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Multiplicity results for mass constrained Allen–Cahn equations on Riemannian manifolds with boundary

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Abstract

We present multiplicity results for mass constrained Allen–Cahn equations on a Riemannian manifold with boundary, considering both Neumann and Dirichlet conditions. These results hold under the assumptions of small mass constraint and small diffusion parameter. We obtain lower bounds on the number of solutions according to the Lusternik–Schnirelmann category of the manifold in case of Dirichlet boundary conditions and of its boundary in the case of Neumann boundary conditions. Under generic non-degeneracy assumptions on the solutions, we obtain stronger results based on Morse inequalities. Our approach combines topological and variational methods with tools from Geometric Measure Theory.

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Contents

1 Introduction

Let (M, g) be a smooth, compact n–dimensional Riemannian manifold with boundary, for $n \geq 2$. Let $W \in \mathcal{C}^3_{\text{loc}}(\mathbb{R}, [0, +\infty))$ a double-well potential such that $W^{-1}(0) = \{0, 1\}$. For fixed $\varepsilon, m > 0$, consider the following Allen–Cahn type equations with a mass constraint:

$$\begin{cases} -\varepsilon \Delta u_{\varepsilon,m} + \frac{1}{\varepsilon} W'(u_{\varepsilon,m}) = \lambda_{\varepsilon,m}, & \text{on } M, \\ \int_M u_{\varepsilon,m} \mathrm{d}v_g = m, \end{cases} \tag{1.1}$$

where the unknown parameter $\lambda_{\varepsilon,m} \in \mathbb{R}$ is a Lagrange multiplier related to the mass constraint $\int_M u_{\varepsilon,m} \mathrm{d}v_g = m$. We study (1.1) under both Neumann and Dirichlet boundary conditions, which leads respectively to the two following problems:

$$\begin{cases} -\varepsilon \Delta u_{\varepsilon,m} + \frac{1}{\varepsilon} W'(u_{\varepsilon,m}) = \lambda_{\varepsilon,m}, & \text{on } M, \\ \int_M u_{\varepsilon,m} = m, \\ \frac{\partial u_{\varepsilon,m}}{\partial \nu} = 0, & \text{on } \partial M, \end{cases} \tag{1.2}$$

where $\nu \in TM\big|_{\partial M}$ is the unit inner normal vector to ∂M in M, and

$$\begin{cases} -\varepsilon \Delta u_{\varepsilon,m} + \frac{1}{\varepsilon} W'(u_{\varepsilon,m}) = \lambda_{\varepsilon,m}, & \text{on } M, \\ \int_M u_{\varepsilon,m} = m, \\ u_{\varepsilon,m} = 0, & \text{on } \partial M. \end{cases} \tag{1.3}$$

A straightforward computation shows that solutions of the Neumann problem (1.2) can be found among the critical points of the functional

$$\mathcal{E}_\varepsilon(u) := \int_M \left(\varepsilon \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon} W(u) \right) \mathrm{d}v_g,$$

in the function space

$$\mathcal{H}_m := \left\{ u \in H^1(M, \mathbb{R}) : \int_M u \,\mathrm{d}v_g = m \right\},$$

while critical points of the restriction of $\mathcal{E}_\varepsilon$ to the subspace

$$\mathcal{H}_{m,0} := \left\{ u \in H_0^1(M, \mathbb{R}) : \int_M u \,\mathrm{d}v_g = m \right\},$$

are solutions of the Dirichlet problem (1.3).

The functional $\mathcal{E}_\varepsilon$ is widely used in the modelling of phase transition phenomena. For small $\varepsilon > 0$, critical points of $\mathcal{E}_\varepsilon$ in $\mathcal{H}_m$ and $\mathcal{H}_{m,0}$ develop transition layers whose profiles are close to solutions of isoperimetric-type problems. This link was first noticed (in the minimizing case, using the ideas of Γ-convergence) in the seminal work of Modica [42] in the case of Neumann boundary conditions and by Owen, Rubinstein and Sternberg [46] in the case of Dirichlet boundary conditions. Extensions to arbitrary critical points have been given by Hutchinson and Tonegawa [35]. Analogously, in the absence of a mass constraint critical points of Allen–Cahn functionals are linked to the min-max theory of minimal surfaces (and, more generally, submanifolds) which was initiated by Almgren [3, 4] and improved later by Pitts [54]. This theory has been object of an extensive study in the past years after the works of Marques and Neves (e.g., [39, 40]). In particular, a min-max theory for isoperimetric-type surfaces was developed by Zhou and Zhu [63]. As a consequence, there has been an increased and renewed interest in the study of existence and multiplicity of critical points for functionals of Allen–Cahn type and their asymptotic behavior as $\varepsilon \to 0$, departing from results of Gaspar and Guaraco [32, 33]. In the presence of a mass constraint, first studies in this direction have been carried out in [15] in the case of manifolds without boundary, while related vectorial problems have been recently studied in [6]. In a related direction, Bellettini and Wickramaseckera [10] studied the existence of solutions for a more general equation of Allen–Cahn-type associated to surfaces of prescribed mean curvature.

The purpose of this paper is to extend the main results of [15] to the case of manifolds with boundary. More precisely, we establish lower bounds on the number of solutions for (1.2) and (1.3) in terms of certain topological invariants of the underlying manifold. These lower bounds hold true for values of the parameters ε and m which are relatively small. We believe that these results on manifolds with boundary are of interest because they include the case of Euclidean domains in $\mathbb{R}^n$ (the one originally considered in classical works such as [42, 46]), which is left out in the setting of manifolds without boundary. Moreover, they require a fine understanding of relative isoperimetric-type problems for small volumes on Riemannian manifolds with boundary, which actually depend heavily on the boundary conditions (Dirichlet or Neumann) that are imposed. Overall, significant differences appear when one moves from the setting without boundary to the setting with boundary.

In order to state and prove our main results, we consider the following typical assumptions on the potential W:

(H1). The wells are non-degenerate global minimizers, i.e. $W''(0), W''(1) > 0$.
(H2). The potential W is coercive, i.e. there exist $R > 0$ and $\alpha > 0$ such that

$$W'(u)u \geq \alpha|u|^2, \quad \text{if } |u| \geq R.$$

(H3). The potential W has subcritical growth at infinity, i.e. there exist $p \in [2, 2^*)$, $\alpha' > 0$ and $R' > 0$ such that

$$|W''(u)| \leq \alpha'|u|^{p-2}, \quad \text{if } |u| \geq R'.$$

Here 2^* denotes the critical Sobolev exponent, i.e. $2^* = 2n/(n-2)$ if $n \geq 3$ and $2^* = +\infty$ if $n = 2$.

For $n = 2, 3$, a standard example of function $W : \mathbb{R} \to [0, +\infty)$ satisfying (H1), (H2) and (H3) is the quartic potential:

$$W_q(u) = u^2(u-1)^2.$$

We will denote by $\mathcal{E}_{\varepsilon,m}$ the restriction of $\mathcal{E}_\varepsilon$ to $\mathcal{H}_m$, and by $\overline{\mathcal{E}}_{\varepsilon,m}$ the restriction to $\mathcal{H}_{m,0}$, that is

$$\mathcal{E}_{\varepsilon,m} := \mathcal{E}_\varepsilon\big|_{\mathcal{H}_m} \quad \text{and} \quad \overline{\mathcal{E}}_{\varepsilon,m} := \mathcal{E}_\varepsilon\big|_{\mathcal{H}_{m,0}}.$$

Our multiplicity results for critical points of $\mathcal{E}_{\varepsilon,m}$ and $\overline{\mathcal{E}}_{\varepsilon,m}$ are established in term of certain topological invariants of M and ∂M that we quickly recall here, referring for instance to Benci [11] for more details on definitions. Given a topological space X, denote by $\mathrm{cat}(X)$ its *Lusternik–Schnirelmann category*, i.e. the smallest number of sets, open and contractible in X, needed to cover X ($\mathrm{cat}(X) = +\infty$ in case such number does not exist). Denote moreover by $\mathscr{P}_1(X)$ the sum of the *Betti numbers* of X or, equivalently, the value at 1 of the *Poincaré polynomial* of X. Following [11], here the Betti numbers are taken with coefficients in $\mathbb{Z}_2$.

The main results of the paper are the following ones.

Theorem A *Assume that (H1), (H2) and (H3) hold. Then, there exists $m^* > 0$ depending on M and g such that for all $m \in (0, m^*)$ there exists $\varepsilon_m, c_m > 0$ depending on M and g as well such that for any $\varepsilon \in (0, \varepsilon_m)$ the Neumann problem (1.2) has at least $\mathrm{cat}(\partial M)$ solutions $u_{\varepsilon,m}$ with $\mathcal{E}_\varepsilon(u_{\varepsilon,m}) \leq c_m$ and at least one solution $v_{\varepsilon,m}$ with $\mathcal{E}_\varepsilon(v_{\varepsilon,m}) > c_m$. Moreover, if m and ε as above are such that all critical points of $\mathcal{E}_{\varepsilon,m}$ are non-degenerate, then (1.2) has at least $\mathscr{P}_1(\partial M)$ solutions $u_{\varepsilon,m}$ with $\mathcal{E}_\varepsilon(u_{\varepsilon,m}) \leq c_m$ and at least $\mathscr{P}_1(\partial M) - 1$ solutions $v_{\varepsilon,m}$ with $\mathcal{E}_\varepsilon(v_{\varepsilon,m}) > c_m$.*

Theorem B *Assume that (H1), (H2) and (H3) hold. Then, there exists* $m^* > 0$ *depending on* M *and* g *such that for all* $m \in (0, m^*)$ *there exist* $\varepsilon_m, c_m > 0$ *depending on* M *and* g *as well such that for any* $\varepsilon \in (0, \varepsilon_m)$ *the Dirichlet problem* (1.3) *has at least* $\operatorname{cat}(M)$ *solutions* $u_{\varepsilon,m}$ *with* $\mathcal{E}_\varepsilon(u_{\varepsilon,m}) \le c_m$ *and, if* $\operatorname{cat}(M) > 1$*, there exists at least one solution* $v_{\varepsilon,m}$ *with* $\mathcal{E}_\varepsilon(v_{\varepsilon,m}) > c_m$*. Moreover, if* m *and* ε *as above are such that all critical points of* $\overline{\mathcal{E}}_{\varepsilon,m}$ *are non-degenerate, then* (1.3) *has at least* $\mathscr{P}_1(M)$ *solutions* $u_{\varepsilon,m}$ *with* $\mathcal{E}_\varepsilon(u_{\varepsilon,m}) \le c_m$ *and at least* $\mathscr{P}_1(M) - 1$ *solutions* $v_{\varepsilon,m}$ *with* $\mathcal{E}_\varepsilon(v_{\varepsilon,m}) > c_m$.

In Sect. 5 we discuss some (essentially known) results which state that solutions of (1.2) are non-degenerate in suitable *generic* situations. Roughly speaking, given the Eq. (1.2), one can perform an arbitrary small perturbation of a key parameter (more precisely, either the Riemannian metric or the boundary of the domain) so that the resulting problem is such that all solutions are non-degenerate for a dense set of ε and m. That is, the second parts of Theorems A and B, which are stronger than the first ones, apply in *almost* all cases.

Remark 1.1 Notice that while the lower bound given in Theorem B for the Dirichlet problem (1.3) is formulated in term of the topology of the whole of M (as in the case without boundary), the lower bound on the number of solutions given by Theorem A for the Neumann problem (1.2) depends solely on the topology of ∂M. This fact is related to the different nature of the limiting isoperimetric-type problem linked to each case in the $m, \varepsilon \to 0$ asymptotics.

Remark 1.2 It is easy to check that one always has $\operatorname{cat}(\partial M) > 1$, as ∂M is a closed and compact manifold. However, this is not always true for M (consider, for instance, a closed ball in $\mathbb{R}^n$).

Example 1.3 Let M be a closed ball in $\mathbb{R}^n$. Then we have $\operatorname{cat}(\partial M) = \mathscr{P}_1(\partial M) = 2$, so that by Theorem A we have at least 3 solutions for the Neumann problem (1.2) (in the appropriate parameter regime). In particular, we do not obtain a better result in the non-degenerate case when M is a ball. As $\operatorname{cat}(M) = 1$ and $\mathscr{P}_1(M) = 1$, Theorem B only guarantees the existence of one solution for the Dirichlet problem (1.3).

Example 1.4 The situation is different if we take M to be equal to an annulus in $\mathbb{R}^2$. In this case, ∂M is the union of two disjoint circles and this implies that $\operatorname{cat}(\partial M) = \mathscr{P}_1(\partial M) = 4$, so Theorem A gives in general at least 5 solutions for the Neumann problem (1.2), and at least 7 solutions in the non-degenerate case. Moreover, as $\operatorname{cat}(M) = \mathscr{P}_1(M) = 2$, Theorem B ensures the existence of at least three solutions for (1.3), with no improvement in the non-degenerate case.

Remark 1.5 The non-degeneracy assumption (H1) for the potential W is standard and, moreover, generic in several suitable senses (see Sect. 5 for more details). The coercivity assumption (H2) is also standard and guarantees that the Γ-convergence results for the energies $(\mathcal{E}_\varepsilon)_\varepsilon$ holds, see, e.g., Fonseca and Tartar [31], Owen, Rubinstein and Sternberg [46] (in the Euclidean case).

Regarding the assumption (H3), it is of more technical nature but also standard in variational problems concerning nonlinear elliptic PDEs or differential geometry. It ensures that these problems are compact or, more precisely, that both $\mathcal{E}_{\varepsilon,m}$ and $\overline{\mathcal{E}}_{\varepsilon,m}$

satisfy the Palais–Smale condition. As we prove in Sect. 6, it is possible to drop (H3) and obtain some partial results. However, it is not clear to us whether Theorems A and B still hold true if (H3) is dropped. Indeed, the coercivity assumption (H2) could be used to construct suitable energy decreasing projections or truncation-type operators in order to restore strong compactness of sublevel sets of $\mathcal{E}_\varepsilon$, but such projections or operators would not respect in general the mass constraint. One could also wonder about the validity of our result without the compactness assumption on the manifold M. In any case, these questions, although interesting, go beyond the scope of this paper.

Remark 1.6 Theorem A and Theorem B extend to *vectorial double-well potentials* $W: \mathbb{R}^k \to \mathbb{R}$ without substantial modifications on the proofs, as long as one assumes that

$$W^{-1}(\{0\}) = \{a_0, a_1\} \text{ and } D^2W(a_i) \text{ is positive definite for } i \in \{0, 1\},$$
$$\nabla W(u) \cdot u \geq \alpha |u|^2, \quad \text{if } |u| \geq R,$$

for some positive quantities α and R and

$$|D^2W(u)| \leq \alpha' |u|^{p-2}, \quad \text{if } |u| \geq R',$$

for some positive quantities α' and R' and for some p in $[2, 2^*)$. These assumptions are the vectorial analogues of (H1), (H2) and (H3). For simplicity, we have chosen to present the results for scalar potentials only. Nevertheless, as noticed in [6] (see also the more recent work [26]) and also as discussed below, the extension to vectorial potentials with three or more phases is not straightforward.

Our approach to prove Theorem A and Theorem B combines some topological and variational methods with Γ-convergence properties of $\mathcal{E}_\varepsilon$ and tools from Geometric Measure Theory. Variational methods relating the multiplicity of solutions of an elliptic problem with the topology of the underlying domain were already used in earlier works by Benci, Cerami and Passaseo [12–14], see also Benci [11] and references therein. The results contained in this paper are mainly inspired by the recent contribution [15], where the manifold M is supposed to be *closed* and compact (cf. also the earlier result [16]).

Roughly speaking, the idea of the proofs both here and in [15] consists on using the Γ-convergence properties of the functionals $\mathcal{E}_{\varepsilon,m}$ and $\overline{\mathcal{E}}_{\varepsilon,m}$ in order to show that under small mass constraint and for $\varepsilon > 0$ small enough the sublevel sets

$$\mathcal{E}^c_{\varepsilon,m} := \{u \in \mathcal{H}_m : \mathcal{E}_\varepsilon(u) \leq c\} \quad \text{and} \quad \overline{\mathcal{E}}^c_{\varepsilon,m} := \left\{u \in \mathcal{H}_{m,0} : \mathcal{E}_\varepsilon(u) \leq c\right\}$$

are, for suitable $c > 0$, homotopically equivalent to ∂M and M, respectively. The topological complexity of the above sublevel sets is hence comparable to that of ∂M and M, and we may apply the infinite-dimensional versions of Lusternik–Schnirelman and Morse theories (see Palais [48, 49]) to establish the stated lower bounds on the number of critical points.

There are other results in related settings which are also obtained by similar strategies. For instance, Jerrard and Sternberg [37] provided an abstract framework which allows to use “critical points” (in a non-smooth sense) of a Γ-limit functional in order to prove existence of critical points for the Γ-converging functionals when they are “close” enough to the Γ-limit. They applied this result to the 2D Allen–Cahn and 3D Ginzburg–Landau (with and without magnetic field) functionals. Improvements of this abstract approach in the concrete setting of Ginzburg–Landau functionals on manifolds without boundary have been made recently by Colinet, Jerrard and Sternberg [25] in 3D and by De Philippis and Pigati [51] in arbitrary dimension. Similar ideas, but also using different methods, had been applied before by Pacard and Ritoré [47] and Kowalczyk [38] in the Allen–Cahn setting. In the same spirit, Gaspar and Guaraco [32, 33] proved that the number of solutions of the Allen–Cahn equation (without volume constraint) in a closed manifold goes to infinity as $\varepsilon \to 0$. Their proof is restricted to even potentials, see also Passaseo [50]. By similar methods, Bellettini and Wickramaseckera [10] proved existence of solutions for a more general equation of Allen–Cahn type. Related results for the Ginzburg–Landau equations on manifolds were obtained by Stern [58], and by Pigati and Stern [52] in the case with magnetic field, see also [21, 22]. Moreover, earlier work of this type exists also in the context of the 2D Ginzburg–Landau functional considered by Bethuel, Brezis and Helein [17]. Almeida and Bethuel [1, 2] proved that there exists a lower bound on the number of critical points according to the topological degree of the boundary data as long as the perturbation parameter $\varepsilon > 0$ is small enough (see also Zhou and Zhou [61] for improvements).

