Mathematische Zeitschrift (2024) 308:19
https://doi.org/10.1007/s00209-024-03565-x

Mathematische Zeitschrift

Stability theory for the NLS equation on looping edge graphs

Jaime Angulo Pava[1]

Received: 29 July 2023 / Accepted: 2 July 2024 / Published online: 11 August 2024

Abstract

The aim of this work is to present new spectral tools for studying the orbital stability of standing waves solutions for the nonlinear Schrödinger equation (NLS) with power nonlinearity on looping edge graphs, namely, a graph consisting of a circle with several half-lines attached at a single vertex. The main novelty of this paper is at least twofold: by considering δ-type boundary conditions at the vertex, the extension theory of Krein&von Neumann, and a splitting eigenvalue method, we identify the Morse index and the nullity index of a specific linearized operator around of *a priori* positive single-lobe state profile for every positive power, this information will be main for a local stability study; and so via a bifurcation analysis on the phase plane we build at least two families of positive single-lobe states and we study the stability properties of these in the subcritical, critical, and supercritical cases. Our results recover some spectral studies in the literature associated to the NLS on looping edge graphs which were obtained via variational techniques.

Keywords Nonlinear Schrödinger equation · Standing waves on metric graphs · Orbital stability · Extension theory of symmetric operators · Sturm Comparison Theorem

Mathematics Subject Classification Primary 35Q51 · 35Q55 · 81Q35 · 35R02; Secondary 47E05

1 Introduction

Evolution models on metric graph (i.e., a network-shaped structure of vertices connected by edges identified as intervals) arise as models in wave propagation, for instance, in a quasi one-dimensional (e.g. meso- or nanoscale) system that looks like a thin neighborhood of a graph. Just to mention a few examples of these systems, we have the nonlinear Schrödinger equation (NLS), the sine-Gordon (s-G) and the Korteweg-de Vries (KdV) models. Physical phenomenas that describe these models we have the following: the NLS on networks of nano-wires: propagation of optical electromagnetic pulses and Bose-Einstein condensation (see [19, 20, 27, 28, 30, 39] and references therein); the s-G on Josephson junction networks and electric circuits (see [14–16, 41, 42] and references therein); the KdV on blood pressure waves in large arteries (see [25, 26] and reference therein).

✉ Jaime Angulo Pava
angulo@ime.usp.br

[1] Department of Mathematics IME-USP, Cidade Universitária, Rua do Matão 1010, CEP 05508-090, São Paulo, SP, Brazil

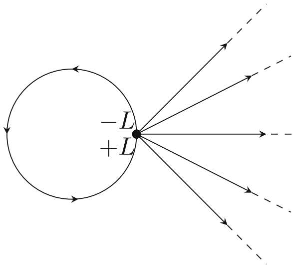

Fig. 1 A looping edge graph with 5 half-lines

Roughly speaking, a metric graph $\mathcal{G}$ is a structure represented by a finite number of *vertices* $V = \{v_i\}$ and a set of adjacent edges at the vertices $E = \{e_j\}$ (for further details see [19, 37]). Each edge e_j can be identified with a finite or infinite interval of the real line, I_e (so the spaces $L^p(\mathcal{G})$, $H^1(\mathcal{G})$, etc, are defined in the natural way. See notation below). Now, the notation $e \in E$ will be used to mean that e is an edge of $\mathcal{G}$. This identification introduces the coordinate x_e along the edge e. We identify any function $\mathbf{U}$ on $\mathcal{G}$ (the *wave functions*) with a collection $\mathbf{U} = (u_e)_{e\in E}$ of functions u_e defined on the edge e of $\mathcal{G}$. Thus, each u_e can be considered as a real or complex-valued function on the interval I_e.

We recall that an evolution model on a metric graph is equivalent to a system of PDEs defined on the edges (intervals) with a coupling given exclusively through the boundary conditions at the vertices (known as the "topology of the graph") and which will determine the dynamic on the network. Moreover, the freedom in setting the topology in a graph allows us to create different dynamics much closer to the real world applications.

In the past years, evolution models on networks or branched structures have attracted much attention in the context of soliton transport (see [1, 2, 9, 11–16, 30, 35, 43] and references therein). Soliton and other nonlinear waves in branched systems appear in different systems and these provide in-depth informations about the dynamic of the model. To study the dynamics of these profiles, in general the problem is difficult to tackle because both the equation of motion and the topology of the graph can be complex. Moreover, a central point that makes this analysis a delicate issue is the presence of a vertex (or several vertices) where a soliton-profile coming into the vertex along one of the bonds shows a complicated motion around the vertex such as reflection and emergence of the radiation there. In particular, one cannot see easily how the energy travels across the network. Thus, the study of soliton propagation through networks can become a challenge. Results on the existence and stability (or instability) of soliton profiles are still unclear for many type of evolution models and/or metric graphs. Thus, one of the objectives of this work is to shed light on these themes in the case of the NLS model on non-compact metric graphs of the type looping edge graph (see Fig. 1 below).

In the last years, the following nonlinear Schrödinger model equation (NLS)

$$i\mathbf{U}_t \ + \Delta\mathbf{U} + (p+1)|\mathbf{U}|^{2p}\mathbf{U} = \mathbf{0} \tag{1.1}$$

has been studied intensively on different type of metric graphs $\mathcal{G}$ (for instance, star graph, tadpole graph, flower graph, dumbbell graphs, double-bridge graphs and periodic ring graphs, see the review paper [35]). For $\mathbf{U}(x_e,t) = (u_e(x_e,t))_{e\in E}$ we have the nonlinearity $|\mathbf{U}|^{2p}\mathbf{U}$, $p > 0$, acting componentwise, i.e., for instance $(|\mathbf{U}|^{p-1}\mathbf{U})_e = |u_e|^{2p}u_e$, and the Laplacian operator Δ will be a self-adjoint operator with $D(\Delta) \subset L^2(\mathcal{G})$ which will give the coupling conditions in the graph-vertices. We recall, that the action of Δ on a metric graph is given by

$$-\Delta : (u_e)_{e\in E} \to (-u_e'')_{e\in E}. \tag{1.2}$$

Now, the study of the existence of standing waves, orbital stability, and well-posedness problems for (1.1) have been studied internsively in the past years, and for a variety of graph-geometry types (see [1, 3, 12, 13, 22, 33–35, 38, 43–46] and reference therein).

The focus of this paper is to shed a new light on the dynamics of standing wave solutions of the NLS model (1.1) posed on looping edge graphs $\mathcal{G}_N$, namely, a ring with N half-lines attached at one vertex point (see Fig. 1). Thus, if the ring is place on the interval $[-L, L]$ and the semi-infinite lines are identified as $[L, +\infty)$, we obtain a metric graph $\mathcal{G}_N$ (by abusing notation) with a structure represented by the set $E = \{e_j\}$ where $e_0 = [-L, L]$ and $e_j = [L, +\infty)$, $j = 1, ..., N$, are the edges of $\mathcal{G}_N$, and they are connected at the unique vertex $\nu = L$. We will denote a wave function $\mathbf{U}$ on the looping edge graph $\mathcal{G}_N$ as $\mathbf{U} = (\phi, (\psi_j)_{j=1}^N)$, where $\phi : [-L, L] \to \mathbb{C}$ and $\psi_j : [L, +\infty) \to \mathbb{C}$, for every $j = 1, ..., N$. The action of $-\Delta$ on $\mathcal{G}_N$ will be considered on the following domains ($Z \in \mathbb{R}$)

$$\begin{aligned} D_{Z,N} = \{\mathbf{U} \in H^2(\mathcal{G}_N) : \phi(L) = \phi(-L) = \psi_1(L) = \cdots = \psi_N(L) \text{ and} \\ \phi'(L) - \phi'(-L) = \textstyle\sum_{j=1}^N \psi_j'(L+) + Z\psi_1(L)\}, \end{aligned} \tag{1.3}$$

where for any $n \geqq 0$,

$$H^n(\mathcal{G}_N) = H^n(-L, L) \oplus \bigoplus_{j=1}^N H^n(L, +\infty).$$

The boundary conditions in (1.3) are called of δ-type if $Z \neq 0$, and of Neuman-Kirchhoff type if $Z = 0$. By using the extension theory for symmetric operators (see Theorem 6.6 in Appendix below) follows that $(-\Delta, D_{Z,N})_{Z\in\mathbb{R}}$ represents a one-parameter family of self-adjoint operators on the looping edge graph $\mathcal{G}_N$. The parameter Z is a coupling constant between the loop and the several half-lines. The choice of the coupling at the vertex $\nu = L$ corresponds to a conceivable quantum-wire experiment (see [27–29] and reference therein). We recall that for the case $N = 1$, $\mathcal{G}_1$, is called a tadpole graph or lasso graph .

Our main interest here will be to study the existence and stability of standing wave solutions for NLS model in (1.1) posed on $\mathcal{G}_N$, namely, solutions given by the profile $\mathbf{U}(x, t) = e^{-i\omega t}\Theta(x)$, with $\omega < 0$, $\Theta = (\Phi, \Psi) \in D_{Z,N}$, $\Psi = (\psi_j)_{j=1}^N$ (real-valued components), and satisfying the stationary NLS vectorial equation

$$-\Delta\Theta - \omega\Theta - (p+1)|\Theta|^{2p}\Theta = \mathbf{0}. \tag{1.4}$$

More explicitly, Φ and Ψ satisfy the following system, one on the ring and the other one on the several half-line,

$$\begin{cases} -\Phi''(x) - \omega\Phi(x) - (p+1)|\Phi(x)|^{2p}\Phi(x) = 0, & x \in (-L, L), \\ -\psi_j''(x) - \omega\psi_j(x) - (p+1)|\psi_j(x)|^{2p}\psi_j(x) = 0, & x \in (L, +\infty),\ j = 1, ..., N, \\ \Phi(L) = \Phi(-L) = \psi_1(L) = \cdots = \psi_N(L), \\ \Phi'(L) - \Phi'(-L) = \sum_{j=1}^N \psi_j'(L+) + Z\psi_1(L), & Z \in \mathbb{R}. \end{cases} \tag{1.5}$$

It is clear that the delicate point in the existence of solutions for (1.5) is given for the component Φ on $[-L, L]$ and satisfying the δ-coupling condition. The components of Ψ will have obviously a soliton profile in the form

$$Q_\omega(x) = (-\omega)^{1/2p} sech^{1/p}(p\sqrt{-\omega}x), \qquad \text{modulo translation and sign.} \tag{1.6}$$

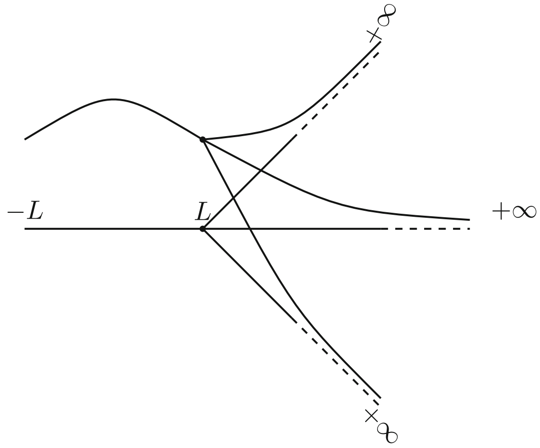

Fig. 2 A positive single-lobe state profile for the NLS model on $\mathcal{G}_N$, $N = 3$

It is an open problem to characterize all solutions to system (1.5) (see Sect. 2 for a brief description on the known results on the existence of solutions for (1.5) in the case $Z \geqq 0$, and as some of these depend on N). Among all profiles for (1.5), we are interested here in the *positive single-lobe states* (see Kairzhan&Noja&Pelinovsky [35] where a more general definition is given for standing waves on quantum graphs). More exactly, we have the following definition.

Definition 1.1 The profile standing wave $\Theta = (\Phi, \Psi) \in D_{Z,N}$, $\Psi = (\psi_j)_{j=1}^N$, is said to be a positive single-lobe state for (1.5) if every component is positive on each edge of $\mathcal{G}_N$, the maximum of Θ is achieved at a single internal point, symmetric on $[-L, L]$, and monotonically decreasing on $[0, L]$. Moreover, every ψ_j is strictly decreasing on $[L, +\infty)$.

Figure 2 shows a profile of a positive single-lobe state on $\mathcal{G}_N$, $N = 3$.

The existence of positive single-lobe states have been studied under rather restrictive geometry and topological conditions via variational techniques (see the review paper of Kairzhan et al. [35] and Sect. 2 below). By instance, for $N \geqq 3$ and $Z = 0$, we can not obtain these profiles as being the ground state associated to a constraint variational problem, and for $N \geqq 1$ and $Z < 0$ (arbitrary), the existence is still open. Here, in Sect. 5, we use a bifurcation analysis on the phase plane for building at least two families of positive single-lobe states of (1.5) on $\mathcal{G}_N$, namely, for any $N \geqq 1$, $Z = 0$ and $p > 0$ (see proof of Theorem 1.5), and for the strength $Z < 0$ satisfying $Z = \omega$ and $N \geqq 1$ (see proof of Theorem 1.6 and Remark 5.2)

For $\Theta = (\Phi, \Psi)$ being a positive single-lobe state for (1.5), we have that every profile ψ_j has the representation (so-called a tail profile)

$$\psi_j(x) = (-\omega)^{1/2p} \psi_0(\sqrt{-\omega}(x - L) + a_j), \quad x \geqq L, \quad a_j > 0,$$

with $\psi_0 = Q_{-1}$ giving by

$$\psi_0(y) = sech^{1/p}(py), \quad \psi_0(0) = 1, \quad \psi_0'(0) = 0, \quad y \in \mathbb{R}. \tag{1.7}$$

Thus, from the continuity condition at $x = L$, we get $a_1 = a_2 = \cdots = a_N$ and hence $\psi_1 = \psi_2 = \cdots = \psi_N$ on $[L, +\infty)$. In our words, our positive single-lobe states are symmetric on $\mathcal{G}_N$ (which can be called a "*octopus state solution with N-tails arms*").

On the other hand, we will provide new spectral tools for studying the orbital stability of *a priori* positive single-lobe state on arbitrary looping edge graph $\mathcal{G}_N$. We are not aware of previous spectral studies for the NLS model on this particular framework, specially in the case $N \geqq 3$ and $Z \leqq 0$. Thus, we will establish new results on the stability of positive single-lobe profiles in the subcritical, critical, and supercritical cases for the NLS on looping edge graphs (see Theorems 1.5–1.6 below).

We note that since our spectral analysis is of a non-variational type, this can be considered as a first step towards studying other type of standing wave profiles on $\mathcal{G}_N$, $N \geqq 1$, and/or with coupling conditions different to δ-interactions, such as δ'-interactions (we recommend to the reader to see reference [10] where a stability theory for two-lobe profil on a tadpole graph (see Fig. 3 below) has been established recently).

1.1 Preliminaries and main results

Next, we give the main results of our work associated to the orbital stability of positive single-lobe states for the NLS model. By convenience of the reader, we establish before the main points in the stability study of standing waves solutions for NLS models on looping edge graphs. So, by starting, we note that the basic symmetry associated to the NLS model (1.1) is the phase invariance, namely, if $\mathbf{U}$ is a solution of (1.1) then $e^{i\theta}\mathbf{U}$ is also a solution for any $\theta \in [0, 2\pi)$. Therefore, it is reasonable to define orbital stability for the model (1.1) as follows (see [31]).

