Topology and its Applications 379 (2026) 109512

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

On $p-$compact topologies on certain Abelian groups ☆

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A R T I C L E I N F O

Article history:
Received 16 November 2023
Received in revised form 24 July 2024
Available online 7 July 2025

Dedicated to István Juhász on his 80th birthday

Keywords:
Topological Abelian group
Direct sum
Selective ultrafilter
p-compactness

A B S T R A C T

Given a selective ultrafilter $p \in \omega^*$, we prove that there exists a p-compact group topology on $\mathbb{Q}^{(\mathfrak{c})}$ without nontrivial convergent sequences and a closed subgroup $H \subseteq \mathbb{Q}^{(\mathfrak{c})}$ which contains an element not divisible by any $n \in \omega$ ($\mathbb{Q}^{(\mathfrak{c})}$ denotes the direct sum of $\mathfrak{c}$ copies of $\mathbb{Q}$). It also shows that $\mathbb{Z} \oplus \mathbb{Q}^{(\mathfrak{c})}$ does not admit a p-compact topology for any $p \in \omega^*$. Given H a group and G a subgroup of $\mathbb{Q}$ which is not n-divisible, for some prime number $n > 1$, but is m-divisible for each prime number $m \neq n$, we show that a group topology compatible with $H \oplus G$ cannot be p-compact for any $p \in \omega^*$.

1. Introduction

P. J. Halmos [11] proved that it is possible to topologize the additive group $\mathbb{R}$ so that it becomes a Hausdorff compact topological group. On the other hand, L. Fuchs [8] (see also [13] and [12]) showed that free Abelian groups do not admit a compact group topology. Moreover, A. H. Tomita [17] showed that these kind of groups do not admit a group topology whose countable power is countably compact. In 1990, M. G. Tkachenko [16] constructed, under the assumption of the Continuum Hypothesis, a countably compact group topology on the free Abelian group of cardinality $\mathfrak{c}$. His construction modifies the construction of A. Hájnal and I. Juhász [10] of a hereditarily finally dense (HFD) group in $\{0,1\}^{\mathfrak{c}}$ with the property that all small projections are full. As a consequence, the resulting group never has non-trivial convergent sequences.

☆ The first author was supported by the research project PAPIIT grant no. IN-100122. The second author received support from FAPESP Processo Número 2019/19924-4 to visit the first listed author where the research for this paper has been conducted. The third author received support from FAPESP Processo Número 2019/12628-0. The second author would like to thank the hospitality and support of Prof. Garcia Ferreira during his stay at UNAM, Campus Morelia.

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https://doi.org/10.1016/j.topol.2025.109512

Several examples were obtained using different forms of Martin's Axiom with [14] using Martinś Axiom for countable posets. In [15] it was shown that the free Abelian group of cardinality $\mathfrak{c}$ admits a countably compact group topology from the existence of $\mathfrak{c}$ incomparable selective ultrafilters and A. C. Boero, I. Castro Pereira and A. H. Tomita obtained such an example using a single selective ultrafilter [6]. Using $2^{\mathfrak{c}}$ selective ultrafilters, the example in [15] showed the consistency of a countably compact group topology on the free Abelian group of cardinality $2^{\mathfrak{c}}$. All forcing examples obtained so far had their cardinalities bounded by $2^{\mathfrak{c}}$. Two years ago it was proved, in [3], that if there are $\mathfrak{c}$ incomparable selective ultrafilters then, for every infinite cardinal κ such that $\kappa^\omega = \kappa$, there exists a group topology on the free Abelian group of cardinality κ without nontrivial convergent sequences and such that every finite power is countably compact. It is still unknown whether or not there exists such topological group in ZFC. A. C. Boero and A. H. Tomita [5] showed from the existence of $\mathfrak{c}$ selective ultrafilters that there exists a free Abelian group of cardinality c whose square is countably compact. A. H. Tomita [18] showed that there exists a group topology on the free Abelian group of cardinality $\mathfrak{c}$ that makes all its finite powers be countably compact. In this context of topological products, we mention the following results.

Proposition 1.1 *([17]). Let G be a non-trivial Abelian semigroup with neutral element 0. Suppose that G is endowed with a semigroup topology which satisfies the following conditions:*

(1) G is torsion-free;
(2) for every $x \neq 0$, there exist infinitely many primes p such that $p^n \nmid x$ for some $n \in \mathbb{N}$; and
(3) there exist $x \in G \setminus \{0\}$, $A \in [\omega]^\omega$ and an enumeration $p_1, p_2, \ldots$ of all but finitely many primes such that $0 \notin \overline{\{(p_1 \ldots p_n)^n x : n \in A\}}$.

Then G^ω is not countably compact.

