DEGREE OF EQUIVARIANT MAPS BETWEEN GENERALIZED G-MANIFOLDS

NORBIL CORDOVA, DENISE DE MATTOS, AND EDIVALDO L. DOS SANTOS

Abstract. Yasuhiro Hara in [10] and Jan Jaworowski in [11] studied, under certain conditions, the degree of equivariant maps between free G-manifolds, where G is a compact Lie group. The main results obtained by them involve data provided by the classifying maps of the orbit spaces. In this paper, we extend these results, by replacing the free G-manifolds by free generalized G-manifolds over $\mathbb{Z}$.

1. Introduction

Let M, N be orientable, connected, compact n-manifolds. By Poincaré duality theorem, the cohomology groups $H^n(M;\mathbb{Z})$ and $H^n(N;\mathbb{Z})$ are infinite cyclic with preferred generators $[M]$ and $[N]$, called fundamental classes of M and N, respectively. Given a continuous map $f : M \to N$, the *degree* of f, denoted by $\deg(f)$, is the integer satisfying the condition

$$f^*[N] = \deg(f)[M],$$

where $f^* : H^n(N;\mathbb{Z}) \to H^n(M;\mathbb{Z})$ is the homomorphism induced by f.

In [10] and [11], Y. Hara and J. Jaworowski independently applied the transfer homomorphism to study the degree of *equivariant maps*[1] $f : M \to N$ by considering both M and N as free G-manifolds, where G is a compact Lie group. To achieve these results, the first author used the definition of ideal-valued cohomological index, introduced by Fadel and Husseini in [9] and the second author used the definition of "G-class". Given a compact Lie group G, let us consider $EG \to BG$ the universal principal G-bundle, where BG is the classifying space of G. Let X be a paracompact free G-space and let $q_X : X/G \to BG$ be a classifying map for the free G-action on X.

The *G-index of Fadell and Husseini of* X, denoted by $\mathrm{Ind}^G(X;\Bbbk)$, is defined to be the kernel of the homomorphism

$$q_X^* : H^*(BG;\Bbbk) \to H^*(X/G;\Bbbk)$$

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[1]A continuous map $f : X \to Y$ between G-spaces is called an equivariant map if $f(gx) = gf(x)$, for all $x \in X$ and for all $g \in G$.

induced by q_X, where $H^*(-;\Bbbk)$ is the Alexander-Spanier total cohomology with coefficients in some field $\Bbbk$. This G-index is a graded ideal in the graded ring $H^*(BG;\Bbbk)$, then for an integer r, we set

$$\begin{aligned}\mathrm{Ind}_r^G(X;\Bbbk) &= \mathrm{Ind}^G(X;\Bbbk)\cap H^r(BG;\Bbbk)\\ &= \ker(q_X^*: H^r(BG;\Bbbk)\longrightarrow H^r(X/G;\Bbbk)).\end{aligned} \tag{1}$$

The *G-class subgroup*, denoted by $\Gamma_r^G(X;\Bbbk)$, is defined to be the image of homomorphism

$$q_X^*: H^{r-\dim G}(BG;\Bbbk)\to H^{r-\dim G}(X/G;\Bbbk)$$

induced from q_X. Its elements are called *G-class*.

Consequently, if $f: X\to Y$ is an equivariant map, then

$$\mathrm{Ind}_r^G(Y;\Bbbk)\subset \mathrm{Ind}_r^G(X;\Bbbk)\quad\text{and}\quad \Gamma_r^G(X;\Bbbk)\cong \bar{f}^*(\Gamma_r^G(N;\Bbbk)).$$

In this paper, we study the degree of equivariant maps between generalized manifolds, obtaining an extension of the Hara and Jaworowski's results.

