)$ is global. Therefore, using the first integral of system (3) in this case, the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1c.

Assume that $0 < b < (1-a)^2/4$ and $a > 1$. The unique finite singular point is the nilpotent center at $(0, 0)$, and there are no infinite singular points. Using the same arguments than in the case $b > (1-a)^2/4$, it follows that the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1a.

Assume that $0 < b < (1-a)^2/4$, $-1 < a < 1$ and $a+b > 0$. The unique finite singular point is the nilpotent center at $(0, 0)$, and has four infinite singular points in the local chart U_1, i.e. $m_\pm$ are hyperbolic nodes, m_- is unstable and m_+ is stable and $r_\pm$ are hyperbolic saddles. Of course, we have the symetric four infinite singular points with respect to the origin of coodinates in the local chart V_1. Hence the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1b.

Assume that $0 < b < (1-a)^2/4$, $-1 < a < 1$ and $a+b = 0$. In this case we get the same phase portrait in the Poincaré disc than in the previous case $0 < b < (1-a)^2/4$, $-1 < a < 1$ and $a+b > 0$, the unique difference is that the infinite singular points $r_\pm$ and their symmetric in the chart V_1 are semihyperbolic saddles.

Assume that $0 < b < (1-a)^2/4$, $-1 < a < 1$ and $a+b < 0$. Then there are five finite singular points, the nilpotent center $(0, 0)$, and the four hyperbolic saddles $Q_{1\pm}$ and $Q_{2\pm}$. And there are four infinite singular points in the chart U_1 that are hyperbolic nodes, m_- and r_+ are unstable, and m_+ and r_- are stable, and of course the symmetric four nodes in the chart V_1. Therefore, using the first integral of system (3) in this case, we obtain the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1e.

Assume that $0 < b < (1-a)^2/4$ and $a = -1$. The proof of this case is exactly the same than the previous case $0 < b < (1-a)^2/4$, $-1 < a < 1$ and $a+b < 0$.

Assume that $0 < b < (1-a)^2/4$, $a < -1$ and $a+b > 0$. Then the unique finite singular point is the nilpotent center at $(0, 0)$, and the four infinite singular points $m_\pm$ and $r_\pm$ in the local chart U_1, $m_\pm$ are saddles, and r_+ is an unstable node and r_- is a stable node. Of course, we also have the symmetric four infinite singular points in the chart V_1. Therefore, using the first integral of system (3) in this case, the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1d.

Assume that $0 < b < (1-a)^2/4$, $a < -1$ and $a+b = 0$. In this case we get the same phase portrait in the Poincaré disc than in the previous case $0 < b < (1-a)^2/4$, $a < -1$ and $a+b > 0$, the unique difference is that the infinite singular points $m_\pm$ and their symmetric in the chart V_1 are semihyperbolic saddles.

Assume that $0 < b < (1-a)^2/4$, $a < -1$ and $a+b < 0$. The proof of this case is exactly the same than the case $0 < b < (1-a)^2/4$, $-1 < a < 1$ and $a+b < 0$.

Assume that $b = 0$ and $a \geq 1$. Then the unique finite singular point is the nilpotent center $(0, 0)$, and system (3) has no infinite singular point in the chart U_1, but the origin of the chart U_2 is an infinite singular point formed by two hyperbolic sectors. Of course, the origin of the chart V_2 is also an infinite singular point formed by two hyperbolic sectors. Using the arguments of the proof of the case $b > (1-a)^2/4$ the nilpotent center is global. So the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1f.

Assume that $b = 0$ and $0 < a < 1$. Then the unique finite singular point is the nilpotent center $(0, 0)$. In the chart U_1 system (3) has the two saddles $p_\pm$, and of course the two symmetric saddles in the chart V_1. Moreover, the origin of the chart U_2 has the local phase portrait of Fig. 2b, and we have its symmetric infinite singular point in the origin of the chart V_2. Therefore, using the first integral of system (3) in this case, the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1g.

Assume that $b = 0$ and $a = 0$. In this case we get the same phase portrait in the Poincaré disc than in the previous case $b = 0$ and $0 < a < 1$, the unique difference is that the infinite singular points $p_\pm$ and their symmetric in the chart V_1 are semihyperbolic saddles.

Assume that $b = 0$ and $a < 0$. Then system (3) has five finite singular points, the nilpotent center at (0, 0) and four saddles at $Q_{1\pm}$ and $Q_{2\pm}$; and at infinity it has two nodes $p_\pm$ in the chart U_1, being p_+ unstable and p_- stable, and their symmetric in the chart V_1. Also the origins of the charts U_2 and V_2 are infinite singularites, their phase portraits are shown in Fig. 2a. Therefore, using the first integral of system (3) in this case, the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1h.

Assume that $b < 0$ and $a \leq -1$. Then system (3) has seven finite singular points, the nilpotent center at (0, 0), two centers at $P_\pm$ and four saddles at $Q_{1\pm}$ and $Q_{2\pm}$; and at infinity it has two nodes $r_\pm$ in the chart U_1, being r_+ unstable and r_- stable; and their symmetric nodes in the chart V_1. Therefore, using the first integral of system (3) in this case, the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1i.

Assume that $b < 0$, $a > -1$ and $a + b > 0$. Then system (3) has three finite singular points, the nilpotent center at (0, 0) and two centers at $P_\pm$; and at infinity it has two saddles $r_\pm$ in the chart U_1; and their symmetric saddles in the chart V_1. Therefore, using the first integral of system (3) in this case, the phase portrait in the Poincaré disc is topologically equivalent to the phase portrait of Fig. 1j.

Assume that $b < 0, a > -1$ and $a+b = 0$. In this case we get the same phase portrait in the Poincaré disc than in the previous case $b < 0$, $a > -1$ and $a + b > 0$, the unique difference is that the infinite singular points $r_\pm$ and their symmetric in the chart V_1 are semihyperbolic saddles.

Assume that $b < 0$, $a > -1$ and $a + b < 0$. The proof of this case is exactely the proof of the case $b < 0$ and $a \leq -1$.

In summary, Theorem 2 is proved when the differential system (1) is equivalent to the differential system (3) of Proposition 1.

