Geometry of the parabolic subset of a generically immersed 3-manifolds in $\mathbb{R}^4$

A. C. Nabarro [*], M. C. Romero Fuster and M. C. Zanardo [†]

Abstract

The parabolic subset of a 3-manifold generically immersed in $\mathbb{R}^4$ is a surface. We analyze in this study the generic geometrical behavior of such surface, considered as a submanifold of $\mathbb{R}^4$. Typical Singularity Theory techniques based on the analysis of the family of height functions are applied in order to describe the geometrical characterizations of the different singularity types.

1 Introduction

The parabolic subset of a 3-manifold M immersed in $\mathbb{R}^4$ is the set of singular points of the Gauss map on M (see, p.e. [3], [6]). For a generically immersed 3-manifold, the parabolic subset is a surface immersed in $\mathbb{R}^4$ with possible isolated singularities corresponding to corank 2 singularities of the Gauss map on M. This study analyzes the generic geometrical behavior of that surface, considered as a submanifold of $\mathbb{R}^4$ (see [1], [3], [4]). Typical Singularity Theory techniques based on the analysis of the family of height functions were applied towards geometrical characterizations of the different singularities and the geometrical behavior of the parabolic subset considered as a submanifold of $\mathbb{R}^4$. Since the corank 2 singularities of the Gauss map are generically isolated singular points at which the parabolic surface is not a smooth submanifold, this analysis was restricted to the (non necessarily closed) surface given by the complement of such points in the parabolic subset.

The results of this study can be applied to that described in [8] which explored the geometry of the canal hypersurfaces of generic curves in $\mathbb{R}^4$.

*Supported by FAPESP grant 2016/19139-7

†Supported by CAPES grant CSF-PVE-S 88887.122686/2016-00

2020 Mathematical subject classification: 58K05, 53B25, 53D10

Key Words and Phrases: Parabolic surface of 3-manifolds in $\mathbb{R}^4$, height functions, geometrical characterizations for local singularities.

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Given an embedded of a hypersurface M in $\mathbb{R}^4$, its parabolic subset is denoted by $\Sigma\Gamma$, and the height functions on M and $\Sigma\Gamma$ in the directions $\mathbf{v}$ for a given unit normal vector $\mathbf{v} \in N_pM$ at $p \in M$ are denoted by $H_\mathbf{v}$ and $\tilde{H}_\mathbf{v}$, respectively.

The aim of this study is to prove the following result:

Theorem 1.1. The following assertions hold for an open and dense set of immersions of a 3-manifold in M in $\mathbb{R}^4$:

(1) A point p is an A_2 singularity of $H_\mathbf{v}$ on M if and only if p is an A_1 singularity of $\tilde{H}_\mathbf{v}$ on $\Sigma\Gamma$,

(2) If p is an A_3 versal singularity of $H_\mathbf{v}$ on M then p is, generically, an A_3 singularity of $\tilde{H}_\mathbf{v}$ on $\Sigma\Gamma$. Moreover, there may be singularities of type A_4 or D_4 at isolated points. If p is an A_3 singularity of $\tilde{H}_\mathbf{v}$, then p is an A_3 versal singularity of $H_\mathbf{v}$.

(3) A point p is an A_4 versal singularity of $H_\mathbf{v}$ on M if and only if p is (generically) an A_4 singularity of $\tilde{H}_\mathbf{v}$ on $\Sigma\Gamma$.

(4) There are no A_2 singularities of $\tilde{H}_\mathbf{v}$ at p, i.e., $\tilde{H}_\mathbf{w}$ may have singularities of type A_2 for $\mathbf{w} \in N_p\Sigma\Gamma$, however, $\mathbf{w} \neq \mathbf{v}$.

Moreover, the parabolic surface $\Sigma\Gamma$ satisfies the following statements:

(5) If p is an A_3 versal singularity of $H_\mathbf{v}$ on M and an A_3 singularity of $\tilde{H}_\mathbf{v}$, then $p \in \Sigma\Gamma$ is either a parabolic, or a hyperbolic point of M. The hyperbolic case occurs if and only if $\tilde{H}_\mathbf{v}$ also has an A_3 versal singularity.

(6) If p is an A_3 versal singularity of $H_\mathbf{v}$ on M and A_4 (resp. D_4) of $\tilde{H}_\mathbf{v}$, then $p \in \Sigma\Gamma$ is a hyperbolic (resp. inflection) point.

(7) If p is an A_4 versal singularity of $H_\mathbf{v}$ on M, then $p \in \Sigma\Gamma$ is a hyperbolic point.

(8) If p is an A_3 versal singularity of $H_\mathbf{v}$ on M and the curvature κ of the A_3 curve in M vanishes at p, then $p \in \Sigma\Gamma$ is an inflection point.

2 Preliminaries

We use mainly the book [3] as reference for this section. Let $\mathcal{E}(n,m)$ denote the set of germs, at the origin 0 in $\mathbb{R}^n$, of smooth functions $(\mathbb{R}^n,0) \to \mathbb{R}^m$, $\mathcal{E}_n = \{f : (\mathbb{R}^n,0) \to \mathbb{R} \mid f \text{ is the germ of a smooth function}\}$. With the addition and multiplication operations, $\mathcal{E}_n$ becomes a commutative $\mathbb{R}$-algebra with a unit. It has a maximal ideal $\mathcal{M}_n$ which is the subset of germs of functions that vanish at the origin. We have $\mathcal{M}_n = \{f \in \mathcal{E}_n \mid f(0) = 0\}$. Since $\mathcal{M}_n$ is the unique maximal ideal of $\mathcal{E}_n$, $\mathcal{E}_n$ is a local algebra. If $(x_1,...,x_n)$ is a system of local coordinates of $(\mathbb{R}^n,0)$, then $\mathcal{M}_n$ is generated by the germs of functions x_i, $i = 1,...,n$, that is, $\mathcal{M}_n = \mathcal{E}_n.\{x_1,...,x_n\}$. For a given positive integer k, the kth-power of the maximal ideal $\mathcal{M}_n$ is denoted by $\mathcal{M}_n^k$. It is the set of germs of functions $f \in \mathcal{M}_n$

with zero partial derivatives of order less or equal to $k-1$ at the origin. We also have $\mathcal{M}_n^k = \mathcal{E}_n.\{x_1^{i_1}\cdots x_n^{i_n} \mid i_1+\cdots+i_n = k\}$.

Let $\mathcal{R}$ denote the group of germs of diffeomorphisms $(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$. This group is labelled the "right group" and acts smoothly on $\mathcal{E}(n,m)$ by $h.f = fh^{-1}$ for any $h\in\mathcal{R}$ and $f\in\mathcal{E}(n,m)$. Suppose that $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ is $\mathcal{R}$-equivalent to $\pm x_1^2\pm\cdots\pm x_{n-1}^2\pm x_n^{k+1}$ then we say that f has an A_k singularity (or has a singularity of type A_k) at 0 and if f is $\mathcal{R}$-equivalent to $\pm x_1^2\pm\cdots\pm x_{n-3}^2+x_{n-2}^2x_{n-1}\pm x_n^{k-1}$ then we say that f has a D_k singularity at 0.

Given a hypersurface M in $\mathbb{R}^m$, the family of height functions (projections to lines) is given by $H: M\times S^{m-1}\to\mathbb{R}$, where $H(p,\mathbf{u}) = \langle p,\mathbf{u}\rangle$ and S^{m-1} is the unit hypersphere in $\mathbb{R}^m$. For a fixed $\mathbf{u}\in S^{m-1}$, the height function $H_\mathbf{u}$ measures the contact of the hypersurface M with the hyperplane normal to $\mathbf{u}$.

Let M be an m-manifold immersed into $\mathbb{R}^{m+k}$ ande $C_\epsilon M = \{p+\epsilon\mathbf{u}\in\mathbb{R}^{m+k} \mid \mathbf{u}\perp T_pM\}$, which for a small enough $\epsilon\in\mathbb{R}_+$ can be seen to be a hypersurface immersed in $\mathbb{R}^{m+k}$. This is also known as the canal hypersurface of M in $\mathbb{R}^{m+k}$. We denote it by CM and observe that the restriction of the natural projection $\pi: M\times S^{m+k-1}\to S^{m+k-1}$ to the submanifold $\Sigma\Lambda\equiv CM$ can be viewed as the normal Gauss map $\Gamma: CM\to S^{m+k-1}$ on the hypersurface CM. This map is also known as the generalized normal Gauss map of M. When M is a hypersurface ($k=1$), we have that M and CM are locally diffeomorphic and hence Γ is locally equivalent to the normal Gauss map on M.