Proposition 1.2 *([17]). Let G be a non-trivial Abelian semigroup with the neutral element 0. Suppose that G is endowed with a semigroup topology which contains a proper open subgroup. If conditions (1) and (2) of Theorem 1.1 are satisfied, then G^ω is not countably compact.*

By showing that an infinite free Abelian topological group contains a proper open subgroup, A. H. Tomita concluded that:

Theorem 1.3 *([17]). Let G be an infinite free Abelian group endowed with a group topology. Then, G^ω is not countably compact.*

Another consequence of Theorem 1.3 is that $\mathbb{Z}^{(\mathfrak{c})}$ does not admit a p-compact topology for any $p \in \omega^*$.

Thus, we observe that clause (2) of Proposition 1.1 suggests that a good candidate for a torsion-free group that admits a p-compact topology might be a divisible group, such as $\mathbb{Q}$. Indeed, in [1] it has been proved that if $p \in \omega^*$ is a selective ultrafilter and $\kappa = \kappa^\omega$, then $\mathbb{Q}^{(\kappa)}$ admits a p-compact group topology without non-trivial convergent sequences.

After these results, divisibility seemed to play a role. Thus, it became interesting to check 'how much' divisibility is necessary to have p-compact group topologies.

We show that full divisibility is not necessary to obtain a compact group nor a p-compact group without non-trivial convergent sequences for p selective ultrafilter. In the third section, we show that the group $\mathbb{Z} \oplus \mathbb{Q}^{(\mathfrak{c})}$ does not admit a p-compact topology for any $p \in \omega^*$. In particular, this answers the following question proposed in [2]:

Question 1.4. *Let $p \in \omega^*$. Is there a p-compact group topology (without non-trivial convergent sequences) compatible with $\mathbb{Z}^{(\mathfrak{c})} \times \mathbb{Q}^{(\mathfrak{c})}$, for some ultrafilter p? A group topology whose ω-th power is countably compact? What about $\mathbb{Z} \times \mathbb{Q}^{\mathfrak{c}}$?*

Besides, we also show that, given H a group and G a subgroup of $\mathbb{Q}$ which is not n-divisible, for some prime number $n > 1$, but is m-divisible for each prime number $m \neq n$, a group topology compatible with $H \oplus G$ cannot be p-compact for any $p \in \omega^*$.

2. Preliminaries

All groups will be assumed to be Abelian. The symbol "|" denotes the division in $\mathbb{Z}$. The unitary circle $\mathbb{T}$ will be identified with the Abelian group $\mathbb{R}/\mathbb{Z}$. Given an infinite cardinal κ, for each $g \in \mathbb{Q}^\kappa$ we define $supp(g) = \{\alpha < \kappa : g(\alpha) = 0\}$ and we will denote the direct sum of κ copies of $\mathbb{Q}$ by

$$\mathbb{Q}^{(\kappa)} := \{g \in \mathbb{Q}^\kappa : |\text{supp}(g)| < \omega\}.$$

We remark that $\mathbb{R}$ can be algebraically identified with the group $\mathbb{Q}^{(\mathfrak{c})}$. An ultrafilter on ω is called *free* if $\cap p = \emptyset$ and ω^* will stand for the set of all free ultrafilters on ω. For an ultrafilter $p \in \omega^*$, we define an equivalence relation on $(\mathbb{Q}^{(\mathfrak{c})})^\omega$ by letting $f \equiv_p g$ if $\{n \in \omega : f(n) = g(n)\} \in p$. We let $[f]_q$ be the equivalence class determined by f and $(\mathbb{Q}^{(\mathfrak{c})})^\omega/p$ be $(\mathbb{Q}^{(\mathfrak{c})})^\omega/\equiv_p$. This set has a natural $\mathbb{Q}$-vector space structure. We call the group $(\mathbb{Q}^{(\mathfrak{c})})^\omega/p$ with this structure the *p−ultrapower* of $\mathbb{Q}^{(\mathfrak{c})}$ and it is denoted by $\text{ult}_p(\mathbb{Q}^{(\mathfrak{c})})$.

We recall the following concepts of Algebra that are remarkable in this paper:

- An Abelian group G is *divisible* if, for each $n \in \omega \setminus \{0\}$ and $g \in G$, there is $y \in G$ so that $g = ny$.
- Given a prime number $n \in \omega$, we say that an Abelian group G is n-divisible if, for each $g \in G$, there is $y \in G$ so that $g = ny$.

The following notion has been around in different topics of mathematics and we may assert that it belongs to the mathematical folklore.

• Let p be an ultrafilter and $(x_n)_{n<\omega}$ be a sequence in a space X. We say that $x \in X$ is a *p-limit point* of $(x_n)_{n<\omega}$, in symbols $x = p - \lim_{n\to\omega} x_n$, if $\{n < \omega : x_n \in V\} \in p$ for every neighborhood V of x.