3 Analysis of system (4)

As in the previous section we divide the study of the differential systems (4) into four subsections.

3.1 First integrals

Proposition 7 *The first integrals $H = H(x, y)$ of the differential systems (4) are:*

(a) *If* $((1 + a)^2 - 4b) \neq 0$, *then*

$$H = 2(a-1)\coth^{-1}\left(\frac{\sqrt{(a+1)^2-4b}\,(y^2(a-b)-1)}{(a+1)y^2(a-b)+2ax^2-a-2bx^2+1}\right)+ \sqrt{(a+1)^2-4b}\,\log\left(x^4(b-a)+x^2\left(-\left((a+1)y^2(a-b)\right)+a-1\right)+ by^4(b-a)-y^2(a-2b+1)+1\right).$$

(b) *If* $(1 + a)^2 - 4b = 0$ *and* $a \neq -1$, *then*

$$H = \frac{(a-1)^2y^2+4}{\left(a^2-1\right)y^2+2(a-1)x^2+4}+\log\left(\left(a^2-1\right)y^2+2(a-1)x^2+4\right).$$

(c) *If* $(1 + a)^2 - 4b = 0$ *and* $a = -1$, *then* $H = x^4 + y^4 + 2y^2(1 + x^2)$.

The proof of Proposition 7 is essentially the proof of Proposition 3.

3.2 The finite singular points

Proposition 8 *The finite singular points of the differential system (4) are:*

(a) the nilpotent center at the origin $(0,0)$*; and*
(b) if $b<0$ *the two additional* $P_\pm = \left(0, \pm\sqrt{\frac{-1}{b}}\right)$ *hyperbolic saddles.*

Proof It is immediate to check that the singular points are the origin and additionally $P_\pm$ when $b<0$. Since the determinant of the Jacobian matrix of the differential system (4) at the singular points $P_\pm$ is $2/b$, by Theorem 2.15 of [10] these two singularitis are hyperbolic saddles. □

3.3 The infinite singular points

Proposition 9 *The infinite singular points in the local chart* U_1 *of system (4) are:*

(a) two infinite singular points $p_\pm = (\pm 1/\sqrt{-1-a}, 0)$ *if* $b=0$ *and* $a<-1$*,* p_- *is a hyperbolic stable node and* p_+ *is a hyperbolic unstable node;*
(b) two infinite singular points $q_\pm = (\pm\sqrt{(1+a+\sqrt{(a+1)^2-4b})/(-2b)}, 0)$ *if either* $b<0$*, or* $a\le 1$ *and* $0<b<(1+a)^2/4$*, then* $q_\pm$ *are hyperbolic saddles if* $0<b<(1+a)^2/4$ *and* $-1<a\le 1$*. While* $q_\pm$ *are hyperbolic nodes if either* $b<0$*, or* $0<b<(1+a)^2/4$ *and* $a<0$*;*
(c) two infinite singular points $r_\pm = (\pm\sqrt{(1+a-\sqrt{(a+1)^2-4b})/(-2b)}, 0)$*, if* $a<-1$ *and* $0<b<(1+a)^2/4$*, then* $r_\pm$ *are hyperbolic saddles.*
(d) two infinite singular points $s_\pm = (\pm\sqrt{2/(-1-a)}, 0)$ *if* $b=(1+a)^2/4$*. Then* $s_\pm$ *are semihyperbolic saddle-nodes.*

Proof From Subsect. 7.3 the Poincaré compactification of system (3) in the local chart U_1 is

$$\dot{u} = -1-(1+a)u^2 - bu^4 - u^2v^2, \qquad \dot{v} = -uv(a+bu^2+v^2). \tag{12}$$

If $b=0$ the possible infinite singular points of the differential system (12) are $p_\pm$. Clearly these points are real if and only if $a<-1$. So statement (a) is proved for $b=0$. For $a<-1$ the eigenvalues of the Jacobian matrix of the differential system (12) at p_- are $-2\sqrt{-a-1}$ and $a/\sqrt{-a-1}$, so by the Hartman-Grobman Theorem (see for instance [8]) or Theorem 2.15 of [10] p_- is a hyperbolic stable node. Since such eigenvalues for the singular point p_+ are $2\sqrt{-a-1}$ and $-a/\sqrt{-a-1}$, hence p_+ is a hyperbolic unstable node. This proves statement (a).

If $b\neq 0$ the possible infinite singular points of the differential system (12) are $q_\pm$ and $r_\pm$. It is easy to verify that the points $q_\pm$ are real if either $b<0$, or $a\le 1$ and $0<b<(1+a)^2/4$, and that the points $r_\pm$ are real if $a<-1$ and $0<b<(1+a)^2/4$.

If the determinant of the Jacobian matrix at an infinite singular point is negative we have a hyperbolic saddle (see Theorem 2.15 of [10]), and if such determinant is positive we have a hyperbolic node because a focus can not exist at infinity because the infinity is invariant (see again Theorem 2.15 of [10]).

At the points $q_\pm$ the mentioned determinant is positive if and only if either $b<0$, or $0<b<(1+a)^2/4$ and $a<0$, and negative if and only if $0<b<(1+a)^2/4$ and $-1<a\le 1$.