If we denote by $h_\mathbf{u}: CM\to\mathbb{R}$ the height function in the direction $\mathbf{u}$ over CM and by I the $(k-1)\times(k-1)$-identity matrix, it is not difficult to check that, in appropriate coordinate systems

$$Hess(h_\mathbf{u})(p,\mathbf{u}) = \begin{bmatrix} Hess(H_\mathbf{u})(p) & X \\ 0 & I\end{bmatrix} = D\Gamma(p,\mathbf{u}).$$

The determinant of $D\Gamma(p,\mathbf{u})$ is the Gauss-Kronnecker curvature function $\mathcal{K}$ of CM at the point $(p,\mathbf{u})$. The singular set $\Sigma\Gamma = \mathcal{K}^{-1}(0)$ is the parabolic set of CM. It follows from the above expression that $(p,v)\in\Sigma\Gamma$ if and only if $(p,\mathbf{u})$ is a degenerate singularity of $h_\mathbf{u}$, which is in turn equivalent to saying that p is a degenerate singularity of $f_\mathbf{u}$ (see [7]).

We write M in Monge form in a neighbourhood of a point p, that is, we consider M given locally by the graph of a function $w=f(x,y,z)$ near the origin, with $w=0$ as the tangent hyperplane at the origin. If we parametrize locally the sphere S^3 by $(a,b,c,1)$ near the normal to the hypersurface at the origin, we obtain the following expression for the family of height functions $H(x,y,z,a,b,c) = ax+by+cz+f(x,y,z)$. In particular, for $\mathbf{v}=(0,0,0,1)$, $H_\mathbf{v}(x,y,z) = f(x,y,z)$.

Let $M=\psi(U)$ be a hypersurface in $\mathbb{R}^4$ parametrized in Monge form and U an open subset of $\mathbb{R}^3$:

$$\begin{aligned} \psi: U &\to \mathbb{R}^4 \\ (x,y,z) &\mapsto \psi(x,y,z) = (x,y,z,f(x,y,z)), \end{aligned}$$

with $f\in\mathcal{M}_3^2$ and $p=\psi(x,y,z)\in M$, where we assume that, up to a convenient translation, p is the origin, $f_x=f_y=f_z=0$ at $(0,0,0)$ and up a convenient rotation,

$f_{xy} = f_{xz} = f_{yz} = 0$ at $(0,0,0)$. Let us consider the k-jet of f, $k = 2, 3\ldots$, given by

$$
\begin{aligned}
j^2 f &= a_{200}x^2 + a_{020}y^2 + a_{002}z^2,\\
j^3 f &= j^2 f + a_{300}x^3 + a_{210}x^2y + a_{120}xy^2 + a_{030}y^3 + a_{003}z^3 + a_{012}yz^2 + a_{021}y^2z + a_{102}xz^2\\
&\quad + a_{201}x^2z + a_{111}xyz,\\
j^4 f &= j^3 f + a_{400}x^4 + a_{310}x^3y + a_{220}x^2y^2 + a_{130}xy^3 + a_{040}y^4 + a_{031}y^3z + a_{022}y^2z^2\\
&\quad + a_{013}yz^3 + a_{004}z^4 + a_{301}x^3z + a_{202}x^2z^2 + a_{103}xz^3 + a_{211}x^2yz + a_{121}xy^2z + a_{112}xyz^2,\\
j^5 f &= j^4 f + a_{500}x^5 + a_{410}x^4y + a_{320}x^3y^2 + a_{230}x^2y^3 + a_{140}xy^4 + a_{050}y^5 + a_{041}y^4z\\
&\quad + a_{032}y^3z^2 + a_{023}y^2z^3 + a_{014}yz^4 + a_{005}z^5 + a_{401}x^4z + a_{302}x^3z^2 + a_{203}x^2z^3 + a_{104}xz^4\\
&\quad + a_{113}xyz^3 + a_{122}xy^2z^2 + a_{131}xy^3z + a_{212}x^2yz^2 + a_{221}x^2y^2z + a_{311}x^3yz
\end{aligned}
$$

and so on. Therefore,

$$
\begin{aligned}
f(x,y,z) &= a_{200}x^2 + a_{020}y^2 + a_{002}z^2 + a_{300}x^3 + a_{210}x^2y + a_{120}xy^2 + a_{030}y^3 + a_{003}z^3\\
&\quad + a_{012}yz^2 + a_{021}y^2z + a_{102}xz^2 + a_{201}x^2z + a_{111}xyz + a_{400}x^4 + a_{310}x^3y\\
&\quad + a_{220}x^2y^2 + a_{130}xy^3 + a_{040}y^4 + a_{031}y^3z + a_{022}y^2z^2 + a_{013}yz^3 + a_{004}z^4 + \cdots .
\end{aligned} \tag{1}
$$

Definition 2.1. ([3], Definition 3.1) A s-parameter unfolding of a germ $f_0 \in \mathcal{M}_n.\mathcal{E}(n,p)$ is a map-germ $F : (\mathbb{R}^n \times \mathbb{R}^s, 0) \longrightarrow (\mathbb{R}^p \times \mathbb{R}^s, 0)$, $(x,u) \mapsto (f(x,u), u)$ such that $f_0(x) = f(x,0)$. We use the notation: $f_u(x) = f(x,u)$, where f_u is an deformation of f_0, parametrized by $u \in \mathbb{R}^s$, and $\dot{F}\ (x) = \frac{\partial f}{\partial u_i}(x,0)$, for $i = 1, \cdots, s$.

Theorem 2.2. ([3], Theorem 3.6) An unfolding F is versal if and only if

$$L\mathcal{A}_e f_0 + \mathbb{R}.\{\dot{F}_1, \cdots, \dot{F}_s\} = \mathcal{E}(n,p).$$

Remark 2.3. If f is k-determined we can show that

$$j^k(L\mathcal{A}(f) + \mathbb{R}.\{\dot{F}_1, \cdots, \dot{F}_p\}) = J^k(n,p)$$

to verify if F is a versal unfolding.

Without loss of generality we can assume that $a_{200}a_{020} \neq 0$ and $a_{002} = 0$, provided **v** is normal to M at p and then p is a corank 1 singularity for height function H_{v} of M. The conditions on the coefficients of f for those singularities to occur and their corresponding versal unfoldings are described in the following proposition.

Proposition 2.4. ([6], Proposition 2.1) Using the notation as above let us suppose $a_{200}a_{020} \neq 0$, then, the height function H_{v} on M in the direction $\mathrm{v} \in N_pM$ has the following generic singularities of corank 1 at p:

$A_2 : a_{002} = 0$ and $a_{003} \neq 0$;

$A_3 : a_{002} = 0,\ a_{003} = 0$ and $4a_{004}a_{020}a_{200} - {a_{012}}^2a_{200} - {a_{102}}^2a_{020} \neq 0$;

$A_4 : a_{002} = 0,\ a_{003} = 0,\ 4a_{004}a_{020}a_{200} - {a_{012}}^2a_{200} - {a_{102}}^2a_{020} = 0$ and

$$4a_{005}{a_{200}}^2{a_{020}}^2 + a_{021}{a_{012}}^2{a_{200}}^2 - 2a_{012}a_{013}a_{020}{a_{200}}^2 + a_{012}a_{020}a_{102}a_{111}a_{200} + a_{201}{a_{102}}^2{a_{020}}^2 - 2a_{103}a_{102}a_{200}{a_{020}}^2 \neq 0.$$

Such singularities are versaly unfolded by the height functions family provided:

$$\begin{aligned}&A_{\leq 2}: \text{ always;}\\&A_3: a_{012}\neq 0 \text{ or } a_{102}\neq 0;\\&A_4: a_{102}\varphi(a_{ijk})\neq 0;\end{aligned}$$

where $\varphi(a_{ijk})$ is a polynomial of degree 12 in coefficients a_{ijk}, with $0 \leq i+j+k \leq 5$ (see [5]).

Proposition 2.5. ([6], Proposition 2.4)

(1) The parabolic set (i.e. set of points in M where the height function along the normal has an A_2-singularity) is locally a smooth 2-dimensional surface.

(2) The A_3-singularities of the height function H occur generically on a smooth curve on the parabolic set, labelled A_3 curve.

(3) The A_4-singularities of H occur generically at isolated points of the A_3 curve.

Remark 2.6. Although umbilic singularities ($D_4^{\pm}$) may occur generically, the parabolic subset is not a regular surface at such points and, therefore, they shall not be considered.

By taking into account the number of necessary conditions, it follows from the Transversality Theorem that non versal singularities of types A_3 and A_4 of the height function cannot occur generically ([6], Proposition 2.1) .

Given a parametrization x for a surface N in $\mathbb{R}^4$, denote by $l_1, m_1, n_1, l_2, m_2, n_2$ the coefficients of the second fundamental form with respect to any basis $\{x_{u_1}, x_{u_2}, f_3, f_4\}$ of $T_pM \to N_pM$, and define Δ by

$$\Delta = \frac{1}{4}\begin{vmatrix} l_1 & 2m_1 & n_1 & 0\\ l_2 & 2m_2 & n_2 & 0\\ 0 & l_1 & 2m_1 & n_1\\ 0 & l_2 & 2m_2 & n_2\end{vmatrix} = \frac{1}{4}(4(l_1m_2 - l_2m_1)(m_1n_2 - m_2n_1)) - (l_1n_2 - l_2n_1)^2).$$

A point p is said to be elliptic/parabolic/hyperbolic if $\Delta < 0/ = 0/ > 0$. The set of points (x, y) where $\Delta = 0$ is called the parabolic set of N and is denoted by Δ.

