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BAYES AND MINIMAX PREDICTION IN FINITE POPULATIONS

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Abstract: We consider Bayesian and minimax prediction of some finite population quantities. The usual best linear unbiased predictor of the population total T is shown to be minimax under normality and the squared error loss function. A general Bayesian predictor is derived for the population variance S_y^2. A minimax predictor of S_y^2 is presented for the location model. Bayes, minimax and best unbiased prediction is also considered for the finite population regression coefficient β_N.

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Key Words: *Finite populations; Bayes predictors; Minimax predictors; Superpopulations.*

1. Introduction

Let $\mathcal{P} = \{1, \cdots, N\}$ denote a finite population of N units, where N is known. Associated with the k-th unit of $\mathcal{P}$, there are $p+1$ quantities, $y_k, x_{k1}, \cdots, x_{kp}$, where all but y_k are known, $k = 1, \cdots, N$. Let $\mathbf{y} = (y_1, \cdots, y_N)$, and $\mathbf{X} = (\mathbf{X}_1, \cdots, \mathbf{X}_N)'$, where $\mathbf{X}_k = (x_{k1}, \cdots, x_{kp})'$, $k = 1, \cdots, N$. Relating the two sets of variables, we consider the linear model

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \tag{1}$$

where $\mathbf{e} \sim N(0, \mathbf{V})$. Let $\psi = (\beta, \mathbf{V})$ denote the parameter in model (1). The Bayes model assumes that β is a normal random vector with mean vector $\mathbf{b}$ and covariance matrix $\mathbf{B}$, that is,

$$\beta \sim N(\mathbf{b}, \mathbf{B}). \tag{2}$$

The Bayes model defined by (1) and (2) is designated by ψ_B.

Let $\theta(\mathbf{y})$ be a population quantity of interest. Examples of such quantities are: the population total, $T = \sum_{i=1}^{N} y_i$; the population variance $S_y^2 = \sum_{i=1}^{N} (y_i - \bar{y}_N)^2/N$, where $\bar{y}_N = T/N$ is the population mean and the finite population regression coefficient

$$\beta_N = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}. \tag{3}$$

A sample $\mathbf{s}$ of size n is selected from $\mathcal{P}$ according to some specified sampling plan in order to obtain information on some $\theta(\mathbf{y})$. Let $\mathbf{r} = \mathcal{P} - \mathbf{s}$. After the sample $\mathbf{s}$ has been selected, we may reorder the elements of $\mathbf{y}$ so that we have the corresponding partitions of $\mathbf{y}$, $\mathbf{X}$ and $\mathbf{V}$; that is,

$$\begin{pmatrix} \mathbf{y}_s \\ \mathbf{y}_r \end{pmatrix}, \quad \begin{pmatrix} \mathbf{X}_s \\ \mathbf{X}_r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{V}_s & \mathbf{V}_{sr} \\ \mathbf{V}_{rs} & \mathbf{V}_r \end{pmatrix}.$$

In Section 2 we present the general framework for Bayesian prediction, with respect to the normal model ψ_B. Sections 3 and 4 are devoted to the Bayesian prediction of the population total, T, the population regression coefficient β_N and the population variance S_y^2. Section 5 presents basic results for minimax prediction of population quantities. In

Section 6 we present examples of Bayes and minimax predictors of T, S_y^2 and β_N for various regression models.

2. Bayesian Prediction of Population Quantities

Let $L(\hat{\theta}(\mathbf{y}_s); \theta(\mathbf{y}))$ be a loss function for predicting $\theta(\mathbf{y})$ by $\hat{\theta}(\mathbf{y}_s)$. For a Bayes model ψ_B, the ψ_B*-Bayes prediction* risk of $\hat{\theta}(\mathbf{y}_s)$ is defined as

$$E_{\psi_B}[L(\hat{\theta}(\mathbf{y}_s); \theta(\mathbf{y}))].$$

Notice that the expectation operator in the above expression is performed with respect to the joint distribution of $\mathbf{y}$ and ψ. The Bayes predictor is the one minimizing the ψ_B-Bayes prediction risk. In particular, for the squared error loss function, the Bayes predictor of $\theta(\mathbf{y})$ is

$$\hat{\theta}_B(\mathbf{y}_s) = E_{\psi_B}[\theta(\mathbf{y}) \mid \mathbf{y}_s]. \tag{4}$$

The Bayes prediction risk is

$$E_{\psi_B}[(\hat{\theta}_B(\mathbf{y}_s) - \theta(\mathbf{y}))^2] = E_{\psi_B}\{\mathrm{Var}_{\psi_B}[\theta(\mathbf{y}) \mid \mathbf{y}_s]\}. \tag{5}$$

The next theorem specifies the Bayes-predictive distribution of $\mathbf{y}_r$ given $\mathbf{y}_s$, for the case where the covariance matrix $\mathbf{V}$ is known. (Bolfarine et al., 1987).

