# Liouville theorem for a quasilinear non-uniformly elliptic equation in half-spaces and applications

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**Abstract.** The purpose of this paper is to study Liouville-type theorems for the equation $2\Delta_\infty u + \Delta u = 0$ in half-spaces. Our main result classifies $C^1$-viscosity solutions that continuously vanish on the flat boundary and grow linearly at infinity. This is somehow equivalent to treating a class of equations in divergence form in Orlicz space without the $\Delta_2$ condition. The ingredients for the proof involve the regularity theory for solutions; the construction of barriers along the boundary; and new up to the boundary gradient estimates. These elements allow to implement a Lipschitz implies $C^{1,\alpha}$ type approach. We obtain some applications that encompass the recovery of classical results like Radó's zero-level set removability result and Schwarz reflection principle yielding substantially more regularity.

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## 1 Introduction

Classification of solutions to elliptic equations plays a pivotal role in studying problems in partial differential equations, geometric analysis, and geometry. A classical result from the theory of harmonic functions establishes that if $u$ is a nonnegative harmonic function in the half-space $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$, continuously vanishing on $\partial\mathbb{R}^n_+$, then $u(x) = u(e_n)x_n$ for every $x \in \mathbb{R}^n_+$. Possibly this result was first stated by Loomis and Wider [20]. Afterward, this was considered by Rudin [25] in the higher dimensional case. For a short proof of this result for harmonic functions in half-space, we refer the reader to [5] and a textbook proof can now be found in Theorem 7.22 in [1]. In [17], Kilpeläinen et al proved this result for $p$-harmonic function. Similar results were also obtained for the infinity Laplace and the minimal surface equations. These results can be found in [4] and [12], respectively. The classification theorem for $g$-harmonic functions in half-spaces was obtained by Braga and Moreira in [7], and for homogeneous differential inequalities involving the extremal Pucci operators ($\mathcal{M}^{\pm}_{\lambda,\Lambda}$) by Braga in [6]. Following [7], let $g : [0, +\infty) \to \mathbb{R}$ such that

($g_1$) $g \in C^0[0,\infty) \cap C^1(0,\infty)$,

($g_2$) $g(0) = 0, \quad g(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} g(t) = +\infty$.

For $g$ satisfying ($g_1$) and ($g_2$), we define the Lieberman's quotient by

$$Q_g(t) := \frac{tg'(t)}{g(t)} \quad \text{for } t > 0.$$

For $\delta_0 > 0$, we say that a function $g$ satisfying ($g_1$) and ($g_2$)

$$g \in \mathcal{C}_{\delta_0} \iff Q_g(t) \geq \delta_0 \quad \forall\, t > 0.$$

For a pair $0 < \delta_0 \leq g_0$, we say that the $g$ belongs to Lieberman's class $\mathcal{C}_{\delta_0,g_0}$ if only if ($g_1$) and ($g_2$) are satisfied and

$$\delta_0 \leq Q_g(t) \leq g_0 \quad \forall\, t > 0. \tag{1}$$

These assumptions on the function $g$ establish for some kind of uniform ellipticity conditions to operators in divergence form of the type

$$\mathcal{L}_g u := \operatorname{div}\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right). \tag{2}$$

Let $\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$, $B^+_r := \{x \in \mathbb{R}^n_+ : |x| < r\}$,

$$\begin{aligned} C^0_{vfb}(\overline{B^+_r}) &:= \{u \in C^0(\overline{B^+_r}) : u = 0 \text{ on } B'_r := \partial\mathbb{R}^n_+ \cap \partial B^+_r\}, \\ C^0_{vfb}(\overline{\mathbb{R}^n_+}) &:= \{u \in C^0(\overline{\mathbb{R}^n_+}) : u = 0 \text{ on } \partial\mathbb{R}^n_+\}, \\ C^k_{fb}(B^+_r) &:= \bigcup_{0<\epsilon<r} C^k(\overline{B^+_{r-\epsilon}}), \text{ for } k \in \mathbb{N} \cup \{\infty\}, \end{aligned}$$

where $r > 0$, $C^0(\overline{\mathbb{R}^n_+})$ denotes the set of continuous functions on $\overline{\mathbb{R}^n_+}$, and the subscripts *vfb* and *fb* stand for vanishing on the flat boundary and the flat boundary, respectively. In [7], the authors also show that if $u \in C^0_{vfb}(\overline{\mathbb{R}^n_+}) \cap W^{1,G}_{loc}(\mathbb{R}^n_+)$ is a nonnegative solution in the sense of distributions to

$$\operatorname{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 \text{ in } \mathbb{R}^n_+, \tag{3}$$

where $G(t) = \int_0^t g(s)ds$ for $t > 0$, $G(0) = 0$, and $g \in \mathcal{C}_{\delta_0, g_0}$, then $u(x) = u(e_n)x_n$ for all $x \in \overline{\mathbb{R}^n_+}$. In [7] it is also proved a version of this result without the sign restriction. More precisely, if $u \in C^0_{vfb}(\overline{\mathbb{R}^n_+}) \cap W^{1,G}_{loc}(\mathbb{R}^n_+)$ is a solution in the sense of distributions to (3) and

$$u(x) = O(|x|) \quad \text{as } |x| \to \infty,\ x \in \mathbb{R}^n_+,$$

then

$$u(x) = u(e_n)x_n \quad \forall\, x \in \overline{\mathbb{R}^n_+}.$$

The proof of this result in [7] is short but far from immediate. It is the combination of a quantitative version of the Hopf-Oleinik Lemma [8, Theorem 3.2] with Carleson estimate, boundary Harnack inequality, and Schwartz reflection principle in the context of non-negative $g$-harmonic functions. Also, it worth observing that if $u$ is a non-negative solution in any of these situations, then $u(x) = O(|x|)$ at infinity.