In our case, the Γ-limit functionals related to $\mathcal{E}_{\varepsilon,m}$ and $\overline{\mathcal{E}}_{\varepsilon,m}$ correspond to different isoperimetric-type problems on M in the small volume regime, for which there are plenty of available results which provide a quite complete picture. In [20], Bérard and Meyer showed that the isoperimetric profile of a n-dimensional compact and smooth Riemannian manifold becomes comparable to the isoperimetric profile of $\mathbb{R}^n$ as the volume tends to zero. Morgan and Johnson [44] took one step further by proving that solutions to the isoperimetric problem with small volume are close to small spheres, so that their diameter tends to zero for vanishing volumes. More precisely, a result in [45] implies that these spheres are exactly those centered at a point of maximal scalar curvature (of the manifold), see also Druet [29]. Analogous results are available in the setting of compact Riemannian manifolds with boundary (without taking the boundary into account): Bayle and Rosales [9] proved the asymptotic equivalence with the relative isoperimetric profile of the *half*-space and Fall [30] showed that relative isoperimetric regions of small volume are close to half-spheres centered at the boundary points of maximal mean curvature. Moreover, one might also consider variants of the isoperimetric problem in which the boundary part of the manifold is also taken into account, still obtaining an asymptotic equivalence with the isoperimetric problem in $\mathbb{R}^n$.

One natural question concerns the extension of the results obtained in Theorem A and Theorem B to *systems* of Allen–Cahn-type. It turns out that the main obstacle for such extensions lies in the fine understanding of the resulting Γ-limit problem. It is known since the work of Baldo [8] that the Γ-limit functional for vectorial multi-well potentials corresponds to a minimization problem for the perimeter of *clusters* (with cardinality corresponding to the number of wells of the potential) under volume constraints. These problems can be seen as vectorial versions of the isoperimetric problem, and they are still poorly understood in the small volume regime. In particular, as pointed out in [6], in the vectorial case it is not known in general whether the diameters of the different cluster components of minimizers are uniformly controlled by their respective volume, which is a crucial ingredient in our approach. However, properties of that kind are available for clusters of three sets on the plane, allowing to extend the results of [15] to the case of triple-well potentials on 2D closed manifolds (see [6]). Further results is this direction have been obtained more recently in [26].

In a different direction, most of the recent results quoted above concerning the existence of critical points of Allen–Cahn and Ginzburg–Landau functionals (e.g., [6, 10, 25, 32, 33, 51, 52, 58]) are proven in the setting of manifolds without boundary. It is natural to wonder about their validity when one considers an underlying manifold which has a non-empty boundary as we do in this paper. Moreover, not much is known concerning Ginzburg–Landau functionals with a flux constraint on the vorticity, which are naturally associated to a limiting isoperimetric-type problem in codimension 2. To our knowledge, the only results in this direction were proven in [18] and by Chiron [24], where they were applied to problems of existence of traveling waves for the Gross–Pitaevskii equation. It is possible that these results could be extended to some degree and then used to obtain multiplicity results in the spirit of the present paper.

2 Sketch of the proofs

In this section we outline the main ingredients used in the proofs of Theorems A and B. We recall that a C^1 functional f on a Banach manifold $\mathfrak{M}$ is said to satisfy the Palais–Smale condition if every Palais–Smale sequence for f admits a converging subsequence, where $(x_k)_{k\in\mathbb{N}}$ is a Palais–Smale sequence for f if $(f(x_k))_{k\in\mathbb{N}}$ is bounded and $(\|\mathrm{d}f(x_k)\|)_{k\in\mathbb{N}}$ tends to 0 as $n \to \infty$. We shall notice here that given $x \in \mathfrak{M}$, the positive quantity $\|\mathrm{d}f(x)\|$ stands for the norm of $\mathrm{d}f(x)$ as an element of $(T_x\mathfrak{M})^*$, the cotangent space of $\mathfrak{M}$ at x. The Palais–Smale condition is a classical and useful sufficient condition in critical point theory. When it holds one only needs to detect a change of topology in the level sets of the functional in order to prove the existence of critical points. The interested reader can find more details in [48, 49]. In our case, the Palais–Smale condition is fulfilled:

Lemma 2.1 *Assume that (H3) holds. Then, for all $\varepsilon > 0$ and $m \in \mathbb{R}$ the functional $\mathcal{E}_\varepsilon$ is C^2 and satisfies the Palais–Smale condition on both $\mathcal{H}_m$ and $\mathcal{H}_{m,0}$. Moreover, if $u_{\varepsilon,m} \in \mathcal{H}_m$ is a critical point of $\mathcal{E}_{\varepsilon,m}$, then there exists $\lambda_{\varepsilon,m}$ such that $(u_{\varepsilon,m}, \lambda_{\varepsilon,m})$ solves (1.2) and if $u_{\varepsilon,m} \in \mathcal{H}_{m,0}$ is a critical point of $\overline{\mathcal{E}}_{\varepsilon,m}$, then there exists $\lambda_{\varepsilon,m}$ such that $(u_{\varepsilon,m}, \lambda_{\varepsilon,m})$ solves (1.3).*

The proof of Lemma 2.1 is standard. In particular, compactness of the Palais–Smale sequences for $\mathcal{E}_\varepsilon$ follows from the compactness of M along with the subcritical behavior of the potential provided by assumption (H3) (for more details see, for instance, [15, Proposition 4.12]).

According to the previous discussion, the proofs of Theorems A and B boil down to determining changes of topology in the sublevel sets of the functionals $\mathcal{E}_{\varepsilon,m}$ and $\overline{\mathcal{E}}_{\varepsilon,m}$. To do this, we employ the so-called *photography method*, which allows to understand some topological invariants of a sublevel of a functional through a homotopy equivalence with another topological space for which these invariants are more easily computable. More precisely, we will use an abstract result from [11, 13]. Before proceeding to the statement, let us recall some notions following [11, Chapter I]. Given $\mathfrak{M}$ a C^2-Hilbert manifold, $f : M \to \mathbb{R}$ of class C^2 and $u \in \mathfrak{M}$ a critical point of f, one can define the Hessian form of f at u, denoted by $H_f(u)$. Moreover, the latter is a bilinear form on $T_u(\mathfrak{M})$. If $\mathfrak{M}$ is an open subset of $\mathbb{R}^n$, then $H_f(u)$ is the usual Hessian of f at u. The Morse index of u (as a critical point of f) is then defined as the maximal dimension for which there exists a subspace of $T_u(\mathfrak{M})$ in which $H_f(u)$ is negative definite. If there exists $H^- \oplus H^+$ a splitting of $T_u(\mathfrak{M})$ along with a positive constant ν such that

$$H_f(u)[v, v] \geq \nu \|v\|^2, \quad \text{for all } v \in H^+$$

and

$$H_f(u)[v, v] \leq -\nu \|v\|^2, \quad \text{for all } v \in H^-,$$

then u is said to be non-degenerate. Finally, if u is isolated, then the multiplicity of u is defined as the formal series

$$\sum_{k=0}^{+\infty} \beta_k(f^c, f^c \setminus \{\overline{x}\}),$$

where $c := f(u)$, $f^c := \{x \in \mathfrak{M} : f(x) \leq c\}$ and $\beta_k(f^c, f^c \setminus \{x\})$ stands for the k-th Betti number of the pair $(f^c, f^c \setminus \{\overline{x}\})$ for all $k \in \mathbb{N}$. It can be then shown that the multiplicity of non-degenerate critical points is well-defined and equal to one. Recall that we are taking the Betti numbers relative to coefficients in $\mathbb{Z}_2$. However, as mentioned in [11], analogous results could be obtained for other choices of fields of coefficients.

Theorem C *Let X be a topological space, $\mathfrak{M}$ be a C^2-Hilbert manifold, $f : \mathfrak{M} \to \mathbb{R}$ be a C^1-functional. For $c \in \mathbb{R}$, let f^c be the c-sublevel set of f as defined above. Assume that*

- *(E1) f is bounded below;*
- *(E2) f satisfies the Palais–Smale condition;*
- *(E3) There exists $c \in \mathbb{R}$ and two continuous maps $\Psi_R : X \to f^c$ and $\Psi_L : f^c \to X$ such that $\Psi_L \circ \Psi_R$ is homotopic to the identity map of X.*

Then, the number of critical points in f^c is at least $\mathrm{cat}(X)$. *If $\mathfrak{M}$ is contractible and* $\mathrm{cat}(X) > 1$, *there is at least another critical point of f outside f^c. Moreover, there exists $c_0 \in (c, \infty)$ such that one of the two following conditions holds:*

(i) f^{c_0} *contains infinitely many critical points;*
(ii) f^c *contains* $\mathscr{P}_1(X)$ *critical points and* $f^{c_0} \setminus f^c$ *contains* $\mathscr{P}_1(X)-1$ *critical points if counted with their multiplicity. More precisely, we have the following relation:*

$$\sum_{u \in \mathrm{Crit}(f)} t^{\mu(u)} = t\mathscr{P}_t(X) + t^2[\mathscr{P}_t(X) - 1] + t(1+t)\mathcal{Q}(t),$$

where $\mathrm{Crit}(f)$ *is the set of critical points of* f, $\mu(u)$ *is the Morse index of the solution* u *and* $\mathcal{Q}(t)$ *is a polynomial with nonnegative integer coefficients. In particular, if all the critical points are non-degenerate, there are at least* $\mathscr{P}_1(X)$ *critical points in* f^c *and at least* $\mathscr{P}_1(X) - 1$ *critical points in* $f^{c_0} \setminus f^c$.

Since $\mathcal{E}_\varepsilon$ is bounded below (as it is nonnegative) and satisfies the Palais–Smale condition by Lemma 2.1, Theorem C implies Theorem A if we are able to show that for all m and ε sufficiently small there exist $c > 0$ and two continuous maps $\mathcal{P}_{\varepsilon,m} : \partial M \to \mathcal{E}^c_{\varepsilon,m}$ (the *photography* map) and $\mathcal{B} : \mathcal{E}^c_{\varepsilon,m} \to \partial M$ (the *barycenter* map) such that $\mathcal{B} \circ \mathcal{P}_{\varepsilon,m} : \partial M \to \partial M$ is homotopic to the identity map. Similarly, Theorem B will follow by showing the existence of two continuous maps $\mathcal{P}_{\varepsilon,m} : M \to \overline{\mathcal{E}}^c_{\varepsilon,m}$ and $\mathcal{B} : \overline{\mathcal{E}}^c_{\varepsilon,m} \to M$ such that $\mathcal{B} \circ \mathcal{P}_{\varepsilon,m} : M \to M$ is homotopic to the identity map of M. Analogously to [15], the two main ingredients that allow to construct the maps $\mathcal{P}_{\epsilon,m}$ and $\mathcal{B}$ are the following:

1. The Γ-convergence, as $\varepsilon \to 0^+$, of the energies $\mathcal{E}_\varepsilon$ towards some form of isoperimetric problem, the latter depending one the chosen boundary condition. In the case of Neumann boundary conditions, the Γ-limit is the classical relative isoperimetric problem, i.e., the boundary of M is not taken into account on the computation of the perimeter. If one considers homogeneous Dirichlet boundary conditions, then the Γ-limit takes into account the perimeter intersecting the boundary ∂M.
2. The fact that the isoperimetric profile and the relative isoperimetric profile on compact manifolds with boundary are asymptotic as $m \to 0^+$ to the isoperimetric profiles on the Euclidean space and half-space, respectively. As a consequence, for sufficiently small volumes any "almost minimizer" (a concept that will be formally defined later) of the relative isoperimetric problem concentrates almost its entire volume around a point on the boundary of M.
   For the isoperimetric problem, the same concentration phenomena occurs but for points on M. See Fig. 1 for an explanatory picture.

Combining this two ingredients, we find that in the Neumann case any function that belongs to the smallest sublevel $\mathcal{E}^c_{\varepsilon,m}$ that contains the image of the photography map has almost all its mass in a geodesic ball centered in a point of ∂M. By composing with the nearest point projection on ∂M, it follows that the barycenter map $\mathcal{B} : \mathcal{E}^c_{\varepsilon,m} \to \partial M$ is well defined and continuous. In the Dirichlet case, we have that an almost minimizer of $\overline{\mathcal{E}}_{\epsilon,m}$ has almost all its mass in a geodesic ball centered in a point of M, so that the barycenter map $\mathcal{B} : \mathcal{E}^c_\varepsilon \to M$ is well defined and continuous. Conversely, one finds the photography maps $\mathcal{P}_{\varepsilon,m}$ by constructing almost minimizers of $\mathcal{E}_\varepsilon$ supported on geodesic balls centered at arbitrary points of ∂M in the Neumann case and of M in the Dirichlet case.

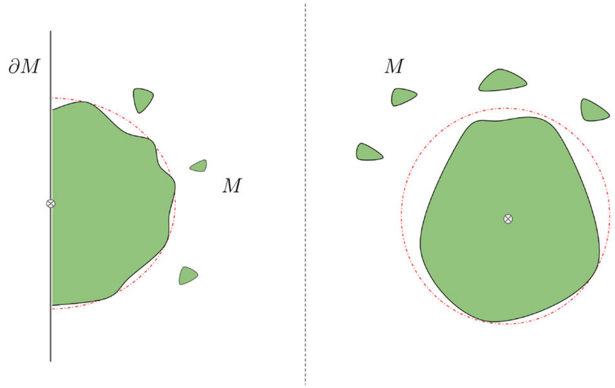

Fig. 1 The filled regions represent an arbitrary almost minimizer of the relative isoperimetric problem (left) and the isoperimetric problem (right). In the former case, most of the volume is concentrated inside a half-ball (dotted line) centered at the boundary. In the latter case, concentration occurs around some point of M. However, some of the volume might be away from these balls, but it is always a small portion of the volume which can be controlled. In particular, half-balls and balls are almost minimizers of the relative isoperimetric and isoperimetric problems, respectively

Remark 2.2 To apply Theorem C in the proof of Theorem A, one needs that $\mathrm{cat}(\partial M) > 1$ and that $\mathcal{H}_m$ is contractible. Since these two conditions are always satisfied (in fact, $\mathcal{H}_m$ is convex), they are omitted in the statement of Theorem A. This is also the reason why the condition $\mathrm{cat}(M) > 1$, not necessarily fulfilled a priori, appears in the statement of Theorem B.

3 Proof of Theorem A

3.1 The relative isoperimetric problem

We recall some standard notations. Let $BV(M, \mathbb{R})$ be the set of functions of bounded variation defined on M. For any $\Omega \subset M$ a measurable set, we denote by $\mathbf{1}_\Omega$ its characteristic function and define its perimeter as the total variation measure of $\mathbf{1}_\Omega$. If the perimeter of Ω is finite, we say that Ω is a set of finite perimeter (or a Cacciopoli set). Let us denote by $\mathcal{C}_g(M)$ the set of all $\Omega \subset M$ of finite perimeter. For Ω in $\mathcal{C}_g(M)$, one can define $\partial^*\Omega$, the reduced boundary of Ω, as the set of points of $\partial\Omega$ in which the notion of measure-theoretic normal vector exists and has length equal to one. The reduced boundary is denoted as $\partial^*\Omega$. Such a notion was introduced by De Giorgi [27], who proved (see also Ambrosio, Fusco and Pallara [5, Theorem 3.59]) that

$$\int_U |\nabla \mathbf{1}_\Omega| = H^{n-1}(\partial^*\Omega \cap U), \tag{3.1}$$

for all $U \subset M$ relatively open in M and where $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure on $\mathbb{R}^{\tilde{n}}$. Let $\sigma := \int_0^1 \sqrt{W(s)}\mathrm{d}s$. For $u \in L^1(M, \mathbb{R})$ we set

$$\mathcal{E}_0(u) := \begin{cases} \sigma \mathcal{H}^{n-1}(\partial^* \Omega \cap \mathrm{int}(M)), & \text{if } u = \mathbf{1}_\Omega, \quad \Omega \in \mathcal{C}_g(M) \\ +\infty, & \text{otherwise.} \end{cases}$$

According to (3.1), one can equivalently define $\mathcal{E}_0$ using the notion of relative perimeter of Ω (up to the multiplicative constant σ), which is defined as the total measure of $|\nabla \mathbf{1}_\Omega|$ in $\mathrm{int}(M)$.

For fixed $m \in [0, \mathrm{vol}(M)]$, we define

$$I_M(m) := \inf \left\{ \mathcal{H}^{n-1}(\partial^* \Omega \cap \mathrm{int}(M)) : \Omega \in \mathcal{C}_g(M) \text{ and } \int_M \mathbf{1}_\Omega \mathrm{d}v_g = m \right\}.$$

Notice that $I_M : [0, \mathrm{vol}(M)] \to \mathbb{R}$ is the *isoperimetric profile function* of M and that

$$I_M(m) = \frac{1}{\sigma} \inf \left\{ \mathcal{E}_0(u) : u \in L^1(M, \mathbb{R}), \int_M u \, \mathrm{d}v_g = m \right\}.$$

The same problem can be stated in the euclidean semi-space

$$\mathbb{R}^n_+ = \left\{ z \in \mathbb{R}^n : z^n \geq 0 \right\} = \mathbb{R}^{n-1} \times [0, +\infty),$$

so that the function $I_{\mathbb{R}^n_+} : (0, +\infty) \to \mathbb{R}$ is defined analogously and there exists a dimensional constant $c_n^+ > 0$ such that

$$I_{\mathbb{R}^n_+}(m) = c_n^+ m^{\frac{n-1}{n}}. \tag{3.2}$$

When m is small, there exists a link between the isoperimetric function I_M and its euclidean counterpart. More formally, Bayle and Rosales [9] (see also Fall [30]) have shown that

$$I_M(m) = \left(1 + \mathcal{O}(m^{1/n})\right) I_{\mathbb{R}^n_+}(m). \tag{3.3}$$

3.2 Γ-convergence

We now state the two main Γ-convergence results we need, starting with the the most celebrated one, first proven by Modica and Mortola [42, 43], and later extended to the vector-valued setting by several authors: Baldo [8], Fonseca and Tartar [31] and Sternberg [59]. In [31], the double-well case was considered without the subcritical growth condition (H3), meaning that their result is the most well adapted to our situation. We also point out that in the above mentioned papers the results are proven in the Euclidean case, but it was later shown (see Benci et al. [15] and the references therein) that they can be extended to the Riemannian setting.