THEOREM 1. *Under the Bayesian model ψ_B, the Bayes predictive distribution of $\mathbf{y}_r$ given $\mathbf{y}_s$, is multivariate normal with mean vector*

$$E_{\psi_B}[\mathbf{y}_r \mid \mathbf{y}_s] = \mathbf{X}_r\hat{\beta}_B + \mathbf{V}_{rs}\mathbf{V}_s^{-1}(\mathbf{y}_s - \mathbf{X}_s\hat{\beta}_B), \tag{6}$$

and covariance matrix

$$\begin{aligned} Var_{\psi_B}[\mathbf{y}_r \mid \mathbf{y}_s] = {} & \mathbf{V}_r - \mathbf{V}_{rs}\mathbf{V}_s^{-1}\mathbf{V}_{sr} \\ & + (\mathbf{X}_r - \mathbf{V}_{rs}\mathbf{V}_s^{-1}\mathbf{X}_s)(\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s + \mathbf{B}^{-1})^{-1}(\mathbf{X}_r - \mathbf{V}_{rs}\mathbf{V}_s^{-1}\mathbf{X}_s)', \end{aligned} \tag{7}$$

where

$$\hat{\beta}_B = (X_s' V_s^{-1} X_s + B^{-1})^{-1}(X_s' V_s^{-1} y_s + B^{-1} b). \tag{8}$$

A result of Royall and Pfeffermann (1982) is a special case of the above theorem, for a "non-informative" prior, obtained as the limit of $N(b; B)$ when $B^{-1} \to 0$.

Suppose now that $V = \sigma^2 W$, where W is known and $W_{rs} = 0$, but σ^2 is unknown.

THEOREM 2. *Under the "noninformative" prior distribution of (β, σ^2), according to which, the prior density is*

$$\zeta(\beta, \sigma^2) \propto \frac{1}{\sigma} \tag{9}$$

the following results hold:

$$E_{\psi_B}[y_r \mid y_s] = X_r \hat{\beta}_s \tag{10}$$

$$Var_{\psi_B}[y_r \mid y_s] = \frac{n-p}{n-p-2}\hat{\sigma}^2\{W_r + X_r(X_s' W_s^{-1} X_s)^{-1} X_r'\}, \tag{11}$$

where

$$\hat{\beta}_s = (X_s' W_s^{-1} X_s)^{-1} X_s' W_s^{-1} y_s, \tag{12}$$

$$\hat{\sigma}^2 = (y_s - X_s \hat{\beta}_s)' W_s^{-1}(y_s - X_s \hat{\beta}_s)/(n-p) \tag{13}$$

PROOF: Formula (10) is obtained from (6) and (8) by letting $B^{-1} \to 0$, since $V_{rs} = 0$. Also, from formula (7) we obtain

$$\mathrm{Var}_{\psi_B}[y_r \mid y_s, \sigma] = \sigma^2(W_r + X_r(X_s' W_s^{-1} X_s)^{-1} X_r').$$

Furthermore,

$$\begin{aligned}\mathrm{Var}_{\psi_B}[y_r \mid y_s] &= E_{\psi_B}\{\mathrm{Var}_{\psi}[y_r \mid y_s, \sigma] \mid y_s\} + \mathrm{Var}_{\psi_B}\{E_{\psi}[y_r \mid y_s, \sigma] \mid y_s\} \\ &= E_{\psi_B}[\sigma^2 \mid y_s]\{W_r + X_r(X_s' W_s^{-1} X_s)^{-1} X_r'\}.\end{aligned}$$

Finally, the result follows from the fact that,

$$E_{\psi_B}[\sigma^2 \mid y_s] = \frac{n-p}{n-p-2}\hat{\sigma}^2.$$

3. Bayes Prediction of Linear Quantities

Let $\theta_L = \mathbf{l}'\mathbf{y}$, where $\mathbf{l}' = (\mathbf{l}'_s, \mathbf{l}'_r)$ is known, be any linear quantity. The next theorem follows immediately from (4) and (5).

THEOREM 3. *For any linear quantity $\theta_L = \mathbf{l}'\mathbf{y}$, the Bayes predictor under the squared error loss and any ψ_B model for which $Var_{\psi_B}[\mathbf{y}_r \mid \mathbf{y}_s]$ exists, is*

$$\hat{\theta}_{BL}(\mathbf{y}_s) = \mathbf{l}'_s\mathbf{y}_s + \mathbf{l}'_r E_{\psi_B}[\mathbf{y}_r \mid \mathbf{y}_s].$$

The Bayes risk of this predictor is

$$E_{\psi_B}[(\hat{\theta}_{BL}(\mathbf{y}_s) - \theta_L(\mathbf{y}))^2] = \mathbf{l}'_r \; Var_{\psi_B}[\mathbf{y}_r \mid \mathbf{y}_s]\mathbf{l}_r.$$

COROLLARY 1. *The Bayes predictor of the population total T under the normal regression model ψ_B and squared error loss function is*

$$\hat{T}_B(\mathbf{y}_s) = \mathbf{1}'_s\mathbf{y}_s + \mathbf{1}'_r\mathbf{X}_r\hat{\beta}_B,$$

where

$$\hat{\beta}_B = (\mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{X}_s + \mathbf{B}^{-1})^{-1}(\mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{y}_s + \mathbf{B}^{-1}\mathbf{b}). \tag{14}$$

The corresponding Bayes prediction risk is

$$E_{\psi_B}[\hat{T}_B(\mathbf{y}_s) - T(\mathbf{y})]^2 = \mathbf{1}'_r\mathbf{V}_r\mathbf{1}_r + \mathbf{1}'_r\mathbf{X}_r(\mathbf{X}'_s\mathbf{V}_s^{-1}\mathbf{X}_s + \mathbf{B}^{-1})^{-1}\mathbf{X}'_r\mathbf{1}_r. \tag{15}$$