It is evident that if X is Hausdorff, then the p-limit point is unique when it exists. It is not hard to prove that a space X is countably compact if every sequence of points in X has a p-limit point in X for some ultrafilter p. In connection with this fact, the following class of spaces was introduced by A. R. Bernstein [4].

Definition 2.1. Let p be an ultrafilter. A space X is said to be p-compact if for every sequence $(x_n)_{n<\omega}$ of points of X there is $x \in X$ such that $x = p - lim_{n\to\omega} x_n$.

It is known and easy to prove that every compact space is p-compact and every p-compact space is countably compact for every ultrafilter p. Also, we know that p-compactness is preserved under arbitrary products, for each ultrafilter p. Hence, a countably compact space whose product with itself is not countably compact is an example of a countably compact space which is not p-compact for any ultrafilter p. J. Ginsburg and V. Saks [9] showed that all powers of a space X are countably compact iff there is free ultrafilter p for which X is p-compact.

3. The results and their proofs

We start this section with a elementary lemma.

Lemma 3.1. *For each $c \in \mathbb{Z}$ there exists $r \in \mathbb{Z}$ so that $r \nmid 2^n + n! - c$ for all but finitely many $n \in \omega$.*

Proof. For the case when $c = 0$ we have that $r \nmid 2^n + n!$ for every odd natural number r and $2 \leq n \in \omega$. First, suppose that $c = 2^k l$ for some $k > 0$ and odd $l \in \mathbb{Z}$. If $n > 2^{k+1}$, we have that $2^{k+1} | 2^n$ and $2^{k+1} | n!$, but $2^{k+1} \nmid 2^k l$, hence $2^{k+1} \nmid 2^n + n! - c$. Finally, if c is an odd number, it is clear that if $r = 2$ and $n > 1$, $r \nmid 2^n + n! - c$. □

Theorem 3.2. *For every topological; Abelian group G, we have that $\mathbb{Z} \oplus G$ does not admit a p-compact topology for any $p \in \omega^*$.*

Proof. Suppose that $\mathbb{Z} \oplus G$ admits a p-compact topology for some $p \in \omega^*$, and consider a non-trivial sequence $(x_n)_{n\in\omega}$ of elements of $\mathbb{Z}$. Choose $g_1 \in G$ and $c \in \mathbb{Z}$ so that

$$p - \lim_{n\in\omega} x_n = c + g_1.$$

Fix $r \in \mathbb{Z}$ with $|c| < \frac{r}{2}$. Now, for each $n \in \omega$, let $e_n \in \mathbb{Z}$, $0 \leq e_n < r$, and $y_n \in \mathbb{Z}$ be so that

$$x_n = ry_n + e_n,$$

and $A \in p$ be so that $e_n = e \in \mathbb{Z}$ for every $n \in A$. We claim $c = e$. Indeed, as $\mathbb{Z} \oplus G$ is a p-compact topological group, there are $g_2 \in G$ and $b \in \mathbb{Z}$ such that

$$p - \lim_{n\in\omega} y_n = b + g_2.$$

Then, we have that

$$c + g_1 = p - \lim_{n\in\omega} x_n = p - \lim_{n\in\omega}(ry_n + e) = (rb + e) + rg_2.$$

Hence, we obtain that $rb + e = c$ and $rg_2 = g_1$. This implies that $r|(c - e)$ and since $|c| < \frac{r}{2}$ and $0 \leq e < r$, we obtain that $c = e$. This shows the claim.

In order to get a contradiction, we consider the sequence $(x_n)_{n\in\omega}$ of elements of $\mathbb{Z}$ where $x_n = 2^n + n!$, for every $n \in \omega$. Let $c \in \mathbb{Z}$ for this sequence $(x_n)_{n\in\omega}$. According to Lemma 3.1, we can choose two integers $r_1, r_2 \in \mathbb{Z}$ so that $r_1 \nmid x_n - c$ and $r_2 \nmid x_n$ for all but finitely many $n \in \omega$. Consider also $r \in \mathbb{Z}$ so that $\frac{r}{2} > |c|$ and $r_1 r_2 | r$. Applying the claim there is $A \in p$ such that $c = e$ and $x_n - c = x_n - e = x_n - e_n = ry_n$ for every $n \in A$. It follows from $r_1 \mid r$ that $r_1 \mid x_n - c$, contradicting the choice of r_1. □

Theorem 3.3. *Let G be a subgroup of $\mathbb{Q}$ and $r > 1$ be a prime number such that G is t-divisible for each prime $t \neq r$ but is not r-divisible. Then, for every group H the group topology compatible with $H \oplus G$ cannot be p-compact for any $p \in \omega^*$.*