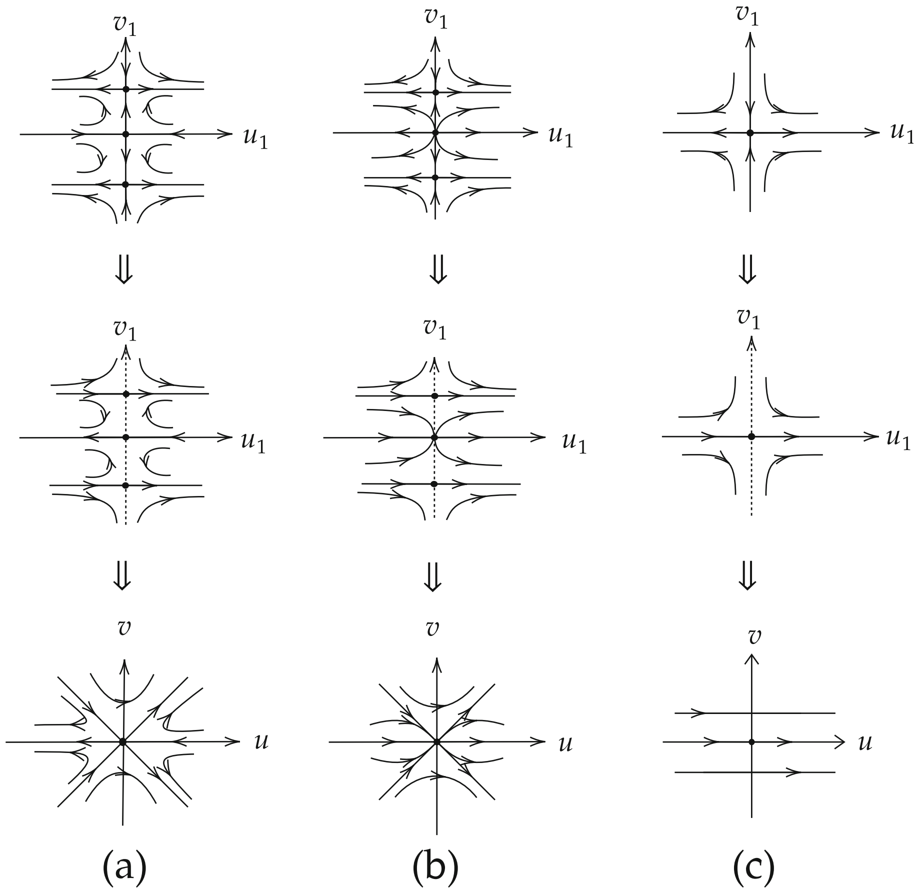

Fig. 3 The blow up of the infinite singular point at the origin of the local chart U_2 of system (13): **a** for $a < -1$, **b** for $-1 \leq a < 0$ and **c** for $a \geq 0$. The three pictures of the first row of this figure correspond to differential system (15). The three pictures of the second row of this figure correspond to differential system (14). Finally the three pictures of the third row of this figure correspond to differential system (13), i.e to the local phase portrait at the origin of the chart U_2

At the points $r_\pm$ the mentioned determinant is always negative. This proves statements (b) and (c).

When $b = (1+a)^2/4$ the uniwue infinite singular points of system (12) are the points $s_\pm$. By Theorem 2.19 of [10] they are semihyperbolic saddle-nodes. This completes the proof of the proposition. □

Proposition 10 *The origin of the local chart U_2 of system (4) is an infinite singular point if and only if $b = 0$. Its local phase portrait is given in Fig. 3a if $a < -1$, in Fig. 3b if $-1 \leq a < 0$, and in Fig. 3c if $a \geq 0$.*

Proof From Subsect. 7.3 the Poincaré compactification of system (4) in the local chart U_2 is

$$\dot{u} = b + (1+a)u^2 + v^2 + u^4, \quad \dot{v} = uv(1+u^2). \tag{13}$$

Thus the origin of U_2 is an infinite singular point if and only if $b = 0$. Then the Jacobian matrix of the differential system (13) at the origin is identically zero. So for studying the local phase portrait of the origin of U_2 we shall use vertical blow ups, see Subsect. 7.2 of the Appendix.

Since $u = 0$ is not a characteristic direction we do the vertical blow up change of variables $(u, v) = (u_1, u_1 v_1)$. In the new variables (u_1, v_1) system (13) writes

$$\dot{u_1} = u_1^2(1 + a + u_1^2 + v_1^2), \quad \dot{v_1} = -u_1 v_1(a + v_1^2). \tag{14}$$

Doing a rescaling of the time variable we eliminate the common factor u_1 between $\dot{u_1}$ and $\dot{v_1}$, and we get the differential system

$$\dot{u_1} = u_1(1 + a + u_1^2 + v_1^2), \quad \dot{v_1} = -v_1(a + v_1^2). \tag{15}$$

The differential system (15) on the vertical axis $u_1 = 0$ has three singular points $(0, 0)$ and $(0, \pm\sqrt{-a})$ if $a < 0$, and only the singular point $(0, 0)$ if $a \geq 0$.

Using Theorems 2.15 and 2.19 of [10] we obtain that the singular point $(0, 0)$ is: a hyperbolic saddle if $a < -1$; a semihyperbolic unstable node if $a = -1$; a hyperbolic unstable node if $-1 < a < 0$; a semihyperbolic saddle if $a = 0$, and a hyperbolic saddle if $a > 0$. While the two singular points $(0, \pm\sqrt{-a})$ are hyperbolic saddles.

With these information going back through the blow ups we obtain Fig. 3, and the proposition is proved. □

3.4 Phase portraits in the Poincaré disc

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From Propositions 8, 9 and 10 we obtain the following for the differential system (4).

Assume that $0 < b < (1 + a)^2/4$ and $a - b \geq 0$. Then the unique finite singular point is the nilpotent center $(0, 0)$ and system (4) has no infinite singular points. Then the local center at the origin is a global center using the arguments of system (3) when $b > (1 - a)^2/4$, see Fig. 1a.

Assume that $0 < b < (1 + a)^2/4$ and $a - b < 0$. Then the unique finite singular point is the nilpotent center $(0, 0)$ and system (4) has the two infinite singular points $s_\pm$ thar are semihyperbolic saddle-nodes. Then using the first integral of system (4) in this case we obtain the phase portrait of system (4) is given in Fig. 1b.

Assume that $0 < b = (1 + a)^2/4$. Then the unique finite singular point is the nilpotent center $(0, 0)$ and system (4) has the two infinite singular points $s_\pm$ that are semihyperbolic saddle-nodes. Then using the first integral of system (4) in this case we obtain the phase portrait of system (4) is given in Fig. 1b.

Assume that $b = 0$ and $a \geq 0$. Then the unique finite singular point is the nilpotent center $(0, 0)$ and the unique infinite singular points are the origins of the local charts U_2 and V_2, whose local phase portraits are given in Fig. 1c. The arguments used in the case $0 < b < (1 + a)^2/4$ and $a - b \geq 0$ show that the $(0, 0)$ is a global center, see Fig. 1c.