COROLLARY 2. *The Bayes predictor of T under the normal regression model ψ_B with $\mathbf{V} = \sigma^2\mathbf{W}$, where $\mathbf{W}$ is known and $\mathbf{W}_{sr} = 0$, and noninformative prior (9) on (β, σ^2) is*

$$\hat{T}_B = \mathbf{1}'_s\mathbf{y}_s + \mathbf{1}'_r\mathbf{X}_r\hat{\beta}_s. \tag{16}$$

The Bayes prediction risk of (16) is

$$E_{\psi_B}[\hat{T}_B - T]^2 = \frac{\nu}{\nu - 2}\hat{\sigma}^2\{\mathbf{1}'_r\mathbf{W}_r\mathbf{1}_r + \mathbf{1}'_r\mathbf{X}_r(\mathbf{X}_s\mathbf{W}_s^{-1}\mathbf{X}_s)^{-1}\mathbf{X}'_r\mathbf{1}_r\}. \tag{17}$$

Another type of vector valued linear quantity is the population regression coefficient β_N. The estimation of β_N was studied by Konijn (1962), Fuller (1975), Hartley and Sielken (1975), Sarndal (1982) and others.

In relation to the model $\psi = (\beta, V)$, with $V_{sr} = 0$, one can write

$$\beta_N = \mathbf{A}_s \hat{\beta}_s + \mathbf{A}_r \beta_r, \tag{18}$$

where

$$\mathbf{A}_s = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s,$$

$$\mathbf{A}_r = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}_r'\mathbf{V}_r^{-1}\mathbf{X}_r$$

and where $\hat{\beta}_s$ is given in (12). β_r is a weighted least squares "estimator" of β based on $\mathbf{X}_r$, $\mathbf{V}_r$ and $\mathbf{y}_r$. Since $\mathbf{y}_r$ has not been observed, β_r is treated as an unknown vector valued quantity. Notice that

$$\mathbf{A}_r + \mathbf{A}_s = \mathbf{I}_p,$$

where I_p is the p-dimensional identity matrix.

We consider here a Bayes predictor for β_N with respect to a ψ_B-generalized prediction risk given by

$$R_{G\psi_B}[\hat{\beta}_N, \beta_N] = E_{\psi_B}[\lambda'(\hat{\beta}_N - \beta_N)(\hat{\beta}_N - \beta_N)'\lambda], \tag{19}$$

for some vector λ. A Bayes predictor with respect to (19) is

$$\hat{\beta}_{BN} = E_{\psi_B}[\beta_N \mid \mathbf{y}_s]. \tag{20}$$

The corresponding Bayes risk is

$$R_{G\psi_B}[\hat{\beta}_{BN}, \beta_N] = \lambda' E_{\psi_B}[\mathrm{Var}_{\psi_B}[\beta_N \mid \mathbf{y}_s]]\lambda. \tag{21}$$

THEOREM 4. *Consider the Bayes normal model ψ_B with $\mathbf{V}_{sr} = 0$. The Bayes predictor of β_N with respect to the generalized risk (19) is*

$$\hat{\beta}_{BN} = \mathbf{A}_s \hat{\beta}_s + \mathbf{A}_r \hat{\beta}_B \tag{22}$$

The corresponding ψ_B-generalized Bayes prediction risk is

$$R_{G\psi_B}[\hat{\beta}_{BN}, \beta_N] = \lambda'(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}_r'\mathbf{V}_r^{-1}\Sigma_r\mathbf{V}_r^{-1}\mathbf{X}_r(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\lambda, \tag{23}$$

where $\hat{\beta}_B$ and $\Sigma_r = \mathrm{Var}_{\psi_B}[\mathbf{y}_r \mid \mathbf{y}_s]$ are given by (8) and (7), respectively.

Notice that (22) yields β_s as a limit of Bayes predictors, when $B^{-1} \to 0$.

4. Bayes Prediction of S_y^2

We present in the present section the general formula of the Bayes predictor of S_y^2, under the squared error loss, for the normal regression model ψ_B. We start with the expression

$$S_y^2 = \frac{n}{N}s_y^2 + \left(1 - \frac{n}{N}\right)\left[S_{ry}^2 + \frac{n}{N}(\bar{y}_s - \bar{y}_r)^2\right], \tag{24}$$

where $\bar{y}_s$ and s_y^2 are the mean and variance of $\mathbf{y}_s$; $\bar{y}_r$ and S_{ry}^2 are the mean and variance of $\mathbf{y}_r$. Accordingly, we should derive the Bayes predictor of $S_{ry}^2 + \frac{n}{N}(\bar{y}_s - \bar{y}_r)^2$. Let $\eta_r(\mathbf{y}_s) = E[\mathbf{y}_r \mid \mathbf{y}_s]$ as in (6), and let Σ_r be the covariance matrix (7). The Bayes predictive distribution of $\bar{y}_r$, given $\mathbf{y}_s$, is normal with mean