Definition 1.2 The standing wave $\mathbf{U}(x,t) = e^{-i\omega t}(\Phi(x), \Psi(x))$ for (1.1) is said to be *orbitally stable* in a Banach space X if for any $\varepsilon > 0$ there exists $\eta > 0$ with the following property: if $\mathbf{U}_0 \in X$ satisfies $||\mathbf{U}_0 - (\Phi, \Psi)||_X < \eta$ then the solution $\mathbf{U}(t)$ of (1.1) with $\mathbf{U}(0) = \mathbf{U}_0$ exists for any $t \in \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} ||\mathbf{U}(t) - e^{i\theta}(\Phi, \Psi)||_X < \varepsilon.$$

Otherwise, the standing wave $\mathbf{U}(x,t) = e^{-i\omega t}(\Phi(x), \Psi(x))$ is said to be *orbitally unstable* in X.

The space X in Definition 1.2 with the action of $-\Delta$ on $D_{Z,N}$, will be the continuous energy-space $\mathcal{E}(\mathcal{G}_N)$ defined by

$$\mathcal{E}(\mathcal{G}_N) = \Big\{(f, \mathbf{g}) \in H^1(\mathcal{G}_N) : \text{for } \mathbf{g} = (g_j)_{j=1}^N,\ f(-L) = f(L) = g_1(L) = \cdots = g_N(L)\Big\}. \tag{1.8}$$

Next, we consider the following two conserved functionals for (1.1) defined in the energy space $\mathcal{E}(\mathcal{G}_N)$,

$$E_Z(\mathbf{U}) = \|\nabla\mathbf{U}\|^2_{L^2(\mathcal{G}_N)} - \|\mathbf{U}\|^{2p+2}_{L^{2p+2}(\mathcal{G}_N)} - Z|u(L)|^2, \qquad \text{(energy)} \tag{1.9}$$

and

$$Q(\mathbf{U}) = \|\mathbf{U}\|^2_{L^2(\mathcal{G}_N)}, \qquad \text{(mass)} \tag{1.10}$$

where $\mathbf{U} = (u, (v_j)_{j=1}^N)$. We observe that $E_Z, Q \in C^2(\mathcal{E}(\mathcal{G}_N), \mathbb{R})$ since $p > 0$. Now, for a fixed $\omega < 0$, let $\mathbf{U}_\omega(x,t) = e^{-i\omega t}(\Phi_\omega(x), \Psi_\omega(x))$ be a standing wave solution for (1.1) with $(\Phi_\omega, \Psi_\omega) \in D_{Z,N}$ being a positive single-lobe state. Then, for the action functional

$$\mathbf{S}(\mathbf{U}) = E_Z(\mathbf{U}) - \omega Q(\mathbf{U}), \qquad \mathbf{U} \in \mathcal{E}(\mathcal{G}_N), \tag{1.11}$$

we have $\mathbf{S}'(\Phi_\omega, \Psi_\omega) = \mathbf{0}$. Next, for $\mathbf{U} = \mathbf{U}_1 + i\mathbf{U}_2$ and $\mathbf{W} = \mathbf{W}_1 + i\mathbf{W}_2$, where the vector functions $\mathbf{U}_j, \mathbf{W}_j$, $j = 1, 2$, are assumed to have real components, it is not difficult to see that the second variation of $\mathbf{S}$ in $(\Phi_\omega, \Psi_\omega)$ is given by

$$\mathbf{S}''(\Phi_\omega, \Psi_\omega)(\mathbf{U}, \mathbf{W}) = \langle \mathcal{L}_{+,Z}\mathbf{U}_1, \mathbf{W}_1\rangle + \langle \mathcal{L}_{-,Z}\mathbf{U}_2, \mathbf{W}_2\rangle, \tag{1.12}$$

where the two $(N+1)\times(N+1)$-diagonal operators $\mathcal{L}_{\pm,Z}$ are given for $\Psi_\omega = (\psi_j)_{j=1}^N$ and $V_j = -(p+1)(2p+1)\psi_j^{2p}$ (obviously, $V_1 = \cdots = V_N$), by

$$\begin{aligned}\mathcal{L}_{+,Z} &= diag(-\partial_x^2 - \omega - (p+1)(2p+1)\Phi^{2p}, -\partial_x^2 - \omega + V_1, ..., -\partial_x^2 - \omega + V_N)\\ \mathcal{L}_{-,Z} &= diag(-\partial_x^2 - \omega - (p+1)\Phi^{2p}, -\partial_x^2 - \omega - (p+1)\psi_1^{2p}, ..., -\partial_x^2 - \omega - (p+1)\psi_N^{2p}).\end{aligned} \tag{1.13}$$

We note that these two diagonal operators are self-adjoint with domain $D(\mathcal{L}_{\pm,Z}) \equiv D_{Z,N}$. We also see that since $(\Phi_\omega, \Psi_\omega) \in D_{Z,N}$ and satisfies system (1.5), $\mathcal{L}_{-,Z}(\Phi_\omega, \Psi_\omega)^t = \mathbf{0}$ and so the kernel of $\mathcal{L}_{-,Z}$ is non-trivial.

Now, from [31, 32] (see Theorem 6.8 in Appendix) we know that the Morse index and the nullity index of the operators $\mathcal{L}_{\pm,Z}$, are a fundamental step in deciding about the orbital stability of standing wave profiles. Thus, our main results are the following,

Theorem 1.3 *Consider the self-adjoint operator $(\mathcal{L}_{+,Z}, D_{Z,N})$ in (1.13) determined by the positive single-lobe state $(\Phi_\omega, \Psi_\omega)$ with $\Psi_\omega = (\psi_j)_{j=1}^N$, $\psi_1 = \psi_2 = \cdots = \psi_N$. Then,*

1) *Perron–Frobenius property: let $\beta_0 < 0$ be the smallest eigenvalue for $\mathcal{L}_{+,Z}$ with associated eigenfunction $(f_{\beta_0}, \mathbf{g}_{\beta_0})$, $\mathbf{g}_{\beta_0} = (g_j)_{j=1}^N$. Then, f_{β_0} is positive and even on $[-L, L]$, and for every $j = 1, ..., N$, $g_j(x) > 0$ with $x \in [L, +\infty)$,*
2) *β_0 is simple,*
3) *the Morse index of $\mathcal{L}_{+,Z}$ is one,*
4) *it defines the quantity*

$$\alpha_N = \frac{N\psi_1''(L)}{\psi_1'(L)} + Z.$$

Then, the kernel associated to $\mathcal{L}_{+,Z}$ is trivial in the following cases: for $\alpha_N \neq 0$ or $\alpha_N = 0$ in the case of admissible parameters Z satisfying $Z \leqq 0$.

The reader is asked to refer to Remark 4.1 for some comments on the conditions in item 4) above.

Theorem 1.4 *Consider the self-adjoint operator $(\mathcal{L}_{-,Z}, D_{Z,N})$ in (1.13) determined by the positive single-lobe state $(\Phi_\omega, \Psi_\omega)$ with $\Psi_\omega = (\psi_j)_{j=1}^N$, $\psi_1 = \psi_2 = \cdots = \psi_N$. Then,*

1) *the kernel of $\mathcal{L}_{-,Z}$, $ker(\mathcal{L}_{-,Z})$, satisfies $ker(\mathcal{L}_{-,Z}) = span\{(\Phi_\omega, \Psi_\omega)\}$.*
2) *$\mathcal{L}_{-,Z}$ is a non-negative operator, $\mathcal{L}_{-,Z} \geqq 0$.*

We note that our results in Theorem 1.3 recover similar spectral results in the literature associated to standing wave solutions on $\mathcal{G}_1$ and $\mathcal{G}_2$ for $Z = 0$, which were obtained via variational and bifurcations techniques (see Sect. 2 below and [2, 4, 5, 22, 44, 45]). Our approach has the advantage of being able to be extended to other types of profiles (for instance, bound states profiles) (see Remark 4.3). Statement 3) for $Z < 0$ and statement 4) for arbitrary Z, in Theorem 1.3, as far as we know, they are the first to be established in the literature on looping edge graphs.

The proof of Theorem 1.3 will be based in a *splitting eigenvalue method* applied to $\mathcal{L}_{+,Z}$ on looping edge graphs (see Lemma 3.5). More exactly, we reduce our eigenvalue problem associated to $\mathcal{L}_{+,Z} \equiv (\mathcal{L}_0, \mathcal{L}_1)$ with domain $D_{Z,N}$ to two classes of eigenvalue problems, one with periodic boundary conditions for $\mathcal{L}_0$, on $[-L, L]$ and the other one for $\mathcal{L}_1$ with δ-type boundary conditions on the metric star graph generated by the set of edges $E_1 = \{e_j\}_{j=1}^N$, $e_j = [L, +\infty)$ for all j, and attached at the unique vertex $\nu = L$.

Next, by using a bifurcation analysis on the phase plane associated to the first elliptic equation in (1.5) we show the existence of at least two families of positive single-lobe states of (1.5) on $\mathcal{G}_N$ (see Sect. 5). More exactly, we have the following existence and stability results.

Theorem 1.5 *Consider $Z = 0$ and $p > 0$ in (1.5). Then, there exists a C^1- mapping $\omega \in (\omega_0, 0) \to \Theta_\omega = (\Phi_\omega, (\psi_\omega)_{j=1}^N)$ of positive single-lobe states on $\mathcal{G}_N$ for any $N \geqq 1$, with $-\omega_0 > 0$ small enough. Moreover, for $\omega \in (\omega_0, 0)$, the orbit*

$$\{e^{i\theta}\Theta_\omega : \theta \in [0, 2\pi)\}$$

is stable for $p \in (0, 2)$ and unstable for $p \in (2, +\infty)$, for any $N \geqq 1$.

Theorem 1.5 with $N = 1, 2$, recovers several results known in the literature (see Sect. 2 and [2, 4, 5, 22, 44, 45]). The case $N \geqq 3$ is novelty in the literature. The stability problem for the case $p = 2$ and $N \geqq 3$ remains open.

Theorem 1.6 *Let $p > 0$ fixed and consider L such that $L > \frac{1}{2p}$. Then there exists a C^1-mapping $\omega \in (\omega_0, 0) \to \Theta_\omega = (\Phi_\omega, (\psi_\omega)_{j=1}^N)$ of positive single-lobe states on $\mathcal{G}_N$ for any $N \geqq 1$, with $-\omega_0 > 0$ small enough, such that $\Theta_\omega \in D_{Z,N}$ with $Z = \omega$. Moreover, for $\omega \in (\omega_0, 0)$, the orbit*

$$\{e^{i\theta}\Theta_\omega : \theta \in [0, 2\pi)\}$$

is stable for $p \in (0, 2]$ and unstable for $p \in (2, +\infty)$, for any $N \geqq 1$.

The proof of Theorems 1.5–1.6 are given in Sect. 5. The statement of the orbital stability follows from Theorems 1.3–1.4 and from the abstract stability framework established by Grillakis&Shatah&Strauss in [31, 32]. By convenience of the reader, we establish in Theorem 6.8 (Appendix) an adaptation of the abstract results in [31, 32] to the case of positive single-lobe states on looping edge graphs.

We note that in Definition 1.2, we need *a priori* information about the local and global well-posedness of the Cauchy problem associated to (1.1). Indeed, by using the results in [22] in the case of looping edge graphs and (1.9)–(1.10), we can see that the NLS is globally well-posed in the energy space $\mathcal{E}(\mathcal{G}_N)$ for every $p \in (0, 2)$, and for $p = 2$ the global solution is defined at least for small initial data in the $L^2(\mathcal{G}_N)$-norm. For $p > 2$ we get local well-posedness in $\mathcal{E}(\mathcal{G}_N)$.

We would like to point out that our approach for studying positive single-lobe states on looping edge graphs could be used in the study of other branches of standing wave profiles for (1.1) such as those shown in Fig. 3 below on a tadpole graph (see Angulo [10]), or in the case that the second component of the profile standing wave, $\Psi = (\psi_j)_{j=1}^N$, has a combination of tails and bumps soliton profiles.

The paper is organized as follows. In Sect. 2, we give a brief review of results in the literature about the existence and stability of standing wave solutions for (1.1) on looping edge graphs. In Sect. 3, we show Theorems 1.3–1.4 on a tadpoles graph via our splitting eigenvalue lemma (Lemma 3.5). In Sect. 4, Theorems 1.3–1.4 will be showed for generic looping edge graphs. In Sect. 5, we give the proof of Theorems 1.5–1.6. Lastly, in the Appendix we establish some tools and results of the extension theory of Krein and von Neumann used in our work, as well as, a Perron–Frobenius property for δ-interactions Schrödinger operators on the line.

Notation. Let $-\infty \leq a < b \leq \infty$. We denote by $L^2(a,b)$ the Hilbert space equipped with the inner product $(u,v)=\int_a^b u(x)\overline{v(x)}dx$. By $H^n(\Omega)$ we denote the classical Sobolev spaces on $\Omega \subset \mathbb{R}$ with the usual norm. We denote by $\mathcal{G}_N$ the looping edge graph parametrized by the set of edges $\mathbf{E}=E_0\cup E_1$ with $E_0=\{e_0\}$, $e_0=[-L,L]$, $E_1=\{e_j\}_{j=1}^N$, $e_j=[L,+\infty)$, for all j, and attached to the common vertex $\nu=L$. On the graph $\mathcal{G}_N$ we define the spaces

$$L^p(\mathcal{G}_N)=L^p(-L,L)\oplus\bigoplus_{j=1}^N L^p(L,+\infty), \quad p>1,$$

with the natural norms. Also, for $\mathbf{U}=(u_1,\mathbf{g}), \mathbf{V}=(v_1,\mathbf{h})\in L^2(\mathcal{G}_N)$, with $\mathbf{g}=(g_j)_{j=1}^N$, $\mathbf{h}=(h_j)_{j=1}^N$, the inner product on $L^2(\mathcal{G}_N)$ is defined by

$$\langle \mathbf{U},\mathbf{V}\rangle=\int_{-L}^{L} u_1(x)\overline{v_1(x)}dx+\sum_{j=1}^N\int_L^\infty g_j(x)\overline{h_j(x)}dx.$$

Let A be a closed densely defined symmetric operator in the Hilbert space H. The domain of A is denoted by $D(A)$. The deficiency indices of A are denoted by $n_\pm(A):=\dim ker(A^*\mp iI)$, with A^* denoting the adjoint operator of A. The number of negative eigenvalues counting multiplicities (or Morse index) of A is denoted by $n(A)$.