Generalized manifolds are a class of spaces which reflect many of the local and global homology properties of the topological manifolds. Such manifolds have recently been studied; for example, it has been known that there exist generalized manifolds which are not homotopy equivalent to topological manifolds (see [7, 8]). Specifically, we prove the following

Theorem 1.1. *Let G be a compact Lie group and let X and Y be orientable, connected, compact generalized n-manifolds over $\mathbb{Z}$ on which G acts freely. Let*

$$q_X^*: H^{n-\dim G}(BG;\mathbb{Z}_p)\to H^{n-\dim G}(X/G;\mathbb{Z}_p)$$
$$q_Y^*: H^{n-\dim G}(BG;\mathbb{Z}_p)\to H^{n-\dim G}(Y/G;\mathbb{Z}_p)$$

be induced by the classifying map for X and Y respectively. Assume that $f: X\to Y$ is an equivariant map. Then we have the following:

1. *If $q_X^*\neq 0$ then $f^*: H^n(Y;\mathbb{Z}_p)\to H^n(X;\mathbb{Z}_p)$ is an isomorphism and the degree of f is congruent to 1 modulo p, p prime.*
2. *If $q_X^*=0$ and $q_Y^*\neq 0$, then $f^*: H^n(Y;\mathbb{Z}_p)\to H^n(X;\mathbb{Z}_p)$ is trivial and the degree of f is congruent to zero modulo p, p prime.*
3. *If $q_X^*=0$ and $q_Y^*=0$, then no assertion can be made about the degree of f.*

Corollary 1.2. *Let G be a compact Lie group and let X and Y be orientable, connected, compact generalized n-manifolds over $\mathbb{Z}$ on which G acts freely. Let $f: X\to Y$ be an equivariant map. Then we have the following:*

1. *If either*

$$\mathrm{Ind}_{n-\dim G}^G(X;\mathbb{Z}_p)\neq H^{n-\dim G}(BG;\mathbb{Z}_p),\qquad \textit{or}\qquad \Gamma_n^G(X;\mathbb{Z}_p)\neq 0$$

then $f^*: H^n(Y;\mathbb{Z}_p)\to H^n(X;\mathbb{Z}_p)$ is an isomorphism and the degree of f is congruent to 1 modulo p, p prime .

2. *If either*

$$\mathrm{Ind}_{n-\dim G}^G(Y,\mathbb{Z}_p)\subsetneq \mathrm{Ind}_{n-\dim G}^G(X;\mathbb{Z}_p)= H^{n-\dim G}(BG;\mathbb{Z}_p),$$
$$\textit{or}\quad \Gamma_n^G(X;\mathbb{Z}_p)=0\ \ \textit{and}\ \ \Gamma_n^G(Y;\mathbb{Z}_p)\neq 0$$

then $f^* : H^n(Y;\mathbb{Z}_p) \to H^n(X;\mathbb{Z}_p)$ *is trivial and the degree of* f *is congruent to zero modulo* p*,* p *prime.*

For the first two cases, given p prime, the degree of equivariante map f is $1+pk$ or pk for some $k \in \mathbb{Z}$.

2. Preliminaries

Throughout this paper, L will always denote a principal ideal domain (PID). As usual, we use $\mathbb{Z}$ and $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ to denote the ring of the integers and the prime field of integers modulo $p > 0$, respectively. We assume that all spaces under considerations are Hausdorff and locally compact spaces. Also, we assume familiarity with sheaf theory (see [6]). In particular, the following notations will be adopted:

(1) For a given space X, let $\mathscr{A}$ be a sheaf of L-modules and let Φ be a family of supports on X. We denote by $H^p_\Phi(X;\mathscr{A})$ (respectively, $H_p^\Phi(X;\mathscr{A})$) the sheaf cohomology (respectively, the Borel-Moore homology [2]) of X with support in Φ and with coefficients in $\mathscr{A}$. The symbol of the family cld of all closed subsets of X will be omitted.
(2) The constant sheaf $X \times L$ will be denoted by L, when the context indicates this as a sheaf.

Remark 2.1. In the case of paracompact spaces, the sheaf cohomology with closed supports and constant coefficients agrees with both the Alexander-Spanier cohomology and Čech cohomology.

For a paracompactifying family Φ of supports on X such that the union of all elements of Φ is X, we define the *cohomological dimension* $\dim_L X$ of X as the least integer n (or ∞) so that

$$H^k_\Phi(X;\mathscr{A}) = 0, \text{ for all sheaves } \mathscr{A} \text{ of } L\text{-modules and all } k > n.$$

Of course, this definition is independent of the choice of the paracompactifying family Φ. When X is a paracompact space, the family cld is paracompactifying, then we can take $\Phi = \text{cld}$.