$$h(\mathbf{y}_s) = \frac{1}{N-n}\mathbf{1}_r'\eta_r(\mathbf{y}_s) \tag{25}$$

and variance

$$D_r^2 = \frac{1}{(N-n)^2}\mathbf{1}_r'\Sigma_r\mathbf{1}_r. \tag{26}$$

It follows that the Bayes predictive distribution of $\frac{n}{N}(\bar{y}_r - \bar{y}_s)^2$, given $\mathbf{y}_s$, is like that of

$$\frac{n}{N}D_r^2\chi^2[1;\lambda],$$

where

$$\lambda = \frac{(h(\mathbf{y}_s) - \bar{y}_s)^2}{2D_r^2}.$$

$\chi^2[1;\lambda]$ denotes the noncentral chi squared distribution with 1 degree of freedom and noncentrality parameter λ. The above implies that,

$$E_{\psi_B}\left[\frac{n}{N}(\bar{y}_r-\bar{y}_s)^2 \mid \mathbf{y}_s\right]=\frac{n}{N}(D_r^2+(h(\mathbf{y}_s)-\bar{y}_s)^2). \tag{27}$$

Using the theorem about the expected value of a symmetric quadratic form (see, Seber, 1977, pp. 13), we obtain

$$E[S_{ry}^2 \mid \mathbf{y}_s]=tr[E_r\Sigma_r]+\eta_r(\mathbf{y}_s)'E_r\eta_r(\mathbf{y}_s), \tag{28}$$

where

$$E_r=\frac{1}{N-n}\left(\mathbf{I}_r-\frac{1}{N-n}\mathbf{J}_r\right),$$

$\mathbf{I}_r$ is the identity matrix of dimension $N-n$ and $\mathbf{J}_r=\mathbf{1}_r\mathbf{1}_r'$. Substituting (27) and (28) above, we obtain

THEOREM 5. *The Bayes predictor of S_y^2 under model ψ_B and squared error loss is*

$$\begin{aligned}\hat{S}_{By}^2=E_{\psi_B}[S_y^2 \mid \mathbf{y}_s]=\frac{n}{N}s_y^2+\left(1-\frac{n}{N}\right)\{&tr[E_r\Sigma_r]+\eta_r(\mathbf{y}_s)'E_r\eta_r(\mathbf{y}_s)\\&+\frac{n}{N}(D_r^2+(h(\mathbf{y}_s)-\bar{y}_s)^2)\}.\end{aligned} \tag{29}$$

5. Minimax Prediction Of Population Quantities

The notion of *minimax predictor* is introduced here. Let $\hat{\theta}$ be a member of a class $\mathcal{H}$ of predictors. Assume that the parameter ψ of the superpopulation model belongs to a parameter space Ψ, such that

$$\rho^*(\hat{\theta})=\sup_{\psi\in\Psi}R_\psi[\hat{\theta},\theta]<\infty.$$

A predictor $\hat{\theta}_*$ is called minimax in $\mathcal{H}$, if

$$\rho^*(\hat{\theta}_*)=\inf_{\hat{\theta}\in\mathcal{H}}\rho^*(\hat{\theta}),$$

and if

$$\inf_{\hat\theta\in\mathcal{H}} \sup_{\psi\in\Psi} R_\psi[\hat\theta,\theta] = \sup_{\psi\in\Psi} \inf_{\hat\theta\in\mathcal{H}} R_\psi[\hat\theta,\theta].$$

We state here, without proof, several key results on minimax prediction in finite populations for which there are equivalent results in minimax estimation theory (see Zacks (1971) Ch. 6-8; Lehmann (1983) Ch. 4). Let $\zeta(\psi)$ be a prior density (or probability function) over Ψ, and let $\hat\theta_B(\mathbf{y}_s;\zeta)$ be the corresponding Bayes predictor, having a Bayes prediction risk $\rho(\hat\theta_B,\zeta)$.

Result 1. *If $\hat\theta_B(\mathbf{y}_s;\zeta)$ is a Bayes predictor and if $R_\psi[\hat\theta_B(\cdot;\zeta),\theta]$ is independent of ψ, then, $\hat\theta_B(\mathbf{y}_s;\zeta)$ is minimax.*

Result 2. *Let $\{\zeta_k; k = 1,2,\cdots\}$ be a sequence of prior densities over Ψ, and let $\{\hat\theta_B(\cdot;\zeta_k),\ k = 1,2,\cdots\}$ and $\{\rho(\hat\theta_B;\zeta_k),\ k = 1,2,\cdots\}$ be the corresponding Bayes predictors and Bayes risks. If $\hat\theta(\mathbf{y}_s)$ is a predictor such that*

$$\sup_{\psi\in\Psi} R_\psi[\hat\theta,\theta] \le \limsup_{k\to\infty} \rho(\hat\theta_B;\zeta_k)$$

then $\hat\theta$ is a minimax predictor.

Result 3. *If the prediction risk of $\hat\theta$ is constant over Ψ, and if there exists a sequence $\{\zeta_k; k = 1,2,\cdots\}$ of prior distributions over Ψ, such that*

$$\lim_{k\to\infty} \rho(\hat\theta_B;\zeta_k) = R_\psi[\hat\theta,\theta] = \rho^*(\hat\theta),$$

then $\hat\theta$ is a minimax predictor.