2 Existence of standing waves on a looping edge graph

In this section, we briefly describe existing results in the literature (to the author's knowledge) for system (1.5). One of the objectives of this review is to show where our results in Theorems 1.5–1.6 fill some gaps in the study of the existence and stability of standing waves for the NLS model on looping edge graphs (see also the review in [35]). So, we consider the two conserved quantities E_Z and Q in (1.9) and (1.10), respectively. If the infimum of the constrained minimization problem:

$$\mathcal{M}_{Z,\mu}=\inf_{\mathbf{U}\in\mathcal{E}(\mathcal{G}_N)}\{E_Z(\mathbf{U}):Q(\mathbf{U})=\mu\},\quad \mu>0 \tag{2.1}$$

is finite and it is attained at $\Theta_\mu=(\Phi,\Psi)\in\mathcal{E}(\mathcal{G}_N)$ so that $\mathcal{M}_{Z,\mu}=E_Z(\Theta_\mu)$ and $Q(\Theta_\mu)=\mu$, we say that this Θ_μ is the *ground state*. By using classical bootstrapping arguments, the same Θ_μ is also a strong solution $\Theta_\mu\in D_{Z,N}$ to the system (1.5) with ω being the corresponding Euler-Lagrange multiplier which depends on μ. Then, we consider the set of minimizers for (2.1), namely,

$$S_{Z,\mu}=\{\Theta\in\mathcal{E}(\mathcal{G}_N):Q(\Theta)=\mu,\ E_Z(\Theta)=\mathcal{M}_{Z,\mu}\}. \tag{2.2}$$

It is well-known that ground states on generic metric graphs with δ-interactions concentrated at vertices of the graph, exist under rather restrictive topological conditions (see [2, 4–6, 23, 35]). More exactly, in the case of a looping edge graph (which is also called a noncompact metric graph or a starlike graph) the existence of a ground state will depend on the sign of Z and the number N. Indeed, by using the concentration compactness method for starlike structures, we obtain in the case of looping edge graphs the following:

a) let $Z > 0$: for $p \in (0, 2)$ (subcritical case), we obtain from [23] that as

$$-\lambda_0 \equiv inf\{\|\nabla \mathbf{U}\|^2 - Z|u(0)|^2 : \mathbf{U} \in \mathcal{E}(\mathcal{G}_N),\, Q(\mathbf{U}) = 1\}$$

satisfies $\lambda_0 > 0$ and it is an isolated eigenvalue for the Schrödinger operator $(-\Delta, D_{Z,N})$, a ground state Θ_μ always exists for μ small enough. Moreover, this ground state belongs to the branch of stationary state $\omega \in (-\lambda_0 - \delta, -\lambda_0) \to \Pi(\omega)$ bifurcating from $(-\lambda_0, 0)$. Thus, via this identification, it is possible to determine the positivity of the ground state Θ_μ and that $S_{Z,\mu} = \{e^{i\theta}\Theta_\mu : \theta \in [0, 2\pi)\}$. Furthermore, by the global well-posedness result in $\mathcal{E}(\mathcal{G}_N)$ of (1.1) (see [23]) and by a classical argument from Cazenave&Lions [24], it follows that Θ_μ is orbitally stable.
For $p = 2$ and $p > 2$ (critical and supercritical cases, respectively), we obtain from [17] (see also [21]) that for

$$B(r) \equiv \{\mathbf{U} \in \mathcal{E}(\mathcal{G}_N) : \|\nabla \mathbf{U}\|^2 - Z|u(0)|^2 + 2\lambda_0 Q(\mathbf{U}) \leqq r\},$$

and any $r > 0$, there exists $\mu^* = \mu^*(r) > 0$ small enough such for any $\mu \in (0, \mu^*)$ there is a ground state $\Theta_{\mu,r}$ for the minimization problem

$$\mathcal{M}_{Z,\mu,r} = \inf_{\mathbf{U} \in \mathcal{E}(\mathcal{G}_N)} \{E_Z(\mathbf{U}) : \mathbf{U} \in B(r) \text{ and } Q(\mathbf{U}) = \mu\}. \tag{2.3}$$

b) let $Z = 0$: in this case we have the pure Kirchhoff boundary condition determined by $D_{Z,N}$ and the existence of ground state will depend of the number N of half-lines essentially. Indeed,

i) the case $N = 1$ corresponds to the tadpole case extensively studied in the literature ([2, 4, 5, 22, 35, 44, 45]). It was shown in [4] that for $p \in (0, 2)$ a ground state Θ_μ always exists for any positive value of μ. Moreover, Θ_μ is a positive single-lobe state for (1.5) (from our Theorem 1.3 we recover the results in [45] associated to the Morse index and the non-degeneracy of the kernel for $\mathcal{L}_{+,Z}$ in (1.13) associated to this Θ_μ). In the critical power $p = 2$ was shown in [5] (Theorem 3.3) that the ground state, on the tadpole graph, is attained if and only if $\mu \in (\mu_{\mathbb{R}^+}, \mu_{\mathbb{R}}]$, where $\mu_{\mathbb{R}^+}$ is the mass of the half-soliton Q_ω in (1.6) on the half-line and $\mu_{\mathbb{R}}$ is the mass of the full-soliton on the full line, namely, $\mu_{\mathbb{R}^+} = \frac{\pi}{4}$ and $\mu_{\mathbb{R}} = \frac{\pi}{2}$. Also, it was shown in the recent work [44] (Theorem 1.1) that for every $\omega < 0$ there exists a global minimizer $\Gamma(\omega)$ of the constrained minimization problem

$$\mathcal{B}(\omega) = \inf_{\mathbf{U} \in \mathcal{E}(\mathcal{G}_1)} \{\|\nabla \mathbf{U}\|^2_{L^2(\mathcal{G}_1)} - \omega Q(\mathbf{U}) : \|\mathbf{U}\|_{L^6(\mathcal{G}_1)} = 1\}, \tag{2.4}$$

which yields a solution Θ_ω being a positive single-lobe state for (1.5). In this case, Theorem 1.3 recovers Theorem 1.2 in [44], it which was based on dynamical system methods and the analytical theory for differential equations. For generalized power p, the existence and stability of solutions for (1.5) with mass small enough, it was obtained in [45] via a bifurcation analysis,

ii) the case $N = 2$ was studied in [2] (Theorem 2.5) and the existence of a ground state for $p \in (0, 2)$ and any positive value of μ in (2.1) holds true. For $p = 2$, the existence of a ground state is restricted to the case of mass $\mu \equiv \mu_{\mathbb{R}}$ (see Theorem 3.2 in [5]).

iii) the case $N \geqq 3$ is much more challenging. Indeed, for $p \in (0, 2]$ there is not ground state according to Theorem 2.5 of [2] and Theorem 3.2 of [5]. Thus, one of the objectives of this paper will be to shed light on the dynamics of positive single-lobe state in this case (see Theorem 1.5).

Now, by the previously results, several conditions on μ in (2.1) have been found for ensuring (or, on the contrary, ruled out) the existence of absolute minimizers of (2.1) (ground states) at least for $Z \geqq 0$, the case $Z < 0$ is still open. But, it may happen that there are solutions to the stationary equation (1.5) without being ground state, in other words, the so-called *bound states* (see [6, 22, 35]), that is, functions with a prescribed mass such that are constrained *critical points* of $E_Z(\mathbf{U})$ and possibly without being absolute minimizers. In [6], it has been showed the existence of these profiles with a mass μ large enough and in the case $Z = 0$.

Therefore, we can say from the review above that for a looping edge graph $\mathcal{G}_N$, with $N \geqq 3$ and/or $Z \leqq 0$, the existence and stability of bound states become more relevant to be studied. In Theorems 1.5–1.6 we establish the first results in the literature related to the existence and stability of single-lobe profiles for $N \geqq 1$, $p > 0$ and $Z \leqq 0$.

3 Morse and nullity indices for operators $\mathcal{L}_{+,Z}$ on tadpole graphs

In this section we initially show Theorem 1.3, in the case of a tadpole graph for greater clarity in the exposition. The case $N \geq 2$ will be established in Sect. 4. Thus, for $N = 1$ we will consider one *a priori* positive single-lobe state $(\Phi_\omega, \Psi_\omega)$ solution for (1.5) with $\omega < 0$. By convenience we denote $\Phi = \Phi_\omega$ and $\Psi = \Psi_\omega$. Thus, the linearized operator $\mathcal{L}_{+,Z}$ in (1.13) becomes as

$$\mathcal{L}_{+,Z} = diag(-\partial_x^2 - \omega - (p+1)(2p+1)\Phi^{2p}, -\partial_x^2 - \omega - (p+1)(2p+1)\Psi^{2p}) \tag{3.1}$$

with domain $D_Z \equiv D_{Z,1}$

$$\begin{aligned} D_Z = \{(f, g) \in H^2(\mathcal{G}_1) : f(L) = f(-L) = g(L), \\ \text{and, } f'(L) - f'(-L) = g'(L+) + Zg(L)\}. \end{aligned} \tag{3.2}$$

Next, for $(f, g) \in D_Z$ we consider $h(x) = g(x + L)$ for $x > 0$. Then $h(0) = g(L)$ and $h'(0) = g'(L)$. Therefore, the eigenvalue problem $\mathcal{L}_{+,Z}(f, g)^t = \lambda(f, g)^t$ will be equivalent to the following one

$$\begin{cases} \mathcal{L}_{0,+} f(x) = \lambda f(x), & x \in (-L, L), \\ \mathcal{L}_{1,+} h(x) = \lambda h(x), & x \in (0, +\infty), \\ (f, h) \in D_{Z,0}, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} \mathcal{L}_{0,+} &= -\partial_x^2 - \omega - (p+1)(2p+1)\Phi^{2p}, \\ \mathcal{L}_{1,+} &\equiv -\partial_x^2 - \omega - (p+1)(2p+1)\psi_{0,a}^{2p}, \end{aligned} \tag{3.4}$$

and $\psi_{0,a}(x) = (-\omega)^{1/2p}\psi_0(\sqrt{-\omega}x + a)$, with $x > 0$, $a > 0$ (a tail-soliton profile). The domain $D_{Z,0}$ is given by

$$\begin{aligned} D_{Z,0} = \{(f, h) \in X^2(-L, L) : f(L) = f(-L) = h(0), \text{ and}, \\ f'(L) - f'(-L) = h'(0) + Zh(0)\}, \end{aligned} \tag{3.5}$$

with $X^n(-L, L) \equiv H^n(-L, L) \oplus H^n(0, +\infty)$, $n \in \mathbb{N}$. We note that $(\Phi, \psi_{0,a}) \in D_{Z,0}$.

For convenience of notation, we will consider $\mathcal{L}_0 \equiv \mathcal{L}_{0,+}$, $\mathcal{L}_1 \equiv \mathcal{L}_{1,+}$, and $\psi_a \equiv \psi_{0,a}$. Thus, we define $\mathcal{L}_+ \equiv diag(\mathcal{L}_0, \mathcal{L}_1)$ with domain $D_{Z,0}$. Therefore, the statements in Theorem 1.3 for $\mathcal{L}_{+,Z}$ will be sufficient to be showed for $(\mathcal{L}_+, D_{Z,0})$.

3.1 Perron–Frobenius property and Morse index for $(\mathcal{L}_+, D_{Z,0})$

In the following we will see that $n(\mathcal{L}_+) \geqq 1$. First, since $(\Phi, \psi_a) \in D_{Z,0}$ and

$$\langle \mathcal{L}_+(\Phi, \psi)^t, (\Phi, \psi)^t \rangle = -2p(p+1)\Big[\int_{-L}^{L} \Phi^{2p+1}(x)dx + \int_0^{+\infty} \psi_a^{2p+1}(x)dx \Big] < 0 \tag{3.6}$$

we obtain from the mini-max principle that $n(\mathcal{L}_+) \geqq 1$. Now, we note that via the extension theory for symmetric operators of Krein-von Neumann, it is possible to show $n(\mathcal{L}_+) \leqq 2$ for any Z.

Theorem 3.1 *The Morse index associated to $(\mathcal{L}_+, D_{Z,0})$ is one. Thus, the Morse index for $(\mathcal{L}_{+,Z}, D_{Z,1})$ is also one.*

The proof of Theorem 3.1 (see Sect. 3.1.3) is based on the Perron–Frobenius property (PF property) for $(\mathcal{L}_+, D_{Z,0})$ for any $Z \in \mathbb{R}$. Our approach associated to the PF property is based on EDO's techniques, and we recover known results in the literature for the case $Z \geqq 0$ (see [17, 23, 29]).

3.1.1 Perron–Frobenius property for $(\mathcal{L}_+, D_{Z,0}), Z \in \mathbb{R}$.

We start our analysis by defining the quadratic form $\mathcal{Q}_Z$ associated to operator $\mathcal{L}_+$ on $D_{Z,0}$, namely, $\mathcal{Q}_Z : D(\mathcal{Q}_Z) \to \mathbb{R}$, with

$$\mathcal{Q}_Z(\phi, \zeta) = \int_{-L}^{L} (\phi')^2 + V_\Phi \phi^2 dx + \int_0^{+\infty} (\zeta')^2 + W_\psi \zeta^2 dx - Z|\zeta(0)|^2, \tag{3.7}$$

$V_\Phi = -\omega - (p+1)(2p+1)\Phi^{2p}$, $W_\psi = -\omega - (p+1)(2p+1)\psi_a^{2p}$ and $D(\mathcal{Q}_Z)$ is defined by

$$D(\mathcal{Q}_Z) = \{(\phi, \zeta) \in X^1(-L, L) : \phi(L) = \phi(-L) = \zeta(0)\}. \tag{3.8}$$

Theorem 3.2 *Let $\lambda_0 < 0$ be the smallest eigenvalue for $\mathcal{L}_+$ on $D_{Z,0}$ with associated eigenfunction $(\phi_{\lambda_0}, \zeta_{\lambda_0})$. Then, $\phi_{\lambda_0}, \zeta_{\lambda_0}$ are positive functions. Moreover, ϕ_{λ_0} is even on $[-L, L]$.*

Proof We split the proof into several steps.

1) The profile ζ_{λ_0} is not identically zero: Indeed, suppose $\zeta_{\lambda_0} \equiv 0$, then ϕ_{λ_0} satisfies

$$\begin{cases} \mathcal{L}_0\phi_{\lambda_0}(x) = \lambda_0\phi_{\lambda_0}(x), & x \in (-L, L), \\ \phi_{\lambda_0}(L) = \phi_{\lambda_0}(-L) = 0 \\ \phi'_{\lambda_0}(L) = \phi'_{\lambda_0}(-L). \end{cases} \tag{3.9}$$

Next, from the Dirichlet condition (which implies that the eigenvalue is simple and so ϕ_{λ_0} is odd or even) and from oscillations theorems of the Floquet theory (which implies that the number of zeros of ϕ_{λ_0} is even on $[-L, L)$), we need to have that ϕ_{λ_0} is odd. Then, by Sturm-Liouville theory there is an eigenvalue θ for $\mathcal{L}_0$, such that $\theta < \lambda_0$, with associated eigenfunction $\xi > 0$ on $(-L, L)$, and $\xi(-L) = \xi(L) = 0$.

Now, let $\mathcal{Q}_{Dir}$ be the quadratic form associated to $\mathcal{L}_0$ with Dirichlet domain, namely, $\mathcal{Q}_{Dir} : H_0^1(-L, L) \to \mathbb{R}$ with

$$\mathcal{Q}_{Dir}(f) = \int_{-L}^{L} (f')^2 + V_\Phi f^2 dx. \tag{3.10}$$

Then, $\mathcal{Q}_{Dir}(\xi) = \mathcal{Q}_Z(\xi, 0) \geqq \lambda_0 \|\xi\|^2$ and so, $\theta \geqq \lambda_0$, which is a contradiction.