Next, we give the precise definition of generalized manifolds (for more details, see [2], [6] and [13]).

Definition 2.2. A space X is called a *homology n-manifold* over L, if it satisfies the following conditions:

(1) The cohomological dimension $\dim_L X$ of X over L is finite ($\leq n$).
(2) The relative Borel-Moore homology

$$\mathscr{H}_q(X;L)_x \cong H_q(X, X-\{x\};L) = \begin{cases} L, & \text{for } q = n \\ 0, & \text{for } q \neq n. \end{cases}$$

These conditions imply that $\dim_L X = n$, and also that the sheaf of the local homology groups $\mathscr{H}_q(X,L)$ are trivial, for $q \neq n$. Bredon in [4] has proved that the orientation sheaf $\mathscr{O} = \mathscr{H}_n(X,L)$ is locally constant with stalks isomorphic to L. We say that X is orientable if $\mathscr{O}$ is constant. By [2, Theorem 7.6] or [6, Chapter V, Theorem 9.3, p. 330], the homology manifolds satisfy the Poincaré duality theorem, that is, if X is a homology n-manifold over L, then for any family of supports Φ on X,

$$H^p_\Phi(X;\mathscr{O}) \cong H^\Phi_{n-p}(X;L).$$

Definition 2.3. A (orientable) generalized n-manifold over L, also called *cohomology n-manifold* in sense of Borel (see [1, Chapter I, § 3, Definition 3.3, p. 9]), is a (orientable) homology n-manifold over L which is also clc_L (cohomology locally connected, see [6, Chapter II, Definition 17.1, p. 126]).

Definition 2.4. Let $f : X \to Y$ be a continuous map. The *Leray sheaf* of f, denoted by $\mathscr{H}^t(f;L)$ is the sheaf over Y generated by the presheaf

$$U \longmapsto H^t(\pi^{-1}(U);L).$$

Definition 2.5. A *cohomology fiber space* over L (abbreviated cfs_L) is a proper and continuous map $f : X \to Y$ such that the Leray sheaf $\mathscr{H}^t(f;L)$ is locally constant for all t. We shall call it a cfs^k_L if $\mathscr{H}^t(f;L) = 0$, for $t > k$ and $\mathscr{H}^k(f;L) \neq 0$.

3. The transfer homomorphism for G-actions

Let G be a compact Lie group acting on a paracompact space X. Let X/G be the orbit space and let $\pi : X \to X/G$ be the orbit map, which is surjective and that take a point x in X into its orbit $\breve{x} = Gx$. It follows from [5, Chapter I, Theorem 3.1, p. 38] that π is a proper and closed map.

By [6, Chapter IV, Proposition 4.2, p. 214], the Leray sheaf $\mathscr{H}^t(\pi;L)$ of π has stalks

$$\mathscr{H}^t(\pi;L)_{\breve{x}} \cong H^t(\pi^{-1}(\breve{x});L) = H^t(Gx;L), \tag{2}$$

for all $\breve{x} \in X/G$, since X is paracompact and π is closed.

For coefficients in the constant sheaf L and closed supports in both X and X/G, there is the Leray spectral sequence for π [6, Chapter IV, Theorem 6.1, p. 222] which has E_2-term

$$E_2^{s,t} = H^s(X/G;\mathscr{H}^t(\pi;L))$$

and converges to $H^{s+t}(X;L)$.

Our purpose this section is to recall the construction of the transfer homomorphism for any compact Lie group G:

$$\tau_X : H^{s+\dim G}(X;L) \longrightarrow H^s(X/G;L). \tag{3}$$

Finite case. For details see [1, Chapter III, § 2, p. 37]. Suppose that G is finite. In this case, $\dim G = 0$ and since Gx is finite $H^t(Gx;L) = 0$, for $t \neq 0$ and hence $\mathscr{H}^t(\pi;L) = 0$, for $t \neq 0$. Then $E_2^{s,t} = 0$, for $t \neq 0$ and therefore, we have

$$H^s(X;L) \overset{\cong}{\twoheadrightarrow} E_\infty^{s,0} \overset{\cong}{\rightarrowtail} E_2^{s,0} = H^s(X/G;\mathscr{H}^0(\pi;L)).$$