As shown by Royall (1976), the minimum variance linear unbiased predictor of the population total, $\hat T_{\mathrm{BLUP}}$, is of the form

$$\hat T_{\mathrm{BLUP}} = \mathbf{1}_s'\mathbf{y}_s + \mathbf{1}_r'[X_r\hat\beta_s + V_{rs}V_s^{-1}(\mathbf{y}_s - X_s\hat\beta_s)].$$

In the following theorem we prove that, when $V_{rs} = 0$, the BLUP of T is a minimax predictor.

THEOREM 6. *Consider the normal superpopulation Bayes model* ψ_B, *with* $\mathbf{V}_{rs} = 0$. *The minimax predictor of* T *with respect to the squared error loss is*

$$\hat{T}_M = \mathbf{1}_s'\mathbf{y}_s + \mathbf{1}_r'\mathbf{X}_r\hat{\beta}_s, \tag{30}$$

with prediction risk

$$E_\psi[\hat{T}_M - T]^2 = \mathbf{1}_r'\mathbf{V}_r\mathbf{1}_r + \mathbf{1}_r'\mathbf{X}_r(\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}\mathbf{X}_r'\mathbf{1}_r. \tag{31}$$

PROOF: Consider a sequence of prior distributions $N(\mathbf{b}; \mathbf{B}_k)$ such that $||\mathbf{B}_k|| = k$, where the norm of the covariance matrix $\mathbf{B}$ is $||\mathbf{B}|| = \sum_{i=1}^{p} B_{ii}$. The corresponding Bayes predictors $\hat{T}_{B_k}$ converge, as $k \to \infty$, to the Best Linear Unbiased Predictor, BLUP, (see Royall, 1976)

$$\hat{T}_{\mathrm{BLUP}} = n\bar{y}_s + \mathbf{1}_r'\mathbf{X}_r\hat{\beta}_s.$$

Moreover, from (5) and (15), the Bayes prediction risks $\rho(\hat{T}_{B_k}; \mathbf{b}, \mathbf{B}_k)$ converge, as $k \to \infty$, to the predicition risk of $\hat{T}_{\mathrm{BLUP}}$, namely

$$E_\psi[\hat{T}_{\mathrm{BLUP}} - T]^2 = \mathbf{1}_r'\mathbf{V}_r\mathbf{1}_r + \mathbf{1}_r'\mathbf{X}_r(\mathbf{X}_s'\mathbf{V}_s^{-1}\mathbf{X}_s)^{-1}\mathbf{X}_r'\mathbf{1}_r.$$

Since this prediction risk is independent of β, $\hat{T}_{\mathrm{BLUP}}$ is, according to Result 2, a minimax predictor of T.

The following result from the theory of minimaxity allows us to show that the above minimax predictors of T are minimax also for distribution free superpopulation models, with bounded variances (could be unknown).

RESULT 4. *Let* $y_1, \cdots, y_N$ *be jointly distributed according to the distribution* F, *belonging to a family of distributions* $\mathcal{F}_1$. *Suppose that* $\hat{T}_M$ *is a minimax predictor of* T *when* $F \in \mathcal{F}_0 \subset \mathcal{F}_1$. *If*

$$\sup_{F\in\mathcal{F}_0} E_F[\hat{T}_M - T]^2 = \sup_{F\in\mathcal{F}_1} E_F[\hat{T}_M - T]^2,$$

then $\hat{T}_{M}$. *is minimax for* $\mathcal{F}_1$.

THEROEM 7. *Under the superpopulation model (1), if the diagonal elements of* $\mathbf{V}$ *belong to a closed interval* $[0, M]$, $0 < M < \infty$, *then* $\hat{T}_M$ *given by (30) is minimax.*

Results 2 and 4 can be extended to vector valued linear predictors, by considering the ψ-generalized risk function (19). We obtain then that $\hat{\beta}_s$ is a minimax predictor of β_N, with respect to (19). Minimax predictor of S_y^2 can be obtained from (29), by letting $B^{-1} \to 0$, as will be shown in some examples.

6. Examples

In the present section we provide several examples in which Bayes and minimax predictors are derived for some special normal regression models.

Example 1. Consider the location model under normality in which $\mathbf{V} = \sigma^2\mathbf{I}$, where σ^2 is known, $\mathbf{X} = \mathbf{1}_N$, β is a scalar with prior distribution $N(b; B)$. The Bayes estimator of β, $\hat{\beta}_B$ is given by

$$\hat{\beta}_B = \frac{\sum_{i \in s} y_i/\sigma^2 + b/B}{n/\sigma^2 + 1/B}.$$

Thus, in this special case, $\eta_r(\mathbf{y}_s) = \hat{\beta}_B \mathbf{1}_r$ and

$$\Sigma_r = \sigma^2(\mathbf{I}_r + \frac{1}{n + \sigma^2/B}\mathbf{J}_r).$$

Moreover, $h(\mathbf{y}_s) = \hat{\beta}_B$,

$$D_r^2 = \frac{\sigma^2}{N-n}\left(1 + \frac{N-n}{n+\sigma^2/B}\right), \; tr[E_r\Sigma_r] = \sigma^2\left(1 - \frac{1}{N-n}\right)$$

and

$$\eta_r(\mathbf{y}_s)' E_r \eta_r(\mathbf{y}_s) = 0.$$

Substituting all these terms into (29), we obtain that the Bayes predictor of S_y^2 is

$$\hat{S}^2_{B_y} = \frac{n}{N}s_y^2 + \left(1-\frac{n}{N}\right)\sigma^2\left[1-\frac{\sigma^2}{(N-n)B}\left(\frac{1}{n}-\frac{1}{N}\right)\left(1+\frac{\sigma^2}{nB}\right)^{-1}+\frac{n}{N}\left(\frac{\hat{\beta}_B-\bar{y}_s}{\sigma}\right)^2\right]. \tag{32}$$