Given a surface in $\mathbb{R}^4$ locally given in the Monge form by $(x, y, Q_1(x, y), Q_2(x, y))$ and a vector $\mathbf{v} = (v_1, v_2, v_3, v_4) \in S^3$, the height function $\tilde{H}_{\mathbf{v}}$ on N is given by

$$\tilde{H}_{\mathbf{v}}(x, y) = v_1x + v_2y + v_3Q_1(x, y) + v_4Q_2(x, y).$$

Therefore, $\tilde{H}_{\mathbf{v}}$ has a singularity at p if and only if $v_1 = v_2 = 0$, i.e., $\mathbf{v} \in N_pN$. By a convenient rotation of the normal plane at the origin , we can assume that $\mathbf{v} = (0, 0, 0, 1)$ is a degenerate direction and, therefore, Q_1 and Q_2 can be written as

$$\begin{aligned}Q_1(x, y) =& a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + \cdots\\ Q_2(x, y) =& b_{20}x^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + \cdots\end{aligned}$$

and the singularities of $\tilde{H}_{\mathbf{v}}$ can be given in terms of the coefficients of Q_1 and Q_2, as in the following proposition.

Proposition 2.7. ([1], Proposition 2.2.2) Using the notation as above, the conditions for the generic singularities of the height function $\tilde{H}_{\mathbf{v}}$ are:

$A_2 : b_{20} \neq 0$ and $b_{03} \neq 0$;
$A_3 : b_{20} \neq 0,\ b_{03} = 0$ and $4b_{20}b_{04} - b_{12}^2 \neq 0$;
$A_4 : b_{20} \neq 0,\ b_{03} = 0,\ 4b_{20}b_{04} - b_{12}^2 = 0$ and $2b_{20}^2 b_{05} - 2b_{20}b_{12}b_{13} + b_{21}b_{12}^2 \neq 0$;
$D_4 : b_{20} = 0$ and $b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3$ is non degenerate.

Such singularities are a versally unfolded height functions family if and only if

A_2 : always;
$A_3 :\ a_{02} \neq 0$ or $b_{12} \neq 0$;

$$A_4 :\ a_{02}\left(b_{20}b_{13} - b_{21}b_{12}\right) - b_{12}\left(b_{20}a_{03} - \frac{1}{2}a_{11}b_{12}\right) \neq 0;$$

$$D_4 :\ 3b_{30}\left(a_{02}b_{12} - \frac{3}{2}a_{11}b_{03}\right) - b_{21}\left(a_{02}b_{21} - \frac{1}{2}a_{11}b_{12}\right) + a_{20}\left(3b_{21}b_{03} - b_{12}^2\right) \neq 0.$$

Definition 2.8. A hyperplane with orthogonal direction **v** is an osculating hyperplane of N at $p = x(u)$ if it is tangent to N at p and $\tilde{H}_{\mathbf{v}}$ has a degenerate (i.e., non Morse, that is $A_{\geq 2}$) singularity at p. We call the direction **v** a binormal direction of N at p.

Proposition 2.9. ([3], Proposition 7.7) Let N be a smooth surface immersed in $\mathbb{R}^4$ and let p be a point on N.

(1) If p is an elliptic point, then $\tilde{H}_{\mathbf{v}}$ has a non degenerate singularity at p of type A_1^- for all **v** in N_pN.

(2) If p is a hyperbolic point, then there are exactly two distinct binormal directions $\mathbf{v}_1$, $\mathbf{v}_2$ in N_pN.

(3) If p is a parabolic point but not an inflection point, then there is a unique binormal direction **v** in N_pN.

(4) If p is non-degenerate inflection point, then there is a unique binormal direction **v** in N_pN and the 2-jet of $\tilde{H}_{\mathbf{v}}$ at p is identically zero.

Singularities of $\tilde{H}_{\mathbf{v}}$ occur only in the non-elliptic region.

3 Characterization of height functions singularities on a hypersurface M and its parabolic subset

The image of $\Sigma\Gamma$ through ψ is generically a smooth surface, known as the parabolic surface ([6]). We shall identify the subsets $\Sigma\Gamma \equiv \psi(\Sigma\Gamma)$. The parabolic surface $\Sigma\Gamma$ can be considered as a surface in $\mathbb{R}^4$ and the following question can be asked: Given the normal vector **v** at a parabolic point $p \in M$, is there a relation between the singularities of the height functions H and $\tilde{H}$ in the direction **v** (denoted by $H_{\mathbf{v}}$ and $\tilde{H}_{\mathbf{v}}$) at p?

Proposition 3.1. Let H_{v} be a height function on the hypersurface M and $\tilde{H}_{\mathrm{v}}$ a height function on the surface $\Sigma\Gamma$ in the normal direction $\mathbf{v}$ to M at p. Then the following assertions hold:

(1) If p is an A_2 singularity of H_{v} then p is a singularity of type A_1 of $\tilde{H}_{\mathrm{v}}$,

(2) If p is an A_3 singularity of H_{v} then p is, generically, an A_3 singularity of $\tilde{H}_{\mathrm{v}}$ although, at isolated points, it may be of either type A_4, or D_4,

(3) If p is an A_4 versal singularity of H_{v} then p is, generically, a singularity of type A_4 for $\tilde{H}_{\mathrm{v}}$.

Moreover, the following conditions hold for item 2:

(i) p is an A_3 singularity of $\tilde{H}_{\mathrm{v}}$ if and only if $a_{012}^2a_{200} + a_{020}a_{102}^2 \neq 0$ and $9a_{004}a_{020}a_{200} - 2a_{200}a_{012}{}^2 - 2a_{020}a_{102}{}^2 \neq 0$;

(ii) p is an A_4 singularity of $\tilde{H}_{\mathrm{v}}$ if and only if $a_{012}^2a_{200} + a_{020}a_{102}^2 \neq 0$, $9a_{004}a_{020}a_{200} - 2a_{200}a_{012}{}^2 - 2a_{020}a_{102}{}^2 = 0$ and $21a_{005}a_{200}{}^2a_{020}{}^2 + 4a_{021}a_{012}{}^2a_{200}{}^2 - 9a_{012}a_{013}a_{020}a_{200}{}^2 + 4a_{012}a_{020}a_{102}a_{111}a_{200} + 4a_{201}a_{102}{}^2a_{020}{}^2 - 9a_{103}a_{102}a_{200}a_{020}{}^2 \neq 0$;

(iii) p is a D_4 singularity of $\tilde{H}_{\mathrm{v}}$ if and only if $a_{012}^2a_{200} + a_{020}a_{102}^2 = 0$ and

$$\begin{aligned}
&a_{012}(48a_{004}a_{012}^4a_{102}a_{200}a_{300} - 96a_{004}a_{012}^4a_{200}^2a_{202} + 96a_{004}a_{012}^4a_{200}a_{201}^2 - \\
&48a_{004}a_{012}^3a_{102}^2a_{200}a_{210} - 96a_{004}a_{012}^3a_{200}a_{111}a_{201}a_{102} + 96a_{004}a_{012}^3a_{200}^2a_{112}a_{102} - \\
&96a_{004}a_{200}^2a_{012}^2a_{022}a_{102}^2 + 48a_{004}a_{012}^2a_{102}^3a_{120}a_{200} + 96a_{004}a_{012}a_{021}a_{102}^3a_{111}a_{200} - \\
&48a_{004}a_{012}a_{030}a_{102}^4a_{200} - 96a_{004}a_{021}^2a_{102}^4a_{200} + 25a_{012}^4a_{201}^2a_{102}^2 + 36a_{012}^4a_{200}^2a_{103}^2 - \\
&60a_{012}^4a_{200}a_{103}a_{201}a_{102} + 60a_{012}^3a_{013}a_{200}a_{201}a_{102}^2 - 72a_{012}^3a_{013}a_{200}^2a_{103}a_{102} - \\
&50a_{012}^3a_{102}^3a_{111}a_{201} + 60a_{012}^3a_{103}a_{111}a_{200}a_{102}^2 + 60a_{012}a_{013}a_{021}a_{102}^4a_{200} + \\
&36a_{013}^2a_{200}^2a_{012}^2a_{102}^2 - 60a_{012}^2a_{013}a_{102}^3a_{111}a_{200} - 60a_{012}^2a_{021}a_{102}^3a_{103}a_{200} + \\
&50a_{012}^2a_{021}a_{102}^4a_{201} + 25a_{012}^2a_{102}^4a_{111}^2 - 50a_{012}a_{021}a_{102}^5a_{111} + 25a_{021}^2a_{102}^6) \neq 0
\end{aligned}$$