Theorem 3.1 (cf. [31], Theorem 3.4 and [15], Proposition 3.3) *Assume that (H1) and (H2) hold. Then, the following statements hold:*

(i) *(Γ-$\liminf$ inequality): If $(\varepsilon_k)_{k\in\mathbb{N}} \subset]0,+\infty[$ is such that $\varepsilon_k \to 0^+$ and $(u_{\varepsilon_k})_{k\in\mathbb{N}} \subset H^1(M,\mathbb{R})$ is such that $u_{\varepsilon_k} \to u_0$ in $L^1(M,\mathbb{R})$, then $\liminf_{k\to\infty} \mathcal{E}_{\varepsilon_k}(u_{\varepsilon_k}) \geq \mathcal{E}_0(u_0)$;*

(ii) (Γ-$\limsup$ (in)equality)*: For any $u_0 \in L^1(M,\mathbb{R})$ such that $u_0 = \mathbf{1}_\Omega$ for some $\Omega \subset M$ and for every sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset]0,+\infty[$ such that $\varepsilon_k \to 0^+$, there exists $(u_{\varepsilon_k})_{k\in\mathbb{N}} \subset H^1(M,\mathbb{R})$ such that $u_{\varepsilon_k} \to u_0$ in $L^1(M,\mathbb{R})$,*

$$\int_M u_{\varepsilon_k} \mathrm{d}v_g = \int_M u_0 \mathrm{d}v_g \quad \textit{and} \quad \limsup_{k\to\infty} \mathcal{E}_{\varepsilon_k}(u_{\varepsilon_k}) \leq \mathcal{E}_0(u_0).$$

Remark 3.2 The interested reader can find in [15, Proposition 3.3] a detailed and explicit construction of the functions u_ε that approximate u_0 as required by the Γ-$\limsup$ statement of Theorem 3.1, which is the Riemannian version of the construction given by Modica in [42] in the Euclidean setting. We give here just the idea of this construction, following [42]. For every $A \subset M$ we denote by $d_A : M \to \mathbb{R}$ the following function:

$$d_A(x) := \begin{cases} -\mathrm{dist}_g(x, \partial A), & \text{if } x \in A, \\ \mathrm{dist}_g(x, \partial A), & \text{if} x \notin A. \end{cases} \tag{3.4}$$

Subsequently, for every $\varepsilon > 0$ one can find a Lipschitz continuous function $\widetilde{q}_\varepsilon : \mathbb{R} \to [0,1]$ such that $\widetilde{q}_\varepsilon(t) = 0$ if $t < 0$, and $\widetilde{q}_\varepsilon(t) = 1$ if $t \geq \eta_\varepsilon > 0$, where $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, such a function is a solution of the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{q}_\varepsilon(t) = \frac{1}{\varepsilon}\sqrt{\varepsilon^{3/2} + 2W(\widetilde{q}_\varepsilon(t))}, \qquad \text{for all } t \in [0, \eta_\varepsilon]. \tag{3.5}$$

Finally, for every $u_0 = \mathbf{1}_\Omega$, u_ε is obtained as

$$u_\varepsilon(x) = \widetilde{q}_\varepsilon(d_{M\setminus\Omega}(x) + \delta_{\varepsilon,\Omega}), \tag{3.6}$$

where $\delta_{\varepsilon,\Omega} \in [0, \eta_\varepsilon]$ is a correction term such that

$$\int_M u_\varepsilon \, \mathrm{d}v_g = \int_M \widetilde{q}_\varepsilon(d_{M\setminus\Omega}(x) + \delta_{\varepsilon,\Omega})\mathrm{d}v_g = \int_M u_0 \, \mathrm{d}v_g. \tag{3.7}$$

Theorem 3.3 (cf. Theorem 4.1 of [31]) *Assume that (H1) and (H2) hold. If $(\varepsilon_k)_k \subset \mathbb{R}$ is a sequence of positive numbers such that $\varepsilon_k \to 0^+$ and $(u_{\varepsilon_k})_k \subset H^1(M,\mathbb{R})$ is a sequence of functions such that*

$$\mathcal{E}_{\varepsilon_k}(u_{\varepsilon_k}) \leq c, \qquad \forall k \in \mathbb{N},$$

for some constant $c > 0$, then there exists a subsequence, still denoted by $(\varepsilon_k)_k$, such that $(u_{\varepsilon_k})_k$ converges to a function $u_0 \in L^1(M,\mathbb{R})$ with respect to the L^1–norm.

Remark 3.4 As a direct consequence of Theorem 3.3 and the lim-inf property of Theorem 3.1, if $\mathcal{E}_{\epsilon_k}(u_{\varepsilon_k}) \le c$ for every $k \in \mathbb{N}$, then the function $u_0 \in L^1(M, \mathbb{R})$ such that $u_{\varepsilon_k} \to u_0$ up to subsequences is such that $\mathcal{E}_0(u_0) \le E^*$, there exists a measurable set $\Omega \subset M$ such that $u_0 = \mathbf{1}_\Omega$ and

$$\lim_{k\to\infty} \int_M u_k \, \mathrm{d}v_g = \int_M u_0 \, \mathrm{d}v_g .$$

3.3 Photography map

In order to obtain the map $\mathcal{P}_{\varepsilon,m}$, we combine the Γ-convergence result given by Theorem 3.1 with the characterization of the isoperimetric regions in a manifold with boundary given by Fall [30]. Roughly, as long as m and ε are sufficiently small, for every point $p \in \partial M$ we define $\mathcal{P}_{\varepsilon,m}(p)$ as the ε–approximation ensured by the lim-sup part of Theorem 3.1 of the characteristic function of a small perturbation of a half-ball centered at p and with volume m, denoted by $E_{p,m} \subset M$. Such a set is defined by employing the normal coordinates at the point p, and by using the implicit function theorem to construct the perturbation that minimizes the perimeter. The details of the construction of $E_{p,m}$ can be explored in [30]. However, for the purposes of this article, we will provide a concise overview of the essential details.

Let $N_{\partial M}$ the unit interior normal vector field along ∂M and, for any $p \in \partial M$, let $\exp_p^{\partial M}$ be the exponential map of ∂M at p. Let us consider an orthonormal frame field $(e_1, \dots, e_{n-1}, N_{\partial M})$ of M along ∂M. We define the map $f^p \colon \mathbb{R}^{n-1} \to \partial M$ as follows:

$$f^p(z) := \exp_p^{\partial M}(z^i e_i),$$

where we used the Einstein summation convention. For every $z \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$, with $t \ge 0$ and sufficiently small, let us define also

$$F^p(z, t) = \exp_{f^p(z)}(t N_{\partial M}) \in M,$$

so that F^p provides a local parametrization of a neighborhood of $p \in M$. Our aim is to define a "semi-sphere" on M with center in p. To this aim, let us denote by B^{n-1} the unit ball in $\mathbb{R}^{n-1}$ and by $S_+^{n-1} \in \mathbb{R}^n$ the upper hemisphere of S^{n-1}, hence

$$S_+^{n-1} := \big\{ z \in \mathbb{R}^n : \|z\| = 1, z^n \ge 0 \big\}.$$

Let us denote by $\Theta \colon B^{n-1} \to S_+^{n-1}$ the inverse of the stereographic projection from the south pole $(0, \dots, 0, -1) \in \mathbb{R}^n$, formally seeing B^{n-1} as $\{z \in \mathbb{R}^n : \|z\| \le 1, z^n = 0\}$. Let $\widetilde{\Theta} \colon B^{n-1} \to B^{n-1}$ and $t \colon B^{n-1} \to [0, 1]$ be such that

$$\Theta(z) = \big(\widetilde{\Theta}(z), t(z)\big).$$

For every r sufficiently small, the hypersurface defined by

$$\left\{F^p\left(r\widetilde{\Theta}(z), rt(z)\right) : z \in B^{n-1}\right\} \subset M$$

can be seen as an semi-sphere centered at p. All the hypersurfaces nearby this semi-sphere centered at $p \in \partial M$ can be obtained by applying a small perturbation $\omega\colon S_+^{n-1} \to \mathbb{R}$, and so we define the hypersurface $\Sigma_{p,r,\omega} \subset M$ as follows:

$$\Sigma_{p,r,\omega} := \left\{F^p\left(\left(r + \omega(\Theta(z))\right)\widetilde{\Theta}(z), \left(r + \omega(\Theta(z))\right)t(z)\right) : z \in B^{n-1}\right\}.$$

Since $t(z) = 0$ for every $z \in \partial B^{n-1}$, $\partial\Sigma_{p,r,\omega} \subset \partial M$, so that $\Sigma_{p,r,\omega}$ and ∂M enclose a set that we denote by $E_{p,r,\omega}$. It can be checked that $\Sigma_{p,r,\omega}$ does not depend on the orthonormal frame field chosen above.

By [30, Lemma 4.6], there exists $r_0 > 0$ such that for any $p \in \partial M$ and $r \in (0, r_0)$ there exists a unique smooth $\omega^{p,r} \in C^{2,\alpha}(S_+^{n-1})$ such that $\Sigma_{p,r,\omega^{p,r}}$ has a mean curvature function that is, modulo an additive constant, an eigenfunction of the first strictly positive eigenvalue of the Laplacian on the hemisphere. In this way, $\Sigma_{p,r,\omega^{p,r}}$ represents an almost CMC (constant mean curvature) half-sphere centered at p, thereby implying that it is the unique competitor among all hypersurfaces $\Sigma_{p,r,\omega}$ to serve as a solution to the isoperimetric problem. Moreover, $\|\omega^{p,r}\|_{C^{1,\alpha}(S_+^{n-1})} \to 0$ as $r \to 0$ and, since it is defined by using the Implicit Function Theorem, the map $(p, r) \mapsto \omega^{p,r}$ is continuous. We denote as *pseudo half-bubbles* the hypersurfaces $\Sigma_{p,r,\omega^{p,r}}$, in analogy with the *pseudo-bubbles* introduced in [45].

Following the notation in [30], we will use the simpler notation $E_{p,r}$ to refer to $E_{p,r,\omega^{p,r}}$. Furthermore, for any $p \in \partial M$ and sufficiently small m, we denote by $r_{p,m} > 0$ the unique radius such that $E_{p,r_{p,m}}$ has volume m, i.e., $\int_M \mathbf{1}_{E_{p,r_{p,m}}} \mathrm{d}v_g = m$. For more detailed information on the existence of such $r_{p,m}$, please refer to [30, Lemma 4.7].

Remark 3.5 By the compactness assumption of M and ∂M, there exists $m_0 > 0$ such that for every $p \in \partial M$ and $m \in (0, m_0)$ we have that $r_{p,m}$ is well defined and less than r_0. As a consequence, $\omega^{p,r_{p,m}}$ is always well defined by [30, Lemma 4.6] for every $p \in \partial M$ and $m \in (0, m_0)$.

For the sake of notation, for every $p \in \partial M$ and $m \in (0, m_0)$ we set

$$E_{p,m} := E_{p,r_{p,m}} \quad \text{and} \quad \Sigma_{p,m} := \Sigma_{p,r_{p,m},\omega^{p,r_{p,m}}} = \partial E_{p,m} \cap M.$$

Let us notice that, since $m < m_0$, the hypersurface $\Sigma_{p,m}$ is smooth. Finally, we will denote by $u_{0,p}^m \in L^1(M, \mathbb{R})$ the characteristic function of $E_{p,m}$, hence

$$u_{0,p}^m(x) := \begin{cases} 1, & \text{if } x \in E_{p,m}, \\ 0, & \text{otherwise.} \end{cases}$$

Before defining the photography map, let us state [30, Lemma 4.8] in accordance with our notation.

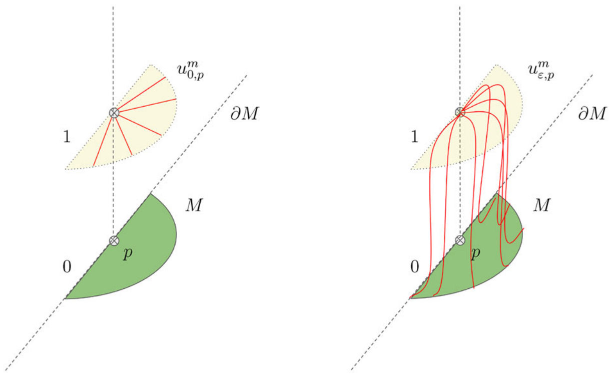

Fig. 2 Depiction of the map $\mathcal{P}_{\varepsilon,m}$ introduced in Definition 3.8. On the left, we see $u^m_{0,p}$, the indicator function of the set $E_{p,m}$, which is the set enclosed by the semi-sphere $\Sigma_{p,m}$ and ∂M. The map $\mathcal{P}_{\varepsilon,m}$ returns $u^m_{\varepsilon,m}$, depicted on the right, which is the smooth approximation of $u^m_{0,p}$ given by the Γ-convergence result (Theorem 3.1)

Lemma 3.6 *For every $p \in \partial M$, let us denote by $H_{\partial M}(p)$ the mean curvature of ∂M at p. Then, there exists a dimensional constant $\gamma_n > 0$ such that*

$$\mathcal{E}_0(u^m_{0,p}) = \sigma \left(I_{\mathbb{R}^n_+}(m) - \gamma_n H_{\partial M}(p) m + \mathcal{O}(m^{\frac{n+1}{n}}) \right), \qquad \forall m \in (0, m_0), \quad (3.8)$$

and, moreover, the following holds:

$$I_M(m) = I_{\mathbb{R}^n_+}(m) - \gamma_n \max_{p \in \partial M} \{H_{\partial M}(p)\} m + \mathcal{O}(m^{\frac{n+1}{n}}). \quad (3.9)$$

Remark 3.7 For our specific purposes, it is possible to avoid the use of the perturbation $\omega^{p,r}$ by setting $\omega \equiv 0$ in the definitions of $\Sigma_{p,r,\omega}$, $E_{p,r}$, and $u^m_{0,p}$. This choice is viable because the primary tool we will employ in the subsequent construction is the estimate (3.8), which remains valid even when $\omega \equiv 0$. However, in order to avoid a detailed proof of this last assertion, as it falls outside the scope of our work, we rely on the estimate provided by [30].

Definition 3.8 For every $m \in (0, m_0)$ and $\varepsilon > 0$, $\mathcal{P}_{\varepsilon,m} \colon \partial M \to \mathcal{H}_m$ is the map that links $p \in \partial M$ to $u^m_{\varepsilon,p}$, that is the ε–approximation of $u^m_{0,p} \in L^1(M, \mathbb{R})$ given by the *lim-sup* property of Theorem 3.1 (see Remark 3.2).

See Fig. 2 for a picture aimed at clarifying Definition 3.8. The main result of this section is the following, which provides an estimate on the smallest sublevel of $\mathcal{E}_{\varepsilon,m}$ that contains the image of the photography map.

Proposition 3.9 *Assume that (H1) and (H2) hold. There exists a constant $\theta = \theta(M, g, W) > 0$ such that there exists $m_1 = m_1(M, g, W, \theta) \in]0, m_0[$ such that for every $m \in]0, m_1[$ there exists $\varepsilon_1 = \varepsilon_1(M, g, W, \theta, m) > 0$ such that for every $\varepsilon \in]0, \varepsilon_1[$ we have*

$$\mathcal{E}_{\varepsilon,m}(\mathcal{P}_{\varepsilon,m}(p)) \leq \sigma I_M(m) + \theta m, \qquad \forall p \in \partial M. \quad (3.10)$$

In other words, the sublevel $\mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m}$ contains the whole image of the photography map $\mathcal{P}_{\epsilon,m}$.

Proof Combining (3.8) and (3.9), we get

$$\mathcal{E}_0(u^m_{0,p}) = \sigma\left(I_M(m) + \gamma_n\big(\max_{p\in\partial M}\{H_{\partial M}(p)\} - H_{\partial M}(p)\big)m\right) + \mathcal{O}(m^{\frac{n+1}{n}}).$$

Hence, setting $\theta = \theta(M, g) > 0$ as

$$\theta = 2\sigma\gamma_n\left(\max_{p\in\partial M}\{H_{\partial M}(p)\} - \min_{p\in\partial M}\{H_{\partial M}(p)\} + 1\right),$$

we have

$$\mathcal{E}_0(u^m_{0,p}) < \sigma I_M(m) + \frac{\theta}{2}m + \mathcal{O}(m^{\frac{n+1}{n}}), \qquad \forall p \in \partial M.$$

As a consequence, there exists $m_1 = m_1(M, g, \theta) \in (0, m_0)$ such that for every $m \in (0, m_1)$ we have

$$\mathcal{E}_0(u^m_{0,p}) < \sigma I_M(m) + \theta m, \qquad \forall p \in \partial M. \tag{3.11}$$

By Theorem 3.1 and Remark 3.2, we also know that

$$\limsup_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(u^m_{\varepsilon,p}) = \limsup_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(\mathcal{P}_{\varepsilon,m}(p)) \le \mathcal{E}_0(u^m_{0,p}), \qquad \forall p \in \partial M. \tag{3.12}$$

By the strict inequality in (3.11) and since M is a compact manifold, by (3.12) we infer the existence of $\varepsilon_1 = \varepsilon_1(M, g, W, \theta, m) > 0$ such that (3.10) holds for every $\varepsilon \in (0, \varepsilon_1)$. □

Remark 3.10 By (3.2) and (3.3), namely that $I_M(m) \approx m^{\frac{n-1}{n}}$ as m goes to zero, we have

$$\lim_{m\to 0^+} \frac{m}{I_M(m)} = 0, \tag{3.13}$$

hence the "width" of the sublevel that contains the photography goes rapidly to 0 as $m \to 0^+$.

In order to use $\mathcal{P}_{\varepsilon,m}$ as a photography map in the sense of Theorem C, it remains to prove its continuity.

Proposition 3.11 *Assume that (H1) and (H2) hold. Let $m_1 > 0$ and $\varepsilon_1 > 0$ be defined as in Proposition 3.9. For every $m \in (0, m_1)$ and $\varepsilon \in (0, \varepsilon_1)$, the map $\mathcal{P}_{\varepsilon,m} \colon \partial M \to \mathcal{H}_m$ is continuous.*

Proof The proof follows the same line of the one of [15, Proposition 4.14], and we report here the main steps.