Proof. Suppose that $H \oplus G$ is endowed with a p-compact group topology for some $p \in \omega^*$. Let $g \in G$ be so that $g \neq rh$ for each $h \in G$. Now, we inductively define a sequence $(m_k)_{k\in\omega}$ of elements of ω inductively as follows: Put $m_0 = 1$ and for $1 < k \in \omega$ we select $m_k \in \omega$ so that

$$r^{m_k} > k \sum_{l=0}^{k-1} r^{m_l} + k,$$

for each $k > 0$. Let $h_1 \in H$ and $c \in \mathbb{Q}$ be such that

$$p-\lim_{n\in\omega}\left(\sum_{k=0}^{n} r^{m_k} g\right)_n = h_1 + cg,$$

where $c = \dfrac{u}{v}$, and $u, v \in \mathbb{Z}$ so that $\gcd(u, v) = 1$. Without loss of generality, we may assume that $v > 0$. Fix $n_0 \in \omega$ so that $n_0 > \text{máx}(|u|, |v|)$ and set

$$S := v \sum_{k=0}^{n_0-1} r^{m_k} - u > 0.$$

Then, for each $n \geq n_0$,

$$v \sum_{k=0}^{n} r^{m_k} - u = v \sum_{k=n_0}^{n} r^{m_k} + S,$$

and

$$r^{m_{n_0}} > n_0 \sum_{k=0}^{n_0-1} r^{m_k} + n_0 \geq v \sum_{k=0}^{n_0-1} r^{m_k} - u = S > 0.$$

Hence, $r^{m_{n_0}} \nmid v\sum_{k=0}^{n} r^{m_k} - u$ for any $n \geq n_0$ since $v\sum_{k=0}^{n} r^{m_k} - u = v\sum_{k=n_0}^{n} r^{m_k} + S$. Let $(y_n)_{n\geq n_0}$ be a sequence in $\mathbb{Z}$ so that, for each $n \geq n_0$,

$$r^{m_{n_0}} y_n = v \sum_{k=n_0}^{n} r^{m_k}.$$

Then, there are $h_2 \in H$ and $d \in \mathbb{Q}$ so that $dg \in G$ and

$$p-\lim_{n\in\omega} \left(\frac{y_n}{v} g\right) = h_2 + dg,$$

with $d = \dfrac{i}{j}$, $\gcd(i, j) = 1$. We assert that $r \nmid j$. Indeed, suppose that $l > 0$ is such that $j = r^l s$, with $s \in \mathbb{Z}$ and $r \nmid s$. As i cannot be divisible by r, there exists $g_1 \in G$ such that $dg = ig_1$. Therefore, we have that

$$\frac{i}{r^l s} g = ig_1 \text{ which implies that } \; g = r^l s g_1,$$

but this is a contradiction. Thus, we obtain that $r \nmid j$. Now, since

$$p-\lim_{n\in\omega}\left(\frac{r^{m_{n_0}} y_n}{v} g\right) = r^{m_{n_0}} h_2 + r^{m_{n_0}} dg,$$

then we have that

$$\begin{aligned} h_1 + cg &= p-\lim_{n\in\omega}\left(\sum_{k=0}^{n} r^{m_k} g\right) \\ &= p-\lim_{n\in\omega}\left(\sum_{k=n_0}^{n} r^{m_k} g + \sum_{k=0}^{n_0-1} r^{m_k} g\right) \end{aligned}$$

$$= \sum_{k=0}^{n_0-1} r^{m_k} g + p - \lim_{n\in\omega} \left(\frac{r^{m_{n_0}} y_n}{v} g \right)$$
$$= \sum_{k=0}^{n_0-1} r^{m_k} g + r^{m_{n_0}} h_2 + r^{m_{n_0}} dg.$$

Hence,

$$c - \sum_{k=0}^{n_0-1} r^{m_k} = \frac{u}{v} - \sum_{k=0}^{n_0-1} r^{m_k} = -\frac{S}{v} = r^{m_{n_0}} d = r^{m_{n_0}} \frac{i}{j},$$

and so $-Sj = r^{m_{n_0}} iv$, which is impossible since $r \nmid j$ and $0 < S < r^{m_{n_0}}$. □

Lemma 3.4. *Let $\psi : \mathbb{Q}^{(\mathfrak{c})} \to \mathbb{T}$ be a nontrivial morphism and $g \in \mathbb{Q}^{(\mathfrak{c})}$ be so that $\psi(g) \neq 0$ has order r. Then,*

$$\psi^{-1}[\{\mathbb{Z}, \frac{1}{r} + \mathbb{Z}, ..., \frac{r-1}{r} + \mathbb{Z}\}]$$

is not a r-divisible group.