Proof. Let $H_{\mathbf{v}}(x, y, z) = f(x, y, z)$, where $\mathbf{v} = (0, 0, 0, 1)$ and f is as in equation (1). The fold singularities of the Gauss map $M \to S^n$ correspond to A_2 singularities of the height function. Then the surface $\Sigma\Gamma$ of M is given by the points (x, y, z) at which the Hessian determinant of f vanishes, then $\Sigma\Gamma$ is equal to $P^{-1}(0)$, where

$$
\begin{aligned}
P(x,y,z) &= 8a_{200}a_{020}a_{002} + (24a_{002}a_{020}a_{300} + 8a_{002}a_{200}a_{120} + 8a_{020}a_{102}a_{200})x + (8a_{002}a_{020}a_{210} \\
&+ 24a_{002}a_{030}a_{200} + 8a_{012}a_{020}a_{200})y + (8a_{002}a_{020}a_{201} + 8a_{002}a_{021}a_{200} + 24a_{003}a_{020}a_{200})z \\
&+ (48a_{002}a_{020}a_{400} + 24a_{002}a_{120}a_{300} + 8a_{002}a_{200}a_{020} - 8a_{002}a_{210}^2 - 8a_{020}{a_{201}}^2 - 2{a_{111}}^2a_{200} \\
&+ 24a_{020}a_{102}a_{300} + 8a_{020}a_{200}a_{202} + 8a_{102}a_{120}a_{200})x^2 + (24a_{002}a_{020}a_{310} + 72a_{002}a_{030}a_{300} \\
&- 8a_{002}a_{120}a_{210}) + 24a_{002}a_{130}a_{200} + 24a_{012}a_{020}a_{300} + 8a_{012}a_{120}a_{200} + 8a_{020}a_{102}a_{210} \\
&- 8a_{020}a_{111}a_{201} + 8a_{020}a_{112}a_{200} - 8a_{021}a_{111}a_{200} + 24a_{030}a_{102}a_{200})xy + (24a_{002}a_{020}a_{301} \\
&+ 24a_{002}a_{021}a_{300} - 8a_{002}a_{111}a_{210} + 8a_{002}a_{120}a_{201} + 8a_{002}a_{121}a_{200} + 72a_{003}a_{020}a_{300} \\
&+ 24a_{003}a_{120}a_{200} - 8a_{012}a_{111}a_{200} - 8a_{020}a_{102}a_{201} + 24a_{020}a_{103}a_{200} + 8a_{021}a_{102}a_{200})xz \\
&+ (8a_{002}a_{020}a_{200} + 24a_{002}a_{030}a_{210} + 48a_{002}a_{040}a_{a200} - 8a_{002}a_{120}^2 - 2a_{020}{a_{111}}^2 - 8{a_{021}}^2a_{200} \\
&+ 8a_{012}a_{020}a_{210} + 24a_{012}a_{030}a_{200} + 8a_{020}a_{022}a_{200})y^2 + (8a_{002}a_{020}a_{211} + 8a_{002}a_{021}a_{210} \\
&+ 24a_{002}a_{030}a_{201} + 24a_{002}a_{031}a_{200} - 8a_{002}a_{111}a_{120} + 24a_{003}a_{020}a_{210} + 72a_{003}a_{030}a_{200} \\
&+ 8a_{012}a_{020}a_{201} - 8a_{012}a_{021}a_{200} + 24a_{013}a_{020}a_{200} - 8a_{020}a_{102}a_{111})yz + (8a_{002}a_{020}a_{202} \\
&+ 8a_{002}a_{021}a_{201} + 8a_{002}a_{022}a_{200} + 48a_{004}a_{020}a_{200} + 24a_{003}a_{020}a_{201} + 24a_{003}a_{021}a_{200} \\
&- 2a_{002}a_{111}^2 - 8{a_{012}}^2a_{200} - 8a_{020}{a_{102}}^2)z^2 + \text{ h.o.t.}
\end{aligned}
$$

(1) Let us suppose that $H_{\mathrm{v}} = f$ has a singularity of type A_2. Then, from Proposition 2.4 we have that $a_{002} = 0$ and $a_{003} \neq 0$. Note that $P_z(0,0,0) = 24a_{003}a_{020}a_{200} \neq 0$ and therefore the variable z can be expressed in terms of x and y, i.e.,

$$z = c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2 + c_6x^3 + c_7x^2y + c_8xy^2 + c_9y^3 + \cdots$$

By substituting the expression of z in $P(x,y,z) = 0$ and calculating the coefficients c_i we obtain that

$$z = \frac{-1}{3a_{003}}\big(a_{102}x + a_{012}y\big) + \varphi_1(x,y),$$

where $\varphi_1 \in \mathcal{M}_2^2$. Therefore, the surface $\Sigma\Gamma$ can be parametrized by

$$(x,y) \mapsto \left(x, y, \frac{-1}{3a_{003}}\big(a_{102}x + a_{012}y\big) + \varphi_1(x,y), \tilde{f}(x,y)\right),$$

and

$$
\begin{aligned}
\tilde{f}(x,y) &= f\left(x, y, \frac{-1}{3a_{003}}\big(a_{102}x + a_{012}y\big) + \varphi_1(x,y)\right) \\
&= a_{200}x^2 + a_{020}y^2 + \text{higher order terms.}
\end{aligned}
$$

Since $a_{002} \neq 0$ and $a_{020} \neq 0$, then $\tilde{H}_{\mathrm{v}} = \tilde{f} \sim_{\mathcal{R}} x^2 \pm y^2$, i.e., $\tilde{H}_{\mathrm{v}}$ has an A_1 singularity.

(2) Let us suppose now that $H_{\mathrm{v}} = f$ has a singularity of type A_3. Therefore, $a_{002} = 0$, $a_{003} = 0$ and $4a_{004}a_{020}a_{200} - {a_{012}}^2a_{200} - {a_{102}}^2a_{020} \neq 0$.

In such case,

$$\nabla P(0,0,0) = a_{200}a_{020}(a_{102}, a_{012}, 0).$$

Since the A_3 singularity is versal, then either $a_{102} \neq 0$, or $a_{012} \neq 0$ and the parabolic subset is a regular surface. Without loss of generality we can assume that $a_{102} \neq 0$ and, therefore, by using calculations analogous to those of the previous case, x can be written in terms of variables y and z. These calculations lead to the following:

$$\begin{aligned}
x = &-\frac{a_{012}}{a_{102}}y - \frac{1}{4{a_{102}}^3a_{020}a_{200}}\Big(4{a_{012}}^2a_{020}a_{200}a_{202} - 4{a_{012}}^2a_{020}{a_{201}}^2 - {a_{012}}^2{a_{111}}^2a_{200} \\
&+ 4a_{012}a_{020}a_{102}a_{111}a_{201} - 4a_{012}a_{020}a_{102}a_{112}a_{200} + 4a_{012}a_{021}a_{102}a_{111}a_{200} \\
&+ 4a_{020}a_{022}{a_{102}}^2a_{200} - a_{020}{a_{102}}^2{a_{111}}^2 - 4{a_{021}}^2{a_{102}}^2a_{200}\Big)y^2 \\
&- \frac{1}{4{a_{102}}^3a_{020}a_{200}}\Big(24a_{004}a_{020}{a_{102}}^2a_{200} - 4{a_{012}}^2{a_{102}}^2a_{200} - 4{a_{102}}^4a_{020}\Big)z^2 \\
&- \frac{1}{2{a_{102}}^3a_{020}a_{200}}\Big(2{a_{012}}^2a_{102}a_{111}a_{200} + 4a_{012}a_{020}{a_{102}}^2a_{201} - 6a_{012}a_{020}a_{102}a_{103}a_{200} \\
&- 4a_{012}a_{021}{a_{102}}^2a_{200} + 6a_{013}a_{020}{a_{102}}^2a_{200} - 2a_{020}{a_{102}}^3a_{111}\Big)yz + \cdots \\
= &-\frac{a_{012}}{a_{102}}y + \varphi_2(y,z),
\end{aligned}$$

where $\varphi_2 \in \mathcal{M}_2^2$. Therefore, the surface $\Sigma\Gamma$ in $\mathbb{R}^4$ is parametrized by

$$(y,z) \mapsto \left(-\frac{a_{012}}{a_{102}}y + \varphi_2(y,z), y, z, \tilde{f}(y,z)\right),$$

where

$$\tilde{f}(y,z) = \left(a_{020} + \frac{{a_{012}}^2a_{200}}{{a_{102}}^2}\right)y^2 + Ay^3 + By^2z + Cyz^2 + \cdots,$$

with

$$\begin{aligned}
A = &-\frac{1}{2{a_{102}}^4a_{020}}\Big(2a_{300}{a_{012}}^3a_{102}a_{020} - 4{a_{012}}^3a_{020}a_{200}a_{202} + 4{a_{012}}^3a_{020}{a_{201}}^2 \\
&- 2{a_{012}}^2a_{210}{a_{102}}^2a_{020} - 4{a_{012}}^2a_{020}a_{102}a_{111}a_{201} + 4{a_{012}}^2a_{020}a_{102}a_{112}a_{200} \\
&- 4{a_{012}}^2a_{021}a_{102}a_{111}a_{200} - 4a_{012}a_{020}a_{022}{a_{102}}^2a_{200} + 2a_{120}a_{012}{a_{102}}^3a_{020} \\
&+ 4a_{012}{a_{021}}^2{a_{102}}^2a_{200} - 2a_{030}{a_{102}}^4a_{020} + a_{012}^3{a_{111}}^2a_{200} + a_{012}a_{020}{a_{102}}^2{a_{111}}^2\Big), \\
B = &\frac{1}{{a_{102}}^3a_{020}}\Big(6a_{012}a_{013}a_{020}a_{102}a_{200} + 5{a_{012}}^2a_{020}a_{102}a_{201} - 6{a_{012}}^2a_{020}a_{103}a_{200} \\
&+ 2{a_{012}}^3a_{111}a_{200} - 3a_{012}a_{020}{a_{102}}^2a_{111} + a_{020}a_{021}{a_{102}}^3 - 4{a_{012}}^2a_{021}a_{102}a_{200}\Big) \text{ and} \\
C = &\frac{2a_{012}(6a_{004}a_{020}a_{200} - {a_{012}}^2a_{200} - a_{020}{a_{102}}^2)}{{a_{102}}^2a_{020}}.
\end{aligned}$$