The action of G on X naturally induces an action on the Leray sheaf $\mathscr{H}^0(\pi;L)$ such that the subspace $\left(\mathscr{H}^0(\pi;L)\right)^G$ of fixed points of G on $\mathscr{H}^0(\pi;L)$ is isomorphic to the constant sheaf L over X/G. Let

$$\mu : \mathscr{H}^0(\pi;L) \longrightarrow \left(\mathscr{H}^0(\pi;L)\right)^G \tag{4}$$

be the homomorphism between sheaves on X/G, defined by $\mu = \sum_{g\in G} g$. Therefore, the transfer homomorphism τ_X in (3), when G is finite, is defined as the homomorphism in sheaf cohomology of μ,

$$\tau_X : H^s(X;L) \cong H^s(X/G;\mathscr{H}^0(\pi;L)) \longrightarrow H^s(X/G;L). \tag{5}$$

Infinite case. First, we suppose that G is connected. For this case, let us recall the transfer homomorphism due to Oliver (for details, see [12]).

Let $\eta_x : G \to Gx$ be the equivariant map given by $\eta_x(g) = gx$, for all $g \in G$. Since G is connected, by using (2), the induced homomorphism

$$\mathscr{H}^t(\pi;L)_{\breve{x}} \cong H^t(Gx;L) \xrightarrow{\eta_x^*} H^t(G;L)$$

is independent of x in its orbit $\breve{x}$. Consider $H^t(G;L)$ as the stalk at $\breve{x}$ of constant sheaf $X/G \times H^t(G;L)$. Then, the collection $\{\eta_x^*\}$ determines a homomorphism of sheaves over X/G

$$\eta^* : \mathscr{H}^t(\pi;L) \longrightarrow H^t(G;L). \tag{6}$$

This homomorphism induces the homomorphism in sheaf cohomology

$$\eta^* : H^s(X/G;\mathscr{H}^t(\pi;L)) \longrightarrow H^s(X/G;H^t(G;L)). \tag{7}$$

Therefore the transfer homomorphism τ_X in (3), when G is connected, is defined by the composition of the following homomorphism

$$H^{s+\dim G}(X;L) \twoheadrightarrow E_\infty^{s,\dim G} \rightarrowtail E_2^{s,\dim G} = H^s(X/G;\mathscr{H}^{\dim G}(\pi;L)), \tag{8}$$

with the homomorphism given in (7), for the case that $t = \dim G$.

If G is not necessarily connected, let G_0 the connected component which includes the identity element. It is interesting to note that G_0 is a normal group and G/G_0 is a finite group acting on X/G_0. For the orbit map

$$\pi_0 : X/G_0 \to \frac{X/G_0}{G/G_0} \cong X/G,$$

we have the transfer homomorphism in the finite case and for the orbit map $\pi_{G_0} : X \to X/G_0$ we have the transfer homomorphism in the connected case. The composition of these transfer homomorphisms

$$H^{s+\dim G}(X;L) \longrightarrow H^s(X/G_0;L) \longrightarrow H^s(X/G;L) \tag{9}$$

determines the transfer homomorphism τ_X in (3), when G is not connected.

This transfer homomorphism was built in a natural way, as shows the following

Proposition 3.1. *Let G be a compact Lie group and $f : X \to Y$ be an equivariant map between paracompact G-spaces. Then, the following diagram*

$$\begin{tikzcd} H^{s+\dim G}(Y;L) \arrow[r, "f^*"] \arrow[d, "\tau_Y"'] & H^{s+\dim G}(X;L) \arrow[d, "\tau_X"] \\ H^s(Y/G;L) \arrow[r, "\bar{f}^*"] & H^s(X/G;L) \end{tikzcd}$$

is commutative, where $\bar{f} : X/G \to Y/G$ is the map induced by f between the orbit spaces.

4. Proof of main theorem

If X is a G-space, with G compact, then X/G is Hausdorff and locally compact [5, Chapter I, Theorem 3.1, p. 38]. The proof of the main theorem lies in showing that the transfer homomorphism is an isomorphism.