2) $\zeta_{\lambda_0}(0) \neq 0$: suppose $\zeta_{\lambda_0}(0) = 0$ and we consider the odd-extension ζ_{odd} for ζ_{λ_0}, and the even-extension ψ_{even} of the tail-profile ψ_a on whole the line. Then, $\zeta_{odd} \in H^2(\mathbb{R})$ and $\psi_{even} \in H^2(\mathbb{R} - \{0\}) \cap H^1(\mathbb{R})$. Next, we consider the following unfold operator $\widetilde{\mathcal{L}}$ associated to $\mathcal{L}_{1,+}$,

$$\widetilde{\mathcal{L}} = -\partial_x^2 - \omega - (p+1)(2p+1)\psi_{even}^{2p} \tag{3.11}$$

on the δ-interaction domain

$$D_{\delta,\gamma} = \{f \in H^2(\mathbb{R} - \{0\}) \cap H^1(\mathbb{R}) : f'(0+) - f'(0-) = \gamma f(0)\} \tag{3.12}$$

for any $\gamma \in \mathbb{R}$. Then, from the extension theory for symmetric operators we have that the family $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})_{\gamma\in\mathbb{R}}$ represents all the self-adjoint extensions of the symmetric operator $(\mathcal{M}_0, D(\mathcal{M}_0))$ defined by

$$\mathcal{M}_0 = \widetilde{\mathcal{L}}, \quad D(\mathcal{M}_0) = \{f \in H^2(\mathbb{R}) : f(0) = 0\}.$$

Moreover, the deficiency indices of $(\mathcal{M}_0, D(\mathcal{M}_0))$, $n_\pm(\mathcal{M}_0)$, are given by $n_\pm(\mathcal{M}_0) = 1$ (see Albeverio et al. [8]). Now, the even tail-profile ψ_{even} satisfies $\psi'_{even}(x) \neq 0$ for all $x \neq 0$, and so from the well-defined relation

$$\mathcal{M}_0 f = -\frac{1}{\psi'_{even}} \frac{d}{dx}\Big[(\psi'_{even})^2 \frac{d}{dx}\Big(\frac{f}{\psi'_{even}}\Big)\Big], \quad x \neq 0, \tag{3.13}$$

we can see easily that $\langle \mathcal{M}_0 f, f\rangle \geqq 0$ for all $f \in D(\mathcal{M}_0)$. Therefore, from the extension theory (see Proposition 6.4 in Appendix) we obtain that the Morse index for the family $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})$ satisfies $n(\widetilde{\mathcal{L}}) \leqq 1$ for all $\gamma \in \mathbb{R}$. So, since $\zeta_{odd} \in D_{\delta,\gamma}$ (for any γ) and $\widetilde{\mathcal{L}}\zeta_{odd} = \lambda_0 \zeta_{odd}$ on $\mathbb{R}$, we have $n(\widetilde{\mathcal{L}}) = 1$. Then, λ_0 will be the smallest negative eigenvalue for $\widetilde{\mathcal{L}}$ on δ-interactions domains and by Theorem 6.7 in Appendix (Perron Frobenius property for $\widetilde{\mathcal{L}}$ with δ-interactions domains on the line), ζ_{odd} needs to be positive which is a contradiction. Therefore, $\zeta_{\lambda_0}(0) \neq 0$.

3) $\zeta_{\lambda_0} : [0, +\infty) \to \mathbb{R}$ can be chosen strictly positive: Without loss of generality we suppose $\zeta_{\lambda_0}(0) > 0$. Then the condition $\phi'_{\lambda_0}(L) - \phi'_{\lambda_0}(-L) = \zeta'_{\lambda_0}(0) + Z\zeta_{\lambda_0}(0)$ implies

$$\zeta'_{\lambda_0}(0) = \Big[\frac{\phi'_{\lambda_0}(L) - \phi'_{\lambda_0}(-L)}{\zeta_{\lambda_0}(0)} - Z\Big]\zeta_{\lambda_0}(0) \equiv \gamma_0 \zeta_{\lambda_0}(0).$$

Next, we consider ζ_{even} being the even-extension of ζ_{λ_0} to whole the line, then $\zeta_{even} \in D_{\delta,2\gamma_0}$ and $\widetilde{\mathcal{L}}\zeta_{even} = \lambda_0 \zeta_{even}$. Thus, by a similar analysis as in item 2) above, it follows that λ_0 is the smallest eigenvalue for $(\widetilde{\mathcal{L}}, D_{\delta,2\gamma_0})$. Hence, by Theorem 6.7 in Appendix, ζ_{even} is strictly positive on $\mathbb{R}$. Therefore, $\zeta_{\lambda_0}(x) > 0$ for all $x \geqq 0$.

4) $\phi_{\lambda_0} : [-L, L] \to \mathbb{R}$ can be chosen strictly positive: initially we have that ϕ_{λ_0} satisfies the following boundary condition:

$$\phi'_{\lambda_0}(L) - \phi'_{\lambda_0}(-L) = \Big[\frac{\zeta'_{\lambda_0}(0)}{\zeta_{\lambda_0}(0)} + Z\Big]\zeta_{\lambda_0}(0) \equiv \alpha_0 \zeta_{\lambda_0}(0) = \alpha_0 \phi_{\lambda_0}(L).$$

Now, we consider the following eigenvalue problem for $\mathcal{L}_0$ in (3.4) with real coupled self-adjoint boundary condition determined by $\alpha \in \mathbb{R}$,

$$(RC_\alpha): \quad \begin{cases} \mathcal{L}_0 y(x) = \eta y(x), & x \in (-L, L), \\ y(L) = y(-L), \\ y'(L) - y'(-L) = \alpha y(L). \end{cases} \tag{3.14}$$

Next, from Theorem 1.35 in Kong&Wu&Zettl [36] or Theorem 4.8.1 in Zettl [49] with $K = [k_{ij}]$ given by $k_{11} = 1$, $k_{12} = 0$, $k_{21} = \alpha_0$ and $k_{22} = 1$, we obtain that the first eigenvalue $\eta_0 = \eta_0(K)$ for (3.14) with $\alpha = \alpha_0$ is simple. Moreover, since the pair $(\phi_{\lambda_0}, \lambda_0)$ is a solution for (3.14), we have $\lambda_0 \geqq \eta_0$. *In the following we show* $\lambda_0 = \eta_0$. Indeed, we consider the quadratic form associated to the (RC_{α_0})-problem in (3.14), $\mathcal{Q}_{RC}$, where for $h \in H^1(-L, L)$ with $h(L) = h(-L)$,

$$\mathcal{Q}_{RC}(h) = \int_{-L}^{L} (h')^2 + V_\Phi h^2 dx - \alpha_0 |h(L)|^2. \tag{3.15}$$

Now, we define $\xi = \nu \zeta_{\lambda_0}$ with ν being a real constant such that $\xi(0) = \nu \zeta_{\lambda_0}(0) = h(L)$. Thus, $(h, \xi) \in D(\mathcal{Q}_Z)$ in (3.8). Now, by using $\mathcal{L}_1 \zeta_{\lambda_0} = \lambda_0 \zeta_{\lambda_0}$ we obtain

$$\begin{aligned} \mathcal{Q}_{RC}(h) &= \mathcal{Q}_Z(h, \xi) - \alpha_0 h^2(L) - \int_0^{+\infty} (\xi')^2 + W_\psi \xi^2 dx + Z|h(L)|^2 \\ &= \mathcal{Q}_Z(h, \xi) - \alpha_0 h^2(L) + \nu^2 \zeta'_{\lambda_0}(0)\zeta_{\lambda_0}(0) - \lambda_0 \nu^2 \|\zeta_{\lambda_0}\|^2 + Z|h(L)|^2 \\ &= \mathcal{Q}_Z(h, \xi) - [\zeta'_{\lambda_0}(0)h(L) + Zh^2(L)] + h(L)\zeta'_{\lambda_0}(0) + Zh^2(L) - \lambda_0 \|\xi\|^2 \\ &= \mathcal{Q}_Z(h, \xi) - \lambda_0 \|\xi\|^2 \geqq \lambda_0[\|h\|^2 + \|\xi\|^2] - \lambda_0 \|\xi\|^2 = \lambda_0 \|h\|^2. \end{aligned} \tag{3.16}$$

Then $\eta_0 \geqq \lambda_0$ and so $\eta_0 = \lambda_0$.

By the analysis above, we get that λ_0 *is the first eigenvalue for the problem* (RC_{α_0}) in (3.14) and so it is simple. Then, ϕ_{λ_0} is odd or even. If ϕ_{λ_0} is odd, then the condition $\phi_{\lambda_0}(L) = \phi_{\lambda_0}(-L)$ implies $\phi_{\lambda_0}(L) = 0$. But, $\phi_{\lambda_0}(L) = \zeta_{\lambda_0}(0) > 0$. So, we need to have that ϕ_{λ_0} is even. Now, from Oscillation Theorem for (RC_{α_0})-problem, the number of zeros of ϕ_{λ_0} on $[-L, L)$ is 0 or 1 (see Theorem 4.8.5 in [49]). Since $\phi_{\lambda_0}(-L) > 0$ and ϕ_{λ_0} is even, we obtain necessarily that $\phi_{\lambda_0} > 0$ on $[-L, L]$. This finishes the proof.

□

Corollary 3.3 *Let* $\lambda_0 < 0$ *be the smallest eigenvalue for* $(\mathcal{L}_+, D_{Z,0})$. *Then,* λ_0 *is simple.*

Proof The proof is immediate. Suppose λ_0 is double. Then, there is $(f_0, g_0)^t$-eigenfunction associated to λ_0 orthogonal to $(\phi_{\lambda_0}, \zeta_{\lambda_0})$. By Theorem 3.2 we have that $f_0, g_0 > 0$. So, we arrive to a contradiction from the orthogonality property of the eigenfunctions. □

Remark 3.4 The strategy for showing the PF property in Theorem 3.2 can be extended to the case of the first component of the standing wave profile (Φ, ψ_a), Φ, to be even and with multiple bumps on $(-L, L)$ (see Fig. 3 in [22] for examples of these profiles in the case $p = 1$) and/or the profile ψ_a to be of tail $(a > 0)$ or bump $(a < 0)$ type (see Fig. 3 below and Angulo [10]).

3.1.2 Splitting eigenvalue method on looping edge graphs

In the following, we establish our main strategy for studying eigenvalue problems on general looping edge graphs $\mathcal{G}_N$. More exactly, we reduce the eigenvalue problem for $\mathcal{L}_{+,Z} \equiv (\mathcal{L}_0, \mathcal{L}_{1,N})$ in (1.13) to two classes of eigenvalue problems, one for $\mathcal{L}_0$ with periodic boundary conditions on $[-L, L]$ and the other one for the diagonal operator $\mathcal{L}_{1,N} = diag(\mathcal{L}, ..., \mathcal{L})$, $\mathcal{L} = -\partial_x^2 - \omega - (p+1)(2p+1)\psi_1^{2p}$ with δ-type boundary conditions on a star graph.

Lemma 3.5 *Let us consider the self-adjoint operator $(\mathcal{L}_{+,Z}, D_{Z,N})$ in (1.13) for any $N \geqq 1$. Suppose $(f, \mathbf{g}) \in D_{Z,N}$ with $\mathbf{g} = (g_j)_{j=1}^N$ and $g_k(L) \neq 0$, for some k. Suppose $\mathcal{L}_{+,Z}(f, \mathbf{g})^t = \gamma(f, \mathbf{g})^t$, for $\gamma \in \mathbb{R}$. Then, we obtain the following two eigenvalue problems:*

$$\begin{cases} \mathcal{L}_0 f(x) = \gamma f(x), \ x \in (-L, L), \\ f(L) = (-L), \quad f'(L) = f'(-L), \end{cases} \qquad \begin{cases} \mathcal{L} g_j(x) = \gamma g_j(x), \ x > L, \ j = 1, ..., N, \\ \sum_{j=1}^N g_j'(L+) = -Z g_1(L+). \end{cases}$$

Proof For $(f, \mathbf{g}) \in D_{Z,N}$ and $g_k(L) \neq 0$, we have

$$f(-L) = f(L), \quad f'(L) - f'(-L) = \Big[\frac{1}{g_k(L+)} \sum_{j=1}^N g_j'(L+) + Z\Big] f(L) \equiv \theta f(L),$$

and so f satisfies the real coupled problem (RC_α) in (3.14) wth $\alpha = \theta$ and $\eta = \gamma$. In the following, we will see that $\theta = 0$ which proves the lemma.

We consider $K_\theta = [k_{ij}]$ the 2×2-matrix associated to (3.14) given by $k_{11} = 1, k_{12} = 0$, $k_{21} = \theta$ and $k_{22} = 1$ ($\det(K_\alpha) = 1$), and by $\eta_n = \eta_n(K_\theta)$, $n \in \mathbb{N}_0$, the eigenvalues for the (BC_θ)-problem in (3.14). We also consider $\mu_n = \mu_n(K_\theta)$ and $\nu_n = \nu_n(K_\theta)$, $n \in \mathbb{N}_0$, the eigenvalues problems in (3.14) induced by K_θ with the following boundary conditions

$$\begin{aligned} &y(-L) = y(L) = 0 && \text{(Dirichlet condition)}, \\ &y'(-L) = 0, \quad \theta y(L) - y'(L) = 0 && \text{(Neumann-type condition)}, \end{aligned} \tag{3.17}$$

respectively. We recall that if y_n is an eigenfunction of μ_n, then y_n is unique up to constant multiples and has exactly n zeros in $(-L, L)$, $n \in \mathbb{N}_0$ (a similar result is obtained for u_n an eigenfunction of ν_n, $n \in \mathbb{N}_0$). Next, by Theorem 4.8.1 in [49] we have that ν_0 and η_0 are simple, and in particular, we have the following partial distribution of eigenvalues

$$\nu_0 \leqq \eta_0 < \{\mu_0, \nu_1\} < \eta_1 \leqq \{\mu_1, \nu_1\} \leqq \eta_2, \tag{3.18}$$

where the notation $\{\mu_n, \nu_m\}$ is used to indicate either μ_n or ν_m but no comparison is made between μ_n and ν_m. In the following, we will see $u_0'(L) = 0$ and so $\theta = 0$ (because by (3.18) we have $\nu_0 < \mu_0$ and so $u_0(\pm L) \neq 0$). Indeed, since ν_0 is simple and the profile-solution Φ is even we need to have that u_0 is even or odd. But as u_0 has not zeros in $[-L, L]$ we get that u_0 even. Thus u_0' is odd and therefore $u_0'(\pm L) = 0$. This finishes the proof.

□

Remark 3.6 a) Lemma 3.5 holds true for the first component of the standing wave profile (Φ, ψ_a), Φ, to be even and with multiple bumps on $(-L, L)$. Moreover, we can also consider generic operators $\mathcal{L}_{+,Z}$ in (1.13) with external potential V_j not all the same, in other words, the components of $\Psi = (\psi_j)_{j=1}^N$ can be a combination of tails and bumps profiles. Therefore, Lemma 3.5 reduces our study to a spectral analysis on a star metric graph with δ-interaction conditions at a single vertex (see [3, 9, 12, 13] for a similar study in the case of the NLS models on star graphs). The existence and stability of these mixed profiles will be pursued in a future work.

b) By Theorem 1.3 and Lemma 3.5, it follows that β_0 being the smallest eigenvalue for $\mathcal{L}_{+,Z} = (\mathcal{L}_0, \mathcal{L})$ matches with the first eigenvalue for $\mathcal{L}_0$ with periodic boundary conditions and with the first eigenvalue for $\mathcal{L}$ with δ-interactions conditions on a star graph. Moreover, from (3.18), $\beta_0 = \nu_0 = \eta_0$.