Let us denote by $coef(q, x_1^{k_1} \dots x_n^{k_n})$ the coefficient of the monomial $x_1^{k_1} \dots x_n^{k_n}$ in a polynomial q and suppose that

$$coef(\tilde{f}, y^2) = \frac{a_{012}^2 a_{200} + a_{020} a_{102}^2}{a_{102}^2} \neq 0.$$

The following coordinate changes can be applied to eliminate degree 3 monomials:

$$y \mapsto\ y - \left(\frac{coef(\tilde{f}, y^3)}{2coef(\tilde{f}, y^2)} y^2 + \frac{coef(\tilde{f}, y^2 z)}{2coef(\tilde{f}, y^2)} yz + \frac{coef(\tilde{f}, yz^2)}{2coef(\tilde{f}, y^2)} z^2 \right)$$

and those of degree 4 can be eliminated by means of

$$y \mapsto\ y - \left(\frac{coef(\tilde{f}, y^4)}{2coef(\tilde{f}, y^2)} y^3 + \frac{coef(\tilde{f}, y^3 z)}{2coef(\tilde{f}, y^2)} y^2 z + \frac{coef(\tilde{f}, y^2 z^2)}{2coef(\tilde{f}, y^2)} yz^2 + \frac{coef(\tilde{f}, yz^3)}{2coef(\tilde{f}, y^2)} z^3 \right),$$

So we have

$$\tilde{f} \sim_{\mathcal{R}} \left(\frac{a_{012}^2 a_{200} + a_{020} a_{102}^2}{a_{102}^2} \right) y^2 + \frac{\left(4a_{004}a_{020}a_{200} - a_{200}{a_{012}}^2 - a_{020}{a_{102}}^2\right)\left(9a_{004}a_{020}a_{200} - 2a_{200}{a_{012}}^2 - 2a_{020}{a_{102}}^2\right)}{a_{020}a_{200}\left(a_{200}{a_{012}}^2 + a_{020}{a_{102}}^2\right)} z^4.$$

Note $4a_{004}a_{020}a_{200} - a_{200}{a_{012}}^2 - a_{020}{a_{102}}^2 \neq 0$, as a consequence of the assumption $H_{\mathrm{v}} = f$ has a singularity of type A_3. Therefore, if $9a_{004}a_{020}a_{200} - 2a_{200}{a_{012}}^2 - 2a_{020}{a_{102}}^2 \neq 0$, then $\tilde{f} = \tilde{H}_{\mathrm{v}} \sim_{\mathcal{R}} y^2 \pm z^4$, i.e., $\tilde{H}_{\mathrm{v}}$ has, generically, a singularity of type A_3. If $9a_{004}a_{020}a_{200} - 2a_{200}{a_{012}}^2 - 2a_{020}{a_{102}}^2 = 0$, $\tilde{f} = \tilde{H}_{\mathrm{v}} \sim_{\mathcal{R}} y^2 \pm z^5$ is obtained through convenient coordinate changes on $\tilde{f}$, if and only if $21a_{005}{a_{200}}^2{a_{020}}^2 + 4a_{021}{a_{012}}^2{a_{200}}^2 - 9a_{012}a_{013}a_{020}{a_{200}}^2 + 4a_{012}a_{020}a_{102}a_{111}a_{200} + 4a_{201}{a_{102}}^2{a_{020}}^2 - 9a_{103}a_{102}a_{200}{a_{020}}^2 \neq 0$, thus leading to a singularity of type A_4 for $\tilde{H}_{\mathrm{v}}$.

Now, if $coef(\tilde{f}, y^2) = 0$, i.e., $a_{012}^2 a_{200} + a_{020} a_{102}^2 = 0$, then $\tilde{f}$ has cubic and higher order terms. Therefore, $\tilde{f} = \tilde{H}_{\mathrm{v}}$ may have singularities of type $D_4^{\pm}$ at isolated points of A_3 curve when the cubic part of $\tilde{f}$ is non degenerate, i.e., when its discriminant δ does not vanish. It is not difficult to check through convenient straightforward calculations that δ does not vanish if and only if

$$\begin{aligned}
&a_{012}(48a_{004}a_{012}^4a_{102}a_{200}a_{300} - 96a_{004}a_{012}^4a_{200}^2a_{202} - 48a_{004}a_{012}^3a_{102}^2a_{200}a_{210} \\
&+ 96a_{004}a_{012}^4a_{200}a_{201}^2 - 96a_{004}a_{012}^3a_{200}a_{111}a_{201}a_{102} + 96a_{004}a_{012}^3a_{200}^2a_{112}a_{102} \\
&- 96a_{004}a_{200}^2a_{012}^2a_{022}a_{102}^2 + 48a_{004}a_{012}^2a_{102}^3a_{120}a_{200} + 96a_{004}a_{012}a_{021}a_{102}^3a_{111}a_{200} \\
&- 48a_{004}a_{012}a_{030}a_{102}^4a_{200} - 96a_{004}a_{021}^2a_{102}^4a_{200} + 25a_{012}^4a_{201}^2a_{102}^2 + 36a_{012}^4a_{200}^2a_{103}^2 \\
&- 60a_{012}^4a_{200}a_{103}a_{201}a_{102} + 60a_{012}^3a_{013}a_{200}a_{201}a_{102}^2 - 72a_{012}^3a_{013}a_{200}^2a_{103}a_{102} \\
&- 50a_{012}^3a_{102}^3a_{111}a_{201} + 60a_{012}^3a_{103}a_{111}a_{200}a_{102}^2 + 36a_{013}^2a_{200}^2a_{012}^2a_{102}^2 + 25a_{021}^2a_{102}^6 \\
&- 60a_{012}^2a_{013}a_{102}^3a_{111}a_{200} + 50a_{012}^2a_{021}a_{102}^4a_{201} - 60a_{012}^2a_{021}a_{102}^3a_{103}a_{200} \\
&+ 25a_{012}^2a_{102}^4a_{111}^2 + 60a_{012}a_{013}a_{021}a_{102}^4a_{200} - 50a_{012}a_{021}a_{102}^5a_{111}) \neq 0.
\end{aligned}$$

(3) Let us suppose now that $H_{\mathbf{v}} = f$ has an A_4 singularity. Therefore, $a_{002} = 0$, $a_{003} = 0$, $4a_{004}a_{020}a_{200} - {a_{012}}^2a_{200} - {a_{102}}^2a_{020} = 0$ and $4a_{005}{a_{200}}^2{a_{020}}^2 + a_{021}{a_{012}}^2{a_{200}}^2 - 2a_{012}a_{013}a_{020}{a_{200}}^2 + a_{012}a_{020}a_{102}a_{111}a_{200} + a_{201}{a_{102}}^2{a_{020}}^2 - 2a_{103}a_{102}a_{200}{a_{020}}^2 \neq 0$. Moreover,

$$\nabla P(0,0,0) = a_{200}a_{020}(a_{102}, a_{012}, 0)$$

and $a_{102} \neq 0$, since the singularity A_4 is versal. Analogously to the previous cases, surface $\Sigma\Gamma$ in $\mathbb{R}^4$ is parametrized by

$$(y,z) \mapsto \left(-\frac{a_{012}}{a_{102}} y + \varphi_3(y,z), y, z, \tilde{f}(y,z) \right),$$

where $\varphi_3, \tilde{f} \in \mathcal{M}_2^2$. Now, through convenient coordinates changes in $\tilde{f}$ towards eliminating some monomials,

$$\tilde{f} = \tilde{H}_{\mathbf{v}} \sim_{\mathcal{R}} \left(\frac{a_{012}^2 a_{200} + a_{020} a_{102}^2}{a_{102}^2} \right) y^2 + Dz^5,$$

where

$$D = \frac{1}{4{a_{020}}^2{a_{200}}^2} \Big(4a_{005}{a_{020}}^2{a_{200}}^2 + {a_{012}}^2 a_{021}{a_{200}}^2 - 2a_{012}a_{013}a_{020}{a_{200}}^2 + a_{012}a_{020}a_{102}a_{111}a_{200} + {a_{020}}^2{a_{102}}^2 a_{201} - 2{a_{020}}^2 a_{102}a_{103}a_{200} \Big) \neq 0$$

by hypothesis. . Therefore, $\tilde{f} = \tilde{H}_{\mathbf{v}} \sim_{\mathcal{R}} y^2 \pm z^5$, i.e., $\tilde{H}_{\mathbf{v}}$ has an A_4 singularity.