We recall that for every $p \in \partial M$ and $\varepsilon > 0$ the function $\mathcal{P}_{\varepsilon,m}(p)$ is defined by using (3.6), hence we have

$$\big(\mathcal{P}_{\varepsilon,m}(p)\big)(x) = \widetilde{q}_\varepsilon\big(d_{M\setminus E_{p,m}}(x) + \delta_{\varepsilon,E_{p,m}}\big),$$

where $d_{M\setminus E_{p,m}}$ is defined by using (3.4) and $\widetilde{q}_\varepsilon$ is the Lipschitz continuous function that solves (3.5). As a consequence, there exists a constant $C > 0$, which depends on M, g, ε and W, such that the following inequality holds:

$$\begin{aligned}&\|\mathcal{P}_{\varepsilon,m}(p_1) - \mathcal{P}_{\varepsilon,m}(p_2)\|_{H^1(M,\mathbb{R})}\\ &\quad\le C\Big[\|d_{M\setminus E_{p_1,m}} - d_{M\setminus E_{p_2,m}}\|_{H^1(M,\mathbb{R})} + |\delta_{\varepsilon,E_{p_1,m}} - \delta_{\varepsilon,E_{p_2,m}}|\Big].\end{aligned}$$

The interested reader can find more details on this last inequality in [15, Proposition 4.14]. Let us observe that, as long as $m < m_0$, the hypersurface $\Sigma_{p,m} = \partial E_{p,m} \cap M$ is smooth and diffeomorphic to a hemisphere. The proof ends by observing that, as $p_1 \to p_2$, we have

$$\|d_{M\setminus E_{p_1,m}} - d_{M\setminus E_{p_2,m}}\|_{H^1(M,\mathbb{R})} \to 0 \quad \text{and} \quad |\delta_{\varepsilon,E_{p_1,m}} - \delta_{\varepsilon,E_{p_2,m}}| \to 0,$$

the last convergence obtained as a consequence of the Implicit Function Theorem applied to (3.7), where $E_{p,m}$ substitutes Ω, to define the C^1 map $p \mapsto \delta_{\varepsilon,E_{p,m}}$. □

3.4 Barycenter map

The map $\mathcal{B}\colon \mathcal{E}^c_{\varepsilon,m} \to \partial M$ is obtained by combining the Γ-convergence result with the information of the isoperimetric problem given by Proposition 3.15, which ensures that the "almost" minimizers of $\mathcal{E}_0$ have almost all their mass inside a small ball centered on a point of the boundary ∂M. While the map $\mathcal{P}_{\varepsilon,m}$ has been constructed by using the lim-sup part of the Γ-convergence result, namely Theorem 3.1, to define $\mathcal{B}$ we rely on the lim-inf statement and on the compactness result ensured by Theorem 3.3.

Let us begin by introducing some new notation. Let $\text{dist}_g\colon M \times M \to \mathbb{R}$ be the distance function induced by g, hence

$$\text{dist}_g(x,y) := \inf\left\{\left(\int_0^1 g(\dot\gamma,\dot\gamma)\mathrm{d}s\right)^{1/2} : \gamma \in H^1([0,1],M), \gamma(0) = x, \gamma(1) = y\right\}.$$

With this notation, we set $\text{diam}_g(M) = \max\{\text{dist}_g(x,y) : x, y \in M\}$, and for any $p \in M$ and $r \in (0, \text{diam}_g(M))$, let

$$B(p,r) := \big\{x \in M : \text{dist}_g(p,x) \le r\big\}.$$

Without loss of generality, from now on we assume that M is isometrically embedded in $\mathbb{R}^{\tilde n}$, for some $\tilde n \ge n$. This will allow us to construct the map $\mathcal{B}$ as the composition of the "extrinsic" barycenter map and the nearest point projection on ∂M. More formally, let $(\partial M)_r \subset \mathbb{R}^{\tilde{n}}$ be the following set

$$(\partial M)_r := \left\{ z \in \mathbb{R}^{\tilde{n}} : \mathrm{dist}_{\mathbb{R}^{\tilde{n}}}(z, \partial M) \leq r \right\},$$

and let us denote by $\pi_{\partial M} \colon (\partial M)_r \to \partial M$ the nearest point projection onto ∂M. Notice that $\pi_{\partial M}$ is well defined when r is sufficiently small, thanks to the compactness assumption on ∂M. We will combine $\pi_{\partial M}$ with the following map

$$\beta^* \colon u \in L^1(M, \mathbb{R}) \backslash \{0\} \to \frac{\int_M x|u(x)|\mathrm{d}v_g}{\int_M |u(x)|\mathrm{d}v_g} \in \mathbb{R}^{\tilde{n}}, \tag{3.14}$$

which gives the center of mass of a function in $\mathbb{R}^{\tilde{n}}$.

Lemma 3.12 *The map $\beta^* \colon L^1(M, \mathbb{R}) \backslash \{0\} \to \mathbb{R}^{\tilde{n}}$ is continuous.*

Proof For every $u, \tilde{u} \in L^1(M, \mathbb{R}) \backslash \{0\}$, set $v = \int_M |u| \mathrm{d}v_g$ and $\tilde{v} = \int_M |\tilde{u}| \mathrm{d}v_g$. Set also $\|x\|_\infty = \max\{\|x\|_{\mathbb{R}^{\tilde{n}}} : x \in M\}$. Then, we have

$$\begin{aligned} \|\beta^*(u) - \beta^*(\tilde{u})\|_{\mathbb{R}^{\tilde{n}}} &= \left\| \frac{\int_M x|u(x)|\mathrm{d}v_g}{v} - \frac{\int_M x|\tilde{u}(x)|\mathrm{d}v_g}{\tilde{v}} \right\|_{\mathbb{R}^{\tilde{n}}} \\ &\leq \|x\|_\infty \left(\left| \frac{1}{v} - \frac{1}{\tilde{v}} \right| \|u\|_{L^1(M,\mathbb{R})} + \frac{1}{\tilde{v}} \|u - \tilde{u}\|_{L^1(M,\mathbb{R})} \right), \end{aligned} \tag{3.15}$$

where we have added and subtracted $(\int_M x|u(x)|\mathrm{d}v_g)/\tilde{v}$. Let now $u \in L^1(M, \mathbb{R}) \backslash \{0\}$ and $(u_k)_{k \in \mathbb{N}}$ a sequence in $L^1(M, \mathbb{R}) \setminus \{0\}$ such that $\|u - u_k\|_{L^1(M,\mathbb{R})}$ tends to zero as $k \to \infty$. In particular, v_k tends to v as $k \to \infty$. Therefore, by applying (3.15) with $\tilde{u} = u_k$ for all $k \in \mathbb{N}$ we deduce that $\beta^*(u_k)$ tends to $\beta^*(u)$ as $k \to \infty$, which completes the proof. □

In order to define $\mathcal{B} \colon \mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m} \to \partial M$, one needs to show that for every $r > 0$ there exist m and ε sufficiently small such that

$$\beta^*(\mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m}) \subset (\partial M)_r.$$

In other words, we need to prove that if m and ε are sufficiently small, then the barycenter of every function in the sublevel $\mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m}$ lies in a small tubular neighbourhood of ∂M. Then, choosing r sufficiently small as well, one can define $\mathcal{B} \colon \mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m} \to \partial M$ as follows:

$$\mathcal{B} = \pi_{\partial M} \circ \beta^*. \tag{3.16}$$

To this end, we need to ensure that if the volume m is sufficiently small then any function in the sublevel $\mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m}$ has almost all its mass in a small ball centered on a point of the boundary ∂M (see Proposition 3.15). The proof of this result is based on [7, Theorem 1.2], which gives a compactness result for sequences of sets with uniformly bounded volume and perimeter. For the sake of presentation, we restate here that theorem. If $(X, \mathrm{d}, \mathcal{H}^n)$ is a metric measure space (where $\mathcal{H}^n$ denotes the n–dimensional Hausdorff measure) and $\Omega \subset X$ is a Borel set, we remind that the definition of perimeter of Ω used in the following theorem is

$$P(\Omega, X) := \inf \left\{ \liminf_i \int_X \mathrm{lip} f_i \, \mathrm{d}\mathcal{H}^n : f_i \in \mathrm{Lip}_{\mathrm{loc}}(X),\ f_i \to \mathbf{1}_\Omega \text{ in } L^1_{\mathrm{loc}}(X, \mathcal{H}^n) \right\}, \tag{3.17}$$

where $\mathrm{lip} f(x) := \limsup_{y \to x} \frac{|f(y)-f(x)|}{\mathrm{d}(x,y)}$. Moreover, we recall that a metric space $(X, \mathrm{d}, \mathcal{H}^n)$ is an $\mathrm{RCD}(\kappa, n)$ space if, loosely speaking, it has synthetic notions of Ricci curvature bounded below by $\kappa \in \mathbb{R}$ and dimension bounded above by $n \in (0, +\infty]$ (see [7] for more details).

Theorem 3.13 (Theorem 1.2 of [7]) *Let $\kappa \in \mathbb{R}$ and $n \geq 2$. Let $(X_k, \mathrm{d}_k, \mathcal{H}^n_k)$ be a sequence of* $\mathrm{RCD}(\kappa, n)$ *spaces and let $\Omega_k \subset X_k$ be bounded sets of finite perimeter such that* $\sup_k \left(P(\Omega_k, X_k) + \mathcal{H}^n_k(\Omega_k) \right) < +\infty$. *Assume there exists $v_0 > 0$ such that $\mathcal{H}^n_k(B(x, 1)) \geq v_0$ for every $x \in X_k$ and for every $k \in \mathbb{N}$. Then, up to subsequences, there exists a nondecreasing sequence $(N_k)_{k \in \mathbb{N}} \subset \mathbb{N}$, $N_k \geq 1$, points $p_{k,i} \in X_k$, with $1 \leq i \leq N_k$ and pairwise disjoint subsets $\Omega_{k,i} \subset \Omega_k$ such that:*

(i) $\lim_{k\to\infty} \mathrm{d}_k(p_{k,i}, p_{k,j}) = +\infty$ *for any $i \neq j \leq \bar{N}$, where* $\overline{N} := \lim_{k\to\infty} N_k \in \mathbb{N} \cup \{+\infty\}$;

(ii) *For every $1 \leq i < \overline{N} + 1$, the sequence $(X_k, \mathrm{d}_k, \mathcal{H}^n, p_{k,i})$ converges in the pointed measured Gromov–Hausdorff topology to a pointed* $\mathrm{RCD}(\kappa, n)$ *space $(Y_i, \mathrm{dist}_{Y_i}, \mathcal{H}^n, p_i)$ as $k \to \infty$;*

(iii) *There exists $F_i \subset Y_i$ such that $\Omega_{k,i} \to F_i$ as $k \to \infty$ in L^1–strong topology and the following relations hold:*

$$\lim_{k\to\infty} \mathcal{H}^n_k(\Omega_k) = \sum_{i=1}^{\overline{N}} \mathcal{H}^n(F_i),$$

and

$$\sum_{i=1}^{\overline{N}} P(F_i, Y_i) \leq \liminf_{k\to\infty} P(\Omega_k, X_k).$$

Moreover, if Ω_k is an isoperimetric set in X_k for any k, then F_i is an isoperimetric set in Y_i for any $i < \overline{N} + 1$ and

$$P(F_i, Y_i) = \lim_{k\to\infty} P(\Omega_{k,i}, X_k)$$

for any $i < \overline{N} + 1$.

Remark 3.14 In the following, we use the previous theorem on the more simple case of Riemannian manifolds. We want to highlight that when the RCD space is $(M, g, \mathcal{H}^n)$, the perimeter of a measurable set $\Omega \subset M$ with finite perimeter coincides with the $(n-1)$–dimensional Hausdorff measure of its relative boundary, hence

$$P(\Omega, M) = \mathcal{H}^{n-1}(\partial^*\Omega \cap \mathrm{int}(M)).$$

Proposition 3.15 *There exists $\mu = \mu(M) > 0$, such that the following property holds. For every almost isoperimetric sequence $(u_k)_{k\in\mathbb{N}} \subset L^1(M, \mathbb{R})$ with volumes $m_k = \int_M u_k \mathrm{d}v_g \to 0$, i.e.,*

$$\lim_{k\to\infty} \frac{\mathcal{E}_0(u_k)}{\sigma I_M(m_k)} = 1, \tag{3.18}$$

there exists a sequence $(p_k)_{k\in\mathbb{N}} \subset \partial M$ such that

$$\lim_{k\to+\infty} \frac{1}{m_k}\left(\int_{M\setminus B(p_k,\mu m_k^{1/n})} u_k \mathrm{d}v_g\right) = 0. \tag{3.19}$$

Proof of Proposition 3.15 For every fixed $k \in \mathbb{N}$, let $\Omega_k \in \mathcal{C}_g(M)$ be such that $u_k = \mathbf{1}_{\Omega_k}$. Note that, given that $u_k \in L^1(M, \mathbb{R})$, we can select any Ω_k in such a way that Ω_k is a subset of the interior of M. Let us define the following sequence of manifolds of bounded geometry:

$$(X_k, g_k) := (M, m_k^{-1/n} g).$$

In other words, we rescale the metric on M by a factor that depends on m_k in such a way that

$$\mathcal{H}_k^n(\Omega_k) = \int_{X_k} u_k \mathrm{d}v_{g_k} = 1, \qquad \forall k \in \mathbb{N},$$

where $\mathcal{H}_k^n$ denotes the n–dimensional Hausdorff measure on (X_k, g_k). Moreover, by (3.3) and (3.18) and recalling that $\mathcal{E}_0(u_k) = \sigma\mathcal{H}^{n-1}(\partial^*\Omega_k \cap \mathrm{int}(M))$, we obtain the existence of a constant C such that

$$P(\Omega_k, X_k) = \mathcal{H}_k^{n-1}(\partial^*\Omega_k \cap \mathrm{int}(M)) < I_{\mathbb{R}^n_+}(1) + C, \qquad \forall k \in \mathbb{N}.$$

Moreover, we have also the following equality:

$$\lim_{k\to\infty} \mathcal{H}_k^{n-1}(\partial^*\Omega_k \cap \mathrm{int}(M)) = I_{\mathbb{R}^n_+}(1). \tag{3.20}$$

This means that the sequence $\{(X_k, g_k, \mathcal{H}_k^n)\}_{k\in\mathbb{N}}$ and $\Omega_k \subset M_k$ satisfy the hypotheses of Theorem 3.13 and so, up to subsequences, for every k there exist $N_k \in \mathbb{N}$, with $N_k \geq 1$, and points $p_{k,i} \in M_k$ such that for every $i \in 1, \ldots, N_k$ the space

$(X_k, \mathrm{dist}_{g_k}, \mathcal{H}^n_k, p_{k,i})$ converges in the Gromov–Hausdorff topology to a pointed $\mathrm{RCD}(\kappa, n)$ as $k \to \infty$, that we denote by $(M^i_\infty, \mathrm{d}^i_\infty, \mathcal{H}^{n,i}_\infty, p^i_\infty)$. This space could either be the Euclidean space $(\mathbb{R}^n, \mathrm{d}, \mathcal{H}^n, 0)$ or an Euclidean semi-space $(\mathbb{R}^n_+, \mathrm{d}, \mathcal{H}^n, p)$, where the center point p could either lie on the boundary of the semi-space or in its interior. Moreover, setting $\overline{N} = \lim_{k\to\infty} N_k$, for every $i = 1, \ldots, \overline{N}$ there exists $\Omega^i_\infty \subset M^i_\infty$ such that

$$\sum_{i=1}^{\overline{N}} \mathcal{H}^{n,i}_\infty(\Omega^i_\infty) = \lim_{k\to\infty} \mathcal{H}^n_k(\Omega_k) = 1 \tag{3.21}$$

and, using also (3.20), we have

$$\sum_{i=1}^{\overline{N}} \mathcal{H}^{n-1,i}_\infty(\partial^*\Omega^i_\infty \cap \mathrm{int}(M^i_\infty)) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}_k(\partial^*\Omega_k \cap \mathrm{int}(M)) = I_{\mathbb{R}^n_+}(1). \tag{3.22}$$

By using contradiction arguments, this last inequality implies both that $\overline{N} = 1$ and that $(M^1_\infty, \mathrm{d}^1_\infty, \mathcal{H}^n_\infty, p^1_\infty)$ is actually the Euclidean semispace $(\mathbb{R}^n_+, \mathrm{d}, \mathcal{H}^n, 0)$. Formally, let us assume by contradiction that $\overline{N} > 1$ (where we recall that $\overline{N}$ can also be equal to ∞) and let us set $m_i = \mathcal{H}^{n,i}_\infty(\Omega^i_\infty)$ for any $i = 1, \ldots, \overline{N}$. Then $m_i \in (0, 1)$ for any i, as if one it is equal to 0 we can remove it from the count. As we have that

$$I_{\mathbb{R}^n_+}(m) < I_{\mathbb{R}^n}(m), \qquad \forall m > 0,$$

and that $I_{\mathbb{R}^n_+}(m) = c^+_n m^{\frac{n-1}{n}}$ is a strictly subadditive function, we then obtain that

$$\sum_{i=1}^{\overline{N}} \mathcal{H}^{n-1,i}_\infty(\partial^*\Omega^i_\infty \cap \mathrm{int}(M^i_\infty)) \geq \sum_{i=1}^{\overline{N}} I_{\mathbb{R}^n_+}(m_i) > I_{\mathbb{R}^n_+}\left(\sum_{i=1}^{\overline{N}} m_i\right) = I_{\mathbb{R}^n_+}(1),$$

which contradicts (3.22). Thus, we have $\overline{N} = 1$. Now, if $(M^1_\infty, \mathrm{d}^1_\infty, \mathcal{H}^n_\infty, p^1_\infty)$ is the Euclidean space, then

$$\mathcal{H}^{n-1,1}_\infty(\partial^*\Omega^1_\infty \cap \mathrm{int}(M^1_\infty)) \geq I_{\mathbb{R}^n}(1) > I_{\mathbb{R}^n_+}(1),$$

which is another contradiction.

In conclusion, by (3.21) and (3.22), we obtain that Ω^1_∞ is an isoperimetric set for the Euclidean half-space. Thus, it must be a hemisphere, and the sequence of subsets $\Omega_k \subset X_k$ converges to it in the L^1-strong topology. Consequently, we can construct a sequence of points on the boundary, denoted by $(p_k)_{k\in\mathbb{N}} \subset \partial M$, such that (3.19) holds, where μ can be chosen as twice the radius of the semisphere in $\mathbb{R}^n_+$ with unit volume. □

Using the same constants θ and μ given by Proposition 3.9 and Proposition 3.15, respectively, we obtain the following result, which ensures that all the functions in the sublevel that contains the image of the photography map have their mass concentrated near the boundary.