Assume that $b = 0$ and $-1 \leq a < 0$. Then the unique finite singular point is the nilpotent center $(0, 0)$ and the unique infinite singular points are the origins of the local charts U_2 and V_2, whose local phase portraits are given in Fig. 1b. In this case we have the two invariant straight lines $x = \pm\sqrt{-1/a}$. So the phase portrait of system (4) is given in Fig. 1k.

Assume that $b = 0$ and $a < -1$. Then the unique finite singular point is the nilpotent center $(0, 0)$ and the infinite singular points are the origins of the local charts U_2 and V_2, whose local phase portraits are given in Fig. 1a, together with the two infinite hyperbolic nodes, $p_{\pm}$ in the chart U_1 and their two symmetric with respect to the origin in the chart V_1. Since in this case system (4) also has the invariant straight lines $x = \pm\sqrt{-1/a}$ its phase portrait is shown in Fig. 1l.

Assume that $b < 0$ and $a > b$. Then system (4) has three finite singular points the center $(0, 0)$, and the two saddles $P_{\pm}$, and the infinite singular points are the two hyperbolic saddles, $q_{\pm}$ in the chart U_1 and their two symmetric with respect to the origin in the chart V_1. Then, using the first integral of system (4) in this case, the phase portrait of system (4) is given in Fig. 1m.

Assume that $b < 0$ and $a \le b$. In this case the proof is exactly the proof of the previous case $b < 0$ and $a > b$.

In summary, Theorem 2 is proved when the differential system (1) is equivalent to the differential system (4) of Proposition 1.

4 Analysis of system (5)

Now we continue the study of the differential system (5) and we divide it into four subsections.

4.1 First integrals

Proposition 11 *The fist integral $H = H(x, y)$ of the differential system (5) are:*

(a) If $a \neq 1/4$, then

$$H(x, y) = -2\coth^{-1}\left(\frac{\sqrt{4a-1}(1+ay^2)}{1+a(2x^2+y^2)}\right) + \sqrt{4a-1}\log\left(ax^4 + x^2(1+ay^2) + (1+ay^2)^2\right).$$

(b) If $a = 1/4$, then

$$H(x, y) = \frac{4+y^2}{4+2x^2+y^2} + \log\left(4+2x^2+y^2\right).$$

The proof of Proposition 11 is essentially the proof of Proposition 3.

4.2 The finite singular points

Proposition 12 *The finite singular points of the differential system (5) are:*

(a) the nilpotent center at the origin $(0, 0)$; and
(b) if $a < 0$ the two additional $P_{\pm} = \left(0, \pm\sqrt{\frac{-1}{a}}\right)$ nilpotent saddles.

Proof It is easy to check that the singular points are the origin and $P_{\pm}$ when $a < 0$. Since the Jacobian matrix of the differential system (5) at the singular points $P\pm$ is a nilpotent matrix, by applying Theorem 3.5 of [10] these two singularities are nilpontent saddles. □

4.3 The infinite singular points

Proposition 13 *The infinite singular points in the local chart U_1 of system (5) are:*

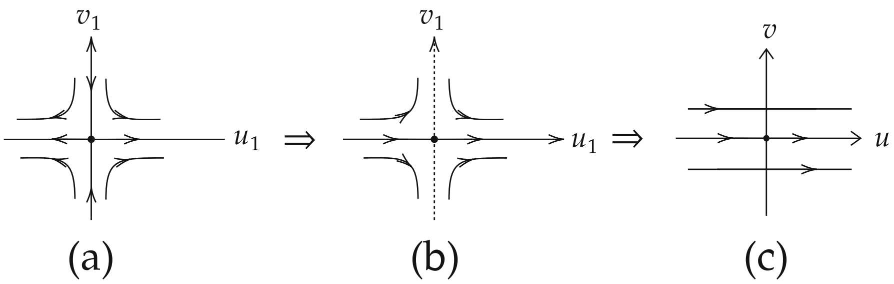

Fig. 4 The blow up of the infinite singular point at the origin of the local chart U_2 of system (17): **a** correspond to differential system (19), **b** correspond to differential system (18) and **c** correspond to differential system (17), i.e to the local phase portrait at the origin of the chart U_2

(a) no infinite singular points if $a \geq 0$; and

(b) two infinite singular points $p_\pm = (\pm\sqrt{-(1+\sqrt{1-4a})/(2a)}, 0)$ if $a < 0$, and p_- is hyperbolic stable node and p_+ is hyperbolic unstable node.

Proof From Subsect. 7.3 the Poincaré compactification of system (5) in the local chart U_1 is

$$\dot{u} = -1 - u^2 - au^4 - u^2v^2, \quad \dot{v} = -uv(1 + au^2 + v^2). \tag{16}$$

The possible infinite singular points of the differential system (16) are $p_\pm$ when $a < 0$ and the points $q_\pm = (\pm(\sqrt{(-1+\sqrt{1-4a})/(2a)}), 0)$, the points $q_\pm$ are not real for all $a \in \mathbb{R}$. So the statement (a) is proved.

For $a < 0$ the only real points are $p_\pm$, and the eigenvalues of the Jacobian matrix of the differential system (16) are $-\sqrt{2}/\sqrt{-((1+\sqrt{1-4a})/a)}$ and $-\sqrt{2-8a}\sqrt{-(1+\sqrt{1-4a})/a}$ for p_-, and $\sqrt{2}/\sqrt{-((1+\sqrt{1-4a})/a)}$ and $\sqrt{2-8a}\ \sqrt{-(1+\sqrt{1-4a})/a}$ for p_+. Therefore by Theorem 2.15 of [10] p_- is hyperbolic stable node and p_+ is hyperbolic unstable node. This proves the statement (b). □

Proposition 14 *The origin of the local chart U_2 of system (5) is an infinite singular point if and only if $a = 0$. Its local phase portrait is given in Fig. 4c.*

Proof From Subsect. 7.3, the Poincaré compactification of system (5) in the local chart U_2 is

$$\dot{u} = a + u^2 + v^2 + u^4, \quad \dot{v} = u^3v. \tag{17}$$

Thus the origin of U_2 is an infinite singular point if and only if $a = 0$. Then the Jacobian matrix of the differential system (17) at the origin is identically zero. In order to study the local phase portrait of the origin in U_2, we shall use vertical blow ups, see Subsect. 7.2 of the Appendix.