□

Corollary 3.2. Singularities of type A_2 of $\tilde{H}_{\mathbf{v}}$, for $\mathbf{v} \in N_pM$ do not occur generically.

Remark 3.3. (1) The condition $a_{012}^2 a_{200} + a_{020}a_{102}^2 = 0$ in the singularity A_4 of Proposition 3.1 leads to a situation that cannot occur generically neither at a point of the hypersurface M in 4-space (for it would be an extra condition on an isolated singularity, which is non generic), nor at a point of the surface $\Sigma\Gamma$. In fact, let us suppose $a_{012}^2 a_{200} + a_{020}a_{102}^2 = 0$; therefore, $\tilde{f}$ has only cubic or higher order terms and the discriminant of its cubic part is given by

$$\delta = \frac{144}{{a_{102}}^{10}} (a_{200}a_{012}a_{004})^2 \tilde{A},$$

where $\tilde{A}$ is given in terms of the sum and the product of coefficients of at least order 2 of f. However, the height function $H_{\mathbf{v}}$ on M is supposed to have an A_4 singularity. From Proposition 2.4 we have that $4a_{004}a_{020}a_{200} - {a_{012}}^2a_{200} - {a_{102}}^2a_{020} = 0$, and therefore we must have $a_{004} = 0$ for $a_{020}a_{200} \neq 0$ and $a_{012}^2a_{200} + a_{020}a_{102}^2 = 0$. Moreover, the discriminant δ is equal to zero, so we have a degenerate cubic and hence a singularity of type $D_{>4}$, which cannot occur generically on surfaces in $\mathbb{R}^4$.

(2) We have from Proposition 3.1 that the points of the A_3 curve of $H_{\mathbf{v}}$ in M is made of points of type A_3 for $\tilde{H}_{\mathbf{v}}$ lying in $\Sigma\Gamma$. Now, we observe that at isolated points of the A_3 curve of $H_{\mathbf{v}}$ may become a singularity of type A_4 for $\tilde{H}_{\mathbf{v}}$. On the other hand, the singularities of type A_4 of $H_{\mathbf{v}}$ correspond to the singularities of type A_4 of $\tilde{H}_{\mathbf{v}}$, i.e., the number of type A_4 singularities in those curves of $\Sigma\Gamma$ is higher than or equal to the number of A_4 singularities on the A_3 curve of M.

The following reciprocal of Proposition 3.1 holds:

Corollary 3.4. Let $\Sigma\Gamma$ be the parabolic subset of a hypersurface M in $\mathbb{R}^4$, where $\Sigma\Gamma$ is a regular surface in $\mathbb{R}^4$. Let $H_{\mathbf{v}}$ be the height function on the hypersurface M and $\tilde{H}_{\mathbf{v}}$ be the height function on the surface $\Sigma\Gamma$ in direction $\mathbf{v}$ normal to M at p. The following assertions hold:

(1) If p is a singularity of type A_1 of $\tilde{H}_{\mathbf{v}}$ then p is a singularity of type A_2 of $H_{\mathbf{v}}$,

(2) If p is a singularity of type A_3 of $\tilde{H}_{\mathbf{v}}$ then p is an A_3 versal singularity of $H_{\mathbf{v}}$,

(3) If p is a singularity of type A_4 of $\tilde{H}_{\mathbf{v}}$ then p is an A_4 versal singularity of $H_{\mathbf{v}}$.

Proof. (1) Let us suppose that $p \in \Sigma\Gamma$ is a type A_1 singularity of $\tilde{H}_{\mathbf{v}}$. Since p is a regular point in the parabolic surface of M, it is necessarily a singularity of type $A_{\geq 2}$ of $H_{\mathbf{v}}$. Observe that if p is of either type A_3 or A_4 this would contradict Proposition 3.1. Therefore, the point p must be a singular point of type A_2 for $H_{\mathbf{v}}$.

(2) If p is a singularity of type A_3 of $\tilde{H}_{\mathbf{v}}$ we have that it cannot be a singularity of type A_1 for $H_{\mathbf{v}}$, for it lies in the parabolic subset; on the other hand, it cannot be a singularity of type A_2 nor of type A_4, for it would contradict Proposition 3.1 and hence p must be a singular point of type A_3 of $H_{\mathbf{v}}$.

(3) The argument in this case runs analogously to the previous one. □

Corollary 3.5. (i) Given a point $\mathbf{v} \in N_pM$ we have that $\mathbf{v}$ is a binormal direction of $\Sigma\Gamma$ if and only if p belongs to the A_3 curve of M. Moreover, this A_3 curve of M is an A_3 curve of the parabolic surface $\Sigma\Gamma$ too.

(ii) Let $\mathbf{v} \in N_p\Sigma\Gamma$ such that $\mathbf{v} \notin N_pM$. Then, the hyperbolic region $\Sigma\Gamma$ may contain other A_3 curves of $\tilde{H}_{\mathbf{v}}$ as well as A_2 type singularities of $\tilde{H}_{\mathbf{v}}$.

Observe in the proof of Proposition 3.1, the coefficient $a_{012}^2 a_{200} + a_{020}a_{102}^2$ was initially supposed to be non zero. Provided it is zero, in an A_3 singularity of $H_{\mathbf{v}}$ the height function $\tilde{H}_{\mathbf{v}}$ on $\Sigma\Gamma$ might have a corank 2 singularity. The geometrical meaning of such a coefficient is analyzed in the next proposition.

Proposition 3.6. Let us suppose p is an A_3 versal singularity of $H_{\mathbf{v}}$. Then, $a_{012}^2 a_{200} + a_{020}a_{102}^2 = 0$ if and only if the curvature κ of the A_3 curve of M vanishes at p.

Proof. The A_3 curve can be parametrized as the set of points (x_0, y_0, z_0) for which the Hessian determinant of f vanishes and the kernel of the Hessian of f is a root of the cubic form f. Therefore, such points are of the form

$$(x_0, y_0, z_0) = (-a_{012}Ay, a_{102}Ay, (a_{012}B + a_{102}C)y) + \text{ h.o.t.},$$

with $a_{102} \neq 0$ or $a_{012} \neq 0$, since A_3 is versal, where

$$\begin{aligned} A &= 4a_{004}a_{020}a_{200} - a_{200}{a_{012}}^2 - a_{020}{a_{102}}^2, \\ B &= a_{102}a_{201}a_{020} + \frac{1}{2}a_{012}a_{200}a_{111} - a_{200}a_{020}a_{103} - a_{200}a_{021}a_{102}, \\ C &= a_{200}a_{020}a_{013} - \frac{1}{2}a_{020}a_{102}a_{111}. \end{aligned}$$

Note that $A \neq 0$ is precisely one of the conditions for an A_3 singularity (Proposition 2.4). So the A_3 curve is given by

$$\gamma(y) = (x_0, y_0, z_0, f(x_0, y_0, z_0))$$

and its curvature is

$$\kappa(y) = 2\sqrt{\frac{A^4(a_{012}^2a_{200} + a_{020}a_{102}^2)^2\left((A)^2(a_{102}^2 + a_{012}^2) + (a_{012}B + a_{102}C)^2\right)}{\left(A^2(a_{102}^2 + a_{012}^2) + (a_{012}B + a_{102}C)^2 + 4(A)^4(a_{012}^2a_{200} + a_{020}a_{102}^2)^2y^2\right)^3}}.$$

Observe that term $(A)^4\left((A)^2(a_{102}^2 + a_{012}^2) + (a_{012}B + a_{102}C)^2\right)$ never vanishes; therefore, $\kappa(y) = 0$ if and only if $a_{012}^2a_{200} + a_{020}a_{102}^2 = 0$.

□

4 Geometry of the parabolic subset $\Sigma\Gamma$ as a surface in 4-space

The previous section addressed to the behavior of the height functions on both, the hypersurface M and its parabolic subset $\Sigma\Gamma$, in a given direction $\mathrm{v} \in N_pM$. Since $\Sigma\Gamma$ is supposed to be a regular surface in $\mathbb{R}^4$ some known geometrical properties for surfaces in 4-space can be applied to it. In particular, the Proposition 3.1 can relate the geometry of the parabolic surface $\Sigma\Gamma$ to the singularity type of the corresponding height function (in the direction v) on M. Let us recall that when $\tilde{H}_{\mathrm{v}}$ has a degenerate (i.e., non Morse) singularity at a point p there exist one or two binormal directions with their respective osculating hyperplanes on the parabolic surface of M (considered a surface in 4-space).

According to Proposition 3.1, at a singularity of type A_2 of H_{v}, $\mathrm{v} \in N_pM$, the function $\tilde{H}_{\mathrm{v}}$ has an A_1 singularity and therefore the binormal direction of $\Sigma\Gamma$ at p does not necessarily coincide with the normal direction of M at p. On the other hand, at the singularities of type A_k, $k = 3, 4$, the direction $\mathrm{v} \in N_pM$ must be a binormal direction of surface $\Sigma\Gamma$ and the following result holds:

Corollary 4.1. Given a hypersurface M such that the height function of M at a point p has a singularity of type A_k, $k = 3, 4$, then one of the osculating hyperplanes of the parabolic surface of M at p coincides with the tangent hyperplane of M at p.