Lemma 3.16 *Assume that (H1) and (H2) hold. For every $\alpha \in (0,1)$, there exists $m_\alpha = m_\alpha(M, g, W, \theta, \alpha) > 0$ such that for every $m \in (0, m_\alpha)$ there exists $\varepsilon_\alpha = \varepsilon_\alpha(M, g, W, \theta, \alpha, m) > 0$ such that for every $\varepsilon \in (0, \varepsilon_\alpha)$ and for any $u \in \mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m}$ there exists a point $p_u \in \partial M$ such that*

$$\int_{M \setminus B(p_u, \mu m^{1/n})} |u| \mathrm{d}v_g \leq \alpha m. \tag{3.23}$$

Proof Arguing by contradiction, there exists a sequence $(m_i)_{i\in\mathbb{N}}$ such that $m_i \to 0^+$ and for every $i \in \mathbb{N}$ there exist two sequences $(\varepsilon_{i,j})_{j\in\mathbb{N}} \in \mathbb{R}$ and $(u_{i,j})_{j\in\mathbb{N}} \in \mathcal{E}_{\varepsilon_{i,j},m_i}^{c_i}$, with $c_i = I_M(m_i) + \theta m_i$, such that for every $i \in \mathbb{N}$ we have $\varepsilon_{i,j} \to 0^+$ as $j \to \infty$ and

$$\int_{M \setminus B_g(p, \mu m_i^{1/n})} |u_{i,j}| \mathrm{d}v_g > \alpha m_i, \qquad \forall p \in \partial M, \ \forall j \in \mathbb{N}. \tag{3.24}$$

Since for any fixed $i \in \mathbb{N}$ we have $\mathcal{E}_{\varepsilon_{i,j},m_i}(u_{i,j}) \leq c_i$ for every $j \in \mathbb{N}$, we can apply Theorem 3.3 to obtain a sequence of characteristic functions $(u_{0,i})_i \subset L^1(M, \mathbb{R})$ such that, for every i, we have $\lim_{j\to\infty} \|u_{i,j} - u_{0,i}\|_{L^1(M,\mathbb{R})} = 0$, up to subsequences (see also Remark 3.4). Hence, for every i there exists j_i such that

$$\int_M \big(|u_{i,j_i}| - u_{0,i}\big) \mathrm{d}v_g \leq \frac{\alpha}{4} m_i, \tag{3.25}$$

and we have also $\int_M u_{0,i} \mathrm{d}v_g = m_i$ and $\mathcal{E}_0(u_{0,i}) \leq c_i$, namely

$$I_M(m_i) \leq \frac{\mathcal{E}_0(u_{0,i})}{\sigma} \leq I_M(m_i) + \theta m_i.$$

Using also (3.13), we obtain

$$\lim_{i\to\infty} \frac{\mathcal{E}_0(u_{0,i})}{\sigma I_M(m_i)} = 1,$$

so we can apply Proposition 3.15 to obtain the existence of a sequence $(p_i)_{i\in\mathbb{N}} \subset \partial M$ such that

$$\lim_{i\to\infty} \frac{1}{m_i} \left(\int_{M \setminus B(p_i, \mu m_i^{1/n})} u_{0,i} \, \mathrm{d}v_g \right) = 0,$$

so there exists i_0 such that

$$\int_{M\setminus B(p_i,\mu m_i^{1/n})} u_{0,i}\,\mathrm{d}v_g \le \frac{\alpha}{4}m_i, \qquad \forall i \ge i_0. \tag{3.26}$$

As a consequence, combining (3.25) and (3.26), for every $i > i_0$ we obtain

$$\begin{aligned}
&\int_{M\setminus B(p_i,\mu m_i^{1/n})} |u_{i,j_i}|\mathrm{d}v_g \\
&\quad = \int_{M\setminus B(p_i,\mu m_i^{1/n})} \Big(|u_{i,j_i}| - u_{0,i}\Big)\mathrm{d}v_g + \int_{M\setminus B(p_i,\mu m_i^{1/n})} u_{0,i}\,\mathrm{d}v_g \\
&\quad \le \int_M \big(|u_{i,j_i}| - u_{0,i}\big)\mathrm{d}v_g + \int_{M\setminus B(p_i,\mu m_i^{1/n})} u_{0,i}\,\mathrm{d}v_g \le \frac{\alpha}{2}m_i,
\end{aligned}$$

which contradicts (3.24). □

Using the previous result, we can ensure that if m and ε are sufficiently small, the extrinsic barycenter of any function in the sublevel of $\mathcal{E}_{\varepsilon,m}$ that contains the image of the photograpy map is near the boundary of the manifold. More formally, setting

$$\mathrm{diam}_{\mathbb{R}^{\tilde n}}(M) := \max\left\{\|x-y\|_{\mathbb{R}^{\tilde n}} : x, y \in M\right\},$$

we have the following result.

Lemma 3.17 *Assume that (H1) and (H2) hold. For every $r > 0$, there exists $m_2 = m_2(M, g, r, \mathrm{diam}_{\mathbb{R}^{\tilde n}}(M)) > 0$ such that for every $m \in (0, m_2)$ there exists $\varepsilon_2 = \varepsilon_2(M, g, r, m) > 0$ such that for every $\varepsilon \in (0, \varepsilon_2)$ and any $u \in \mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m}$ we have $\beta^*(u) \in (\partial M)_r$.*

Proof Recalling the definition of β^* given by (3.14), for every $u \in H^1(M,\mathbb{R})$ and $y \in M$, let us define

$$\rho(u, y) = \frac{|u(y)|}{\int_M |u(x)|\mathrm{d}v_g},$$

so that

$$\beta^*(u) = \int_M x\rho(u, x)\mathrm{d}v_g \quad \text{and} \quad \int_M \rho(u, x)\mathrm{d}v_g = 1, \ \forall u \in H^1(M,\mathbb{R}).$$

Let us choose $\alpha > 0$ such that

$$\alpha \le \frac{r}{2\mathrm{diam}_{\mathbb{R}^{\tilde n}}(M)},$$

and let m_α be given by Lemma 3.16. Let us choose $m \in (0, m_\alpha)$ and let $\varepsilon \in (0, \varepsilon_\alpha)$. By Lemma 3.16, for every $u \in \mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m}$ there exists $p_u \in \partial M$ such that (3.23) holds. As a consequence, for every $u \in \mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m}$ we obtain

$$
\begin{aligned}
\|\beta^*(u) - p_u\|_{\mathbb{R}^{\tilde{n}}} &= \left\| \int_M (x - p_u)\rho(u,x)\mathrm{d}v_g \right\|_{\mathbb{R}^{\tilde{n}}} \\
&\leq \left\| \int_{B(p_u,\mu m^{1/n})} (x - p_u)\rho(u,x)\mathrm{d}v_g \right\|_{\mathbb{R}^{\tilde{n}}} + \left\| \int_{M \setminus B(p_u,\mu m^{1/n})} (x - p_u)\rho(u,x)\mathrm{d}v_g \right\|_{\mathbb{R}^{\tilde{n}}} \\
&\leq \mu m^{1/n} + \alpha \mathrm{diam}_{\mathbb{R}^{\tilde{n}}}(M) \leq \mu m^{1/n} + \frac{r}{2}.
\end{aligned}
$$

As a consequence, choosing $m_2 \in (0, m_\alpha)$ such that

$$\mu m_2^{1/n} \leq \frac{r}{2},$$

for all $m \in (0, m_2)$ and for all $\varepsilon \in (0, \varepsilon_2)$, with $\varepsilon_2 = \varepsilon_\alpha$, we obtain

$$\|\beta^*(u) - p_u\|_{\mathbb{R}^{\tilde{n}}} \leq r, \qquad \forall u \in \mathcal{E}_{\varepsilon,m}^{I_M(m)+\theta m},$$

and we are done. □

Remark 3.18 By Lemma 3.17 and the compactness of ∂M, there exists $r_1 > 0$ such that, choosing $m_2(M, g, r_1, \mathrm{diam}_{\mathbb{R}^{\tilde{n}}}) > 0$ and $\varepsilon_2(M, g, r_1, m) > 0$ as in the lemma, for every $m \in (0, m_2)$ and $\varepsilon \in (0, \varepsilon_2)$, the map $\mathcal{B} = \pi_{\partial M} \circ \beta^* \colon \mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m} \to \partial M$ is well defined and continuous.

3.5 Conclusion of the proof

The following two results, namely, Lemma 3.19 and Lemma 3.20, demonstrate that the barycenter of a function obtained by applying the photography map to a point $p \in \partial M$ is in close proximity to p itself when m and ε are sufficiently small. As a result, taking into account the definition of $\mathcal{B}$ provided in (3.16), the composition $\mathcal{B} \circ \mathcal{P}_{\varepsilon,m} \colon \partial M \to \partial M$ closely approximates the identity map.

Lemma 3.19 *Assume that (H1) and (H2) hold. For every $r \in]0, r_1[$ there exists $m_3 = m_3(M, g, r) > 0$ such that for every $m \in (0, m_3)$ there exists $\varepsilon_3 = \varepsilon_3(M, g, r, m) > 0$ such that*

$$\|\beta^*(\mathcal{P}_{\varepsilon,m}(p)) - p\|_{\mathbb{R}^{\tilde{n}}} \leq r, \qquad \forall p \in \partial M.$$

Proof We recall that for every $p \in \partial M$ and every $m \in (0, \mathrm{vol}(M))$, the radius $r_{p,m}$ is the one such that

$$\int_{E_{p,r_{p,m}}} 1 \mathrm{d}v_g = m.$$

By the compactness of ∂M, there exists $m_3 = m_3(M, g, r) > 0$ such that for every $m \in (0, m_3)$ and for every $p \in \partial M$ we have $r_{p,m} < r/2$. As a consequence, for every $x \in E_{p,r_{p,m}} \subset M$ we have

$$\|x - p\|_{\mathbb{R}^{\tilde{n}}} < \frac{r}{2}. \tag{3.27}$$

Moreover, we recall that for every $m, \varepsilon > 0$ the function $\mathcal{P}_{\varepsilon,m}(p)$ is defined as the ε–approximation of $u^m_{0,x} = \mathbf{1}_{E_{p,m}}$ and that $u^m_{\varepsilon,p}$ converges to $u^m_{0,x}$ in the $L^1(M, \mathbb{R})$ norm as $\varepsilon \to 0$. Hence, using also the continuity of β^* with respect to the L^1–norm ensured by Lemma 3.12 and again the compactness of ∂M, we obtain the existence of a constant $\varepsilon_3 = \varepsilon_3(M, g, r, m) > 0$ such that for every $\varepsilon \in (0, \varepsilon_3)$ and for every $p \in \partial M$ the following holds:

$$\|\beta^*(\mathcal{P}_{\varepsilon,m}(p)) - \beta^*(u^m_{0,p})\|_{\mathbb{R}^{\tilde{n}}} = \|\beta^*(u^m_{\varepsilon,p}) - \beta^*(u^m_{0,p})\|_{\mathbb{R}^{\tilde{n}}} \leq \frac{r}{2}. \tag{3.28}$$

As a consequence, using both (3.27) and(3.28), for every $m \in (0, m_3)$ and for every $\varepsilon \in (0, \varepsilon_3)$ we obtain

$$\begin{aligned}\|\beta^*(\mathcal{P}_{\varepsilon,m}(p)) - p\|_{\mathbb{R}^{\tilde{n}}} &\leq \|\beta^*(\mathcal{P}_{\varepsilon,m}(p)) - \beta^*(u^m_{0,p})\|_{\mathbb{R}^{\tilde{n}}} + \|\beta^*(u^m_{0,p}) - p\|_{\mathbb{R}^{\tilde{n}}} \\ &\leq \frac{r}{2} + \frac{1}{\int_M |u^m_{0,p}| \mathrm{d}v_g} \int_M \Big(\|x - p\|_{\mathbb{R}^{\tilde{n}}} |u^m_{0,p}|\Big) \mathrm{d}v_g \leq r, \quad \forall p \in \partial M,\end{aligned}$$

which concludes the proof. □

Lemma 3.20 *Assume that (H1) and (H2) hold. There exists $m^* = m^*(M, g, W) > 0$ such that for any $m \in (0, m^*)$ there exists $\varepsilon^* = \varepsilon^*(M, g, W, m) > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$ the composition map*

$$\mathcal{B} \circ \mathcal{P}_{\varepsilon,m} \colon \partial M \to \partial M$$

is well defined and homotopic to the identity map.

Proof For $p \in \partial M$, let $\exp^{\partial M}_p$ be the exponential map of ∂M at p, and let $B^{\partial M}_g(p, R)$ be the geodesic ball of center p and radius R on ∂M. We denote by $\mathrm{inj}(\partial M)$ the *injectivity radius* of ∂M, that is

$$\mathrm{inj}(\partial M) := \inf_{p \in \partial M} \sup \left\{ R > 0 \text{ s.t. } \exp^{\partial M}_p : B^{\partial M}_g(p, R) \to \partial M \text{ is a diffeomorphism} \right\}$$

which is positive since ∂M is a compact manifold. Moreover, the compactness of ∂M implies that there exists a constant $C_{\partial M} > 0$ such that

$$\mathrm{dist}_{(\partial M, g)}(p, q) \leq C_{\partial M} \|p - q\|_{\mathbb{R}^{\tilde{n}}} \qquad \forall p, q \in \partial M,$$

where $\mathrm{dist}_{(\partial M, g)}$ stands for the Riemannian distance on $(\partial M, g)$. Recalling the definition of $r_1 > 0$ given in Remark 3.18, let

$$r^* = \min \left\{ r_1, \mathrm{inj}(\partial M)/(4 C_{\partial M}) \right\},$$

let $m_2 = m_2(M, g, r^*, \mathrm{diam}_{\mathbb{R}^{\tilde{n}}}(M)) > 0$ be defined by Lemma 3.17, and for every $m \in (0, m_2)$, set $\varepsilon_2 = \varepsilon_2(M, g, r^*, m) > 0$ in the same way. Moreover, let $m_3 = m_3(M, g, r^*) > 0$ and $\varepsilon_3 = \varepsilon_3(M, g, r^*, m)$ be similarly defined by Lemma 3.19.

Recalling Proposition 3.9, let $m^* = \min\{m_1, m_2, m_3\}$ and for every $m \in (0, m^*)$ set $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

By Proposition 3.9, for any $p \in \partial M$ we have $\mathcal{P}_{\varepsilon,m}(p) \in \mathcal{E}_{\varepsilon,m}^{\sigma I_M(m)+\theta m}$, so we obtain

$$\begin{aligned} \mathrm{dist}_{(\partial M,g)}\big(\mathcal{B}(\mathcal{P}_{\varepsilon,m}(p)), p\big) &\le C_{\partial M}\|\pi_{\partial M}(\beta^*(\mathcal{P}_{\varepsilon,m}(p))) - p\|_{\mathbb{R}^{\tilde{n}}} \\ &\le C_{\partial M}\big(\|\pi_{\partial M}(\beta^*(\mathcal{P}_{\varepsilon,m}(p))) - \beta^*(\mathcal{P}_{\varepsilon,m}(p))\|_{\mathbb{R}^{\tilde{n}}} + \|\beta^*(\mathcal{P}_{\varepsilon,m}(p)) - p\|_{\mathbb{R}^{\tilde{n}}}\big). \end{aligned} \tag{3.29}$$

Since m^* and ε^* are, respectively, less than m_2 and ε_2 defined as in Lemma 3.17, we have

$$\|\pi_{\partial M}(\beta^*(\mathcal{P}_{\varepsilon,m}(p))) - \beta^*(\mathcal{P}_{\varepsilon,m}(p))\|_{\mathbb{R}^{\tilde{n}}} \le r^*,$$

and, by applying in an analogous manner Lemma 3.19, we obtain also

$$\|\beta^*(\mathcal{P}_{\varepsilon,m}(p)) - p\|_{\mathbb{R}^{\tilde{n}}} \le r^*.$$

Hence, from (3.29) we infer

$$\mathrm{dist}_{(\partial M,g)}\big(\mathcal{B}(\mathcal{P}_{\varepsilon,m}(p)), p\big) \le 2C_{\partial M} r^* \le \frac{1}{2}\mathrm{inj}(\partial M).$$

As a consequence, the map $F\colon [0,1] \times \partial M \to \partial M$, given by

$$F(t,p) := \exp_p^{\partial M}\Big(t\big(\exp_p^{\partial M}\big)^{-1}\big(\mathcal{B}(\mathcal{P}_{\varepsilon,m}(p))\big)\Big),$$

is well-defined, continuous and gives a homotopy equivalence between $\mathcal{B} \circ \mathcal{P}_{\varepsilon,m}$ and the identity map in ∂M. □

Finally, we are ready to prove Theorem A.

Proof of Theorem A For every $m, \varepsilon > 0$, the functional $\mathcal{E}_{\varepsilon,m}$ is clearly bounded below and, by Lemma 2.1, it is of class C^1 and satisfies the Palais–Smale condition.

Let us choose m^* as in Lemma 3.20 and, for every $m \in (0, m^*)$, let ε_m be equal to ε^* of the same lemma and set $c_m = \sigma I_M(m) + \theta m$. By Proposition 3.11 and Remark 3.18, for every $\varepsilon \in (0, \varepsilon_m)$ both $\mathcal{P}_{\varepsilon,m}\colon \partial M \to \mathcal{E}_{\varepsilon,m}^{c_m}$ and $\mathcal{B}\colon \mathcal{E}_{\varepsilon,m}^{c_m} \to \partial M$ are continuous and, by Lemma 3.20, their composition is homotopic to the identity. Then, Theorem A directly follows from Theorem C. In particular, there exist at least $\mathrm{cat}(\partial M)$ critical points of $\mathcal{E}_{\varepsilon,m}$ in $\mathcal{E}_{\varepsilon,m}^{c_m}$ and, recalling that $\mathrm{cat}(\partial M) > 1$ and $\mathcal{H}_m$ is a contractible set (see Remark 2.2), there exist at least one critical point with energy larger than c_m. By Lemma 2.1, these critical points are solutions of (1.2). Moreover, if all the critical points of $\mathcal{E}_{\varepsilon,m}$ are non-degenerate, then Theorem C and Lemma 2.1 ensure that (1.2) has at least $\mathscr{P}_1(\partial M)$ solutions with energy less than c_m and $\mathscr{P}_1(\partial M) - 1$ solutions with energy larger than c_m, and we are done. □

4 Proof of Theorem B

The proof of Theorem B follows essentially the same structure of the proof of Theorem A, by applying again Theorem C. Relying on some Γ–convergence results for $\overline{\mathcal{E}}_{\varepsilon,m}$ [36, 46], we can construct the photography map $\mathcal{P}_{\varepsilon,m}\colon M \to \overline{\mathcal{E}}^{c}_{\varepsilon,m}$, where the value of c can be estimated exploiting the convergence of the isoperimetric problem on M that takes into account also the boundary ∂M to the standard Euclidean case as m goes to 0.

4.1 The isoperimetric problem

We define the functional $\overline{\mathcal{E}}_0\colon L^1(M,\mathbb{R}) \to \mathbb{R}$ as

$$\overline{\mathcal{E}}_0(u) := \begin{cases} \sigma\mathcal{H}^{n-1}(\partial^*\Omega), & \text{if } u = \mathbf{1}_\Omega, \quad \Omega \in \mathcal{C}_g(M), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\sigma \in \mathbb{R}$ is as in Sect. 3.1. Let $\overline{I}\colon]0, \mathrm{vol}(M)] \to \mathbb{R}$ be the following function:

$$\overline{I}_M(m) := \inf\left\{\mathcal{H}^{n-1}(\partial^*\Omega) : \Omega \in \mathcal{C}_g(M) \text{ and } \int_M \mathbf{1}_\Omega \mathrm{d}v_g = m\right\}.$$

In other words, the quantity $\overline{I}_M$ is the isoperimetric profile of M that takes also into account the boundary ∂M.