In the case of "noninformative" prior we obtain as a limiting case (when $B \to \infty$),

$$\hat{S}_{My} = \frac{n}{N}s_y^2 + \left(1-\frac{n}{N}\right)\sigma^2. \tag{33}$$

In addition, if σ^2 is also unknown, the noninformative prior (9) yields the Bayes predictor

$$\hat{S}_y = \frac{N-3}{N}\,\frac{n}{n-3}\,s_y^2.$$

This predictor was derived by Ericson (1969) and Zacks and Solomon (1981).

We show here that the predictor (33) of S_y^2 is minimax, for the squared error loss. Under this model, the unknown parameter is β (σ^2 is known). For this purpose we derive the Bayes prediction risk $\rho(\hat{S}^2_{B_y}; b, B)$. As shown above, the Bayes predictive distribution of $\mathbf{y}_r$, given $\mathbf{y}_s$, is $N(\hat{\beta}_B\mathbf{1}_r;\Sigma_r)$, where $\Sigma_r = \sigma^2(\mathbf{I}_r+\mathbf{J}_r/(n+\sigma^2/B))$. The Bayes prediction risk of $\hat{S}^2_{By}$ is the expected value of the posterior variance of S_y^2, i.e.,

$$\rho(\hat{S}^2_{By}; b, B) = E_{\psi_B}\left\{\mathrm{Var}_{\psi_B}\left[\left(1-\frac{n}{N}\right)\left[S^2_{ry}+\frac{n}{N}(\bar{y}_r-\bar{y}_s)^2 \mid \mathbf{y}_s\right]\right]\right\}.$$

Let $\mathbf{y}'\mathbf{A}\mathbf{y}$ be a symmetric quadratic form and $\mathbf{1}'\mathbf{y}$ be a linear form. We have

(i) $\mathbf{y}'\mathbf{A}\mathbf{y} \sim \chi^2[p;\lambda]$ if, and only if $\mathbf{A}\Sigma$ is idempotent of rank p. Moreover, $\lambda = \frac{1}{2}\mu'\mathbf{A}\mu$.

(ii) If $\mathbf{A}\Sigma\mathbf{1} = 0$, then $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{1}'\mathbf{y}$ are independent.

Accordingly, since $S^2_{ry} = \mathbf{y}_r'E_r\mathbf{y}_r/(N-n)$, with $E_r = (\mathbf{I}_r - \mathbf{J}_r/(N-n))$, and since

$$\frac{1}{\sigma^2}E_r\Sigma_r = (\mathbf{I}_r - \mathbf{J}_r/(N-n))\left(\mathbf{I}_r + \frac{1}{n+\sigma^2/B}\mathbf{J}_r\right)$$

is idempotent of rank $N-n$, the Bayes predictive distribution of S^2_{ry}, given $\mathbf{y}_s$, is like that of $\frac{\sigma^2}{(N-n)}\chi^2[N-n-1]$. Indeed,

$$\lambda = \frac{1}{2\sigma^2}\hat{\beta}_B^2\mathbf{1}_r'E_r\mathbf{1}_r = 0.$$

Hence,

$$\text{Var}_{\psi_B}[S^2_{ry} \mid \mathbf{y}_s] = \frac{2\sigma^2}{(N-n)^2}(N-n-1).$$

Moreover, since $\bar{y}_r = \mathbf{1}'_r \mathbf{y}_r/(N-n)$, and $E_r\Sigma_r\mathbf{1}_r = 0$, S^2_{ry} and $\bar{y}_r$ are conditionally independent, given $\mathbf{y}_s$. Thus,

$$\text{Cov}_{\psi_B}[S^2_{ry}(\bar{y}_s - \bar{y}_r)^2 \mid \mathbf{y}_s] = 0.$$

It remains to compute $\text{Var}_{\psi_B}[(\bar{y}_s - \bar{y}_r)^2 \mid \mathbf{y}_s]$. The Bayes predictive distribution of $\bar{y}_r - \bar{y}_s$, given $\mathbf{y}_s$, is normal with mean and variance given, repsectively, by

$$\hat{\beta}_B - \bar{y}_s \quad \text{and} \quad \frac{\sigma^2}{N-n} + \frac{\sigma^2}{n+\sigma^2/B}.$$

Therefore,

$$(\bar{y}_r - \bar{y}_s)^2 \mid \mathbf{y}_s \sim \sigma^2\left(\frac{1}{N-n} + \frac{1}{n+\sigma^2/B}\right)\chi^2\left[1, \frac{(\hat{\beta}_B - \bar{y}_s)^2}{2\sigma^2\left(\frac{1}{N-n} + \frac{1}{n+\sigma^2/B}\right)}\right]. \tag{34}$$