Lemma 4.1. *Assume that G is a compact Lie group acting freely over an orientable homology n-manifold X over L. If G is connected and L is a field then X/G is an orientable homology $(n - \dim G)$-manifold over L.*

Proof. If G acts freely on X, then the equivariant map $\eta_x : G \to Gx$, given by $\eta_x(g) = gx$, for all $g \in G$ is a homeomorphism, for any $x \in X$. Thus $\dim G \leq n$. Now, if G is connected, then the homomorphism η^* in (6) is an isomorphism of sheaves on X/G, since η_x is a homeomorphism. Therefore, the Leray sheaves $\mathscr{H}^t(\pi; L)$ of the orbit map are constants, with $\mathscr{H}^t(\pi; L) = 0$, for $t > \dim G$ and $\mathscr{H}^{\dim G}(\pi; L) = L$, that is, $\pi : X \to X/G$ is a $\mathrm{cfs}_L^{\dim G}$. It follows from [6, ChapterV, Theorem 18.5, p. 397], that X/G is an orientable homology $(n - \dim G)$-manifold over L. □

If a space Y is compact and connected, then $H_c^0(Y; L) = L$. Let now X be an orientable, connected, compact homology n-manifold over a field L, then $\mathscr{O}$ is isomorphic to L and by Poincaré duality

$$H^n(X; L) \cong H_0(X; L) \cong \operatorname{Hom}(H_c^0(X; L), L) \cong L, \tag{10}$$

where the second isomorphism follows from [2, Theorem 3.3, p. 144].

Proposition 4.2. *Assume that G is a compact Lie group acting freely over an orientable homology n-manifolds X over $\mathbb{Z}_p$. If X is connected and compact, then the transfer homomorphism $\tau_X : H^n(X; \mathbb{Z}_p) \to H^{n-\dim G}(X/G; \mathbb{Z}_p)$ is an isomorphism.*

Proof. For the finite case, we consider the short exact sequence

$$0 \longrightarrow \ker\mu \xrightarrow{\imath} \mathscr{H}^0(\pi; \mathbb{Z}_p) \xrightarrow{\mu} \left(\mathscr{H}^0(\pi; \mathbb{Z}_p)\right)^G \longrightarrow 0 \tag{11}$$

of sheaves on X/G, where μ is defined in (4) and $\imath$ is the inclusion. The exact cohomology sequence

$$\cdots \longrightarrow H^n(X/G; \ker\mu) \longrightarrow H^n(X/G; \mathscr{H}^0(\pi; \mathbb{Z}_p)) \xrightarrow{\tau_X} H^n(X/G; \left(\mathscr{H}^0(\pi; \mathbb{Z}_p)\right)^G)$$
$$\longrightarrow H^{n+1}(X/G; \ker\mu) \longrightarrow H^{n+1}(X/G; \mathscr{H}^0(\pi; \mathbb{Z}_p)) \longrightarrow \cdots$$

of (11) becomes

$$\cdots \longrightarrow H^n(X/G; \ker\mu) \longrightarrow H^n(X/G; \mathscr{H}^0(\pi; \mathbb{Z}_p)) \xrightarrow{\tau_X} H^n(X/G; \mathbb{Z}_p) \longrightarrow 0$$

since that $\dim_{\mathbb{Z}_p} X/G = n$ (See [1, Chapter III, Proposition 5.1, p. 43]. Therefore,

$$\tau_X : H^n(X; \mathbb{Z}_p) \cong H^n(X/G; \mathscr{H}^0(\pi; \mathbb{Z}_p)) \longrightarrow H^n(X/G; \mathbb{Z}_p) \neq 0$$

is surjective and using the fact in (10), we conclude that τ_X is an isomorphism.