Remark 4.1 In the following, it is convenient to extend (M, g) to a closed Riemannian manifold $(\widetilde{M}, \tilde{g})$ of the same dimension, in such a way that

$$M \subset \widetilde{M}, \quad \text{and} \quad \tilde{g}|_M = g.$$

In this way, the notion of perimeter given by (3.17), when applied to the subsets of M, takes into account also the contribution along the boundary of the manifold M. In other words, any Caccioppoli set $\Omega \in \mathcal{C}_g(M)$ can be naturally regarded as a subset of $\widetilde{M}$, and $P(\Omega, \widetilde{M}) = \mathcal{H}^{n-1}(\partial^*\Omega)$. Such a manifold can be constructed, for instance, by gluing together two copies of M along their boundaries. We refer to [53, Theorem A] and its proof for further details on this construction.

For small volumes, the function $\overline{I}_M$ converges to its Euclidean counterpart, which is $I_{\mathbb{R}^n}(m) = c_n m^{\frac{n-1}{n}}$, where c_n is the Euclidean isoperimetric constant. More formally, we have the following result.

Lemma 4.2 *The following equality holds:*

$$\lim_{m\to 0^+} \frac{\overline{I}_M(m)}{I_{\mathbb{R}^n}(m)} = \lim_{m\to 0^+} \frac{\overline{I}_M(m)}{c_n m^{\frac{n-1}{n}}} = 1. \tag{4.1}$$

Proof Thanks to the construction given in the Remark 4.1, the thesis is a direct consequence of [45, Theorem 3]. Indeed, since every Caccioppoli subset of M is also a subset of $\widetilde{M}$, it follows that

$$\overline{I}_{\widetilde{M}}(m) \leq \overline{I}_M(m), \qquad \forall m \in (0, \mathrm{vol}(M)).$$

We can then apply [45, Theorem 3] to obtain

$$\lim_{m \to 0^+} \frac{\overline{I}_{\widetilde{M}}(m)}{c_n m^{\frac{n-1}{n}}} = 1.$$

On the other hand, estimating the perimeter of geodesic balls of volume m centered at any point $p \in M$, we also get

$$\overline{I}_M(m) \leq c_n m^{\frac{n-1}{n}} + o(m),$$

as $m \to 0^+$, where this estimate can be found in [45, Lemma 3.10]. Combining these inequalities, a standard limit argument leads to (4.1). □

4.2 Gamma convergence

Proposition 4.3 (cf. Proposition A and Proposition B of [36]) *Assume that (H1) and (H2) hold. Then, the following statements hold:*

(i) ***Lim-inf***: *If* $(\varepsilon_k)_{k\in\mathbb{N}} \subset]0, +\infty[$ *is such that* $\varepsilon_k \to 0^+$ *and* $(u_{\varepsilon_k})_{k\in\mathbb{N}} \subset H_0^1(M, \mathbb{R})$ *is such that* $u_{\varepsilon_k} \to u_0$ *in* $L^1(M, \mathbb{R})$, *then* $\liminf_{k\to\infty} \overline{\mathcal{E}}_{\varepsilon_k}(u_{\varepsilon_k}) \geq \overline{\mathcal{E}}_0(u_0)$;

(ii) ***Lim-sup***: *For any* $u_0 \in L^1(M, \mathbb{R})$ *such that* $u_0 = \mathbf{1}_\Omega$ *for some finite perimeter measurable set* $\Omega \subset M$ *and for every sequence* $(\varepsilon_k)_{k\in\mathbb{N}} \subset]0, +\infty[$ *such that* $\varepsilon_k \to 0^+$, *there exists* $(u_{\varepsilon_k})_{k\in\mathbb{N}} \subset H_0^1(M, \mathbb{R})$ *such that* $u_{\varepsilon_k} \to u_0$ *in* $L^1(M, \mathbb{R})$,

$$\int_M u_{\varepsilon_k} \mathrm{d}v_g = \int_M u_0 \mathrm{d}v_g \quad \textit{and} \quad \limsup_{k\to\infty} \overline{\mathcal{E}}_{\varepsilon_k}(u_{\varepsilon_k}) \leq \overline{\mathcal{E}}_0(u_0).$$

Remark 4.4 Noticing that the compactness result ensured by Theorem 3.3 refers to the functional $\mathcal{E}_\varepsilon \colon H^1(M, \mathbb{R}) \to \mathbb{R}$, as for the case of Neumann boundary condition we have that if $(\varepsilon_k)_{k\in\mathbb{N}} \subset]0, +\infty[$ is a sequence such that $\varepsilon_k \to 0^+$ and if a sequence $(u_{\varepsilon_k})_{k\in\mathbb{N}} \subset \mathcal{H}_{m,0}$ satisfies $\overline{\mathcal{E}}_{\varepsilon_k}(u_{\varepsilon_k}) \leq E^*$ for some constant E^*, then, up to subsequences, $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ converges to a function $u_0 \in L^1(M, \mathbb{R})$. By applying the lim-inf property of Proposition 4.3, we have that $\overline{\mathcal{E}}_0(u_0) \leq E^*$, hence there exists a measurable and finite perimeter set $\Omega \subset M$ such that $u_0 = \mathbf{1}_\Omega$ and $\mathrm{vol}(\Omega) = \int_M u_0 \mathrm{d}v_g = m$.

4.3 Photography map

Since now we are dealing with the Dirichlet boundary condition, namely $u \equiv 0$ on ∂M, the photography map is defined by utilizing a *boundary layer*. The main idea is to associate to each point of the manifold the ε–approximation of the characteristic function of the geodesic ball with prescribed volume centered in the point. However, when the point is near the boundary, such a ball may intersect with the boundary and the ε–approximation does not satisfy the boundary condition. To avoid this problem, we construct a boundary layer map $\mathcal{L}\colon M \to M$ that slightly moves inside the points that lie on tubular neighbourhood of ∂M. In this way, the geodesic ball of prescribed volume m centered at $\mathcal{L}(p)$ is in the interior of M for every $p \in M$, provided m sufficiently small.

Let us proceed with a formal construction. Since the boundary of M is smooth and compact, there exists $\delta_M > 0$ such that the map

$$\partial M \times [0, \delta_M] \ni (Q, t) \mapsto \exp_Q(tN_{\partial M})$$

provides a coordinate system in a neighbourhood of ∂M, where we recall that $N_{\partial M}$ stands for the unit interior normal vector field along ∂M. Now, for any $p \in M$ such that $\mathrm{dist}(p, \partial M) \le \delta_M$, we denote by (Q_p, t_p) the unique element in $\partial M \times [0, \delta_M]$ such that

$$p = \exp_{Q_p}(t_p N_{\partial M}).$$

In other words, t_p is the distance of p to the boundary, while Q_p is its projection on it. Now, let us choose a C^∞ function $h\colon [0, \delta_M] \to [0, \delta_M]$ that is strictly increasing (hence invertible), $h(0) = \delta_M/2$, $h(\delta_M) = \delta_M$ and $h'(\delta_M) = 1$. Our boundary-layer function $\mathcal{L}\colon M \to M$ is defined as follows:

$$\mathcal{L}(p) := \begin{cases} \exp_{Q_p}\big(h(t_p)N_{\partial M}\big), & \text{if } \mathrm{dist}(p, \partial M) \in [0, \delta_M], \\ p, & \text{if } \mathrm{dist}(p, \partial M) \ge \delta_M. \end{cases}$$

Notice that $\mathcal{L}$ is a C^1–map homotopic to the identity.

Since $\mathrm{dist}(\mathcal{L}(p), \partial(M)) \ge \delta_M/2$ for every $p \in M$ and M is compact, there exists a sufficiently small volume, say $m_0 > 0$, such that for any $p \in M$ and $m \in (0, m_0)$ the geodesic ball centered at $\mathcal{L}(p)$ and with volume m doesn't intersect the boundary of the manifold. More formally, denoting by $r_{q,m} > 0$ the radius of the geodesic ball centered at $q \in M$ with volume m, hence $\int_{B(q, r_{q,m})} 1 \mathrm{d}v_g = m$, we have

$$B(\mathcal{L}(p), r_{\mathcal{L}(p),m}) \cap \partial M = \emptyset, \qquad \forall p \in M, m \in (0, m_0).$$

For any $p \in M$ and $m \in (0, m_0)$ let us denote by $u^m_{0,\mathcal{L}(p)}$ the characteristic function of $B(\mathcal{L}(p), r_{\mathcal{L}(p),m})$. Moreover, for every $\varepsilon > 0$ let $u^m_{\varepsilon,\mathcal{L}(p)} \in \mathcal{H}_m$ be ε–approximation of $u^m_{0,\mathcal{L}(p)}$ given by the lim-sup property of Proposition 4.3. Since M is compact and $\mathrm{supp}\, u^m_{0,\mathcal{L}(p)} \subset\subset \mathrm{int}(M)$ for every $p \in M$ and $m \in (0, m_0)$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in]0, \varepsilon_0[$ we have $u^m_{\varepsilon,\mathcal{L}(p)} \in \mathcal{H}_{m,0}$, hence $u_{\varepsilon,\mathcal{L}(p)} \equiv 0$ on ∂M (see Remark 3.2 for the details about the construction of the ε–approximation).

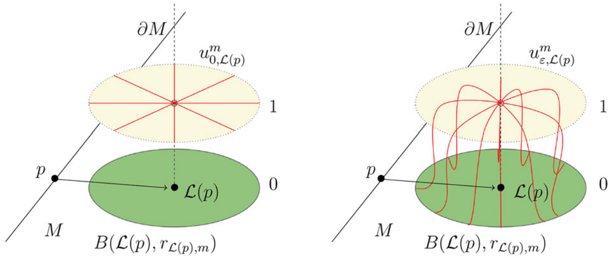

Fig. 3 The map introduced in Definition 4.5 turns the indicator function $u^m_{0,\mathcal{L}(p)}$ into the smooth approximation $u^m_{\varepsilon,\mathcal{L}(p)}$ given by Γ-convergence (Proposition 4.3). The functions are supported on the balls $B(\mathcal{L}(p), r_{\mathcal{L}(p),m})$, whose center $\mathcal{L}(p)$ is obtained by sending a point p away from the boundary through the boundary layer map $\mathcal{L}$

Definition 4.5 For every $m \in (0, m_0)$ and $\varepsilon \in (0, \varepsilon_0)$, we define the photography map $\mathcal{P}_{\varepsilon,m} : M \to \mathcal{H}_{m,0}$ as follows:

$$\mathcal{P}_{\varepsilon,m}(p) = u^m_{\varepsilon,\mathcal{L}(p)}.$$

See Fig. 3 for a representation of Definition 4.5.

Proposition 4.6 *Assume that (H1) and (H2) hold. There exists a function $\tau : \mathbb{R}^+ \to \mathbb{R}$ such that*

$$\lim_{m\to 0^+} \frac{\tau(m)}{m^{\frac{n-1}{n}}} = 0$$

and there exists $m_1 = m_1(M, g, W, \tau) \in (0, m_0)$ such that for every $m \in (0, m_1)$ there exists $\varepsilon_1 = \varepsilon_1(M, g, W, m) \in (0, \varepsilon_0)$ such that for every $\varepsilon \in (0, \varepsilon_1)$ we have

$$\overline{\mathcal{E}}_{\varepsilon,m}(\mathcal{P}_{\varepsilon,m}(p)) \leq \sigma \overline{I}_M(m) + \tau(m), \qquad \forall p \in M. \tag{4.2}$$

Proof By applying some standard results (see, e.g., [45, Lemma 3.10]), since $u^m_{0,\mathcal{L}(p)}$ has compact support in the interior of M for every $m \in (0, m_0)$ we have

$$\overline{\mathcal{E}}_0(u^m_{0,\mathcal{L}(p)}) = \sigma \left(c_n m^{\frac{n-1}{n}} - \gamma_n \mathrm{Sc}_g(\mathcal{L}(p)) m^{\frac{n+1}{n}} \right) + \mathcal{O}(m^{\frac{n+3}{n}}), \qquad \text{as } m \to 0^+,$$

where c_n is the Euclidean isoperimetric constant, γ_n is a constant which depends only on the dimension of the manifold and $\mathrm{Sc}_g(\mathcal{L}(p))$ denotes the scalar curvature of the metric tensor g at the point $\mathcal{L}(p)$. Since M is compact, there exists a constant $\omega > 0$ and $m_1 \in (0, m_0)$ such that for every $m \in (0, m_1)$ we have

$$\overline{\mathcal{E}}_0(u^m_{0,\mathcal{L}(p)}) < \sigma c_n m^{\frac{n-1}{n}} + \omega m^{\frac{n+1}{n}}, \qquad \forall p \in M.$$

Since the last inequality is strict and using again the compactness of M, by the lim-sup property of Proposition 4.3 there exists $\varepsilon_1 = \varepsilon_1(M, g, m) \in (0, \varepsilon_0)$ such that for every $\varepsilon \in (0, \varepsilon_1)$ we have

$$\overline{\mathcal{E}}_{\varepsilon,m}(u^m_{\varepsilon,\mathcal{L}(p)}) \leq \sigma c_n m^{\frac{n-1}{n}} + \omega m^{\frac{n+1}{n}}, \qquad \forall p \in M. \tag{4.3}$$

By (4.1), the function $\tau_0 \colon \mathbb{R}^+ \to \mathbb{R}$ given by $\tau_0(m) = c_n m^{\frac{n-1}{n}} - \overline{I}_M(m)$ is a $o(m^{\frac{n-1}{n}})$ as m goes to 0. As a consequence, setting $\tau(m) = \sigma\tau_0(m) + \omega m^{\frac{n+1}{n}}$, (4.3) is equivalent to (4.2). □

Since $\mathcal{L} \colon M \to M$ is a continuous function, one can employ the same construction of the proof of Proposition 3.11 to obtain the continuity of the photography map, which is stated by the next result.

Proposition 4.7 *Assume that (H1) and (H2) hold. Let $m_1 > 0$ and $\varepsilon_1 > 0$ be defined as in Proposition 4.6. For every $m \in (0, m_1)$ and $\varepsilon \in (0, \varepsilon_1)$ the map $\mathcal{P}_{\varepsilon,m} \colon M \to \mathcal{H}_{m,0}$ is continuous.*

Proof Cf. Proposition 3.11. □

4.4 Barycenter map

The construction of the barycenter map is analogous to the case of Neumann boundary condition. In particular, we will use again the function $\beta^* \colon L^1(M, \mathbb{R}) \to \mathbb{R}^{\tilde{n}}$ defined in (3.14), which gives the "extrinsic" center of mass of a function in the Euclidean space $\mathbb{R}^{\tilde{n}}$ where the manifold M is isometrically embedded. In this case, we will compose this map with the nearest point projection map $\pi_M \colon \mathbb{R}^{\tilde{n}} \to M$. We need to prove that, when m and ε are sufficiently small, $\beta^*(u)$ is near the manifold for any function u that belongs to the sublevel of $\overline{\mathcal{E}}_{\varepsilon,m}$ that contains the image of the photography map, namely for every $u \in \overline{\mathcal{E}}^c_{\varepsilon,m}$ with $c = \sigma\overline{I}_M(m) + \tau(m)$ (see Proposition 4.6). To obtain the last result, we rely on the "concentration property", namely that as m goes to 0 the support of every function in $\overline{\mathcal{E}}^c_{\varepsilon,m}$ is inside a small ball, up to a negligible part.

Proposition 4.8 *There exists $\mu = \mu(M, g) > 0$ such that the following property holds. For every almost isoperimetric sequence $(u_k)_{k\in\mathbb{N}} \subset L^1(M, \mathbb{R})$ with volumes $m_k = \int_M u_k \mathrm{d}v_g \to 0$, i.e.,*

$$\lim_{k\to\infty} \frac{\overline{\mathcal{E}}_0(u_k)}{\sigma \overline{I}_M(m_k)} = 1, \tag{4.4}$$

there exists a sequence $(p_k)_{k\in\mathbb{N}} \subset M$ such that

$$\lim_{k\to+\infty} \frac{1}{m_k}\left(\int_{M\setminus B(p_k,\mu m_k^{1/n})} u_k \mathrm{d}v_g\right) = 0. \tag{4.5}$$

Proof The proof is analogous to that of Proposition 3.15, and it relies on the compactness result provided by Theorem 3.13. However, in this case we cannot work directly on M, since the notion of perimeter for Caccioppoli sets accounts only for the portion lying in the interior of M (see Remark 3.14). To overcome this issue, we consider the closed manifold $\widetilde{M}$, introduced in Remark 4.1, which contains M as a subset. Accordingly, we may regard the almost isoperimetric sequence $(u_k)_{k\in\mathbb{N}} \subset L^1(M,\mathbb{R})$ as functions on $\widetilde{M}$, extended by zero outside M. We then denote by $(\Omega_k)_{k\in\mathbb{N}} \subset \mathcal{C}_g(\widetilde{M})$ the associated finite perimeter subsets of $\widetilde{M}$ such that $u_k = \mathbf{1}_{\Omega_k}$.

Let $(X_k, g_k) = (\widetilde{M}, m^{-1/k} g)$, so that $\mathcal{H}^n_k(\Omega_k) = 1$ for every k, where $\mathcal{H}^n_k$ denotes the n–dimensional Hausdorff measure on (X_k, g_k). By (4.1) and (4.4), we have also that

$$\lim_{k\to\infty} P(\Omega_k, X_k) = \lim_{k\to\infty} \mathcal{H}^{n-1}_k(\partial^*\Omega_k) = I_{\mathbb{R}^n}(1).$$

This means that the sequence $(\Omega_k)_{k\in\mathbb{N}}$ has uniformly bounded volumes and perimeters and we can apply Theorem 3.13 to obtain a number $\overline{N} \in \mathbb{N}\cup\{+\infty\}$, a family of pointed RCD(κ, n) spaces $(M^i_\infty, \mathrm{d}^i_\infty, \mathcal{H}^n_\infty, p^i)$, for $i = 1, \dots, \overline{N}$, and subsets $\Omega^i_\infty \subset M^i_\infty$ such that

$$\sum_{i=1}^{\overline{N}} \mathcal{H}^n_\infty(\Omega^i_\infty) = \lim_{k\to\infty} \mathcal{H}^n_k(\Omega_k) = 1, \tag{4.6}$$

and

$$\sum_{i=1}^{\overline{N}} \mathcal{H}^{n-1}_\infty(\partial^*\Omega^i_\infty) \le \liminf_{k\to\infty} \mathcal{H}^{n-1}_k(\partial^*\Omega_k) = I_{\mathbb{R}^n}(1). \tag{4.7}$$

Since $\widetilde{M}$ is a closed manifold (and therefore each rescaled space X_k is also a closed manifold), every limit space $(M^i_\infty, \mathrm{d}^i_\infty, \mathcal{H}^n_\infty, p^i)$ must be the Euclidean space $(\mathbb{R}^n, \mathrm{d}, \mathcal{H}^n, 0)$.