3.1.3 Morse index for $(\mathcal{L}_+, D_{Z,0})$

In the following, we give the proof of Theorem 3.1.

Proof We consider $\mathcal{L}_+$ on $D_{Z,0}$ and suppose $n(\mathcal{L}_+) = 2$ without loss of generality (we note that via extension theory we can show $n(\mathcal{L}_+) \leqq 2$). From Theorem 3.2 and Corollary 3.3, the first negative eigenvalue λ_0 for $\mathcal{L}_+$ is simple with associated eigenfunction $(\phi_{\lambda_0}, \zeta_{\lambda_0})$ having positive components and ϕ_{λ_0} being even on $[-L, L]$. Therefore, for λ_1 being the second negative eigenvalue for $\mathcal{L}_+$, we need to have $\lambda_1 > \lambda_0$.

Let $(f_1, g_1) \in D_{Z,0}$ be an associated eigenfunction to λ_1. In the following, we divide our analysis in several steps.

1) Suppose $g_1 \equiv 0$: then $f_1(-L) = f_1(L) = 0$ and f_1 odd (see step 1) in the proof of Theorem 3.2). Now, our profile-solution Φ satisfies

$$\mathcal{L}_0\Phi' = 0, \;\; \Phi' \;\text{ is odd, }\;\; \Phi'(x) > 0, \;\text{ for } x \in [-L, 0),$$

thus since $\lambda_1 < 0$ we obtain from the Sturm Comparison Theorem that there is $r \in (-L, 0)$ such that $\Phi'(r) = 0$, which is a contradiction. Then, g_1 is non-trivial.
2) Suppose $g_1(0+) = 0$: we consider the odd-extension $g_{1,odd} \in H^2(\mathbb{R})$ of g_1 and the unfold operator $\widetilde{\mathcal{L}}$ in (3.11) on the δ-interaction domains $D_{\delta,\gamma}$ in (3.12). Then, $g_{1,odd} \in D_{\delta,\gamma}$ for any γ and so by Perron-Frobenius property for $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})$ (Theorem 6.7 in Appendix) we need to have that $n(\widetilde{\mathcal{L}}) \geqq 2$. But, by step 2) in the proof of Theorem 3.2 we obtain $n(\widetilde{\mathcal{L}}) \leqq 1$ for all γ, and so we get a contradiction.
3) Suppose $g_1(0+) > 0$ (without loss of generality): we will see that $g_1(x) > 0$ for all $x > 0$. Indeed, by Lemma 3.5 we get $g_1'(0+) = -Zg_1(0+)$. Thus, by considering the even-extension $g_{1,even}$ of g_1 on whole the line and the unfold operator $\widetilde{\mathcal{L}}$ in (3.11) on $D_{\delta,-2Z}$, we have that $g_{1,even} \in D_{\delta,-2Z}$ and so $n(\widetilde{\mathcal{L}}) \geqq 1$. But, we know that $n(\widetilde{\mathcal{L}}) \leqq 1$ and so λ_1 is the smallest eigenvalue for $(\widetilde{\mathcal{L}}, D_{\delta,-2Z})$. Therefore, by Theorem 6.7 in Appendix, g_1 is strictly positive.
4) Lastly, since the pairs $(\zeta_{\lambda_0}, \lambda_0)$ and (g_1, λ_1) satisfy the eigenvalue problem

$$\begin{cases} \mathcal{L}_1 g(x) = \gamma g(x), \;\; x > 0, \\ g'(0+) = -Zg(0+), \end{cases} \tag{3.19}$$

then ζ_{λ_0} and g_1 need to be orthogonal (which is a contradiction). This finishes the proof.

□

3.2 Kernel for $(\mathcal{L}_+, D_{Z,1})$

In the next, we study the nullity index for $\mathcal{L}_{+,Z}$ on $D_{Z,1}$ (see item 4) in Theorem 1.3 for the case of a tadpole graph). By using the notations at the beginning of this section, it is sufficient to show the following.

Theorem 3.7 *Consider $\mathcal{L}_+ = diag(\mathcal{L}_0, \mathcal{L}_1)$ on $D_{Z,0}$, for Z fixed, and let the quantity*

$$\alpha_1 = \frac{\psi_a''(0)}{\psi_a'(0)} + Z.$$

Then, the kernel associated to $\mathcal{L}_+$ on $D_{Z,0}$ is trivial in the following cases:

1) for $\alpha_1 \neq 0$, or,
2) $\alpha_1 = 0$ in the case of some admissible parameter Z satisfying $Z \leqq 0$.

Remark 3.8 By Sect. 2, for the case $Z = 0$ we always have the existence of positive single-lobe states on a tadpole graph. Hence, by Theorem 3.7 the associated linearized operator $\mathcal{L}_+$ has a trivial kernel (in this case, we recover similar results established in [2, 4, 5, 22, 44, 45] which were obtained by different techniques).

Proof Let $(f, h) \in D_{Z,0}$ such that $\mathcal{L}_+(f, h)^t = \mathbf{0}$. Thus, since $\mathcal{L}_1 h = 0$ and $\mathcal{L}_1 \psi_a' = 0$, we obtain from classical Sturm-Liouville theory on half-lines ([18]) that there is $c \in \mathbb{R}$ with

$$h = c\psi_a', \quad \text{on } (0, +\infty).$$

1) Suppose $c = 0$: then $h \equiv 0$ and f satisfies $\mathcal{L}_0 f = 0$ with Dirichlet-periodic conditions

$$f(L) = f(-L) = 0 \;\text{ and }\; f'(L) = f'(-L).$$

Suppose $f \neq 0$. From Floquet's oscillation theory, we need to have that zero is not the first eigenvalue for $\mathcal{L}_0$ with periodic conditions. Therefore f needs to change of sign. Now, from Sturm-Liouville theory for Dirichlet conditions we have that f is even or odd (as 0 will be a simple eigenvalue and Φ has an even profile). Next, since $\mathcal{L}_0 \Phi' = 0$ on $[-L, L]$, Φ' odd and $\Phi'(L) \neq 0$, we get that f is even (indeed, from classical ODE's theory $f = a\Phi' + bP$, where P is the even-solution of $\mathcal{L}_0 P = 0$ with $P(0) = 1$, $P'(0) = 0$, and so $a = 0$). Moreover, the number of zeros of f on $(-L, L)$ is *even*. Lastly, again from Floquet's oscillation theory, f also needs to have an *even* number of zeros on $[-L, L)$. Hence f has an odd number of zeros on $(-L, L)$ (as $f(-L) = 0$), which is a contradiction. Therefore, $f \equiv 0$ and so $ker(\mathcal{L}_+)$ is trivial.

2) Suppose $c \neq 0$: then $h(0) \neq 0$ ($h(x) > 0$ without loss of generality with $c < 0$). Hence, from the splitting eigenvalue result in Lemma 3.5 in the case of a tadpole graph, we obtain that f satisfies

$$\begin{cases} \mathcal{L}_0 f(x) = 0, \; x \in [-L, L], \\ f(L) = f(-L) = h(0) > 0, \quad \text{and} \quad f'(L) = f'(-L), \end{cases} \tag{3.20}$$

and $h'(0) = -Zh(0)$. The last equality implies immediately

$$Z + \frac{\psi_a''(0)}{\psi_a'(0)} = 0,$$

therefore if we have by hypotheses that $\alpha_1 \neq 0$, we obtain a contradiction. Then, $c = 0$ and by item 1) above $ker(\mathcal{L}_+) = \{\mathbf{0}\}$.

Next, we consider the case $\alpha_1 = 0$ with some $Z \leqq 0$. Then, initially by Floquet theory and oscillation theory, we have the following partial distribution of eigenvalues, η_n and μ_n, associated to $\mathcal{L}_0$ with periodic and Dirichlet conditions, respectively,

$$\eta_0 < \mu_0 < \eta_1 \leqq \mu_1 \leqq \eta_2 < \mu_2 < \eta_3. \tag{3.21}$$

In the next, we will prove $\eta_1 = 0$ in (3.21) and that it is simple. Indeed, initially we suppose that $0 > \mu_1$. Then, we know that the profile Φ satisfies $\mathcal{L}_0 \Phi' = 0$ on $[-L, L]$,

Φ' is odd, $\Phi'(x) > 0$ for $[-L, 0)$, and that the eigenfunction associated to μ_1 is odd, therefore from Sturm Comparison Theorem we get that Φ' needs to have one zero on $(-L, 0)$, which is impossible. Hence, $0 \leqq \mu_1$. Next, it supposes that $\mu_1 = 0$ and let χ_1 be an odd eigenfunction for μ_1. Let $\{\Phi', P\}$ be a base of solutions for the problem $\mathcal{L}_0 g = 0$ (we recall that P can be chosen being even, satisfying $P(0) = 1$ and $P'(0) = 0$). Then, $\chi_1 = a\Phi'$ with $0 = \chi_1(L) = a\Phi'(L)$. Hence, $\chi_1 \equiv 0$ which is not possible. Therefore, $0 < \mu_1$ and so $\eta_1 = 0$ is simple with eigenfunction f (being even or odd). We note that η_2 is also simple.
Lastly, since $f(-L) = f(L) > 0$ it follows that f is even and by Floquet theory f has exactly two different zeros $-a, a$ $(a > 0)$ on $(-L, L)$. Hence, $f(0) < 0$. Next, we consider the Wronskian function (constant) of f and Φ', namely,

$$W(x) = f(x)\Phi''(x) - f'(x)\Phi'(x) \equiv C, \quad \text{for all } \; x \in [-L, L].$$

Then, $C = f(0)\Phi''(0) > 0$. Therefore, by hypotheses ($\alpha_1 = 0$ with some $Z \leqq 0$) we obtain

$$C = f(L)\Phi''(L) = h(0)\psi''_{0,a}(0) = -cZ[\psi'_{0,a}(0)]^2 \leqq 0, \tag{3.22}$$

which is a contradiction. Then, $c = 0$ and by item 1) above we get again $ker(\mathcal{L}_+) = \{\mathbf{0}\}$. This finishes the proof.

□

4 Morse and nullity indices for operators $\mathcal{L}_{\pm,Z}$ on general looping edge graphs

In this section, we show Theorems 1.3–1.4 on looping edge graphs $\mathcal{G}_N$, $N \geqq 2$.

Proof (**Theorem** 1.3) The proof follows the same strategy as in the case of a tadpole graph. By convenience of the reader, we give the main highlight in the analysis.

1) Perron-Frobenius property: consider $(f_{\beta_0}, \mathbf{g}_{\beta_0}) \in D_{Z,N}$, $\mathbf{g}_{\beta_0} = (g_j)_{j=1}^N$, an eigenfunction for the smallest eigenvalue β_0 of $\mathcal{L}_{+,Z}$. Then at least one component, g_k, is not identically zero and $g_k(L) \neq 0$, because otherwise steps 1) $-$ 2) in the proof of Theorem 3.2 imply a contradiction. Thus, g_k satisfies $\mathcal{L}g_k = \beta_0 g_k$, with $\mathcal{L} = -\partial_x^2 - \omega - (p+1)(2p+1)\psi_1^{2p}$, and

$$g_k'(L) = \Big[\frac{f'_{\beta_0}(L) - f'_{\beta_0}(-L)}{g_k(L)} - \frac{1}{g_k(L)}\sum_{j\neq k}^{N} g_j'(L) - Z\Big] g_k(L) \equiv \gamma_k g_k(L).$$

Therefore, from the Perron-Frobenius property (Theorem 6.7 in Appendix) we get $g_k > 0$ on $[0, +\infty)$. Now, since $g_j(L) = g_k(L)$, for $j \neq k$, we obtain similarly that $g_j > 0$. Lastly, from relation

$$f'_{\beta_0}(L) - f'_{\beta_0}(-L) = \Big[\frac{1}{g_1(L)}\sum_{j=1}^{N} g_j'(L) + Z\Big] g_1(L) \equiv \theta_N f_{\beta_0}(L),$$

and step 4) in the proof of Theorem 3.2 imply that f_{β_0} is positive and even on $[-L, L]$.
2) The property of β_0 to be simple, follows from Corollary 3.3.

3) The Morse index of $\mathcal{L}_{+,Z}$ is one: initially, as $(\Phi, \Psi) \in D_{Z,N}$ and $\langle \mathcal{L}_{+,Z}(\Phi, \Psi)^t, (\Phi, \Psi)^t \rangle < 0$, we known $n(\mathcal{L}_{+,Z}) \geqq 1$. In the following, suppose $n(\mathcal{L}_{+,N}) = 2$, and β_0, β_1 $(\beta_0 < \beta_1)$ the two negative eigenvalues with associated eigenfunctions $(f_{\beta_0}, \mathbf{g}_{\beta_0})$, $\mathbf{g}_{\beta_0} = (g_j)_{j=1}^N$, and $(f_{\beta_1}, \mathbf{h}_{\beta_1})$, $\mathbf{h}_{\beta_1} = (h_j)_{j=1}^N$, respectively. Next, we show that every component of $\mathbf{h}_{\beta_1}$ can be chosen positive. Suppose $\mathbf{h}_{\beta_1} \equiv \mathbf{0}$, then f_{β_1} satisfies periodic-Dirichlet boundary conditions and so its profile is odd. Then, by the proof of Theorem 3.1 (step 1)) we get a contradiction. Hence, there is at least one component h_k, which is not-trivial and satisfying $h_k > 0$ on $(L, +\infty)$ (see item 1) above and item 3) in the proof of Theorem 3.1). Hence, by continuity at $x = L$ it follows $h_j(L) > 0$ for every $j \neq k$ and therefore $h_j > 0$ on $(L, +\infty)$ for every j. Now, by the splitting eigenvalue result in Lemma 3.5, we get that for every j,

$$\mathcal{L}h_j = \beta_1 h_j, \quad \mathcal{L}g_j = \beta_0 g_j \quad on \quad (L, +\infty),$$

with $\sum_{j=1}^N h_j'(L) = -Zh_1(L)$ and $\sum_{j=1}^N g_j'(L) = -Zg_1(L)$. Then for $g \equiv \sum_{j=1}^N g_j$ and $h \equiv \sum_{j=1}^N h_j$, we have that the pairs (g, β_0) and (h, β_1) satisfy the eigenvalue problem

$$\begin{cases} \mathcal{L}\chi(x) = \gamma\chi(x), & x > L, \\ \chi'(L+) = -\frac{Z}{N}\chi(L+). \end{cases} \tag{4.1}$$

Therefore, because g and h are positive and $\beta_0 \neq \beta_1$, we get a contradiction. Then, $n(\mathcal{L}_{+,Z}) = 1$.

4) Kernel for $\mathcal{L}_{+,Z}$: Suppose $(p, \mathbf{q}) \in ker(\mathcal{L}_{+,Z})$, $\mathbf{q} = (q_j)_{j=1}^N$. Then, from $\mathcal{L}q_j = 0$ we get that there are constants c_j such that $q_j = c_j\psi_1'$, and so by continuity at $x = L$ we get $c_1 = c_2 = \cdots = c_N \equiv C$. Hence, $\mathbf{q} = C\Psi'$. In the following we consider two cases:

   a) Suppose $C = 0$: then p satisfies $\mathcal{L}_0 p = 0$, $p(L) = p(-L) = 0$ and $p'(L) = p'(-L)$. Thus, from step 1) in the proof of Theorem 3.7 follows $p \equiv 0$.
   b) Suppose $C \neq 0$ and $C < 0$: then $q_j > 0$ for every j and so by the splitting eigenvalue result in Lemma 3.5, p satisfies the eigenvalue problem (3.20) (with $h = q_1$) and $\sum_{j=1}^N q_j'(L) = -Zq_1(L)$. The last equality implies

$$\frac{Z}{N} + \frac{\psi_1''(L)}{\psi_1'(L)} = 0.$$

   Then, by following a similar analysis as in step 2) in the proof of Theorem 3.7, we finish the statements about the non-degeneracy of $\mathcal{L}_{+,Z}$.