It follows that

$$\text{Var}_{\psi_B}[(\bar{y}_r - \bar{y}_s)^2 \mid \mathbf{y}_s] = 2\sigma^4\left(\frac{1}{N-n} + \frac{1}{n+\sigma^2/B}\right)^2\left(1 + \frac{2(\hat{\beta}_B - \bar{y}_s)^2}{\sigma^2\left(\frac{1}{N-n} + \frac{1}{n+\sigma^2/B}\right)}\right). \tag{35}$$

Taking the expected value of (35) with respect to the marginal distribution of $\mathbf{y}_s$, we obtain from (32)-(34) that, the Bayes prediction risk of $\hat{S}^2_{By}$ is

$$\begin{aligned}\rho(\hat{S}^2_{By}; b, B) = \left(1 - \frac{n}{N}\right)^2 \frac{2\sigma^4}{(N-n)^2}\Bigg\{ & N-n-1 \\ & + \left(\frac{n}{N}\right)^2\left(1 + \frac{N-n}{n+\sigma^2/B}\right)^2\left(1 + \frac{2(N-n)}{\sigma^2\left(1 + \frac{N-n}{n+\sigma^2/B}\right)\left(\frac{\sigma^2}{n} + B\right)}\right)\Bigg\}.\end{aligned}$$

Hence,

$$\lim_{B\to\infty} \rho(\hat{S}^2_{By}; b, B) = \frac{2\sigma^4(N-n)}{N^2}. \tag{36}$$

Finally, the right hand side of (36) is the risk function of the predictor (33), and is independent of β. Hence, according to Result 2, $\hat{S}^2_{M_y}$ is a minimax predictor of S^2_y.

Example 2. Suppose that the superpopulation model (1) is such that $\mathbf{X} = (x_1, \cdots, x_N)'$ and that $\mathbf{V} = \sigma^2 \mathrm{diag}(x_1, \cdots, x_N)$, where σ^2 is known. Considering a noninformative prior on β, it follows that the posterior distribution of β is normal with mean and variance given, respectively, by

$$\hat{\beta}_s = \frac{\bar{y}_s}{\bar{x}_s} \quad \text{and} \quad V(\hat{\beta}) = \frac{\sigma^2}{\sum_{i \in s} x_i}.$$

It can be shown, after some algebraic manipulations that

$$\begin{aligned} D_r^2 &= \frac{\sigma^2 N}{n(N-n)} \frac{\bar{x}\bar{x}_r}{\bar{x}_s}, \\ h(\mathbf{y}_s) - \bar{y}_s &= \hat{\beta}_s \bar{x}_r - \bar{y}_s, \\ \eta_r(\mathbf{y}_s) = \mathbf{X}_r \hat{\beta}_s, &\quad \eta_r'(\mathbf{y}_s) E_r \eta_r(\mathbf{y}_s) = \hat{\beta}_s^2 S_{rx}^2, \\ tr[E_r \Sigma_r] &= \left(1 - \frac{1}{N-n}\right) \bar{x}_r + \frac{S_{rx}^2}{n\bar{x}_s}, \end{aligned}$$

where $\bar{x}$ is the mean of $\mathbf{X}$, $\bar{x}_s$ is the mean of $\mathbf{X}_s$, $\bar{x}_r$ and S_{rx}^2 are the mean and variance of $\mathbf{X}_r$. Collecting all the above results, it follows from (29) that

$$\begin{aligned} \hat{S}_{By}^2 = \frac{n}{N} s_y^2 + \left(1 - \frac{n}{N}\right) \Big\{ \left(1 - \frac{1}{N}\right) \sigma^2 \bar{x}_r + S_{rx}^2 [\hat{\beta}_s^2 + V(\hat{\beta}_s)] \\ + \frac{n}{N} [(\bar{y}_s - \hat{\beta}_s \bar{x}_r)^2 + \bar{x}_r^2 V(\hat{\beta}_s)] \Big\}. \end{aligned} \tag{37}$$

Let $\mathbf{Z}_r = \mathbf{W}_r^{-1/2} \mathbf{y}_r$, where $\mathbf{W}_r = \mathrm{diag}(x_i, i \in \mathbf{r})$, then

$$\mathbf{Z}_r \sim N(\mathbf{Q}_r \beta; \sigma^2 \mathbf{I}),$$

where $\mathbf{Q}_r = \mathbf{X}_r^{1/2}$. Hence according to Seber (1977),

$$\mathrm{Var}_\psi [S_{ry}^2] = 2\sigma^4 tr[\mathbf{B}_r^2] + 4\sigma^2 \beta^2 \mathbf{Q}_r' \mathbf{B}_r^2 \mathbf{Q}_r,$$

with

$$\mathbf{B}_r = \mathbf{W}_r^{1/2} (\mathbf{I}_r - \mathbf{J}_r/(N-n)) \mathbf{W}_r^{1/2}.$$