By arguing as in the final part of the proof of Proposition 3.15, and using the fact that the Euclidean isoperimetric profile $I_{\mathbb{R}^n}(m) = c_n m^{\frac{n-1}{n}}$ is strictly subadditive, we conclude that the conditions (4.6) and (4.7) can only be satisfied if $\overline{N} = 1$. Moreover, $\Omega^1_\infty \subset \mathbb{R}^n$ is an isoperimetric set, so it must be an Euclidean ball of unit volume. Since by Theorem 3.13 we know that $\Omega_k \to \Omega^1_\infty$ in the L^1–strong topology as $k \to \infty$, and each $\Omega_k \subset \widetilde{M}$ is contained in M, that is,

$$\mathcal{H}^n(\Omega_k \setminus M) = 0, \qquad \forall k \in \mathbb{N},$$

we conclude that, also when regarded as subsets of the rescaled spaces X_k, the sets Ω_k concentrate inside M. Therefore, we can find a sequence of points $(p_k)_{k\in\mathbb{N}} \subset M$ such that (4.5) holds, where μ can be chosen as twice the radius of the unit-volume Euclidean ball in $\mathbb{R}^n$. □

Lemma 4.9 *Assume that (H1) and (H2) hold and let $\tau : \mathbb{R}^+ \to \mathbb{R}$ be given by Proposition 4.6. For every $\alpha \in (0,1)$, there exists $m_\alpha = m_\alpha(M, g, W, \theta, \alpha) > 0$ such that for every $m \in (0, m_\alpha)$ there exists $\varepsilon_\alpha = \varepsilon_\alpha(M, g, W, \theta, \alpha, m) > 0$ such that for every $\varepsilon \in (0, \varepsilon_\alpha)$ and for any $u \in \mathcal{E}_{\varepsilon,m}^{\sigma \overline{I}_M(m)+\tau(m)}$ there exists a point $p_u \in M$ such that*

$$\int_{M \setminus B(p_u, \mu m^{1/n})} |u| \mathrm{d}v_g \le \alpha m.$$

Proof Relying on the concentration result ensured by Proposition 4.8, the proof is analogous to the one of Lemma 3.16. For the sake of clarity, we report here the main steps.

Arguing by contradiction, there exists a sequence $(m_i)_{i\in\mathbb{N}}$ that goes to 0 and for every $i \in \mathbb{N}$ there exists two sequences $(\varepsilon_{i,j})_{j\in\mathbb{N}}$ and $(u_{i,j})_{j\in\mathbb{N}} \subset \overline{\mathcal{E}}_{\varepsilon_{i,j},m_i}^{c_i}$, with $c_i = \sigma \overline{I}_M(m_i) + \tau(m_i)$ such that $\varepsilon_{i,j} \to 0^+$ as $j \to \infty$ and

$$\int_{M \setminus B_g(p, \mu m_i^{1/n})} |u_{i,j}| \mathrm{d}v_g > \alpha m_i, \qquad \forall p \in M, \ \forall j \in \mathbb{N}.$$

Since $\overline{\mathcal{E}}_{\varepsilon_{i,j},m_i}(u_{i,j}) \le c_i$ for every j, by Theorem 3.3 we have that for every fixed $i \in \mathbb{N}$ there exists a characteristic function $u_{0,i} \in L^1(M, \mathbb{R})$ such that $u_{i,j}$ converges to it, up to subsequences. Therefore, for every $i \in \mathbb{N}$ there exists j_i sufficiently large such that

$$\int_M \left(|u_{i,j_i} - u_{0,i}|\right) \mathrm{d}v_g \le \frac{\alpha}{4} m_i. \tag{4.8}$$

Moreover, by Remark 4.4 we have

$$\overline{I}_M(m_i) \le \frac{\overline{\mathcal{E}}_0(u_{0,i})}{\sigma} \le \overline{I}_M(m_i) + \tau(m_i).$$

Since $\tau(m) = o(m^{\frac{n-1}{n}})$ and (4.1) holds, the last chain of inequality implies that

$$\lim_{i\to\infty} \frac{\overline{\mathcal{E}}_0(u_{0,i})}{\sigma \overline{I}_M(m_i)} = 1,$$

so we can apply Proposition 4.8 and obtain a sequence of points $(p_i)_{i\in\mathbb{N}} \subset M$ such that

$$\lim_{i\to+\infty} \frac{1}{m_i} \left(\int_{M \setminus B(p_i, \mu m_i^{1/n})} u_{0,i} \mathrm{d}v_g \right) = 0. \tag{4.9}$$

Utilizing the same reasoning employed in the concluding section of the proof for Lemma 3.16, (4.8) and (4.9) lead us to the intended contradiction. □

As we have done for the case of Neumann conditions, now that we have the "concentration" result ensured by Lemma 4.9, it is possible to prove that if m and ε are sufficiently small for any $u \in \overline{\mathcal{E}}_{\varepsilon,m}^{\sigma \overline{I}_M(m)+\tau(m)}$ we have that $\beta^*(u)$ is in a neighbourhood of $M \subset \mathbb{R}^{\tilde{n}}$. More formally, setting

$$M_r = \left\{ x \in \mathbb{R}^{\tilde{n}} : \text{dist}_{\mathbb{R}^{\tilde{n}}}(x, M) < r \right\}, \qquad \forall r > 0,$$

we have the following result.

Lemma 4.10 *Assume that (H1) and (H2) hold. For any $r > 0$, there exists $m_2 = m_2(M, g, r, \text{diam}_{\mathbb{R}^{\tilde{n}}}(M)) > 0$ such that for every $m \in (0, m_2)$ there exists another positive constant $\varepsilon_2 = \varepsilon_2(M, g, r, m)$ such that for every $\varepsilon \in (0, \varepsilon_2)$ and any $u \in \overline{\mathcal{E}}_{\varepsilon,m}^{\sigma \overline{I}_M(m)+\tau(m)}$ we have $\beta^*(u) \in M_r$.*

Proof Cf. Lemma 3.17. □

Remark 4.11 By the previous lemma and the compactness of M, there exists $r_1 > 0$ such that for every $r \in]0, r_1[$, setting $m_2 = m_2(M, g, r, \text{diam}_{\mathbb{R}^{\tilde{n}}}(M))$ and $\varepsilon_2 = \varepsilon_2(M, g, r, m)$ as in the lemma, the map $\mathcal{B} = \pi_M \circ \beta^* : \overline{\mathcal{E}}_{\varepsilon,m}^{c} \to M$ is well defined and continuous, with $c = \sigma \overline{I}_M(m) + \tau(m)$.

4.5 Conclusion of the proof

In this section we show that if m and ε are sufficiently small then $\mathcal{B} \circ \mathcal{P}_{\varepsilon,m} : M \to M$ is homotopic to the identity map. Then, Theorem B follows as an application of Theorem C.

Recalling the definition of $\delta_M > 0$ given in Sect. 4.3 for the purpose of defining the photography map $\mathcal{P}_{\varepsilon,m} : M \to \mathcal{H}_{m,0}$, let us also define the map $\widetilde{\mathcal{P}}_{\varepsilon,m} : M^{\delta_M/2} \to \mathcal{H}_{m,0}$ as follows:

$$\widetilde{\mathcal{P}}_{\varepsilon,m}(p) = u_{\varepsilon,p}^{m},$$

where $M^{\delta_M/2} = \{p \in M : \text{dist}_g(p, \partial M) \geq \delta_M/2\}$. With this notation, $\mathcal{P}_{\varepsilon,m}$ is given by $\widetilde{\mathcal{P}}_{\varepsilon,m} \circ \mathcal{L}$.

Lemma 4.12 *Assume that (H1) and (H2) hold. For every $r \in]0, \delta_M/2[$ there exists $m_3 = m_3(M, g, r) > 0$ such that for every $m \in (0, m_3)$ there exists $\varepsilon_3 = \varepsilon_3(M, g, r, m) > 0$ such that*

$$\|\beta^*(\widetilde{\mathcal{P}}_{\varepsilon,m}(p)) - p\|_{\mathbb{R}^{\tilde{n}}} \leq r, \qquad \forall p \in M^{\delta_M/2}.$$

Proof Cf. Lemma 3.19. □

Lemma 4.13 *Assume that (H1) and (H2) hold, let $\mathcal{P}_{\varepsilon,m} : M \to \mathcal{H}_{m,0}$ be given by Definition 4.5 and let $\mathcal{B} : H^1(M, \mathbb{R}) \to M$ be defined as $\pi_M \circ \beta^*$. There exists $m^* =$*

$m^*(M, g, W) > 0$ such that for any $m \in (0, m^*)$ there exists $\varepsilon^* = \varepsilon^*(M, g, W, m) > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$ the composition map

$$\mathcal{B} \circ \mathcal{P}_{\varepsilon,m} \colon M \to M$$

is well defined and homotopic to the identity map.

Proof The proof is very similar to the one of Lemma 3.20, but it has to take into account the action of the boundary layer map $\mathcal{L}$. However, since $\mathcal{L}$ is homotopic to the identity map, it suffices to show that $\mathcal{B} \circ \widetilde{\mathcal{P}}_{\varepsilon,m} \colon M^{\delta_M/2} \to M$ is homotopic to the identity map as well, provided that m and ε are sufficiently small.

By the compactness of M, there exists a positive constant C_M such that $\mathrm{dist}_{(M,g)}(p, q) \le C_M \|p - q\|_{\mathbb{R}^{\tilde{n}}}$, for every $p, q \in M$, where $\mathrm{dist}_{(M,g)}$ stands for the Riemannian distance on (M, g). Therefore,. We denote by $\mathrm{inj}(M_{\delta_M/2})$ the *injectivity radius* of $M^{\delta_M/2}$ in M, that is

$$\mathrm{inj}(M^{\delta_M/2}) := \inf_{p \in M^{\delta_M/2}} \sup \Big\{ R > 0 \text{ s.t. } \exp_p^M \colon B_g^M(p, R) \to M \text{ is a diffeomorphism} \Big\}.$$

Note that by this construction we have $\mathrm{inj}(M^{\delta_M/2}) \le \delta_M/2$.

Recalling the definition of r_1 given in Remark 4.11, we set

$$r^* = \frac{1}{2} \min \left\{ \frac{\mathrm{inj}(M^{\delta_M/2})}{2C_M}, r_1 \right\} \le \frac{\delta_M}{4}.$$

Let $m_1 = m_1(M, g, W, \tau)$, $m_2 = m_2(M, g, r^*, \mathrm{diam}_{\mathbb{R}^{\tilde{n}}}(M))$ and $m_3 = m_3(M, g, r^*)$ be the positive constants given by Proposition 4.6, Lemma 4.10 and Lemma 4.12, respectively. Let $m^* = \min\{m_1, m_2, m_3\} > 0$ and for every $m \in (0, m^*)$ let $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, where ε_i is defined by using the same results of m_i, for $i = 1, 2, 3$. Since $m < m_1$, for every $p \in M^{\delta_M/2}$ we have that $\widetilde{\mathcal{P}}_{\varepsilon,m}(p) \in \overline{\mathcal{E}}^c_{\varepsilon,m}$, with $c = \sigma \overline{I}_M(m) + \tau(m)$. Since $m < m_3$, this implies that

$\|\beta^*(\widetilde{\mathcal{P}}_{\varepsilon,m}(p)) - p\|_{\mathbb{R}^{\tilde{n}}} \le r^*$ for every $p \in M_{\delta_M/2}$. Therefore, we have

$$\begin{aligned} \mathrm{dist}_{(M,g)}\big(\mathcal{B}(\widetilde{\mathcal{P}}_{\varepsilon,m}(p)), p\big) &\le C_M \|\pi_M(\beta^*(\widetilde{\mathcal{P}}_{\varepsilon,m}(p))) - p\|_{\mathbb{R}^{\tilde{n}}} \\ &\le C_M \big(\|\pi_M(\beta^*(\widetilde{\mathcal{P}}_{\varepsilon,m}(p))) - \beta^*(\widetilde{\mathcal{P}}_{\varepsilon,m}(p))\|_{\mathbb{R}^{\tilde{n}}} + \|\beta^*(\widetilde{\mathcal{P}}_{\varepsilon,m}(p)) - p\|_{\mathbb{R}^{\tilde{n}}} \big) \\ &\le 2C_M r^* \le \frac{\delta_M}{2}, \end{aligned}$$

so the map $F \colon [0, 1] \times M_{\delta_M/2} \to M$, given by

$$F(t, p) := \exp_p^M \Big(t \big(\exp_p^M\big)^{-1} \big(\mathcal{B}(\widetilde{\mathcal{P}}_{\varepsilon,m}(p))\big) \Big),$$

is well-defined, continuous and gives a homotopy equivalence between $\mathcal{B} \circ \widetilde{\mathcal{P}}_{\varepsilon,m}$ and the identity map in $M^{\delta_M/2}$. □

Proof of Theorem B As previously stated, this proof is based on Theorem C. Indeed, for every $m, \varepsilon > 0$ the functional $\overline{\mathcal{E}}_{\varepsilon,m}$ is bounded below and it satisfies the Palais–Smale condition (see Lemma 2.1). Moreover, let $m^* > 0$ be given by Lemma 4.13 and for every $m \in (0, m^*)$ let $\varepsilon^* > 0$ be given by the same lemma. Setting $c = \sigma \overline{I}_M(m) + \tau(m)$, where $\tau : \mathbb{R}^+ \to \mathbb{R}$ is defined in Proposition 4.6, we establish that the photography map $\mathcal{P}_{\varepsilon,m} : M \to \overline{\mathcal{E}}^c_{\varepsilon,m}$ (as defined in Definition 4.5) and the barycenter map $\mathcal{B} : \overline{\mathcal{E}}^c_{\varepsilon,m} \to M$ (defined as $\pi_M \circ \beta^*$) are well-defined, for every $\varepsilon \in (0, \varepsilon^*)$. Furthermore, according to Lemma 4.13, their composition $\mathcal{B} \circ \mathcal{P}_{\varepsilon,m}$ is homotopic to the identity map. Then, all the conclusions of Theorem B can be derived by applying Theorem C. □

5 Generic non-degeneracy

The conclusions of Theorems A and B are stronger if one can ensure that for given m and ε all the solutions of (1.2) and (1.3) are nondegenerate. In many situations, non-degeneracy of solutions is a generic property, in the sense that it is obtained after some arbitrary small perturbation of a key parameter of the problem. This is the case for semilinear elliptic equations as the ones considered in this paper, meaning that the stronger existence statements of Theorem A and B, based on Morse inequalities, hold in generic situations. The purpose of this section is to give some insight on how these genericity properties are obtained from the available literature, which contains analogous results in similar settings. We present generic non-degeneracy results with respect to perturbations of the Riemannian metric (Theorem 5.2 below) as well as with respect to boundary perturbations (Theorem 5.3 below). As it has been noted by the referee, it would also be interesting in view of possible applications to study whether one has generic non-degeneracy with respect to perturbations on the mass constraint, in the spirit of Zhou's work on the multiplicity one conjecture [62]. The answer to this interesting question does not seem obvious to us at first glance; therefore, we leave it open here.

5.1 Abstract transversality result

Generic non-degeneracy is usually obtained by applying Sard–Smale Theorem [57] and abstract transversality results on infinite-dimensional Banach manifolds, see Quinn [55], Saut and Temam [56] and Uhlenbeck [60]. Following [56], we now introduce some objects that will be fixed for the rest of this section and recall some standard definitions. Let X, Y and Z be Banach spaces and $U \subset X$, $V \subset Y$ open subsets. Let $F : U \times V \to Z$ be of class C^k for $k \geq 1$. We will denote by $DF(x_0, y_0)$ the (total) differential of F at the point $(x_0, y_0) \in U \times V$ and by $D_X F(x_0, y_0)$, $D_Y F(x_0, y_0)$ the differentials with respect to the x and the y components, respectively. Given Z' a Banach space, $U' \subset Z'$ an open subset and $g : Z' \to Z$ of class C^1, we say that $z_0 \in Z$ is a regular value of g if $Dg(z'_0)$ (the differential of g at z'_0) is onto for any $z'_0 \in U'$ such that $g(z'_0) = z_0$. The abstract result we use is the following:

Theorem 5.1 (Theorem 1.1 in [56]) *Assume that $z_0 \in Z$ is a regular value of F and that, moreover:*

1. *For any $x \in U$ and $y \in V$, $D_X F(x, y) : X \to Z$ is a Fredholm map of index $l < k$.*
2. *The set of $x \in U$ such that $F(x, y) = z_0$ for all y in a compact subset of V is a relatively compact set of U.*

Then, the set

$$\mathcal{O}{:=}\{y \in V : z_0 \text{ is a regular value of } F(\cdot, y)\},$$

is a dense open subset of V.

5.2 Generic non-degeneracy with respect to the Riemannian metric

Theorem 5.1 was used in [28] by de Paula Ramos in order to prove generic non-degeneracy for solutions of (1.1) (i.e., critical points of the C^2 functionals $\mathcal{E}_{\varepsilon,m}$ and $\overline{\mathcal{E}}_{\varepsilon,m}$) with respect to the Riemannian metric. Analogous results for systems were proven in [6], see also Micheletti and Pistoia [41] for earlier results in related settings. The result proven in [28] reads as follows:

Theorem 5.2 *Let $\mathrm{Met}^\infty(M)$ be the space of C^∞ Riemannian metrics on M and $m \in \mathbb{R}$. For $\hat{g} \in \mathrm{Met}^\infty(M)$ and $\varepsilon \in (0, +\infty)$, we denote by $\mathcal{E}_{\varepsilon,m,\hat{g}}$ and $\overline{\mathcal{E}}_{\varepsilon,m,\hat{g}}$ the functionals $\mathcal{E}_{\varepsilon,m}$ and $\overline{\mathcal{E}}_{\varepsilon,m}$ with respect to the metric $\hat{g}$. Then, for any $g_0 \in \mathrm{Met}^\infty(M)$ the sets*

$$\left\{ \begin{array}{r} (\varepsilon, \hat{g}) \in (0, +\infty) \times \mathrm{Met}^\infty(M) : \textit{ any critical point} \\ \textit{of } \mathcal{E}_{\varepsilon,m,\hat{g}} \textit{ in } H^1_{g_0}(M, \mathbb{R}) \textit{ is non-degenerate} \end{array} \right\}$$

and

$$\left\{ \begin{array}{r} (\varepsilon, \hat{g}) \in (0, +\infty) \times \mathrm{Met}^\infty(M) : \textit{ any critical point} \\ \textit{of } \overline{\mathcal{E}_{\varepsilon,m,\hat{g}}} \textit{ in } H^1_{g_0}(M, \mathbb{R}) \textit{ is non-degenerate} \end{array} \right\}$$

can be written as a countable interesection of open and dense subsets of $(0, +\infty) \times \mathrm{Diff}^\infty(M)$. In particular, they are residual subsets of $(0, +\infty) \times \mathrm{Diff}^\infty(M)$, which are dense.