Proof. Set

$$G := \psi^{-1}[\{\mathbb{Z}, \frac{1}{r} + \mathbb{Z}, ..., \frac{r-1}{r} + \mathbb{Z}\}].$$

It is clear that $g \in G$ and as $\{mg : m \in \mathbb{Z}\}$ is an infinite subset of G, we must have that G is an infinite group. Suppose that there exists $h \in G$ so that $rh = g$. Then,

$$r\psi(h) = \psi(g) \neq 0,$$

and hence $\psi(h) \neq 0$. Since $\psi(h) \in \{\mathbb{Z}, \frac{1}{r} + \mathbb{Z}, ..., \frac{r-1}{r} + \mathbb{Z}\}$, the order of $\psi(h)$ must be r, a contradiction. Therefore, G is not a r-divisible group. □

Theorem 3.5. *There is an Abelian, torsion-free, non-divisible topological group which is compact.*

Proof. It is known that the additive group $\mathbb{R}$ admits a compact topological group topology (see [11]). Hence, by algebraically identifying $\mathbb{R}$ with $\mathbb{Q}^{(\mathfrak{c})}$, we may consider $\mathbb{Q}^{(\mathfrak{c})}$ as a compact group topological group. Since $\mathbb{Q}^{(\mathfrak{c})}$ is, in particular, totally bounded, we can fix a nontrivial homeomorphism $\psi : \mathbb{Q}^{(\mathfrak{c})} \to \mathbb{T}$ (see [7]).

First, we claim that there exist a prime number $k \in \omega$ and $g \in \mathbb{Q}^{(\mathfrak{c})}$ so that $\psi(g) \neq 0$ has order k. Indeed, assume that there exists an irrational ζ so that $\zeta + \mathbb{Z} \in \psi[\mathbb{Q}^{(\mathfrak{c})}]$. In this case, $\psi[\mathbb{Q}^{(\mathfrak{c})}]$ is dense in $\mathbb{T}$. Since $\psi[\mathbb{Q}^{(\mathfrak{c})}]$ is compact and therefore closed in $\mathbb{T}$, it follows that $\psi[\mathbb{Q}^{(\mathfrak{c})}] = \mathbb{T}$, and hence we can take any $n > 1$ and find $g \in \mathbb{Q}^{(\mathfrak{c})}$ so that $\psi(g) \neq 0$ and $n\psi(g) = 0$. In the case when such irrational does not exist, it is clear that there is a prime number $k \in \omega$ and an element $g \in \mathbb{Q}^{(\mathfrak{c})}$ so that $\psi(g)$ has order k.

Consider the prime number k from above and the element $g \in \mathbb{Q}^{(\mathfrak{c})}$ found in the previous paragraph. Our group will be

$$G := \psi^{-1}[\{\mathbb{Z}, \frac{1}{k} + \mathbb{Z}, ..., \frac{k-1}{k} + \mathbb{Z}\}].$$

According to Lemma 3.4, we have that G is infinite and it is not k-divisible. In the topological context, it is evident that G is a closed subgroup of $\mathbb{Q}^{(\mathfrak{c})}$, and so it is a compact topological group. □

To show next the main results of the paper we will use a construction similar to the one made in [1]. For this purpose we shall use some notation and results from this paper:

Let $\mathcal{C}$ be the set of all homomorphisms $\phi : \mathbb{Q}^{(C)} \to \mathbb{T}$, where $C \in [\mathfrak{c}]^\omega$ is a suitably closed set, satisfying the condition:

$$p - \lim_{n\in\omega} \phi(\tfrac{1}{N} f_\xi(n)) = \phi(\tfrac{1}{N}\chi_\xi),$$

for each $\xi \in I \cap C$ and $N \in \omega$.

- For each $C \subseteq \mathfrak{c}$, we define

$$\mathbb{Q}^{(C)} := \{g \in \mathbb{Q}^{(\mathfrak{c})} : \text{supp}(g) \subseteq C\}$$

  which is a subgroup of $\mathbb{Q}^{(\mathfrak{c})}$.
- Let $\{J_0, J_1\}$ be a partition of $[\omega, \mathfrak{c})$ such that $\omega + \omega \in J_1$ and $|J_0| = |J_1| = \mathfrak{c}$.
- Let $\{f_\xi : \xi \in J_0\}$ be an enumeration of $(\mathbb{Q}^{(\mathfrak{c})})^\omega$ such that

$$\bigcup_{n\in\omega} \text{supp}(f_\xi(n)) \subset \xi, \text{ for each } \xi \in J_0.$$