This expression is reduced to

$$\mathrm{Var}_\psi[S_{ry}^2] = \frac{2\sigma^4}{N-n}\left[\left(1-\frac{2}{N-n}\right)m_{rx}^{(2)} + \frac{\bar{x}_r^2}{N-n}\right] + \frac{4\beta^2\sigma^2}{N-n}[m_{rx}^{(3)} - \bar{x}_r m_{rx}^{(2)} - \bar{x}_r S_{rx}^2],$$

where $m_{rx}^{(j)} = \sum_{i\in r} x_i^{(j)}/(N-n)$, $j = 2,3$. Furthermore, since

$$(\bar{y}_s - \bar{y}_r)^2 \sim \sigma^2\left(\frac{\bar{x}_s}{n} + \frac{\bar{x}_r}{N-n}\right)\chi^2[1,\lambda^*],$$

with

$$\lambda^* = \frac{\beta^2(\bar{x}_s - \bar{x}_r)^2}{2\sigma^2\left(\frac{\bar{x}_s}{n} + \frac{\bar{x}_r}{N-n}\right)},$$

it follows that

$$\mathrm{Var}_\psi[(\bar{y}_s - \bar{y}_r)^2] = 2\sigma^4\left(\frac{\bar{x}_s}{n} + \frac{\bar{x}_r}{N-n}\right)^2\left(1 + \frac{2\beta^2(\bar{x}_s - \bar{x}_r)^2}{\sigma^2\left(\frac{\bar{x}_s}{n} + \frac{\bar{x}_r}{N-n}\right)}\right).$$

Similarly, it can be shown that

$$\mathrm{Cov}_\psi[S_{ry}^2; (\bar{y}_s - \bar{y}_r)^2] = \frac{2\sigma^4 S_{rx}^2}{(N-n)^2} + \frac{4\beta^2\sigma^2 S_{rx}^2}{N-n}(\bar{x}_r - \bar{x}_s)$$

and

$$\mathrm{Cov}_\psi[\hat{\beta}_s^2; (\bar{y}_s - \bar{y}_r)^2] = \frac{2\sigma^4}{n^2} + \frac{4\sigma^2\beta^2}{n}(\bar{x}_s - \bar{x}_r).$$

Collecting all the above results, we obtain that the prediction risk of $\hat{S}_{By}^2$ is, for large N,

$$E_\psi[\hat{S}_{By}^2 - S_y^2]^2 \simeq \frac{4\sigma^2 S_{rx}^4}{n\bar{x}_s}\left[\frac{\sigma^2}{n\bar{x}_s} + \beta^2\right].$$

This prediction risk is minimized by a sample maximizing $\sum_{i\in s} x_i$. If σ^2 is unknown and the noninformative prior (9) is used for (β, σ^2), the Bayes predictor of S_y^2 is as given by (37) with σ^2 replaced by

$$\frac{n-1}{n-3}\hat{\sigma}^2, \quad \text{where } \hat{\sigma}^2 = \frac{1}{n-1}\sum_{i\in s}\frac{1}{x_i}(y_i - \hat{\beta}_s x_i)^2.$$

Example 3. Consider the superpopulation model (1) with $\mathbf{X} = (x_1, \cdots, x_N)'$ and $\mathbf{V} = \sigma^2 \mathbf{I}_N$, where $\mathbf{I}_N$ is the identity matrix of dimension N. With noninformative prior on β, it follows that the posterior distribution of β is normal with mean and variance given respectively by

$$\hat{\beta}_s = \frac{\sum_{i\in s} x_i y_i}{\sum_{i\in s} x_i^2} \quad \text{and} \quad V(\hat{\beta}_s) = \frac{\sigma^2}{\sum_{i\in s} x_i^2}. \tag{38}$$

Thus, in this special case, $\eta_r(\mathbf{y}_s) = \mathbf{X}_r \hat{\beta}_s$ and

$$\Sigma_r = \sigma^2 \left\{ \mathbf{I}_r + \frac{\mathbf{X}_r \mathbf{X}_r'}{\sum_{i\in s} x_i^2} \right\}.$$

Moreover, $h(\mathbf{y}_s) = \hat{\beta}_s \bar{x}_r$, $\eta_r(\mathbf{y}_s)' E_r \eta_r(\mathbf{y}_s) = \hat{\beta}_s^2 S_{rx}^2$,

$$D_r^2 = \sigma^2 \left\{ \frac{1}{N-n} + \frac{\bar{x}_r^2}{\sum_{i\in s} x_i^2} \right\} \quad \text{and} \quad tr[E_r \Sigma_r] = \sigma^2 \left\{ 1 - \frac{1}{N-n} + \frac{S_{rx}^2}{\sum_{i\in s} x_i^2} \right\}.$$

It follows from (29), that the Bayes predictor of S_y^2 is

$$\begin{aligned} \hat{S}_{By}^2 = {} & \frac{n}{N} s_y^2 + (1 - \frac{n}{N}) \{ (1 - \frac{1}{N}) \sigma^2 + S_{rx}^2 [\hat{\beta}_s^2 + V(\hat{\beta}_s)] \\ & + \frac{n}{N} [(\bar{y}_s - \hat{\beta}_s \bar{x}_r)^2 + \bar{x}_r^2 V(\hat{\beta}_s)] \}, \end{aligned} \tag{39}$$

where $\hat{\beta}_s$ and $V(\hat{\beta}_s)$ are given in (38). If σ^2 is unknown, the Bayes predictor of S_y^2, under the noninformative prior (9), is again given by formula (39), with $\hat{\beta}_s$ and $\sigma^2(\hat{\beta}_s)$ given in (38) and σ^2 replaced by

$$\frac{n-1}{n-3} \hat{\sigma}^2, \quad \text{where} \quad \hat{\sigma}^2 = \frac{1}{n-1} \sum_{i\in s} (y_i - \hat{\beta}_s x_i)^2.$$

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