The proof of Theorem 5.2 is obtained as an application of the abstract Theorem 5.1 and can be found in [28].

5.3 Generic non-degeneracy with respect to the domain

Notice that Theorem 5.2 has a drawback: In some applications, one might not want to modify the metric of the problem. For instance, if M is an Euclidean domain $\Omega \subset \mathbb{R}^n$ then it would not make much sense to replace the Euclidean metric on Ω by another one. It is therefore reasonable to be interesected in generic non-degeneracy with respect to a different key parameter. It might be tempting to consider a genericity result based on modifying the potential instead of the Riemannian metric. However, Brunovský and Poláčik [19] produced a counterexample for semilinear elliptic equations on the Euclidean ball with homogeneous Dirichlet boundary conditions. Our setting is slightly different than theirs (in particular, more restrictive) meaning that one could still hope to obtain genericity results with respect to the potential. However, the investigation of this question goes beyond the scope of this paper. Instead, we will present a genericity result based on perturbation of the domain, obtained by adapting a result due to Saut and Temam [56] and Henry [34, Chapter 6].

Following [34], let us give the precise statement of the result. For $l \in \{1, \dots, +\infty\}$, let $\mathrm{Diff}^l(M)$ be the set of maps in $\mathcal{C}^l(M, \mathbb{R}^{\tilde{n}})$ such that $h\colon M \to h(M)$ is a C^l-diffeomorphism (recall that M is isometrically embedded into $\mathbb{R}^{\tilde{n}}$). In particular, $\mathrm{Id} \in \mathrm{Diff}^l(M)$. For $l < +\infty$, let us endow $\mathcal{C}^l(M, \mathbb{R}^n)$ with the usual sup norm, so it is a Banach space. This induces the usual Whitney topology on $\mathcal{C}^\infty(M, \mathbb{R}^n)$. Then, with these choices of topologies, $\mathrm{Diff}^l(M)$ is open in $\mathcal{C}^l(M, \mathbb{R}^n)$. For $h \in \mathrm{Diff}^l(M)$ and $\varepsilon > 0$, let $\mathcal{E}_{\varepsilon,h}$ denote the energy functional with parameter ε in the domain $h(M)$. For $m \in \mathbb{R}$, define $\mathcal{E}_{\varepsilon,m,h}$ and $\overline{\mathcal{E}}_{\varepsilon,m,h}$ as the restriction of $\mathcal{E}_{\varepsilon,h}$ to $\mathcal{H}_m$ and $\mathcal{H}_{m,0}$ respectively. The main results then read as follows:

Theorem 5.3 *Let $m \in \mathbb{R}$. The sets*

$$\left\{(\varepsilon, h) \in (0, +\infty) \times \mathrm{Diff}^\infty(M) : \textit{any critical point of } \mathcal{E}_{\varepsilon,m,h} \textit{ is non-degenerate}\right\}$$

and

$$\left\{(\varepsilon, h) \in (0, +\infty) \times \mathrm{Diff}^\infty(M) : \textit{any critical point of } \overline{\mathcal{E}}_{\varepsilon,m,h} \textit{ is non-degenerate}\right\}$$

can be written as a countable interesection of open and dense subsets of $(0, +\infty) \times \mathrm{Diff}^\infty(M)$*. In particular, they are residual subsets of* $(0, +\infty) \times \mathrm{Diff}^\infty(M)$*, which are dense.*

The proof of Theorem 5.3 follows by combining the ideas in [6, 28] with those in [34, 56]. We give a sketch below.

Sketch of the proof of Theorem 5.3 By Baire's Theorem, it suffices to prove that for any $l \in \mathbb{N}^*$ such that $l \geq 3$ and $K > 0$, the sets

$$\Big\{(\varepsilon, h) \in (0, +\infty) \times \mathrm{Diff}^l(M) : \text{ any critical point of } \mathcal{E}_{\varepsilon,m,h} \text{ in } B_\infty(0, K) \text{ is non-degenerate}\} \tag{5.1}$$

and

$$\Big\{(\varepsilon, h) \in (0, +\infty) \times \mathrm{Diff}^l(M) : \text{ any critical point of } \overline{\mathcal{E}}_{\varepsilon,m,h} \text{ in } B_\infty(0, K) \text{ is non-degenerate}\} \tag{5.2}$$

are open and dense in $(0,+\infty)\times \mathrm{Diff}^l(M)$, where we have set

$$B_\infty(0,K) := \{v \in L^\infty(M) : \|v\|_{L^\infty(M)} \le K\}.$$

Let l and C be as above and fixed. Given $\overline{M}$ a Riemannian manifold with boundary of class C^l isometrically embedded into $\mathbb{R}^{\tilde{n}}$, set

$$\hat{F}_{\overline{M}} : H^2(\overline{M}) \times \mathbb{R} \times (0,+\infty) \to L^2(\overline{M}) \times \mathbb{R}$$

as

$$\hat{F}(u,\lambda,\varepsilon) := \left(-\varepsilon\Delta_{\overline{M}} u + \frac{1}{\varepsilon}W'(u) - \lambda, \int_{\overline{M}} u - m\right)$$

for $(u,\lambda,\varepsilon) \in H^2(\overline{M}) \times \mathbb{R} \times (0,+\infty)$. Subsequently, consider

$$F : H^2(M) \times \mathbb{R} \times \mathrm{Diff}^l(M) \times (0,+\infty) \to L^2(M) \times \mathbb{R},$$

defined as

$$F(u,\lambda,\varepsilon,h) := h^*\hat{F}_{h(M)}((h^{-1})^*u,\lambda,\varepsilon)$$

where h^* stands for the composition map defined as

$$h^*(u(x)) := u(h(x)), \quad \text{for a. e. } x \in M,$$

whenever $u \in L^1_{\mathrm{loc}}(M)$ and where $h(M)$ has been endowed with the pull-back metric. We begin by proving that F satisfies the assumptions of Theorem 5.1 with $X = H^2(M) \times \mathbb{R}$, $Y = \mathrm{Diff}^l(M) \times (0,+\infty)$, $Z = L^2(M) \times \mathbb{R}$, $U = (H^2(M)\backslash\mathbb{R}) \times \mathbb{R}$ (so that we are first considering the case of non-constant solutions), $V = Y$ and $z_0 = (0,0) \in Z$. One easily checks that F is of class C^1. It is also easy to see that $DF_X(u,\lambda,\varepsilon,h)$ is a Fredholm operator of index 0 on $U \times V$, which establishes 1 in Theorem 5.1. Condition 2 follows from the fact that the energy functional satisfies the Palais–Smale condition. It remains only to prove that $(0,0)$ is a regular value of F. As in [34, Chapter 6], we argue by contradiction. Assume that there exists $(u,\lambda,\varepsilon,h)$ such that $DF(u,\lambda,\varepsilon,h)$ is not surjective. Arguing as in [34, Page 81], one can assume that $h = \mathrm{Id}_M$ by considering $(h^{-1})^*u$ instead of u. By assumption, one has a nonzero element (ψ,a) in $L^2(M) \times \mathbb{R}$ which is orthogonal to the image of $DF(u,\lambda,\varepsilon,h)$. After some computations along the lines of [34, Chapter 6], one is lead to the following overdetermined problem:

$$-\varepsilon\Delta\psi + \frac{1}{\varepsilon}W''(u)\psi = 0, \quad \text{on } M, \tag{5.3}$$

$$\frac{\partial\psi}{\partial\nu} = \psi(W'(u)-\lambda) = 0, \quad \text{on } \partial M. \tag{5.4}$$

Moreover, one also obtains that $a = 0$, which implies that ψ is nonzero. By a contradiction argument based on the unique continuation principle, one obtains from (5.3) and (5.4) that $W'(u) - \lambda = 0$ on ∂M and hence $\Delta u = 0$ on ∂M. This implies that u is constant on ∂M, which yields once again the same contradiction. As a consequence, we can apply Theorem 5.1 for non-constant solutions. In order to prove generic non-degeneracy for constant solutions, one argues in a direct manner and the statement then follows from the fact that the set of the eigenvalues of the Laplacian (with either Dirichlet or Neumann boundary conditions) is discrete, see [28, Proposition B] for details. This proves the result for the set (5.1). In order to prove the property for the set (5.2), one argues similarly, see [34, Example 6.5]. □

6 A partial result without the subcritical growth assumption

The sole purpose of the subcritical growth assumption (H3) is to ensure that the variational problems under consideration are compact (more precisely, that the functionals $\mathcal{E}_{\varepsilon,m}$ and $\overline{\mathcal{E}}_{\varepsilon,m}$ satisfy the Palais–Smale condition). However, as it was already observed in [15], it is possible to drop (H3) and obtain weaker versions of Theorems A and B. More precisely, one has:

Theorem 6.1 *Assume that (H1) and (H2) hold. Then, there exists $m^* > 0$ such that for all $m \in (0, m^*)$ there exist $\varepsilon_m, c_m > 0$ such that for any $\varepsilon \in (0, \varepsilon_m)$ the Neumann problem (1.2) has at least $\mathrm{cat}(\partial M)$ solutions $u_{\varepsilon,m}$ with $\mathcal{E}_\varepsilon(u_{\varepsilon,m}) \leq c_m$. Moreover, if ε and m as above are such that all critical points of $\mathcal{E}_{\varepsilon,m}$ are non-degenerate, then (1.2) has at least $\mathscr{P}_1(\partial M)$ solutions $u_{\varepsilon,m}$ with $\mathcal{E}_\varepsilon(u_{\varepsilon,m}) \leq c_m$.*

Theorem 6.2 *Assume that (H1) and (H2) hold. Then, there exists $m^* > 0$ such that for all $m \in (0, m^*)$ there exist $\varepsilon_m, c_m > 0$ such that for any $\varepsilon \in (0, \varepsilon_m)$ the Dirichlet problem (1.3) has at least $\mathrm{cat}(M)$ solutions $u_{\varepsilon,m}$ with $\mathcal{E}_\varepsilon(u_{\varepsilon,m}) \leq c_m$. Moreover, if ε and m as above are such that all critical points of $\overline{\mathcal{E}}_{\varepsilon,m}$ are non-degenerate, then (1.3) has at least $\mathscr{P}_1(M)$ solutions $u_{\varepsilon,m}$ with $\mathcal{E}_\varepsilon(u_{\varepsilon,m}) \leq c_m$.*

The proofs of Theorems 6.1 and 6.2 work essentially like that of [15, Theorem 5.9]. However, the assumptions on the potential that we take in this paper are slightly different than those in [15]. Therefore, we include the proofs of Theorems 6.1 and 6.2 for completeness. The crucial ingredient is that for small ε one can obtain a priori bounds on the L^∞ norm of any solution $(u_\varepsilon, \lambda_\varepsilon)$ of the equation

$$-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}\hat{W}'(u_\varepsilon) = \lambda_\varepsilon \text{ on } M. \tag{6.1}$$

according to its energy (see Proposition 6.3 below). Such ingredient allows to modify potentials close to infinity to turn them into potentials with subcritical growth at infinity to which one may apply Theorems A and B. The a priori bounds imply that the low energy solutions of the modified potential are also solutions of the original potential. However, no such thing can be said at this point regarding high energy solutions, hence the weaker statements of Theorems 6.1 and 6.2.

Proposition 6.3 *Let* $\hat{W} \in \mathcal{C}^3_{\mathrm{loc}}(\mathbb{R}, [0, +\infty])$ *be a double-well potential vanishing exactly on* $\{0, 1\}$ *and satisfying* (H1) *and* (H2)*. There exists* $\varepsilon_{ub} > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_{\mathrm{ub}}]$ *and any* $(u_\varepsilon, \lambda_\varepsilon)$ *solution of* (6.1) *such that* $\mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon) < +\infty$ *we have*

$$\|u_\varepsilon\|_{L^\infty(M,\mathbb{R})} \leq C_{\mathrm{ub}}(\varepsilon_{ub}, \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon)),$$

with $C_{\mathrm{ub}}(\varepsilon_{ub}, \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon)) > 0$ *a constant depending on the quantities* ε_{ub} *and* $\mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon)$ *but independent on* ε*.*

The proof of Proposition 6.3 relies on the following result.

Lemma 6.4 *Let* $\hat{W} \in \mathcal{C}^3_{\mathrm{loc}}(\mathbb{R}, [0, +\infty])$ *be a double-well potential vanishing exactly on* $\{0, 1\}$ *and satisfying* (H1) *and* (H2)*. There exist* $\varepsilon_{\mathrm{est}} > 0$ *and* $C_{\mathrm{est}} > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_{\mathrm{est}})$*,* $m \in [0, 1]$ *and* $(u_\varepsilon, \lambda_\varepsilon)$ *a solution of* (6.1) *we have*

$$|\lambda_\varepsilon| \leq C_{\mathrm{est}} \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon).$$

Lemma 6.4 was proven by Chen in [23, Lemma 3.4] in the case of an Euclidean domain of $\mathbb{R}^n$ and then extended to the setting of closed manifolds in [15, Proposition 5.3]. In both cases, the proof is obtained by combining standard elliptic estimates with mollification arguments. As the arguments carry on directly to our setting, we skip the proof. We are now ready to prove Proposition 6.3, which relies on standard arguments (see, for instance, [15, Theorem 5.9]).

Proof of Proposition 6.3 Assume that $\varepsilon \in (0, \varepsilon_{\mathrm{est}})$, with $\varepsilon_{\mathrm{est}}$ as in Lemma 6.4. By (H2), we find $u^*(\varepsilon_{\mathrm{est}}, \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon))$ depending only on $\varepsilon_{\mathrm{est}} > 0$ and $\mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon)$ such that

$$\frac{1}{\varepsilon}\hat{W}'(u) > C_{\mathrm{est}} \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon), \qquad \text{for all } u \geq u^*(\varepsilon_{\mathrm{est}}, \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon))$$

and

$$\frac{1}{\varepsilon}\hat{W}'(u) < -C_{\mathrm{est}} \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon), \qquad \text{for all } u \leq u^*(\varepsilon_{\mathrm{est}}, \_\varepsilon(u_\varepsilon)),$$

with $C_{\mathrm{est}} > 0$ as in Lemma 6.4. Let $x_{\max} \in M$ be a maximum point for u_ε and assume by contradiction that $\|u_\varepsilon\|_{L^\infty(M,\mathbb{R})} > u^*(\varepsilon_{\mathrm{est}}, \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon))$. Then, using (6.1) and Lemma 6.4 it follows

$$-\varepsilon \Delta u_\varepsilon(x_{\max}) = \lambda_\varepsilon - \frac{1}{\varepsilon}\hat{W}'(u_\varepsilon(x_{\max})) < 0,$$

which gives the contradiction. Hence, one has that $\max_M u \leq u^*(\varepsilon_{\mathrm{est}}, \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon))$ and, in an analogous fashion, one finds that $\min_M u \geq u^*(\varepsilon_{\mathrm{est}}, \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon))$. This establishes the result by choosing $\varepsilon_{\mathrm{ub}} = \varepsilon_{\mathrm{est}}$ and $C_{\mathrm{ub}}(\varepsilon_{ub}, \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon)) = u^*(\varepsilon_{\mathrm{est}}, \mathcal{E}_{\varepsilon,\hat{W}}(u_\varepsilon))$. □

At this point, the proof of Theorem 6.1 can be completed as follows.

Proof of Theorem 6.1 Let $\varepsilon \in (0, \varepsilon_{\text{ub}})$, with ε_{ub} as in Lemma 6.4 and $C_{\text{ub}}(\varepsilon_{\text{ub}}, 1) > 0$ the constant given by Proposition 6.3. We can find $u^* \geq C_{\text{ub}}(\varepsilon_{\text{ub}}, 1)$ and $\hat{W} \in \mathcal{C}^3_{\text{loc}}(\mathbb{R}, [0, +\infty))$ such that $\hat{W}^{-1}(0) = \{0, 1\}$ which satisfies (H1), (H2) and (H3) with $\hat{W}(u) = W(u)$ for all $u \in [-u^*, u^*]$. We now apply Theorem A to $\hat{W}$. In particular, there exists $\hat{m}^* > 0$ such that for all $m \in (0, \hat{m}^*)$ there exist $\hat{\varepsilon}_m, \hat{c}_m > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon}_m)$ we have that there exist $\text{cat}(\partial M)$ solutions of (1.2) (for $\hat{W}$) $(\hat{u}_{\varepsilon,m}, \hat{\lambda}_{\varepsilon,m})$ with $\mathcal{E}_{\varepsilon,\hat{W}}(\hat{u}_{\varepsilon,m}) \leq \hat{c}_m$. Moreover $\hat{c}_m$ can be chosen such that $\hat{c}_m \to 0$ as $m \to 0$, see the end of Sect. 3.5. Therefore, choose $m^* \leq \hat{m}^*$ such that $\hat{c}_m \leq 1$ for all $m \in (0, m^*)$. For all such m, let $\varepsilon_m := \frac{1}{2} \min\{\hat{\varepsilon}_m, \varepsilon_{\text{est}}\} > 0$, $\varepsilon \in (0, \varepsilon_m)$ and $\hat{u}_{\varepsilon,m}$. Since $\mathcal{E}_\varepsilon(\hat{u}_{\varepsilon,m}) \leq 1$ whenever $\hat{u}_{\varepsilon,m}$ is a solution as above, we can apply Proposition 6.3 and find that $\|\hat{u}_{\varepsilon,m}\|_{L^\infty(M,\mathbb{R})} \leq C_{\text{ub}}(\varepsilon_{\text{ub}}, 1) \leq u^*$ which implies that $\hat{u}_{\varepsilon,m}$ is a solution of (1.2) for W and hence the result. □

The proof of Theorem 6.2 works in the same way, so we skip it.

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Declarations

Conflict of interest The authors declare that there is no conflict of interest.

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