- For $\xi \in \mathfrak{c}$, we denote χ_ξ the element of $\mathbb{Q}^{(\mathfrak{c})}$ so that $\text{supp}(\chi_\xi) = \{\xi\}$ and $\chi_\xi(\xi) = 1$.
- For $\mu \in \mathfrak{c}$, we define $\vec{\mu} : \omega \to \mathfrak{c}$ by $\vec{\mu}(n) = \mu$ for every $n \in \omega$.
- For $A \subseteq \omega$ and $\zeta : A \to \mathfrak{c}$, we define $\chi_\zeta : A \to \mathbb{Q}^{(\mathfrak{c})}$ by $\chi_\zeta(n) = \chi_{\zeta(n)}$ for every $n \in A$. Observe that if $\mu \in \mathfrak{c}$, then $\chi_{\vec{\mu}}(n) = \chi_\mu$ for every $n \in \omega$.
- Let $I \subseteq J_0$ be such that $\{[f_\xi]_p : \xi \in I\} \cup \{[\chi_{\vec{\mu}}]_p : \mu \in \mathfrak{c}\}$ is a $\mathbb{Q}$-basis for $\text{ult}_p(\mathbb{Q}^{(\mathfrak{c})})$.
- A set $C \in [\mathfrak{c}]^\omega$ is called *suitably closed* if $\bigcup_{n\in\omega} \text{supp}(f_\xi(n)) \subseteq C$ for every $\xi \in C \cap I$.

Next we shall extend each homomorphism $\phi \in \mathcal{C}$ to a homomorphism $\overline{\phi}$ defined in $\mathbb{Q}^{(\mathfrak{c})}$ satisfying that

$$p - \lim_{n\in\omega} \overline{\phi}(\tfrac{1}{N} f_\xi(n)) = \overline{\phi}(\tfrac{1}{N}\chi_\xi),$$

for each $\xi \in I$ and $N \in \omega$, in a similar way to what is done in Lemma 3.6 of [1]. The difference will be that we wish to control the value of the homomorphisms ϕ_β, when $\beta \in [\omega, \omega + \omega)$, in the element $\chi_{\omega+\omega}$.

Lemma 3.6. *Let p be a free ultrafilter and $C \in [\mathfrak{c}]^\omega$ be a suitably closed set. Then, every homomorphism $\phi : \mathbb{Q}^{(C)} \to \mathbb{T}$ satisfying*

$$p - \lim_{n\in\omega} \phi(\tfrac{1}{N} f_\xi(n)) = \phi(\tfrac{1}{N}\chi_\xi),$$

for each $\xi \in I \cap C$ and $N \in \omega$, can be extended to a homomorphism $\overline{\phi} : \mathbb{Q}^{(\mathfrak{c})} \to \mathbb{T}$ satisfying that

$$p - \lim_{n\in\omega} \overline{\phi}(\tfrac{1}{N} f_\xi(n)) = \overline{\phi}(\tfrac{1}{N}\chi_\xi),$$

for each $\xi \in I$ and $N \in \omega$. And if $1 \leq m \in \omega$ and $\omega + \omega \notin C$, then we may add the condition $\overline{\phi}(\chi_{\omega+\omega}) = \dfrac{1}{m} + \mathbb{Z}$.

Proof. To have this done let $\{\xi_\alpha : \alpha < \mathfrak{c}\}$ be a strictly increasing enumeration of $\mathfrak{c} \setminus C$. For each $\alpha \leq \mathfrak{c}$, let $C_\alpha := C \cup \{\xi_\gamma : \gamma < \alpha\}$. Notice, $C_0 = C$, $C_\mathfrak{c} = \mathfrak{c}$ and recall that $\bigcup_{n\in\omega} \text{supp}(f_{\xi_\alpha}(n)) \subseteq \xi_\alpha$ for each

$\alpha < \mathfrak{c}$. In order to get the required extension for every ordinal $\alpha \leq \mathfrak{c}$ we shall recursively define a group homomorphism $\sigma_\alpha : \mathbb{Q}^{(C_\alpha)} \to \mathbb{T}$ satisfying:

a) $\sigma_0 = \phi$;
b) $\sigma_\delta \subseteq \sigma_\alpha$ whenever $\delta \leq \alpha \leq \mathfrak{c}$; and
c) $p - \lim_{n\in\omega} \sigma_\alpha(\frac{1}{N} f_\xi(n)) = \sigma_\alpha(\frac{1}{N}\chi_\xi)$, for each $\alpha \leq \mathfrak{c}$, $\xi \in C_\alpha \cap I$ and $N \in \omega$.