Since $u = 0$ is not a characteristic direction, we do a vertical blow up, i.e. the change of variables $(u, v) = (u_1, u_1v_1)$. In the new variables (u_1, v_1), system (17) writes

$$\dot{u_1} = u_1^2(1 + u_1^2 + v_1^2), \quad \dot{v_1} = -u_1v_1(1 + v_1^2). \tag{18}$$

Doing a rescaling of the time variable we eliminate the common factor u_1 between $\dot{u_1}$ and $\dot{v_1}$, and we get the differential system

$$\dot{u_1} = u_1(1 + u_1^2 + v_1^2), \quad \dot{v_1} = -v_1(1 + v_1^2). \tag{19}$$

The differential system (19) on the vertical axis $u_1 = 0$ has only the origin as singular point, and the eigenvalues of the Jacobian matrix of the differential system (19) at the origin are ± 1, so by Theorem 2.15 of [10] the singular point $(0, 0)$ is a hyperbolic saddle.

Putting together these information and undoing the blow ups we obtain Fig. 4 and the proof is completed. □

4.4 Phase portraits in the Poincaré disc

Combining the information of Propositions 11, 12, 13 and 14 we obtain the following for the differential system (5).

Assume that $a > 0$. Then the unique finite singular point is the nilpotent center $(0, 0)$ and the system (5) has no infinite singular points. In this case the phase portrait is given in Fig. 1a.

Assume that $a = 0$. Then the unique finite singular point is the nilpotent center $(0, 0)$ and the infinite singular points are the origins of the local chart U_2 and V_2 whose local phase portraits are formed by two hyperbolic sectors. Then the behavior of system (5) in the Poincaré disc in this case is the one of Fig. 1b.

Assume that $a < 0$. Then system (5) has the nilpotent center $(0, 0)$ and the nilpotent saddles $P_{\pm}$ and two infinite points $p_{\pm}$ in the chart U_1, p_- is a stable node and p_+ is an unstable node, and their two symmetric with respect to the origin in the chart V_1. In this case taking into account the first Integral of system (5) the phase portrait is the same as the one of Fig. 1m.

In summary, Theorem 2 is proved when the differential system (1) is equivalent to the differential system (5) of Proposition 1.

5 Analysis of system (6)

Here the work is essentially the same then for system (5), and as before we have four subsections.

5.1 First integrals

Proposition 15 *The first integral $H = H(x, y)$ of the differential system (6) are:*

(a) If $a \neq 1/4$, then

$$H(x, y) = -2\coth^{-1}\left(\frac{\sqrt{4a-1}(1+ay^2))}{-1+2ax^2-ay^2}\right) - (\sqrt{4a-1})\log\left(ax^4 - x^2(1+ay^2) + (1+ay^2)^2\right).$$

(b) If $a = 1/4$, then

$$H(x, y) = 4 + y^2 + (4 - 2x^2 + y^2)\log\left(4 - 2x^2 + y^2\right).$$

The proof of Proposition 15 is essentially the proof of Proposition 3.

5.2 The finite singular points

Proposition 16 *The finite singular points of the differnetial system (6) are:*

(a) the nilpotent center at the origin $(0, 0)$*; and*

(b) if $a < 0$ *the two additional* $P_\pm = \left(0, \pm\sqrt{\frac{-1}{a}}\right)$ *nilpotent saddles.*

The proof of Proposition 16 is the same than the proof of Proposition 12.

5.3 The infinite singular points

Proposition 17 *The infinite singular points in the local chart* U_1 *of system (6) are:*

(a) no infinite singular points if $a > 1/4$*;*

(b) two infinite singular points $p_\pm = \left(\pm\sqrt{2}/\sqrt{1+\sqrt{1-4a}}, 0\right)$ *if* $a \le 1/4$*,* p_- *is a hyperbolic stable node and* p_+ *is a hyperbolic unstable node when* $a < 1/4$*, and* $p_\pm$ *are semihyperbolic saddle nodes when* $a = 1/4$*;*

(c) two infinite singular points $q_\pm = \left(\pm\sqrt{(1+\sqrt{1-4a})/(2a)}, 0\right)$ *if* $0 < a < 1/4$*, and the points* $q_\pm$ *are hyperbolic saddles.*

Proof From Subsect. 7.3 the Poincaré compactification of the differential system (6) in the local chart U_1 is

$$\dot{u} = -1 + u^2 - au^4 - u^2v^2, \quad \dot{v} = -uv(-1 + au^2 + v^2). \tag{20}$$

The possible infinite singular points of the differential system (20) are $p_\pm$ and $q_\pm$. But $p_\pm$ are real if and only if $a \le 1/4$, meanwhile $q_\pm$ are real if and only if $0 < a \le 1/4$. Note that $p_\pm = q_\pm$ if $a = 1/4$. So statement (a) is proved.

For $0 < a < 1/4$, the eigenvalues of the Jacobian matrix of the differential system (20) at p_- are $-\sqrt{1+\sqrt{1-4a}}/\sqrt{2}$ and $-2\sqrt{2}(1+\sqrt{1-4a}-4a)/(1+\sqrt{1-4a})^{\frac{3}{2}}$ and at p_+ are $\sqrt{1+\sqrt{1-4a}}/\sqrt{2}$ and $2\sqrt{2}(1+\sqrt{1-4a}-4a)/(1+\sqrt{1-4a})^{\frac{3}{2}}$. So by applying Theorem 2.15 of [10], we obtain that $p_\pm$ are hyperbolic nodes, unstable in the case of p_- and stable for p_+. In a similar way for $0 < a < 1/4$, by using the same theorem we get that points $q_\pm$ are hyperbolic saddles. This proves statement (b) for $a < 1/4$ and statement (c) for $0 < a < 1/4$.