Proof. We can represent the hypersurface M in the Monge form $\psi(x, y, z) = (x, y, z, f(x, y, z))$, with the point p at the origin, where f is given by (1). So the tangent hyperplane to M at p, T_pM, is orthogonal to the vector $\mathbf{v} = (0, 0, 0, 1)$. Now, since the parabolic surface $\Sigma\Gamma$ is contained in M, we then have that T_pM is also a tangent hyperplane to $\Sigma\Gamma$ at p. Moreover, since the height function $H_{\mathbf{v}}$ on M in the direction $\mathbf{v} = (0, 0, 0, 1)$ has a singularity of type A_k, $k = 3, 4$, at p, we get from Proposition 3.1 that the height function on $\Sigma\Gamma$ in the direction $\mathbf{v} = (0, 0, 0, 1)$, $\tilde{H}_{\mathbf{v}}$, has an A_k, $k = 3, 4$, singularity at p, i.e., $\mathbf{v}$ is a binormal direction at p and hence T_pM must coincide with one of the osculating hyperplanes of the parabolic surface $\Sigma\Gamma$ at p. $\square$

As a consequence of Propositions 3.1 and 2.7, the versality conditions for the singularities of the height function $\tilde{H}_{\mathbf{v}}$ on $\Sigma\Gamma$ can be obtained in terms of the coefficents of f on M. The most interesting case occurs when the height function $H_{\mathbf{v}}$ on M has an A_3 versal singularity, as described in the following proposition. According to the previous notations, we have the following:

Proposition 4.2. Let us suppose that the function $H_{\mathbf{v}}$ on M has an A_3 versal singularity. Then, the function $\tilde{H}_{\mathbf{v}}$ on $\Sigma\Gamma$ has an A_3 versal singularity if and only if $6\, a_{004}\, a_{020}\, a_{200} - a_{200}\, {a_{012}}^2 - a_{020}\, {a_{102}}^2 \neq 0$.

Proof. According to the proof of Proposition 3.1, parabolic surface $\Sigma\Gamma$ of M can be parametrized as

$$g : (y, z) \mapsto \left(-\frac{a_{012}}{a_{102}} y + \varphi_2(y, z), y, z, \tilde{f}(y, z) \right).$$

The aim now to eliminate the linear term in the first coordinate and apply the typical methods of the theory of surfaces to $\mathbb{R}^4$ for studying the geometry of $\Sigma\Gamma$. By applying a rotation of matrix

$$A = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta & 0 & 0 \\ -\operatorname{sen}\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to g and taking $\theta = \arctan\left(\frac{a_{012}}{a_{102}}\right)$, the parabolic surface is given by $(Q_1(y, z), y, z, Q_2(y, z))$. Now, let us suppose that $H_{\mathbf{v}}$ has an A_3 versal singularity, therefore, $Q_1,\ Q_2 \in \mathcal{M}_2^2$, with

$$Q_2 \sim_{\mathcal{R}} \left(\frac{a_{012}^2 a_{200} + a_{020} a_{102}^2}{a_{102}^2} \right) y^2 + \frac{\left(4a_{004}a_{020}a_{200} - a_{200}{a_{012}}^2 - a_{020}{a_{102}}^2\right)\left(9a_{004}a_{020}a_{200} - 2a_{200}{a_{012}}^2 - 2a_{020}{a_{102}}^2\right)}{a_{020}a_{200}\left(a_{200}{a_{012}}^2 + a_{020}{a_{102}}^2\right)} z^4,$$

considering $a_{012}^2a_{200} + a_{020}a_{102}^2 \neq 0$ and $9a_{004}a_{020}a_{200} - 2a_{200}{a_{012}}^2 - 2a_{020}{a_{102}}^2 \neq 0$. By using Proposition 2.7, singularity A_3 is versal unfolding of $\tilde{H}_{\mathrm{v}}$ if and only if $a_{02} \neq 0$ or $b_{12} \neq 0$. In this case the coefficients are given by

$$a_{02} = -\frac{6\,a_{004}\,a_{020}\,a_{200} - a_{200}\,{a_{012}}^2 - a_{020}\,{a_{102}}^2}{a_{020}\,a_{102}\,a_{200}\sqrt{\frac{{a_{012}}^2+{a_{102}}^2}{{a_{102}}^2}}}$$

$$b_{12} = \frac{2\,a_{012}\,(2\,a_{200}{a_{102}}^2 + a_{200}\,{a_{012}}^2 + a_{020}\,{a_{102}}^2)\,(6\,a_{004}\,a_{020}\,a_{200} - a_{200}\,{a_{012}}^2 - a_{020}\,{a_{102}}^2)}{a_{200}a_{020}{a_{102}}^2\,({a_{012}}^2 + {a_{102}}^2)}$$

Therefore, singularity A_3 is non versal if and only if $6\,a_{004}\,a_{020}\,a_{200} - a_{200}\,{a_{012}}^2 - a_{020}\,{a_{102}}^2 = 0$. □

Observe that this case is generic for the surface $\Sigma\Gamma$, since these are isolated points on the A_3 curve. The fact that the singularity of $\tilde{H}_{\mathrm{v}}$ is of type A_3 (versal or non versal), coming from a versal singularity of type A_3 for H_{v}, provides interesting information on the geometry of $\Sigma\Gamma$ at the considered point.

Proposition 4.3. Let $p \in M$ be an A_3 versal singularity of H_{v} on M and a singularity of type A_3 of $\tilde{H}_{\mathrm{v}}$ on $\Sigma\Gamma$. Then, we have the following:

(1) The point p is a hiperbolic point of $\Sigma\Gamma$ if and only if p is an A_3 versal singularity of $\tilde{H}_{\mathrm{v}}$.

(2) The point p is a parabolic point of $\Sigma\Gamma$ if and only if p is an A_3 non versal singularity of $\tilde{H}_{\mathrm{v}}$ and ${a_{012}}^2a_{111}\,a_{200} + 2\,a_{012}\,a_{020}\,a_{102}\,a_{201} - 3\,a_{012}\,a_{020}\,a_{103}\,a_{200} - 2\,a_{012}\,a_{021}\,a_{102}\,a_{200} + 3\,a_{013}\,a_{020}\,a_{102}\,a_{200} - a_{020}\,{a_{102}}^2a_{111} \neq 0$.

Proof. Suppose that $p \in M$ is an A_3 versal singularity of H_{v} and of type A_3 for $\tilde{H}_{\mathrm{v}}$. According to Proposition 3.1 we have that $a_{012}^2a_{200} + a_{020}a_{102}^2 \neq 0$ and $9a_{004}a_{020}a_{200} - 2a_{200}{a_{012}}^2 - 2a_{020}{a_{102}}^2 \neq 0$.

On the other hand, from the proof of Proposition 4.2 we have that $\Sigma\Gamma$ can be written in the form $g(y,z) = (Q_1(y,z), y, z, Q_2(y,z))$ and only the calculation of the discriminant Δ of the surface $\Sigma\Gamma$ at p is required. The coefficients of the second fundamental form of g are given by:

$$\begin{aligned} l_1 &= Q_{1_{yy}}(0,0), \quad m_1 = Q_{1_{yz}}(0,0), \quad n_1 = Q_{1_{zz}}(0,0)\\ l_2 &= Q_{2_{yy}}(0,0), \quad m_2 = Q_{2_{yz}}(0,0), \quad n_2 = Q_{2_{zz}}(0,0) \end{aligned}$$

Therefore, the discriminant Δ of g at p is

$$\Delta = \frac{1}{4}\begin{vmatrix} l_1 & 2m_1 & n_1 & 0\\ l_2 & 2m_2 & n_2 & 0\\ 0 & l_1 & 2m_1 & n_1\\ 0 & l_2 & 2m_2 & n_2 \end{vmatrix}$$

$$= -\frac{16\,\left(6\,a_{004}\,a_{020}\,a_{200} - a_{200}\,{a_{012}}^2 - a_{020}\,{a_{102}}^2\right)^2\left(a_{200}\,{a_{012}}^2 + a_{020}\,{a_{102}}^2\right)^2}{{a_{020}}^2{a_{102}}^2{a_{200}}^2\left({a_{012}}^2 + {a_{102}}^2\right)^2}.$$

Moreover,

(1) p is a hiperbolic point of $\Sigma\Gamma$ if and only if $\Delta < 0$, if and only if $6\, a_{004}\, a_{020}\, a_{200} - a_{200}\, {a_{012}}^2 - a_{020}\, {a_{102}}^2 \neq 0$, which, from Proposition 4.2, provides the condition for an A_3 versal singularity.