□

Remark 4.1 a) From condition $2\Phi'(L) = N\psi_1'(L) + Z\psi_1(L) < 0$ and (1.7) we need to have *a priori* that the strength Z satisfies $Z < N\sqrt{-\omega}\ tanh(pa) < N\sqrt{-\omega}$, with $a = a(\omega, Z) > 0$.

b) Some comments deserve to be established about the conditions in item 4) of Theorem 1.3 for the kernel of $\mathcal{L}_{+,Z}$ to be trivial. We consider the mapping

$$\gamma(a) = N\frac{\psi_1''(L)}{\psi_1'(L)} = N\sqrt{-\omega}\,\frac{(p+1)sech^2(pa) - 1}{tanh(pa)}, \quad a > 0, \tag{4.2}$$

related to the quantity α_N in Theorem 1.3. Then, γ is strictly decreasing, with $\gamma(a) \to +\infty$ as $a \to 0^+$, and $\gamma(a) \to -N\sqrt{-\omega}$ as $a \to +\infty$. Moreover, there is an unique a^*

such that $\gamma(a^*) = 0$, which matches with the only inflection-point for ψ_0 in (1.7). Thus, we have that the equation $\gamma(a) = -Z$ always has a unique solution a for $Z < N\sqrt{-\omega}$.

Proof (**Theorem** 1.4) We consider a positive single-lobe state $(\Phi_\omega, \Psi_\omega)$. Then, from (1.5) we get $(\Phi_\omega, \Psi_\omega) \in ker(\mathcal{L}_{-,Z})$. Next, we consider

$$\mathcal{M} = -\partial_x^2 - \omega - (p+1)\Psi_\omega^{2p}, \qquad \mathcal{N} = -\partial_x^2 - \omega - (p+1)\psi_1^{2p},$$

then for any $\mathbf{V} = (f, \mathbf{g}) \in D_{Z,N}$, $\mathbf{g} = (g_j)_{j=1}^N$ we obtain

$$\mathcal{M}f = -\frac{1}{\Phi_\omega}\frac{d}{dx}\Big[\Phi_\omega^2\frac{d}{dx}\Big(\frac{f}{\Phi_\omega}\Big)\Big], \qquad x \in (-L, L)$$

$$\mathcal{N}g_j = -\frac{1}{\psi_j}\frac{d}{dx}\Big[\psi_j^2\frac{d}{dx}\Big(\frac{g_j}{\psi_j}\Big)\Big], \qquad x > L.$$

Thus, we get immediately

$$\langle \mathcal{L}_{-,Z}\mathbf{V}, \mathbf{V}\rangle = \int_{-L}^{L} \Phi_\omega^2\Big(\frac{d}{dx}\Big(\frac{f}{\Phi_\omega}\Big)\Big)^2 dx + \sum_{j=1}^{N}\int_{L}^{+\infty} \psi_j^2\Big(\frac{d}{dx}\Big(\frac{g_j}{\psi_j}\Big)\Big)^2 dx \geqq 0.$$

Moreover, since $\langle \mathcal{L}_{-,Z}\mathbf{V}, \mathbf{V}\rangle = 0$ if and only if $f = c\Phi_\omega$ and $g_j = d_j\psi_j$, we obtain from the continuity property at $x = L$ that $c = d_1 = \cdots = d_N$. Then, $ker(\mathcal{L}_{-,Z}) = span\{(\Phi_\omega, \Psi_\omega)\}$. This finishes the proof. □

Remark 4.2 There is other strategy for showing Theorem 1.4: we can also see that $(\mathcal{L}_{-,Z}, D_{Z,N})$ satisfies the Perron-Frobenius property just like $(\mathcal{L}_{+,Z}, D_{Z,N})$ does (see Theorem 3.2). Hence, since $(\Phi_\omega, \Psi_\omega)$ is a positive single-lobe state and $\mathcal{L}_{-,Z}(\Phi_\omega, \Psi_\omega)^t = \mathbf{0}$, we conclude that zero is a simple eigenvalue and $\mathcal{L}_{-,Z} \geqq 0$.

Remark 4.3 In the following we establish some consequences and comments associated to our results.

1) Let (Φ, Ψ) be a solution for (1.5). If we consider a positive even-profile Φ having multiple bumps on $(-L, L)$ and $\Psi = (\psi_j)_{j=1}^N$ with components being a mixed of tails and bumps profiles, the strategy for showing Theorem 1.4 implies that the associated operator $\mathcal{L}_{-,Z}$ in (1.13) is also non-negative and with one-dimensional kernel.
2) The existence of positive solutions (Φ, Ψ) for (1.5) on a tadpole graph with a profile shown in Fig. 3 (a "positive two-lobe" state), was showed explicitly at least for $p = 1$ in [22, 45]. Now, by using some strategies used in our work, in particular, the splitting eigenvalue result in Lemma 3.5, a stability theory for these profiles has been established recently in Angulo [10].

5 Applications

In this section we show Theorems 1.5–1.6. As far as we know, our stability results in Theorem 1.5 (at least for $N \geqq 3$) and in Theorem 1.6 (for any $N \geqq 1$ and specific $Z < 0$), are the first to be established in the literature (see [6, 35]).

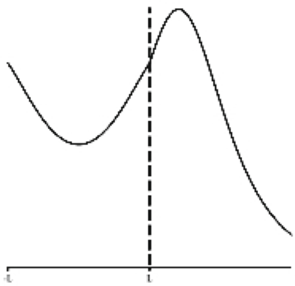

Fig. 3 A positive two-lobe state profile for the NLS model on a tadpole graph

Proof (**Theorem** 1.5) Our basic strategy for obtaining a family of positive single-lobe states on $\mathcal{G}_N$ is the following: by using a bifurcation analysis on the phase plane associated to the the first elliptic equation in (1.5), we get initially an unique branch of profiles $\omega \to \Phi_\omega$, with $-\omega$ small, satisfying the continuous property at the vertex $\nu = L$, $\Phi_\omega(L) = \Phi_\omega(-L) = \psi_1(L)$ and $\Phi'_\omega(L) - \Phi'_\omega(-L) = N\psi'_1(L)$. Thus, we get a C^1-mapping $\omega \in (\omega_0, 0) \to \Theta_\omega = (\Phi_\omega, (\psi_1)_{j=1}^N)$, with $-\omega_0 > 0$ small enough, of positive single-lobe states on $\mathcal{G}_N$. We divide our analysis in several steps:

a) By following an similar analysis as in [45], we have on a tadpole graph for $\omega = -\epsilon^2$, $\epsilon > 0$, $\psi_\epsilon(x) = \epsilon^{\frac{1}{p}}\psi_0(\epsilon(x - L) + a)$, and $\Phi(x) = \epsilon^{\frac{1}{p}}\phi(z)$, $z = \epsilon x$, that the boundary-value problem (1.5) is rewrite in the form

$$\begin{cases} -\phi''(z) + \phi(z) - (p+1)|\phi(z)|^{2p}\phi(z) = 0, \ z \in (-\epsilon L, \epsilon L), \\ \phi(\epsilon L) = \phi(-\epsilon L) = \psi_0(a), \\ \phi'(\epsilon L) - \phi'(-\epsilon L) = N\psi'_0(a). \end{cases} \tag{5.1}$$

Next, from symmetry of trajectories on the phase plane (ϕ, ϕ'), it follows that for $\epsilon > 0$ small enough, the condition $\phi(\epsilon L) = \phi(-\epsilon L)$ can be satisfied if and only if the function ϕ is even in z. Moreover, by the condition of single-lobe profile, we will looking for trajectories inside the homoclinic orbit $x \to (\psi_0(x), \psi'_0(x))$. Therefore, we can consider an initial-value problem for the second-order differential equation in system (5.1) with an initial data $(\phi(0), \phi'(0)) \equiv (\phi_0, 0)$, $\phi_0 > 0$, $\phi_0 < 1$ and $\phi_0 \approx 1$. Then, by edo's theory there exists a $\delta = \delta(\phi_0) > 0$ and a unique local solution $\phi \in C^\infty(-\delta, \delta)$ such that $\phi(0) = \phi_0$ and $\phi'(0) = 0$. Moreover, $\phi(z) > 0$ for all $z \in (-\delta, \delta)$. Thus, being L fixed, we have $\epsilon L < \delta$ for sufficiently small ϵ, so that ϕ is the unique even positive local solution on $[-\epsilon L, \epsilon L]$ with a profile being of single-lobe type. Moreover, by the continuity property of the data-solution mapping, we can choose ϵ such that for ζ in a neighborhood of ϕ_0 the associated solution ϕ_ζ with $\phi_\zeta(0) = \zeta$ can be also defined on $[-\epsilon L, \epsilon L]$.

With regard to the boundary conditions $\phi(-\epsilon L) = \psi_0(a)$ and $2\phi'(\epsilon L) = N\psi'_0(a)$ we have the following: for the first condition we consider the Taylor expansion (with Lagrange remainder) $\phi(-\epsilon L) = \phi_0 + \frac{1}{2}\phi''(0)\epsilon^2 L^2 + \mathcal{O}(\epsilon^4)$ with ϵ small and arbitrary, and $\phi''(0) = g(\phi_0)$ giving by the differential equation in system (5.1). Then, for $\psi_0(a) \approx 1$ arbitrary, but fixed, with $a > 0$, and by considering $F(\phi_0, \epsilon) = \phi_0 + \frac{1}{2}g(\phi_0)\epsilon^2 L^2 + \mathcal{O}(\epsilon^4) - \psi_0(a)$, for $(\phi_0, \epsilon) \in B_\eta(\psi_0(a), 0)$ we obtain $F(\psi_0(a), 0) = 0$ and $\partial_{\phi_0} F(\psi_0(a), 0) = 1$. Therefore, the implicit function theorem implies the existence of a smooth mapping of "initial conditions", $\epsilon \to \phi_0 = \phi_0(\epsilon)$ for ϵ small, such that $F(\phi_0(\epsilon), \epsilon) = 0$. Thus, by uniqueness in the Taylor expansion, we get $\psi_0(a) = \phi_0 + \frac{1}{2}g(\phi_0)\epsilon^2 L^2 + \mathcal{O}(\epsilon^4) = \phi(-\epsilon L)$. Moreover, since $\phi'_0(\epsilon)|_{\epsilon=0} = 0$

follows $\phi_0(\epsilon) = \psi_0(a) + \mathcal{O}(\epsilon^2)$ (thus $\phi_0(\epsilon) < 1$).
Now, for the second condition we get the right shift-value a and we will show that

$$a(\epsilon) = \frac{2L}{N}\epsilon + \mathcal{O}(\epsilon^3).$$

Indeed, from Taylor expansion we get $2\phi'(-\epsilon L) = 2\,g(\phi_0)\epsilon L + \mathcal{O}(\epsilon^3)$ with ϵ small. Next, for $G(a,\epsilon) = 2\,g(\phi_0)\epsilon L + \mathcal{O}(\epsilon^3) - N\psi_0'(a)$ we have $G(0,0) = 0$ and $\partial_a G(0,0) = -N\psi_0''(0) \neq 0$. Therefore, there is a smooth mapping of shift-values, $\epsilon \to a = a(\epsilon)$, such that $G(a(\epsilon),\epsilon) = 0$, and so $2\phi'(\epsilon L) = N\psi_0'(a(\epsilon))$.
Next, we have from $a'(0) = \frac{2g(\phi_0)L}{N\psi_0''(0)}$, the second-order differential equation in (5.1), $\psi_0''(0) = (1-(p+1)\psi_0^{2p}(0))\psi_0(0)$, and $\phi_0(\epsilon) = \psi_0(a) + \mathcal{O}(\epsilon^2) = 1 + \mathcal{O}(\epsilon^2)$ (since $\psi_0(0) = 1$ and $\psi_0'(0) = 0$) that

$$a(\epsilon) = \frac{2g(\phi_0)L}{N\psi_0''(0)}\epsilon + \mathcal{O}(\epsilon^3) = \frac{2L}{N}\epsilon + \mathcal{O}(\epsilon^3). \tag{5.2}$$

Therefore, by analysis above, for $p > 0$ and every $\epsilon > 0$ sufficiently small, there exists a unique positive-lobe solution $\Phi_\epsilon \in C^\infty(-L,L)$ and $a = a(\epsilon) > 0$ of the following boundary-value problem

$$\begin{cases} -\Phi_\epsilon''(x) + \epsilon^2\Phi_\epsilon(x) - (p+1)|\Phi_\epsilon(x)|^{2p}\Phi_\epsilon(x) = 0, \quad x \in (-L,L), \\ \Phi_\epsilon(L) = \Phi_\epsilon(-L) = \epsilon^{\frac{1}{p}}\psi_0(a), \\ \Phi_\epsilon'(L) - \Phi_\epsilon'(-L) = N\epsilon^{1+\frac{1}{p}}\psi_0'(a). \end{cases} \tag{5.3}$$

Moreover, $\Phi_\epsilon = \epsilon^{\frac{1}{p}}(1 + O_{C^\infty(-L,L)}(\epsilon^2))$, $\|\Phi_\epsilon\|_{L^\infty(-L,L)} = O(\epsilon)$ and $a = O(\epsilon)$ as $\epsilon \to 0$.

b) From $\|\Phi_\epsilon\|^2_{L^2(-L,L)} = 2L\epsilon^{\frac{2}{p}}(1 + O(\epsilon^2))$ and $\|\psi_\epsilon\|^2_{L^2(L,\infty)} = \epsilon^{\frac{2}{p}-1}\|\psi_0\|^2_{L^2(a(\epsilon),\infty)}$ where ψ_0 is ϵ-independent, we obtain for $p \in (0,2)$ and ϵ small enough

$$\partial_\epsilon(\|\Phi_\epsilon\|^2_{L^2(-L,L)} + N\|\psi_\epsilon\|^2_{L^2(L,\infty)}) > 0,$$

and for $p \in (2,+\infty)$, ϵ small enough

$$\partial_\epsilon(\|\Phi_\epsilon\|^2_{L^2(-L,L)} + N\|\psi_\epsilon\|^2_{L^2(L,\infty)}) < 0. \tag{5.4}$$

c) From $\omega = -\epsilon^2$ we get immediately a C^1- mapping $\omega \in (\omega_0, 0) \to (\Phi_\omega, \psi_\omega)$ on $\mathcal{G}_1$ with $-\omega_0 > 0$ small enough, such that

$$\partial_\omega(\|\Phi_\omega\|^2_{L^2(-L,L)} + N\|\psi_\omega\|^2_{L^2(L,\infty)}) < 0, \qquad p \in (0,2),$$

$$\partial_\omega(\|\Phi_\omega\|^2_{L^2(-L,L)} + N\|\psi_\omega\|^2_{L^2(L,\infty)}) > 0, \qquad p \in (2,+\infty).$$

d) Define $\omega \in (\omega_0, 0) \to \Theta_\omega = (\Phi_\omega, (\psi_\omega)_{j=1}^N) \in D_{Z,N}$, of positive single-lobe states on $\mathcal{G}_N$. Then, $\|\Theta_\omega\|^2_{L^2(\mathcal{G}_N)} = \|\Phi_\omega\|^2_{L^2(-L,L)} + N\|\psi_\omega\|^2_{L^2(L,\infty)}$ and so,

$$\partial_\omega\|\Theta_\omega\|^2_{L^2(\mathcal{G}_N)} < 0, \qquad p \in (0,2),$$

$$\partial_\omega\|\Theta_\omega\|^2_{L^2(\mathcal{G}_N)} > 0, \qquad p \in (2,+\infty).$$

By Theorems 1.3–1.4, and Theorem 6.8, $N \geqq 1$ fixed, we obtain that the orbit $\{e^{i\theta}\Theta_\omega : \theta \in [0, 2\pi)\}$ is orbitally unstable for $p > 2$ and orbitally stable for $p < 2$. This completes the assertions of the Theorem. □

Remark 5.1 The case $N = 1$ (tadpole case) and $p = 2$ was studied in [44]. In this case were found a C^1- mapping $\omega \in (-\infty, 0) \to (\Phi_\omega, \Psi_\omega)$ of positive single-lobe on $\mathcal{G}_1$ and a threshold value $\omega^* < 0$ of the frequency such that for $\omega > \omega^*$ the standing wave is orbitally stable, and for $\omega < \omega^*$ we get orbital instability.