Indeed, we start setting $\sigma_0 = \phi$. Let $\alpha < \mathfrak{c}$ and suppose that σ_δ as been defined satisfying the above conditions for each $\delta < \alpha$. First, if $g \in \mathbb{Q}^{(\bigcup_{\delta<\alpha} C_\delta)}$, then we define

$$\sigma_\alpha(g) = \big(\bigcup_{\delta<\alpha} \sigma_\delta\big)(g).$$

For the case when $\alpha \in J_1$ and $1 \leq m \in \omega$ is given, we define

$$\begin{cases} \sigma_\alpha(q\chi_{\xi_\alpha}) = 0, & \text{if } \xi_\alpha \neq \omega+\omega \text{ and } q \in \mathbb{Q} \\ \sigma_\alpha(q\chi_{\xi_\alpha}) = q\cdot(\dfrac{1}{m}+\mathbb{Z}), & \text{if } \xi_\alpha = \omega+\omega \text{ and } q \in \mathbb{Q}. \end{cases}$$

Finally, if $\alpha \in J_0$, we set

$$\sigma_\alpha(q\chi_{\xi_\alpha}) = q\cdot\big(p - \lim_{n\in\omega}\big(\bigcup_{\delta<\alpha}\sigma_\delta\big)(\tfrac{1}{N} f_{\xi_\alpha}(n))\big),$$

for every $q \in \mathbb{Q}$. It is clear that we can uniquely extend $\sigma_\mathfrak{c}$ to a group homomorphism $\overline{\phi} : \mathbb{Q}^{(\mathfrak{c})} \to \mathbb{T}$. Observe that $\overline{\phi}$ satisfies the requires conditions since it extends σ_α for every $\alpha \leq \mathfrak{c}$; that is,

$$p - \lim_{n\in\omega} \overline{\phi}(\tfrac{1}{N} f_\xi(n)) = \overline{\phi}(\tfrac{1}{N}\chi_\xi),$$

for each $\xi \in I$ and $N \in \omega$ and if $1 \leq m \in \omega$, then $\overline{\phi_\beta}(\chi_{\omega+\omega}) = \sigma_\mathfrak{c}(\chi_{\omega+\omega}) = \dfrac{1}{m} + \mathbb{Z}$. □

Lemma 3.7 *([1], Lemma 3.5). Fix a selective ultrafilter p. Let $d \in \mathbb{Q}^{(\mathfrak{c})} \setminus \{0\}$, $r \in \mathbb{Q}^{(I)} \setminus \{0\}$ and $B \in p$. Let C be a countably infinite subset of $\mathfrak{c}$ such that $\omega \cup supp(r) \cup supp(d) \subseteq C$ and $\bigcup_{n\in\omega} supp(f_\xi(n)) \subseteq C$ for every $\xi \in C \cap I$. Then there exists a homomorphism $\phi : \mathbb{Q}^{(C)} \to \mathbb{T}$ such that*

a) $\phi(d) \neq 0$;
b) $p - \lim_{n\in\omega} \phi(\frac{1}{N} f_\xi(n)) = \phi(\frac{1}{N}\chi_\xi)$, for each $\xi \in I \cap C$ and $N \in \omega$;
c) the sequence $\big(\phi(\sum_{\mu\in supp(r)} r(\mu) f_\mu(n))\big)_{n\in B}$ does not converge.

To prove the following result we follow ideas similar to the ones used in the proof of Theorem 3.7 of [1]. But, for the purposes of this article, we have to make some changes to find a suitable element in $\mathbb{Q}^{(\mathfrak{c})}$ which is not divisible by any $n \in \omega$.

Theorem 3.8. *If $p \in \omega^*$ is a selective ultrafilter, then there exists a p-compact group topology on $\mathbb{Q}^{(\mathfrak{c})}$ without nontrivial convergent sequences and a closed subgroup $H \subseteq \mathbb{Q}^{(\mathfrak{c})}$ which contains an element not divisible by any $n \in \omega$.*

Proof. Fix a selective ultrafilter $p \in \omega^*$. We shall equip $\mathbb{Q}^{(\mathfrak{c})}$ with a topology via special characters, part of them are given by Lemma 3.7 (see [1]). Enumerate the set $\mathcal{C}$ by $\{\phi_\beta : \beta \in [\omega, \mathfrak{c})\}$ assuming, without loss of generality, that given $\beta \in [\omega, \mathfrak{c})$, $\phi_\beta : \mathbb{Q}^{(C)} \to \mathbb{T}$ is such that $C \subseteq \beta$. By Lemma 3.6, we shall extend each homomorphism $\phi \in \mathcal{C}$ to a homomorphism $\overline{\phi}$ defined in $\mathbb{Q}^{(\mathfrak{c})}$ satisfying that

$$p - \lim_{n \in \omega} \overline{\phi}(\tfrac{1}{N} f_\xi(n)) = \overline{\phi}(\tfrac{1}{N} \chi_\xi),$$

for each $\xi \in I$ and $N \in \omega$. Now, we control the value of the homomorphisms ϕ_β, for $\beta \in [\omega, \omega + \omega)$, in the element $\chi_{\omega+\omega}$. For that, consider $\{\beta_n : n \in \omega\}$ an enumeration of $[\omega, \omega + \omega)$.