In order to complete the proof of statement (b) we assume that $a = 1/4$. Then the second eigenvalue of both points $p_\pm$ are zero, therefore these two singular points are semihyperbolic. By using Theorem 2.19 of [10] we obtain that $p_\pm$ are saddle nodes. And this completes the proof of the proposition. □

Proposition 18 *The origin of the local chart* U_2 *of system (6) is an infinite singular point if and only if* $a = 0$*. Its local phase portrait is given in Fig. 5c.*

Proof From Subsect. 7.3 the Poincaré compactificaton of system (6) in the local chart U_2 is

$$\dot{u} = a - u^2 + u^4 + v^2, \quad \dot{v} = u^3v. \tag{21}$$

Therefore the origin of U_2 is an infinite singular point if and only if $a = 0$. Since the Jacobian matrix of the differential system (21) at the origin is identically zero we proceed to use vertical blow ups.

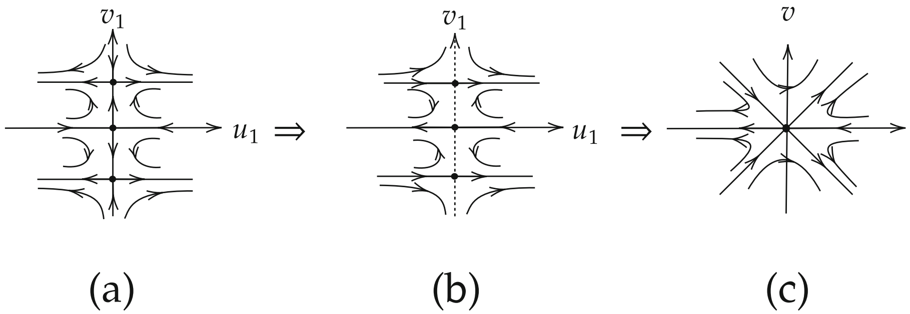

Fig. 5 The blow up of the infinite singular point at the origin of the local chart U_2 system (21): **a** correspond to differential system (23), **b** correspond to differential system (22) and (*c*) correspond to differential system (21), i.e to the local phase portrait at the origin of the chart U_2

Since $u = 0$ is not a characteristic direction we do the vertical blow up $(u, v) = (u_1, u_1 v_1)$. Rewriting system (21) in the variables (u_1, v_1) we have

$$\dot{u_1} = u_1^2(-1 + u_1^2 + v_1^2), \quad \dot{v_1} = -u_1 v_1(v_1 - 1)(1 + v_1). \tag{22}$$

If we eliminate the common factor u_1 between $\dot{u_1}$ and $\dot{v_1}$ by doing a rescale of the time variable, we get the differential system

$$\dot{u_1} = u_1(-1 + u_1^2 + v_1^2), \quad \dot{v_1} = -v_1(v_1 - 1)(1 + v_1). \tag{23}$$

So the differential system (23) has three singular points on the vertical axis: (0, 0) and (0, ±1). By using Theorem 2.15 and 2.19 of [10] we get that (0, 0) is a hyperbolic saddle and the points (0, ±1) are semihyperbolic saddles.

Therefore, using these information and undoing the blow ups we obtain Fig. 5 and this finish the proof. □

5.4 Phase portraits in the Poincaré disc

Combining the information of Propositions 15, 16, 17 and 18 we get the following for the differential system (6).

Assume that $a > 1/4$. Then the unique finite singular point is the nilpotent center and there is no infinite singular point. In this case the phase portrait in the Poincaré disc is given in Fig. 1a.

Assume that $a = 1/4$. Then the unique finite singular point is the nilpotent center, and at infinity system (6) has the infinite semihyperbolic saddles nodes $p_{\pm}$. In this case using the first initegral of system (6) the phase portrait in the Poincaré disc is given in the Fig. 1b.

Assume that $0 < a < 1/4$. Then the unique finite singular point is the nilpotent center and in the local chart U_1 there are four infinite singular points the $p_{\pm}$ and $q_{\pm}$, p_- is a stable node and p_+ is an unstable node, and $q_{\pm}$ are saddles, and of course there are their symmetric with respect to the origin in the chart V_1. In this case using the first integral of system (6) the phase portrait in the Poincaré disc is given in the Fig. 1d.

Assume that $a = 0$. Then the only finite singular point is the nilpotent center at (0, 0). While the infinite singular points are located at the origins of the local charts U_2 and V_2. Their corresponding local phase portraits are displayed in Fig. 5c, in the local chart U_1 there are the two infinite hyperbolic nodes $p_{\pm}$, p_- is a stable node and p_+ is an unstable node, and their symmetric counterparts with respect to the origin in the chart V_1. In this case system (6) has the invariant straight lines $x = \pm 1$. Using these informations the phase portrait in the Poincaré disc is the one of Fig. 1l.

Assume that $a < 0$. Then system (6) has three finite singular points, the nilpotent center $(0, 0)$ and the two saddles $P_\pm$. At infinity system (6) has the two nodes $p_\pm$, p_- is a stable node and p_+ is an unstable node. In this case using the first integral of system (6) the phase portrait in the Poincaré disc is given in the Fig. 1m.

In summary, Theorem 2 is proved when the differential system (1) is equivalent to the differential system (6) of Proposition 1.

6 Analysis of system (7)

Lastly for the differential system (7) we have the following subsections.

6.1 First integrals

Proposition 19 *The function $H(x, y) = x^4 + 2y^2 + ay^4$ is a first integral of the differential system (7).*

The proof of Proposition 19 is the same than the proof of Proposition 3.

6.2 The finite singular points

Proposition 20 *The finite singular points of the differential system (7) are:*

(a) the nilpotent center at the origin $(0, 0)$; and
(b) if $a < 0$ the two additional $P_\pm = \left(0, \pm\sqrt{\frac{1}{-a}}\right)$ nilpotent saddles.

The proof of Proposition 20 follows from the proof of Proposition 12.

6.3 The infinite singular points

Proposition 21 *The infinite singular points in the local chart U_1 of system (7) are:*

(a) no infinite singular points if $a \geq 0$; and
(b) two infinite points $p_\pm = \left(\pm(\frac{1}{-a})^{\frac{1}{4}}, 0\right)$ if $a < 0$, and p_- is a hyperbolic stable node and p_+ is a hyperbolic unstable node.