Proof (**Theorem** 1.6) We will use the same strategy as in Theorem 1.5. Thus, for $\omega = -\epsilon^2$, $\epsilon > 0$, $\psi_\epsilon(x) = \epsilon^{\frac{1}{p}}\psi_0(\epsilon(x - L) + a)$, and $\Phi(x) = \epsilon^{\frac{1}{p}}\phi(z)$, $z = \epsilon x$, the boundary-value problem (1.5) is rewrite in the form

$$\begin{cases} -\phi''(z) + \phi(z) - (p+1)|\phi(z)|^{2p}\phi(z) = 0, \ \ z \in (-\epsilon L, \epsilon L), \\ \phi(\epsilon L) = \phi(-\epsilon L) = \psi_0(a), \\ \epsilon\phi'(\epsilon L) - \epsilon\phi'(-\epsilon L) = N\epsilon\psi_0'(a) + Z\psi_0(a) \end{cases} \tag{5.5}$$

Thus, for $Z = -\epsilon^2$ we get the condition $\phi'(\epsilon L) - \phi'(-\epsilon L) = N\psi_0'(a) - \epsilon\psi_0(a)$. Next, by a phase plane analysis we get a unique even positive local solution ϕ (of single-lobe type) for the second-order differential equation in (5.5) on $[-\epsilon L, \epsilon L]$, ϵ small, such that $\phi(0) = \phi_0 \approx 1$ and $\phi'(0) = 0$. Then for $\psi_0(a) \approx 1$ arbitrary, but fixed, with $a > 0$, the implicit function theorem implies the existence of a smooth mapping of "initial conditions", $\epsilon \to \phi_0 = \phi_0(\epsilon) \approx \psi_0(a)$ such that the associated solutions $\phi = \phi_\epsilon$ with $\phi(0) = \phi_0(\epsilon)$, satisfy for ϵ small that $\psi_0(a) = \phi(-\epsilon L)$. Moreover, $\phi_0(\epsilon) = \psi_0(a) + \mathcal{O}(\epsilon^2)$ (thus $\phi_0(\epsilon) < 1$).

With regard to the boundary condition $2\phi'(\epsilon L) = N\psi_0'(a) - \epsilon\psi_0(a)$ we have the following: we consider

$$H(a, \epsilon) = -\epsilon\psi_0(a) - 2\phi'(\epsilon L) + N\psi_0'(a) = -\epsilon\psi_0(a) - 2\phi''(0)\epsilon L + N\psi_0'(a) + \mathcal{O}(\epsilon^3).$$

Then, $H(0, 0) = 0$ and $\partial_a H(0, 0) = N\psi_0''(0) \neq 0$. Therefore, there is a smooth mapping of shift-values, $\epsilon \to a = a(\epsilon)$, such that $H(a(\epsilon), \epsilon) = 0$, and so $2\phi'(\epsilon L) = N\psi_0'(a(\epsilon)) - \epsilon\psi_0(a(\epsilon))$. Moreover, since $a'(0) = \frac{1+2g(\phi_0)L}{N\psi_0''(0)}$ follows from the second-order differential equation in (5.5), $\psi_0''(0) = (1 - (p+1)\psi_0^{2p}(0))\psi_0(0)$, and $\phi_0(\epsilon) = \psi_0(a) + \mathcal{O}(\epsilon^2) = 1 + \mathcal{O}(\epsilon^2)$ (since $\psi_0(0) = 1$ and $\psi_0'(0) = 0$) that

$$a(\epsilon) = \frac{1 + 2g(\phi_0)L}{N\psi_0''(0)}\epsilon + \mathcal{O}(\epsilon^3) = \frac{1}{N}\Big[2L - \frac{1}{p}\Big]\epsilon + \mathcal{O}(\epsilon^3).$$

From $2L > \frac{1}{p}$, we get $a(\epsilon) > 0$ for ϵ small. Therefore, for every $\epsilon > 0$ sufficiently small, there exists a unique positive-lobe solution $\Phi_\epsilon \in C^\infty(-L, L)$ and $a = a(\epsilon) > 0$ of the following boundary-value problem

$$\begin{cases} -\Phi_\epsilon''(x) + \epsilon^2\Phi_\epsilon(x) - (p+1)|\Phi_\epsilon(x)|^{2p}\Phi_\epsilon(x) = 0, \quad x \in (-L, L), \\ \Phi_\epsilon(L) = \Phi_\epsilon(-L) = \epsilon^{\frac{1}{p}}\psi_0(a), \\ \Phi_\epsilon'(L) - \Phi_\epsilon'(-L) = N\epsilon^{1+\frac{1}{p}}\psi_0'(a) - \epsilon^2\epsilon^{\frac{1}{p}}\psi_0(a). \end{cases} \tag{5.6}$$

Moreover, $\Phi_\epsilon = \epsilon^{\frac{1}{p}}(1 + O_{C^\infty(-L,L)}(\epsilon^2))$, $\|\Phi_\epsilon\|_{L^\infty(-L,L)} = O(\epsilon)$ and $a = O(\epsilon)$ as $\epsilon \to 0$.

Next, from $\|\Phi_\epsilon\|^2_{L^2(-L,L)} = 2L\epsilon^{\frac{2}{p}}(1+O(\epsilon^2))$ and $\|\psi_\epsilon\|^2_{L^2(L,\infty)} = \epsilon^{\frac{2}{p}-1}\|\psi_0\|^2_{L^2(a(\epsilon),\infty)}$ where ψ_0 is ϵ-independent, we obtain for $P(\epsilon) = \|\Phi_\epsilon\|^2_{L^2(-L,L)} + N\|\psi_\epsilon\|^2_{L^2(L,\infty)}$ that

$$\begin{aligned}P'(\epsilon) =&\epsilon^{\frac{2}{p}-1}\Big[\frac{4L}{p} + \frac{1}{\epsilon}N(\frac{2}{p}-1)\|\psi_0\|^2_{L^2(a(\epsilon),\infty)} - \psi_0^2(a(\epsilon))\Big(\Big[2L-\frac{1}{p}\Big]+O(\epsilon^2)\Big)\Big] + O(\epsilon^{\frac{2}{p}+1})\\ =&\epsilon^{\frac{2}{p}-1}\Big[\frac{4L}{p} + \frac{1}{\epsilon}N(\frac{2}{p}-1)\|\psi_0\|^2_{L^2(a(\epsilon),\infty)} - \Big[2L-\frac{1}{p}\Big]+O(\epsilon^2)\Big] + O(\epsilon^{\frac{2}{p}+1}).\end{aligned}$$

Therefore, for $p \in (0,2)$ and ϵ small enough, we have $P'(\epsilon) > 0$, and $P'(\epsilon) < 0$ for $p \in (2,+\infty)$, ϵ small enough. Lastly, for $p = 2$ we obtain $P'(\epsilon) = \frac{1}{2} + O(\epsilon^2)$ and so $P'(\epsilon) > 0$.

Then, from $\omega = -\epsilon^2$ we obtain a C^1- mapping $\omega \in (\omega_0, 0) \to \Theta_\omega = (\Phi_\omega, (\psi_\omega)_{j=1}^N) \in D_{Z,N}$, with $-\omega_0 > 0$ small enough, of positive single-lobe states on $\mathcal{G}_N$ and satisfying

$$\partial_\omega\|\Theta_\omega\|^2_{L^2(\mathcal{G}_N)} < 0, \qquad p \in (0,2],$$

$$\partial_\omega\|\Theta_\omega\|^2_{L^2(\mathcal{G}_N)} > 0, \qquad p \in (2,+\infty).$$

Hence, the orbit $\{e^{i\theta}\Theta_\omega : \theta \in [0,2\pi)\}$ is orbitally unstable for $p > 2$ and orbitally stable for $p \leqq 2$. This completes the assertions of the Theorem. □

Remark 5.2 If we choose $Z = -(-\omega)^{n/2} = -\epsilon^n, n > 2, n \in \mathbb{N}$, in the proof of Theorem 1.6, we obtain an similar stability result as in Theorem 1.5, because in this case the shift parameter $a = a(\epsilon)$ satisfies the same equation as in (5.2). This consideration shows that for an arbitrary $Z < 0$, the dynamics of positive single-lobe states on $\mathcal{G}_N$ can become a tricky problem.

6 Appendix

in this Appendix we formulate some tools from the extension theory of symmetric operators of Krein&von Neumann suitable for our needs (see [40, 48] for further information). Also, we establish a Perron-Frobenius property for Schrödinger operators with δ-interactions domains on whole line.

6.1 Classical extension theory results

The following two results are classical and can be found in [48].

Theorem 6.1 (von-Neumann decomposition) *Let A be a closed, symmetric operator, then*

$$D(A^*) = D(A) \oplus \mathcal{N}_{-i} \oplus \mathcal{N}_{+i}. \tag{6.1}$$

with $\mathcal{N}_{\pm i} = \ker(A^* \mp iI)$. *Therefore, for* $u \in D(A^*)$ *and* $u = x+y+z \in D(A) \oplus \mathcal{N}_{-i} \oplus \mathcal{N}_{+i}$,

$$A^*u = Ax + (-i)y + iz. \tag{6.2}$$

Remark 6.2 The direct sum in (6.1) is not necessarily orthogonal.

Proposition 6.3 *Let A be a densely defined, closed, symmetric operator in some Hilbert space H with deficiency indices equal* $n_\pm(A) = 1$. *All self-adjoint extensions* A_θ *of A may* *be parametrized by a real parameter* $\theta \in [0, 2\pi)$ *where*

$$\begin{aligned} D(A_\theta) &= \{x + c\phi_+ + \zeta e^{i\theta}\phi_- : x \in D(A), \zeta \in \mathbb{C}\}, \\ A_\theta(x + \zeta\phi_+ + \zeta e^{i\theta}\phi_-) &= Ax + i\zeta\phi_+ - i\zeta e^{i\theta}\phi_-, \end{aligned}$$

with $A^*\phi_\pm = \pm i\phi_\pm$, *and* $\|\phi_+\| = \|\phi_-\|$.

The next proposition provides a strategy for estimating the Morse-index of the self-adjoint extensions (see [40], [48]-Chapter X).

Proposition 6.4 *Let* A *be a densely defined lower semi-bounded symmetric operator (that is,* $A \geq mI$*) with finite deficiency indices,* $n_\pm(A) = k < \infty$*, in the Hilbert space* H*, and let* $\widehat{A}$ *be a self-adjoint extension of* A*. Then the spectrum of* $\widehat{A}$ *in* $(-\infty, m)$ *is discrete and consists of, at most,* k *eigenvalues counting multiplicities.*

Next Proposition can be found in Naimark [40] (see Theorem 9).

Proposition 6.5 *All self-adjoint extensions of a closed, symmetric operator which has equal and finite deficiency indices have one and the same continuous spectrum.*

6.2 Extension theory for the Laplacian operator on a looping edge graph

In this subsection, by convenience of the reader, we establish an extension theory for the Laplacian operator $-\Delta$ in (1.2) on a looping edge graph which will imply the domains $D_{Z,N}$ in (1.3).

Theorem 6.6 *Let* $\mathcal{G}_N$ *be a looping edge graph with* $N \geqq 1$*. The Schrödinger type operator* $-\Delta$ *in (1.2) on* $L^2(\mathcal{G}_N)$*, with domain*

$$\begin{aligned} D(-\Delta) = \Big\{(\phi, (\psi_j)_{j=1}^N) \in H^2(\mathcal{G}_N) :\ & \phi(-L) = \phi(L) = \psi_1(L) = \cdots = \psi_N(L) = 0, \\ & \phi'(L) - \phi'(-L) = \sum_{j=1}^N \psi_j'(L)\Big\}, \end{aligned} \tag{6.3}$$

is a densely defined symmetric operator with deficiency indices $n_\pm(-\Delta) = 1$*. Therefore,* $(-\Delta, D(-\Delta))$ *has a one-parameter family of self-adjoint extensions defined by* $(-\Delta, D_\gamma)_{\gamma\in\mathbb{R}}$ *with*

$$\begin{aligned} D_\gamma = \Big\{(f, (g_j)_{j=1}^N) \in H^2(\mathcal{G}_N) :\ & f(L) = f(-L) = g_1(L) = \cdots = g_N(L), \\ & f'(L) - f'(-L) = \sum_{j=1}^N g_j'(L) + \gamma g_1(L)\Big\}. \end{aligned} \tag{6.4}$$

Proof The symmetric property of $(-\Delta, D(-\Delta))$ is immediate. Since, $C_c^\infty(-L, L) \oplus \bigoplus_{j=1}^N C_c^\infty(L, +\infty) \subset D(-\Delta)$ we obtain the density property of $D(-\Delta)$ in $L^2(\mathcal{G}_N)$. Now, by following a similar analysis as in Proposition A.6 in [14], it is not difficult to see that the adjoint operator $(-\Delta^*, D(-\Delta^*))$ of $(-\Delta, D(-\Delta))$ is given by

$$\begin{aligned} -\Delta^* = -\Delta, \quad & D(-\Delta^*) = \{(u, (v_j)_{j=1}^N) \in H^2(\mathcal{G}_N) : \\ & u(-L) = u(L) = v_1(L) = \cdots = v_N(L)\}. \end{aligned} \tag{6.5}$$

Next, the deficiency subspaces $\mathcal{D}_\pm = ker(-\Delta^* \mp i)$ have dimension one. Indeed, by using the definition of $D(-\Delta^*)$ in (6.5) we can see that $\mathcal{D}_\pm = span\{(f_{\pm i}, g_{\pm i})\}$ with

$$\begin{cases} f_{+i}(x) = e^{\sqrt{-i}(x+L)} + e^{-\sqrt{-i}(x-L)}, & x \in (-L, L), \quad Im(\sqrt{-i}) > 0 \\ g_{+i}(x) = (e^{\sqrt{-i}(x-L)} + e^{\sqrt{-i}(x+L)})_{j=1}^N, & x > L, \end{cases} \tag{6.6}$$

and

$$\begin{cases} f_{-i}(x) = e^{\sqrt{i}(x+L)} + e^{-\sqrt{i}(x-L)}, & x \in (-L, L), \quad Im(\sqrt{i}) < 0 \\ g_{-i}(x) = (e^{\sqrt{i}(x-L)} + e^{\sqrt{i}(x+L)})_{j=1}^N, & x > L. \end{cases} \tag{6.7}$$

Thus, from Proposition 6.3 we can deduce that $(-\Delta, D(-\Delta))$ has a one-parameter family of self-adjoint extensions $(-\Delta, D_\gamma)_{\gamma\in\mathbb{R}}$ with D_γ defined in (6.4). This finishes the proof. □

6.3 Perron–Frobenius property for δ-interaction Schrödinger operators on the line

In this section we establish the Perron-Frobenius property for the unfold self-adjoint operator $\widetilde{\mathcal{L}}$ in (3.11),

$$\widetilde{\mathcal{L}} = -\partial_x^2 - \omega - (p+1)(2p+1)\psi_{even}^{2p}, \qquad \omega < 0, \tag{6.8}$$

on δ-interaction domains, namely,

$$D_{\delta,\gamma} = \{f \in H^2(\mathbb{R} - \{0\}) \cap H^1(\mathbb{R}) : f'(0+) - f'(0-) = \gamma f(0)\} \tag{6.9}$$

for any $\gamma \in \mathbb{R}$. Here, ψ_{even} is the even extension to whole the line of the tail-soliton profile $\psi_{0,a}(x) = (-\omega)^{1/2p}\psi_0(\sqrt{-\omega}x + a)$, with $x, a > 0$, and ψ_0 defined in (1.7).