Consider $\mathbb{Q}^{(\mathfrak{c})}$ endowed with the group topology generated by the homomorphisms from $\overline{\mathcal{C}} := \{\overline{\phi} : \phi \in \mathcal{C}\}$. We denote this topological group by G. It follows from the construction that

$$r\chi_\xi = p - \lim_{n \in \omega} r f_\xi(n),$$

for each $\xi \in I$ and $r \in \mathbb{Q}$. Let $h := (h_n)_{n \in \omega}$ be a sequence of elements in $\mathbb{Q}^{(\mathfrak{c})}$. Choose sets $\{r_\xi : \xi \in I\}$ and $\{s_\mu : \mu \in \mathfrak{c}\}$ of rational numbers, where all but finitely many are 0, such that

$$[h]_p = \sum_{i \in I} r_\xi [f_\xi]_p + \sum_{\mu \in \mathfrak{c}} s_\mu [\chi_{\vec{\mu}}]_p.$$

Hence, we obtain that

$$p - \lim_{n \in \omega} h_n = \sum_{i \in I} r_\xi \chi_\xi + \sum_{\mu \in \mathfrak{c}} s_\mu \chi_\mu$$

This shows that G is p-compact.

Now, let g be a one-to-one sequence of elements in G. Again, there are $r \in \mathbb{Q}^{(I)} \setminus \{0\}$ and $s \in \mathbb{Q}^{(\mathfrak{c})}$ such that

$$[g]_p = \sum_{i \in I} r_\xi [f_\xi]_p + \sum_{\mu \in \mathfrak{c}} s_\mu [\chi_{\vec{\mu}}]_p.$$

Set $D := \operatorname{supp}(r)$. Then, there exists $B \in p$ so that

$$g(n) = \sum_{\xi \in D} r_\xi f_\xi(n) + \sum_{\mu \in \mathfrak{c}} s_\mu \chi_\mu(n),$$

for every $n \in B$. Let $d \in G \setminus \{0\}$ arbitrary, and C be a suitably closed set containing $\omega \cup D \cup \operatorname{supp}(d)$. Applying Proposition 3.7, we find a homomorphism $\phi : \mathbb{Q}^{(C)} \to \mathbb{T}$ so that the sequence $\big(\phi(\sum_{\xi \in D} r_\xi f_\xi(n))\big)_{n \in B}$ does not converge in $\mathbb{T}$. As a consequence we have that the sequence $(\sum_{\xi \in D} r_\xi f_\xi(n))_{n \in B}$ does not converge in G, and since $\sum_{\mu \in \mathfrak{c}} s_\mu \chi_\mu$ is constant, we conclude that the sequence $(g(n))_{n \in \omega}$ does not converge either. Therefore, G does not have non-trivial convergent sequences.

Finally, by using Lemmas 3.6 and 3.7, for every positive $n \in \omega$ we can define a group homomorphism $\phi_n : \mathbb{Q}^{(\mathfrak{c})} \to \mathbb{T}$ in $\overline{\mathcal{C}}$ so that $\phi_n(\chi_{\omega+\omega}) = \dfrac{1}{n} + \mathbb{Z}$. It is evident that

$$H := \bigcap_{n \in \omega} \phi_n^{-1}[\{\mathbb{Z}, \frac{1}{n} + \mathbb{Z}, ..., \frac{n-1}{n} + \mathbb{Z}\}]$$

is a nontrivial closed subgroup of G; hence, it is also p-compact. Besides, $\chi_{\omega+\omega} \in H$. Now, fix $n \in \omega$ and suppose that there is $h \in H$ for which $\chi_{\omega+\omega} = nh$. Then,

$$0 \neq \phi_n(\chi_{\omega+\omega}) = n\phi_n(h) = 0,$$

but this is a contradiction. Therefore, the element $\chi_{\omega+\omega}$ in H is non-divisible in H for every $n \in \omega$. This ends the proof. □

4. Open questions

We end this paper with the next problems that we consider be interesting.

Question 4.1. *If H is a non-divisible subgroup of $\mathbb{Q}$ and H_0 is a divisible group, does $H \oplus H_0$ admit a topology whose ω-th power is countably compact?*

Question 4.2. *Is there, in ZFC, an Abelian torsion-free topological group which has an element that is not n-divisible for any $n > 1$ and admits a p-compact topology for some $p \in \omega^*$?*

The authors would like to thank the referee for the careful reading of the article that contributed to improving it.

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