The proof of Proposition 21 essentially follows the steps of the previous propositions about the infinite singular points in the local chart U_1.

Proposition 22 *The origin of the local chart U_2 of system (7) is an infinite singular point if and only if $a = 0$. Its local phase portrait is given in Fig. 4c.*

Proof From Subsect. 7.3 the Poincaré compactification of the differential system (7) in the local chart U_2 is

$$\dot{u} = a + v^2 + u^4, \quad \dot{v} = u^3 v. \tag{24}$$

Therefore the origin of U_2 is an infinite singular point if and only if $a = 0$. Since the Jacobian matrix of the differential system (24) at the origin is identically zero, we study the local phase portrait at this point using vertical blow ups.

Since $u = 0$ is not a characteristic direction we shall do the vertical blow up $(u, v) = (u_1, u_1 v_1)$. Therefore the differential system (24) in the new coordinates is

$$\dot{u_1} = u_1^2(u_1^2 + v_1^2), \quad \dot{v_1} = -u_1 v_1^3. \tag{25}$$

Doing a rescaling of the time variable we are able to eliminate the common factor u_1 between $\dot{u_1}$ and $\dot{v_1}$, and we get

$$\dot{u_1} = u_1(u_1^2 + v_1^2), \quad \dot{v_1} = -v_1^3. \tag{26}$$

The differential system (26) on the vertical axis $u_1 = 0$ has only the origin $(0, 0)$ as singular point. However, the Jacobian matrix of the differential system (26) at the origin is again identically zero.

Since $u_1 = 0$ is a characteristic direction of system (26), in order to do a vertical blow up, we first need to do a twist, i.e. the change of coordinates $(u_1, v_1) = (u_2 - v_2, v_2)$. In the new variables (u_2, v_2) the differential system (26) writes

$$\dot{u_2} = u_2^3 - 3u_2^2 v_2 + 4u_2 v_2^2 - 3v_2^3, \quad \dot{v_2} = -v_2^3. \tag{27}$$

As before we do the vertical blow up $(u_2, v_2) = (u_3, u_3 v_3)$, and we get the system

$$\dot{u_3} = -u_3^3(3v_3 - 1 - 4v_3^2 + 3v_3^3), \quad \dot{v_3} = u_3^2 v_3(v_3 - 1)(1 - 2v_3 + 3v_3^2). \tag{28}$$

We proceed the analysis by eliminating the common factor u_3^2, by rescaling the time variable, getting the system

$$\dot{u_3} = -u_3(3v_3 - 1 - 4v_3^2 + 3v_3^3), \quad \dot{v_3} = v_3(v_3 - 1)(1 - 2v_3 + 3v_3^2). \tag{29}$$

The infinite singular points of the differential system (29) which lies on the vertical axis $u_3 = 0$ are the points $(0, 0)$ and $(0, 1)$, and by Theorem 2.15 of [10] we obtain that both points are hyperbolic saddles. Therefore undoing the blow ups the local phase portrait of the differential system (24) at the origin is given in Fig. 4c. □

6.4 Phase portraits in the Poincaré disc

The Propositions 20, 21 and 22 together with the first integral of the system provide the necessary information for obtain the phase portrait in the Poincaré disc for the differential system (7).

Assume that $a > 0$. Then the unique finite singular point is the nilpotent center $(0, 0)$ and system (7) has no infinite singular points. In this case the phase portrait in the Poincaré disc is the one of Fig. 1a.

Assume that $a = 0$. Then the unique finite singular point is the nilpotent center $(0, 0)$ and the infinite singular points are the origins of the local chart U_2 and V_2, the phase portrait of system (7) in the Poincaré disc in this case is given in Fig. 1c.

Assume that $a < 0$. Then system (7) has the nilpotent center $(0, 0)$ and the nilpotent saddles $P_\pm$ and two infinite singular points $p_\pm$ in the chart U_1, p_- is a stable node and p_+ is an unstable node, and their two symmetric with respect to the origin in the chart V_1. In this case the phase portrait in the Poincaré disc is the one of Fig. 1m.

In summary, Theorem 2 is proved when the differential system (1) is equivalent to the differential system (7) of Proposition 1. This completes the proof of Theorem 2.

7 Appendix

In this appendix we summarize some basic results that we use in this paper.

7.1 Singular points

A point (x_0, y_0) is a singular point of the differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, if $P(x_0, y_0) = Q(x_0, y_0) = 0$. The singular (x_0, y_0) is *hyperbolic* if all eigenvalues of the Jacobian matrix of the differential system evaluated at (x_0, y_0) have real part different from zero, and it is *semi-hyperbolic* if the Jacobian matrix presents only one eigenvalue equal to zero. The singular (x_0, y_0) is *nilpotent* if both eigenvalues of the Jacobian matrix of the differential system evaluated at (x_0, y_0) are zero, but the Jacobian matrix is not identically zero.

The classification of the local phase portraits of the hyperbolic, semi-hyperbolic and nilpotent singular points can be found in Theorems 2.15, 2.19 and 3.5 (or [2]) of [10], respectively.

When the Jacobian matrix evaluated at a singular point is identically zero, the local phase portrait at this singular point must be analysed through special changes of variables called blow up's (for more details on the blow up method, see for instance [1]).

7.2 Vertical blow up

Consider a real planar polynomial differential system given by

$$\dot{x} = P(x, y) = P_n(x, y) + \ldots, \quad \dot{y} = Q(x, y) = Q_n(x, y) + \ldots, \tag{30}$$

with P and Q being coprime polynomials, P_n and Q_n being homogeneous polynomials of degree $n \in \mathbb{N}$ and the dots representing higher order terms in x and y. Since $n > 0$ the origin is a singular point of system (30). Then the *characteristic directions* at the origin are given by the straight lines trought the origin defined by the real linear factors of the homogeneous polynomial $P_n(x, y)y - Q_n(x, y)x$. It is known that the orbits that start or end at the origin start or end tangent to the straight lines given by the characteristic directions. For more details on the characteristic directions see for instance [3].