We note that there are several results in the literature related to the Perron-Frobenius property for Schrödinger operator $-\Delta + V(x)$ with a external potential. In the case of metric graphs, some results have been obtained depending of the topology (see [23, 29], and reference therein). Here, by convenience of the reader, we give an unified proof of this property for $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})$ with any value of γ.

So, we start with the following two remarks: by Weyl's essential spectrum theorem ([47]), we have that the essential spectrum, $\sigma_{ess}(\widetilde{\mathcal{L}})$, of $\widetilde{\mathcal{L}}$ satisfies $\sigma_{ess}(\widetilde{\mathcal{L}}) = [-\omega, +\infty)$. Moreover, from the extension theory, we can see that the Morse index of $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})$ satisfies $n(\widetilde{\mathcal{L}}) \leqq 1$ for any γ (this a consequence of Proposition 6.4 and that ψ_{even} has a tail-profile, see [9], [13]).

Theorem 6.7 (Perron-Frobenius property) *Consider the family of self-adjoint operators* $(\widetilde{\mathcal{L}}, D_{\delta,\gamma})_{\gamma\in\mathbb{R}}$. *For* γ *fixed, assume that* $\beta = \inf \sigma(\widetilde{\mathcal{L}}) < -\omega$ *is the smallest eigenvalue. Then,* β *is simple, and its corresponding eigenfunction* ζ_β *is positive (after replacing* ζ_β *by* $-\zeta_\beta$ *if necessary) and even.*

Proof This result follows by a slight twist of standard abstract Perron-Frobenius arguments (see Proposition 2 in Albert&Bona&Henry [7]). The basic point in the analysis is to show that the Laplacian operator $-\Delta_\gamma \equiv -\frac{d^2}{dx^2}$ on the domain $D_{\delta,\gamma}$ has its resolvent $R_\mu = (-\Delta_\gamma + \mu)^{-1}$ represented by a positive kernel for some $\mu > 0$ sufficiently large. Namely, for $f \in L^2(\mathbb{R})$

$$R_\mu f(x) = \int_{-\infty}^{+\infty} K(x, y) f(y) dy$$

with $K(x, y) > 0$ for all $x, y \in \mathbb{R}$. By convenience of the reader we show this main point, the remainder of the proof follows the same strategy as in [7]. Thus, for γ fixed, let $\mu > 0$ be sufficiently large (with $-2\sqrt{\mu} < \gamma$ in the case $\gamma < 0$), then from the Krein formula (see Theorem 3.1.2 in [8]) we obtain

$$K(x, y) = \frac{1}{2\sqrt{\mu}}\Big[e^{-\sqrt{\mu}|x-y|} - \frac{\gamma}{\gamma + 2\sqrt{\mu}}e^{-\sqrt{\mu}(|x|+|y|)}\Big].$$

Moreover, for every x fixed, $K(x, \cdot) \in L^2(\mathbb{R})$. Thus, the existence of the integral above is guaranteed by Holder's inequality. Now, since $K(x, y) = K(y, x)$, it is sufficient to show that $K(x, y) > 0$ in the following cases.

(1) Let $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$: for $\gamma \geqq 0$, we obtain from $\frac{\gamma}{\gamma+2\sqrt{\mu}} < 1$ and $|x - y| \leqq |x|+|y|$, that $K(x, y) > 0$. For $\gamma < 0$ and $-2\sqrt{\mu} < \gamma$, it follows immediately $K(x, y) > 0$.

(2) Let $x > 0$ and $y < 0$: in this case,

$$K(x, y) = \frac{1}{\gamma + 2\sqrt{\mu}}e^{-\sqrt{\mu}(x-y)} > 0$$

for any value of γ (where again $-2\sqrt{\mu} < \gamma$ in the case $\gamma < 0$) .

This finishes the proof. □

6.4 Orbital stability criterion

By convenience of the reader, in this subsection we adapted the abstract stability results from Grillakis&Shatah&Strauss in [31, 32] for the case of a looping edge graph and standing waves being positive single-lobe states. This criterion was used in the proof of Theorems 1.5–1.6.

Theorem 6.8 *Suppose that there is C^1-mapping $\omega \to (\Phi_\omega, \Psi_\omega)$ of positive single-lobe states for the NLS model (1.1) on a looping edge graph $\mathcal{G}_N$. We consider the assertions in Theorems 1.3–1.4 associated to the Morse index and the nullity index for the operators $\mathcal{L}_{+,Z}$ and $\mathcal{L}_{-,Z}$. Then, for $Ker(\mathcal{L}_{+,Z}) = \{\mathbf{0}\}$ we have*

1) if $\partial_\omega||(\Phi_\omega, \Psi_\omega)||^2_{L^2(\mathcal{G}_N)} < 0$, then $e^{-i\omega t}(\Phi_\omega, \Psi_\omega)$ is orbitally stable in $\mathcal{E}(\mathcal{G}_N)$,

2) if $\partial_\omega||(\Phi_\omega, \Psi_\omega)||^2_{L^2(\mathcal{G}_N)} > 0$, then $e^{-i\omega t}(\Phi_\omega, \Psi_\omega)$ is orbitally unstable in $\mathcal{E}(\mathcal{G}_N)$.

Acknowledgements The author was supported in part by CNPq/Brazil Grant.

Data availability No data was used for the research described in the article.

References

1. Adami, R., Cacciapuoti, C., Finco, D., Noja, D.: Variational properties and orbital stability of standing waves for NLS equation on a star graph. J. Differ. Equ. **257**, 3738–3777 (2014)
2. Adami, R., Serra, E., Tilli, P.: NLS ground states on graphs. Calc. Var. **54**(1), 743–761 (2015)
3. Adami, R., Cacciapuoti, C., Finco, D., Noja, D.: Stable standing waves for a NLS on star graphs as local minimizers of the constrained energy. J. Differ. Equ. **260**, 7397–7415 (2016)
4. Adami, R., Serra, E., Tilli, P.: Threshold phenomena and existence results for NLS ground states on graphs. J. Funct. Anal. **271**, 201–223 (2016)
5. Adami, R., Serra, E., Tilli, P.: Negative energy ground states for the L^2-critical NLSE on metric graphs. Commun. Math. Phys. **352**, 387–406 (2017)

6. Adami, R., Serra, E., Tilli, P.: Multiple positive bound states for the subcritical NLS equation on metric graphs. Calc. Var. **58**(1,5), 16 (2019)
7. Albert, J.P., Bona, J.L., Henry, D.: Sufficient conditions for stability of solitary-wave solutions of model equations for long waves. Physica D **24**, 343–366 (1987)
8. Albeverio, S., Gesztesy, F., Hoegh-Krohn, R., Holden, H.: Solvable models in quantum mechanics, 2nd edn. AMS Chelsea Publishing, Providence (2005)
9. Angulo, J., Cavalcante, M.: Nonlinear Dispersive Equations on Star Graphs, 32^o Colóquio Brasileiro de Matemática, IMPA, (2019)
10. Angulo, J.: Stability theory for two-lobe states on the tadpole graph for the NLS equation. Nonlinearity **37**(4), 045015 (2024)
11. Angulo, J., Cavalcante, M.: Linear instability of stationary solitons for the Korteweg-de Vries equation on a star graph. Nonlinearity **34**, 3373–3410 (2021)
12. Angulo, J., Goloshchapova, N.: On the orbital instability of excited states for the NLS equation with the δ-interaction on a star graph. Discrete Contin. Dyn. Syst. A. **38**(10), 5039–5066 (2018)
13. Angulo, J., Goloshchapova, N.: Extension theory approach in the stability of the standing waves for the NLS equation with point interactions on a star graph. Adv. Differ. Equ. **23**(11–12), 793–846 (2018)
14. Angulo, J., Plaza, R.: Instability of static solutions of the sine-Gordon equation on a Y-junction graph with δ interaction. J. Nonlinear Sci. **31**, 50 (2021)
15. Angulo, J., Plaza, R.: Instability theory of kink and anti-kink profiles for the sine-Gordon on Josephson tricrystal boundaries. Physica D **427**, 133020 (2021)
16. Angulo, J., Plaza, R.: Unstable kink and anti-kink profiles for the sine-Gordon on a Y -junction graph with δ interaction at the vertex. Math. Z. **300**, 2885–2915 (2022)
17. Ardila, A.H.: Orbital stability of standing waves for supercritical NLS with potential on graphs. Appl. Anal. **99**(8), 1359–1372 (2020)
18. Berezin, F.A., Shubin, M.A.: The Schrödinger Equation, Mathematics and its Applications (Soviet Series), 66. Kluwer Acad. Publ, Dordrecht (1991)
19. Berkolaiko, G., Kuchment, P.: Introduction to Quantum Graphs, Mathematical Surveys and Monographs, 186. Amer. Math. Soc, Providence, RI (2013)
20. Burioni, R., Cassi, D., Rasetti, M., Sodano, P., Vezzani, A.: Bose-Einstein condensation on inhomogeneous complex networks. J. Phys. B At. Mol. Opt. Phys. **34**, 4697–4710 (2001)
21. Cacciapuoti, C.: Existence of the ground state for the NLS with potential on graphs, contemporary mathematics, mathematical problems in quantum. Physics **717**, 155–172 (2018)
22. Cacciapuoti, C., Finco, D., Noja, D.: Topology induced bifurcations for the NLS on the tadpole graph. Phys. Rev. E **91**, 013206 (2015)
23. Cacciapuoti, C., Finco, D., Noja, D.: Ground state and orbital stability for the NLS equation on a general starlike graph with potentials. Nonlinearity **30**(8), 3271–3303 (2017)
24. Cazenave, T.: Semilinear Schrödinger Equations, American Mathematical Society, AMS. Lecture Notes, v. 10, (2003)
25. Chuiko, G.P., Dvornik, O.V., Shyian, S.I., Baganov, Y.A.: A new age-related model for blood stroke volume. Comput. Biol. Med. **79**, 144–148 (2016)
26. Crépeau, E., Sorine, M.: A reduced model of pulsatile flow in an arterial compartment. Chaos Solitons Fractals **34**(2), 594–605 (2007)
27. Exner, P., Keating, J., Kuchment, P., Sunada, T., Teplyaev, A.: Analysis on graphs and its applications, xiii. In Proc. Sympos. Pure Math., 77, Amer. Math. Soc., Providence, RI, (2008)
28. Exner, P.: Magnetoresonance on a lasso graph. Found. Phys. **27**, 171 (1997)
29. Exner, P., Jex, M.: On the ground state of quantum graphs with attractive δ-coupling. Phys. Lett. A **376**, 713–717 (2012)
30. Fidaleo, F.: Harmonic analysis on inhomogeneous amenable networks and the Bose-Einstein condensation. J. Stat. Phys. **160**, 715–759 (2015)
31. Grillakis, M., Shatah, J., Strauss, W.: Stability theory of solitary waves in the presence of symmetry, I. J. Funct. Anal. **74**, 160–197 (1987)
32. Grillakis, M., Shatah, J., Strauss, W.: Stability theory of solitary waves in the presence of symmetry, II. J. Funct. Anal. **94**, 308–348 (1990)
33. Kairzhan, A., Pelinovsky, D.E., Goodman, R.: Drift of spectrally stable shifted states on star graphs. SIAM J. Appl. Dyn. Syst. **18**, 1723–1755 (2019)
34. Kairzhan, A., Marangell, R., Pelinovsky, D.E., Xiao, K.: Existence of standing waves on a flower graph. J. Differ. Equ. **271**, 719–763 (2021)
35. Kairzhan, A., Noja, D., Pelinovsky, D.E.: Standing waves on quantum graphs. J. Phys. A Math. Theor. **55**, 243001 (2022). ((**51pp**))

36. Kong, Q., Wu, H., Zettl, A.: Dependence of the n-th Sturm-Liouville eigenvalue problem. J. Differ. Equ. **156**, 328–354 (1999)
37. Kuchment, P.: Quantum graphs, I. Some basic structures, Waves Random Media **14**, 107–128 (2004)
38. Marzuola, J.L., Pelinovsky, D. E.: Ground state on the dumbbell graph, Appl. Math. Res. Express, 98–145 (2016)
39. Mugnolo, D.: Mathematical Technology of Networks, Bielefeld, December 2013, Springer Proceedings in Mathematics & Statistics 128, (2015)
40. Naimark, M.A.: Linear Differential Operators, (Russian), 2nd edition, revised and augmented., Izdat. “Nauka”, Moscow, (1969)
41. Nakajima, K., Onodera, Y.: Logic design of Josephson network. J. Appl. Phys. **47**(4), 1620–1627 (1976)
42. Nakajima, K., Onodera, Y.: Logic design of Josephson network. II. J. Appl. Phys. **49**(5), 2958–2963 (1978)
43. Noja, D.: Nonlinear Schrödinger equation on graphs: recent results and open problems. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **372**, 20130002 (2014). (**20 pp**)
44. Noja, D., Pelinovsky, D.E.: Standing waves of the quintic NLS equation on the tadpole graph. Calc. Var. **59**, 173 (2020)
45. Noja, D., Pelinovsky, D.E., Shaikhova, G.: Bifurcations and stability of standing waves in the nonlinear Schrödinger equation on the tadpole graph. Nonlinearity **28**, 2343–2378 (2015)
46. Pankov, A.: Nonlinear Schrödinger equations on periodic metric graphs. Discrete Contin. Dyn. Syst.-A **38**(4), 697–714 (2018)
47. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Analysis of operators, Academic Press, New York, IV (1975)
48. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Fourier Analysis and Self-Adjoitness, Academic Press, New York, II (1978)
49. Zettl, A.: Sturm-Liouville Theory, Mathematical Surveys and Monographs (SURV), 121, AMS, (2005)