Now, we suppose that G is a connected group. As in the proof of Lemma 4.1, the homomorphism η^* in (6) is an isomorphism. Thus, the homomorphism in (7)

$$\eta^* : H^s(X/G; \mathscr{H}^t(\pi; \mathbb{Z}_p)) \longrightarrow H^s(X/G; H^t(G; \mathbb{Z}_p))$$

is an isomorphism, since $H^p(X/G, -)$ is a functor. By Lemma 4.1, X/G is an homology $(n - \dim G)$-manifold, hence $H^s(X/G; \mathscr{H}^t(\pi; \mathbb{Z}_p)) = 0$ for $s > n - \dim G$. Thus, $E_2^{s,t} = H^s(X/G; \mathscr{H}^t(\pi; \mathbb{Z}_p)) = 0$, for $s > n - \dim G$ and for $t > \dim G$. Then, the transfer homomorphism

$$H^n(X; \mathbb{Z}_p) \overset{\cong}{\twoheadrightarrow} E_\infty^{n-\dim G, \dim G} \overset{\cong}{\rightarrowtail} E_2^{n-\dim G, \dim G} \xrightarrow{\eta^*} H^{n-\dim G}(X/G; \mathbb{Z}_p)$$

is a composition of isomorphism, and hence it is an isomorphism.

If G is not connected, we have that τ_X is given by the composition

$$H^n(X; \mathbb{Z}_p) \longrightarrow H^{n-\dim G}(X/G_0; \mathbb{Z}_p) \longrightarrow H^{n-\dim G}(X/G; \mathbb{Z}_p).$$

The fact that the first arrow is an isomorphism follows of the connected case, previously shown. By Lemma 4.1, X/G_0 is an orientable, connected, compact homology $(n-\dim G)$-manifold over $\mathbb{Z}_p$ and it follows from the finite case that the second arrow is an isomorphism, where $\dim G_0 = \dim G$. $\square$

By [6, Chapter V, Theorem 16.16, p. 380], for every orientable, connected, compact generalized n-manifold on $\mathbb{Z}$, the cohomology group $H^n(X; \mathbb{Z})$ is infinite cyclic with preferred generator $[X]$.

Definition 4.3. Let $f : X \to Y$ be a continuous map between orientable, connected, compact generalized n-manifolds over $\mathbb{Z}$. The *degree* of f, denoted by $\deg(f)$, is the integer satisfying the condition

$$f^*[Y] = \deg(f)[X],$$

where $f^* : H^n(Y; \mathbb{Z}) \to H^n(X; \mathbb{Z})$ is the homomorphism induced by f.

The degree $\deg_L(f)$ over any ring L can be defined in exactly the same way. Moreover for $1 \in L$

$$\deg_L(f) := \deg(f) \cdot 1 \tag{12}$$

Remark 4.4. Let X be an (orientable) homology n-manifold over $\mathbb{Z}$. It follows from [6, Chapter II, Theorem 16.15, p. 115] that $\dim_{\mathbb{Z}_p} X \leq \dim_{\mathbb{Z}} X \leq n$. Now, if we consider the exact sequence in [6, Chapter V, (13), p. 294] for the constant sheaf $\mathbb{Z}_p$ we obtain

$$\mathscr{H}_k(X; \mathbb{Z}) \otimes \mathbb{Z}_p \cong \mathscr{H}_k(X; \mathbb{Z}_p). \tag{13}$$

On the stalks at x, we have

$$\mathscr{H}_k(X; \mathbb{Z}_p)_x = 0, \text{ for all } k \neq n \text{ and } \mathscr{H}_n(X; \mathbb{Z}_p)_x \cong \mathbb{Z}_p.$$

Therefore, we conclude that X is a homology n-manifold over $\mathbb{Z}$.

In addition, if X is orientable (over $\mathbb{Z}$), then the orientation sheaf $\mathscr{O} = \mathscr{H}_n(X; \mathbb{Z})$ is the constant sheaf with stalks $\mathbb{Z}$. It follows from (13) that $\mathscr{H}_n(X; \mathbb{Z}_p)$ is the constant sheaf $\mathscr{O} \otimes \mathbb{Z}_p$ (as $\mathbb{Z}_p$-module), with stalks $\mathbb{Z}_p$. Thus, X is also an orientable homology n-manifold over $\mathbb{Z}_p$.

Proof of Theorem 1.1

Proof. It follows from Remark 4.4 that X and Y are orientable, connected, compact homology n-manifolds over $\mathbb{Z}_p$.