Suppose that we have a singular point at the origin of coordinates, as in the differential system (30) and that this singular point is linearly zero. Then for studying its local phase portrait we will do vertical blow up's.

We define the vertical blow up in the y direction as the change of variables $(u, v) = (x, y/x)$. This change transforms the origin of system (30) in the straight line $u = 0$, analyzing the dynamics of the differential system $(\dot{u}, \dot{v})$ in a neighbourhood of this straight line we are analyzing the local phase portrait of the singular point at the origin of system (30). But before doing a vertical blow up in order that we do not lost information we must avoid that the direction $x = 0$ be a characteristic direction of the origin of system (30). If $x = 0$ is a characteristic direction we do a convenient twist $(x, y) = (u, u + \alpha v)$ with $\alpha \neq 0$ in order that the new vertical straight line $u = 0$ not be a characteristic direction.

7.3 The Poincaré compactification

In order to study the dynamics of a polynomial differential system in the plane $\mathbb{R}^2$ near infinity we need its Poincaré compactification. This tool was created by Poincaré in [17], for more details see Chapter 5 of [10].

Consider the polynomial differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{31}$$

where P and Q are polynomial being d the maximum of the degrees of the polynomials P and Q.

We consider the plane $\{(x_1, x_2, 1); x_1, x_2 \in \mathbb{R}\}$ of $\mathbb{R}^3$ identified with the plane $\mathbb{R}^2$, where we have the differential system (31). This plane is tangent at the nord pole $(0, 0, 1)$ of the 2-dimensional sphere $\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 1\}$. We define the northern hemisphere $H_+ = \{(x_1, x_2, x_3) \in \mathbb{S}^2; x_3 > 0\}$, the southern hemisphere $H_- = \{(x_1, x_2, x_3) \in \mathbb{S}^2; x_3 < 0\}$ and the equator $\mathbb{S}^1 \equiv \{(x_1, x_2, x_3) \in \mathbb{S}^2; x_3 = 0\}$ of the sphere $\mathbb{S}^2$.

Let $f^{\pm} : \mathbb{R}^2 \longrightarrow H_{\pm}$ be the central projections from the tangent plane $\mathbb{R}^2$ at the point $(0, 0, 1)$ of the sphere $\mathbb{S}^2$ to $\mathbb{S}^2$ given by

$$f^{\pm}(x_1, x_2) = \pm\left(\frac{x_1}{\Delta(x_1, x_2)}, \frac{x_2}{\Delta(x_1, x_2)}, \frac{1}{\Delta(x_1, x_2)}\right)$$

where $\Delta(x_1, x_2) = \sqrt{x_1^2 + x_2^2 + 1}$. In other words $f^{\pm}(x_1, x_2)$ is the intersection of the straight line through the points $(0, 0, 0)$ and $(x_1, x_2, 1)$ with $H_{\pm}$. Moreover, the maps $f^{\pm}$ induces over $H_{\pm}$ vector fields analytically conjugate with the vector field of the differential system (31). Indeed, f^+ induces on $H_+ = U_3$ the vector field $X_1(y) = Df^+(\varphi_3(y))X(\varphi_3(y))$, and f^- induces on $H_- = V_3$ the vector field $X_2(y) = Df^-(\psi_3(y))X(\psi_3(y))$. Note that $f^+ = \varphi_3^{-1}$ and $f^- = \psi_3^{-1}$. Thus we obtain a vector field on $\mathbb{S}^2 \backslash \mathbb{S}^1$ that admits an analytic extension $p(X)$ on $\mathbb{S}^2$. The vector field $p(X)$ on $\mathbb{S}^2$ is called the *Poincaré compactification* of the vector field $X = (P, Q)$.

In order to study a vector field over $\mathbb{S}^2$ we consider six local charts that cover the whole sphere $\mathbb{S}^2$. So, for $i = 1, 2, 3$, let

$$U_i = \{(x_1, x_2, x_3) \in \mathbb{S}^2; x_i > 0\} \text{ and } V_i = \{(x_1, x_2, x_3) \in \mathbb{S}^2; x_i < 0\}.$$

Consider the diffeomorphisms $\varphi_i : U_i \longrightarrow \mathbb{R}^2$ and $\psi_i : V_i \longrightarrow \mathbb{R}^2$ given by

$$\varphi_i(x_1, x_2, x_3) = \psi_i(x_1, x_2, x_3) = \left(\frac{x_j}{x_i}, \frac{x_k}{x_i}\right)$$

with $j, k \neq i$ and $j < k$. The sets (U_i, φ_i) and (V_i, ψ_i) are the *local charts* over $\mathbb{S}^2$.

Denote $(u, v) = \varphi_i(x_1, x_2, x_3) = \psi_i(x_1, x_2, x_3)$. Then the expression of the differential system associated to the vector field $p(X)$ in the chart U_1 is

$$u' = v^d\left[Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right], v' = -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right).$$

The expression of $p(X)$ in U_2 is

$$u' = v^d\left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right], v' = -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right).$$

The expression of $p(X)$ in U_3 is

$$u' = P(u, v), v' = Q(u, v).$$

For $i = 1, 2, 3$ the expression of $p(X)$ in the chart V_i differs of the expression in U_i only by the multiplicative constant $(-1)^{d-1}$.

Note that we can identify the infinity of $\mathbb{R}^2$ with the equator $\mathbb{S}^1$. Two points for each direction in $\mathbb{R}^2$ provide two antipodal points of $\mathbb{S}^1$. A singular point of $p(X)$ on $\mathbb{S}^1$ is called *infinite singular point* and a singular point on $\mathbb{S}^2 \backslash \mathbb{S}^1$ is called a *finite singular point*. Observe that the coordinates of the infinite singular points are of the form $(u, 0)$ on the charts U_1, V_1, U_2 and V_2. Thus, if $(x_1, x_2, 0) \in \mathbb{S}^1$ is an infinite singular point, then its antipode $(-x_1, -x_2, 0)$ is also a infinite singular point.

The image of the closed northern hemisphere of $\mathbb{S}^2$ under the projection $(x_1, x_2, x_3) \rightarrow (x_1, x_2, 0)$ is the *Poincaré disc*, denoted by $\mathbb{D}^2$.

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