(2) Analogously, p is a parabolic point of $\Sigma\Gamma$ if and only if $\Delta = 0$ and rank $\alpha = 2$ ([3]), where α is the matrix of the second fundamental form of g. The 2×2 minor determinants of α vanish simultaneously if and only if

$$\begin{cases} (6\, a_{004}\, a_{020}\, a_{200} - a_{200}\, {a_{012}}^2 - a_{020}\, {a_{102}}^2)\, (a_{200}\, {a_{012}}^2 + a_{020}\, {a_{102}}^2) = 0, \\ A\, (a_{200}\, {a_{012}}^2 + a_{020}\, {a_{102}}^2) = 0, \end{cases}$$

where

$$\begin{aligned} A = {} & {a_{012}}^2 a_{111}\, a_{200} + 2\, a_{012}\, a_{020}\, a_{102}\, a_{201} - 3\, a_{012}\, a_{020}\, a_{103}\, a_{200} - 2\, a_{012}\, a_{021}\, a_{102}\, a_{200} \\ & + 3\, a_{013}\, a_{020}\, a_{102}\, a_{200} - a_{020}\, {a_{102}}^2 a_{111}. \end{aligned}$$

Therefore, p is a parabolic point of $\Sigma\Gamma$ if and only if $A \neq 0$.

□

Remark 4.4. The case in which an A_3 versal singularity of H_{v} is an A_3 singularity of $\tilde{H}_{\mathrm{v}}$, either versal or not, was considered in the last proposition. In Proposition 3.1, A_3 versal singularities of H_{v} may also be A_4 or D_4 for $\tilde{H}_{\mathrm{v}}$. Even if height function H_{v} on M does not have corank 2 singularities, height function $\tilde{H}_{\mathrm{v}}$ on $\Sigma\Gamma$ may have inflection points, meaning $\tilde{H}_{\mathrm{v}}$ has umbilic singularities.

Proposition 4.5. Let $p \in M$ be an A_3 versal singularity of H_{v}. If p is not an A_3 singularity of $\tilde{H}_{\mathrm{v}}$, then p is either a hyperbolic point or an inflection point of $\Sigma\Gamma$.

Proof. The point p is an A_3 versal singularity of H_{v}. Since it is not A_3 singularity for $\tilde{H}_{\mathrm{v}}$, according to Prop. 3.1, p may be of either type A_4 or D_4.

(i) If p is an A_4 singularity for $\tilde{H}_{\mathrm{v}}$, then the conditions $a_{012}^2 a_{200} + a_{020} a_{102}^2 \neq 0$ and $9a_{004}a_{020}a_{200} - 2a_{200}{a_{012}}^2 - 2a_{020}{a_{102}}^2 = 0$ hold. By using them and writing the surface $\Sigma\Gamma$ as in the proof of Proposition 4.2, the discriminant Δ at p is given by

$$\Delta = -\frac{16\, \left(a_{200}\, {a_{012}}^2 + a_{020}\, {a_{102}}^2\right)^4}{9\, {a_{200}}^2 {a_{102}}^2 {a_{020}}^2 \left({a_{012}}^2 + {a_{102}}^2\right)^2}$$

i.e. , $\Delta < 0$ and p is a hyperbolic point of $\Sigma\Gamma$.

(ii) If p is an D_4 singularity of $\tilde{H}_{\mathrm{v}}$, then we have $a_{012}^2 a_{200} + a_{020} a_{102}^2 = 0$, implying that $\Delta = 0$ and rank $\alpha < 2$ at p, i.e., p is an inflection point of $\Sigma\Gamma$.

□

The same arguments can be applied when point p is an A_4 singularity of H_{v}. In such a case, hyperbolic points of $\Sigma\Gamma$ are obtained.

Proposition 4.6. Let $p \in M$ be an A_4 versal singularity for $H_{\mathbf{v}}$. Then p is a hyperbolic point of $\Sigma\Gamma$.

Proof. Since $p \in M$ is an A_4 versal singularity of $H_{\mathbf{v}}$, we get from Proposition 3.1 that p must be a singularity of type A_4 also for $\tilde{H}_{\mathbf{v}}$. Therefore, $\Sigma\Gamma$ can be parametrized as

$$\left(-\frac{a_{012}}{a_{102}}y + \varphi_3(y,z), y, z, \tilde{f}(y,z)\right).$$

Following an analogous argument to that of the proof 4.2, the same rotation matrix can be applied in order to eliminate the linear term and parametrize $\Sigma\Gamma$ as $(Q_1(y,z), y, z, Q_2(y,z))$ and hence, the discriminant Δ at the point p is given by

$$\Delta = -\frac{\left(a_{012}{}^2 a_{200} + a_{020}\, a_{102}{}^2\right)^4}{a_{200}{}^2 a_{102}{}^4 a_{020}{}^2 \left(a_{012}{}^2 + a_{102}{}^2\right)}.$$

Since from the hypothesis we have that $a_{012}{}^2 a_{200} + a_{020}\, a_{102} \neq 0$, we can conclude that $\Delta < 0$ and thus p is a hyperbolic point of $\Sigma\Gamma$. □

According to Proposition 3.6, when p is an A_3 versal singularity of $H_{\mathbf{v}}$, then $a_{012}^2 a_{200} + a_{020} a_{102}^2 = 0$ if and only if curvature κ of A_3 curve vanishes at p. Therefore, as a consequence of Proposition 3.6 and Corollary 4.5 we can state the following:

Corollary 4.7. Let p be an A_3 versal singularity of $H_{\mathbf{v}}$. If the curvature κ at p of the A_3 curve in M vanishes, then p is an inflection point of the parabolic surface $\Sigma\Gamma$.

Remark 4.8. The study of hypersurfaces in $\mathbb{R}^4$ has been lately very important. "In medicine, 4D models can be used in magnetic resonance imaging, computed tomography and ultrasound. In the case of magnetic resonance imaging methods that use 4D images have proven to be effective in facilitating the diagnosis of cardiovascular diseases. They differ from previous methods, both in terms of greater accuracy in obtaining a 3D model of the heart and in their ability to calculate blood flow in all directions. In the case of computed tomography a new scanning protocol was created to generate 4D images of the lung. Compared to previous protocols this new one has a shorter scanning time and obtains images of the entire respiratory cycle. In ultrasound applied to prenatal exams a 4D method allows the dynamic visualization of images of the fetal heart at different levels of depth and facilitates the diagnosis of congenital anomalies". See [2].

"It is of interest for several applications such as event detection, robotics, electronic games, animation, human-computer interaction, etc., the reconstruction of 3D computational models from physical objects. This is usually done by using computer vision techniques from 2D images or sensor data, which can be also interpreted as binary images. When the objects to be reconstructed are in motion, 4D images can be used to identify dynamics and occlusions of the objects and thus enable the creation of efficient computational models. Additional techniques can be used to complement the reconstruction, such as topology and mass conservation properties". See [2] for more references.

ACKNOWLEDGEMENTS. The authors would like to thank the referees for useful suggestions.

No conflict of interest was reported by the authors.

References

[1] Bruce, J.W. and Tari, F., *Families of surfaces in* $\mathbb{R}^4$, Proceedings of the Edinburgh Mathematical Society,45,1,181–203,2002,Cambridge University Press

[2] Castelo Filho,,A. and Bueno L. M. Aproximações de variedades definidas implicitamente utilizando técnicas de contagem e enumeração, Sociedade Brasileira de Matemática Aplicada e Computacional, 88, 2019

[3] Izumiya, S.; Romero-Fuster, M. C.; Ruas, M. A. S. and Tari, F., *Differential geometry from a singularity theory viewpoint*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2016).

[4] Mochida, D. and Romero- Fuster, M. C. and Ruas, M. A. S., *The geometry of surfaces in 4-space from a contact viewpoint*, Geometriae Dedicata, 54, 3, 323–332, 1995, Springer

[5] Nabarro, A. C., Sobre a geometria local de hipersuperfícies em $\mathbb{R}^4$, 2000, Universidade de São Paulo
`http://www.teses.usp.br/teses/disponiveis/55/55135/tde-12042016-101148`

[6] Nabarro, A. C., *Duality and contact of hypersurfaces in* $\mathbb{R}^4$ *with hyperplanes and lines*, Proceedings of the Edinburgh Mathematical Society, 46, 3, 637–648, 2003, Cambridge University Press

[7] Nabarro, A. C., Romero-Fuster, M. C., *3-manifolds in Euclidean space from a contact viewpoint*, Communications in analysis and geometry, 17, 4, 755–776, 2009.

[8] Nabarro, A. C., Romero-Fuster, M. C. and Zanardo, M. C., *Gauss maps on Canal Hypersurfaces of generic curves in* $\mathbb{R}^4$, Differential geometry and its applications, v. 79, p. 101816, 2021.

Ana Claudia Nabarro, Maria Carolina Zanardo
Departamento de Matemática, ICMC
Universidade de São Paulo - Campus de São Carlos
Avenida Trabalhador São-carlense, 400 - Centro,
São Carlos, SP, CEP 13566-590, Brazil
E-mail: anaclana@icmc.usp.br
E-mail: mcarolz@alumni.usp.br

Maria del Carmen Romero-Fuster
Departament de Matematiques
Universitat de València
46100 Burjassot (València), Espanya;
E-mail: carmen.romero@uv.es