Let $q_X : X/G \to BG$ and $q_Y : Y/G \to BG$ be classifying maps for the principal G-bundles $X \to X/G$ and $Y \to Y/G$, respectively. Let us consider the following diagram

$$\begin{tikzcd}
H^n(Y;\mathbb{Z}_p) \arrow[rr, "f^*"] \arrow[d, "\tau_Y"'] & & H^n(X;\mathbb{Z}_p) \arrow[d, "\tau_X"] \\
H^{n-\dim G}(Y/G;\mathbb{Z}_p) \arrow[rr, "(\bar{f})^*"] & & H^{n-\dim G}(X/G;\mathbb{Z}_p) \\
& H^{n-\dim G}(BG;\mathbb{Z}_p) \arrow[ul, "q_Y^*"] \arrow[ur, "q_X^*"'] &
\end{tikzcd} \tag{14}$$

where $\bar{f} : X/G \to Y/G$ is the map induced by f between the orbit spaces. By Proposition 3.1, the rectangle in diagram (14) is commutative. Since q_X is homotopic to $q_Y \circ \bar{f}$, then the triangle is also commutative. Moreover, by Proposition 4.2, the transfer homomorphisms τ_X and τ_Y are isomorphism and hence

$$H^n(X;\mathbb{Z}_p) \cong H^{n-\dim G}(X/G;\mathbb{Z}_p) \cong \mathbb{Z}_p$$
$$H^n(Y;\mathbb{Z}_p) \cong H^{n-\dim G}(Y/G;\mathbb{Z}_p) \cong \mathbb{Z}_p$$

Let u, v be generators of $H^n(Y;\mathbb{Z}_p)$ and $H^n(X;\mathbb{Z}_p)$, respectively. Then, by (12), we have $f^*(v) = \deg(f)\, u$. It follows from the diagram (14) that

$$(\bar{f})^*(\bar{v}) = \deg(f)\, \bar{u} \tag{15}$$

for any generators of $H^{n-\dim G}(Y/G;\mathbb{Z}_p)$ and $H^{n-\dim G}(X/G;\mathbb{Z}_p)$.

(1) If $q_X^* \neq 0$ then $(\bar{f})^* \neq 0$ and $q_Y^* \neq 0$. From diagram (14), we conclude that $f^* \neq 0$. Therefore, f^* is an isomorphism.

Now, let $b \in H^{n-\dim G}(BG;\mathbb{Z}_p)$ such that $q_X^*(b) \neq 0$ and $q_Y^*(b) \neq 0$ and consider $\bar{u} = q_X^*(b)$ and $\bar{v} = q_Y^*(b)$ as the generators of $H^{n-\dim G}(X/G;\mathbb{Z}_p)$ and $H^{n-\dim G}(Y/G;\mathbb{Z}_p)$, respectively. Since

$$(\bar{f})^*(\bar{v}) = (\bar{f})^*(q_Y^*(b)) = q_X^*(b) = \bar{u},$$

it follows from (15) that $(\deg(f))\, \bar{u} = \bar{u}$, and hence $\deg(f) \equiv 1 \mathrm{mod}\ p$.

(2) If $q_X^* = 0$ and $q_Y^* \neq 0$ then $q_Y^*(b) \neq 0$ for some $b \in H^{n-\dim G}(BG;\mathbb{Z}_p)$. Since $(\bar{f})^*(q_Y^*(b)) = 0$ then $(\bar{f})^*$ is not injective. From diagram (14), it follows that f^* is not injective and hence f^* is trivial.

Let u, v be a generators of $H^n(Y;\mathbb{Z}_p)$ and $H^n(X;\mathbb{Z}_p)$, respectively. Then $f^*(v) = 0 = (\deg(f))u$ and hence $\deg(f) \equiv 0 \bmod p$.

□

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Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Departamento de Matemática, Caixa Postal 668, São Carlos-SP, Brazil, 13560-970.

E-mail address: `norbil@icmc.usp.br`
E-mail address: `deniseml@icmc.usp.br`

Universidade Federal de São Carlos, UFSCAR, Departamento de Matemática, Caixa Postal 676, São Carlos-SP, Brazil, 13565-905.

E-mail address: `edivaldo@dm.ufscar.br`