The main purpose of this paper is to present a proof of the classification theorem for functions in half-spaces that vanish continuously on the flat boundary and are also solutions to the following quasilinear non-uniformly elliptic equation in nondivergence form

$$\mathcal{L}_\infty u := 2\Delta_\infty u + \Delta u = 0 \text{ in } \mathbb{R}^n_+,$$

where

$$\Delta_\infty u := \sum_{i,j=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = \langle D^2u(x)\nabla u(x), \nabla u(x)\rangle.$$

This is somehow equivalent to treating (3) for the case where $g(t) = 2te^{t^2}$. In this case $\delta_0 = 1$ and $g_0 = \infty$. Hence, the so called $\Delta_2$ condition is violated in the context of elliptic equations (in divergence form) in Orlicz spaces. To see this, letting $\phi(t) = 2te^{t^2}$, we have

$$Q_\phi(t) = \frac{\phi'(t)t}{\phi(t)} = 1 + 2t^2 \geq 1, \quad \forall\, t > 0. \tag{4}$$

Thus, $\phi \in \mathcal{C}_1$ and

$$\lim_{t \to +\infty} \frac{\phi'(t)t}{\phi(t)} = +\infty.$$

Consequently, $\phi \notin \mathcal{C}_{\delta_0, g_0}$ for any $g_0 \geq 1$, and the degenerate operator defined by

$$\mathcal{L}_\phi(u) := \operatorname{div}\left(\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = \operatorname{div}\left(2e^{|\nabla u|^2}\nabla u\right) \tag{5}$$

is a quasilinear non-uniformly elliptic operator. A straightforward computation on the divergence above shows that this operator can be written in nondivergence form as

$$\mathcal{L}_\phi u = 2e^{|\nabla u|^2}\left\{2\Delta_\infty u + \Delta u\right\} = 2e^{|\nabla u|^2}\mathcal{L}_\infty u. \tag{6}$$

Hence,

$$\mathcal{L}_\phi u = 0 \iff \mathcal{L}_\infty u = 0.$$

Motivated by [7], we obtain the following classification result:

**Theorem 1.1.** *Let $u \in C^1(\mathbb{R}^n_+) \cap C^0_{vfb}(\overline{\mathbb{R}^n_+})$ be a viscosity solution to*

$$\mathcal{L}_\infty u = 0 \ \textit{in}\ \mathbb{R}^n_+. \tag{7}$$

*Assume that*

$$u(x) = O(|x|) \quad \textit{as}\ |x| \to \infty,\ x \in \mathbb{R}^n_+. \tag{8}$$

*Then,*

$$u(x) = u(e_n)x_n \quad \forall\, x \in \overline{\mathbb{R}^n_+}.$$

In the proof of Theorem 1.1, we start by showing the equivalence among $C^1$ viscosity, classical, and distributional solutions to (7). In turn, the construction of suitable barriers renders the control of the solution by the distance up to the flat boundary. Now, the interior gradient estimate kicks in yielding global Lipschitz regularity up to the boundary. This allows us to truncate the equation recovering Lieberman's ellipticity conditions in the whole half-space. Finally, a Lipschitz implies $C^{1,\alpha}$ type result finishes the proof.

As a matter of fact, the equivalence of solutions mentioned above goes further dealing with local minimizers of

$$J_\Omega(u) = \int_\Omega e^{|\nabla u|^2}\, dx \tag{9}$$

The study of the regularity of the solutions to (10) was first considered by G. M. Lieberman to answer a question posed by M. Giaquinta. In [19], Lieberman proved that if $u$ is a local minimizer of (9), then $u \in C^2(\Omega)$ and it is a classical solution to

$$\mathcal{L}_\infty u = 0 \quad \text{in } \Omega. \tag{10}$$

This way, our equivalence regularity result (Theorem 1.2) can be seen as a complement to the theory developed by Lieberman in [19]. We should also mention the paper by Marcellini [22], where the local Lipschitz regularity of minimizers is established for a more general class of functionals containing the ones treated here. The results in [22] are also valid for vector valued maps.

The following result deals with the regularity theory for solutions (see Definition 2.1) to (10).

**Theorem 1.2.** *Let $\Omega$ is an open subset of $\mathbb{R}^n$. Assume $u \in C^1(\Omega)$. The following statements are equivalent:*

*(a) $u$ is a classical solution to (10);*

*(b) $u$ is a strong solution to (10);*

*(c) $u$ is a weak solution to (10);*

*(d) $u$ is a local minimizer of (9);*

*(e) $u$ is a viscosity solution to (10).*

*Moreover, if any of the statements above holds then $u \in C^\infty(\Omega)$.*

We finish this section with some applications. The first one is the version of Theorem 1.1 in the whole space:

**Theorem 1.3.** *Let $u \in C^1(\mathbb{R}^n)$ be a viscosity solution to*

$$\mathcal{L}_\infty u = 0 \text{ in } \mathbb{R}^n. \tag{11}$$

*Assume that*

$$u(x) = O(|x|) \quad \text{as } |x| \to \infty. \tag{12}$$

*Then, $u$ is an affine function.*

As an immediate consequence, we have the following.

**Corollary 1.4.** *Let $u \in C^1(\mathbb{R}^n)$ be a viscosity solution to*

$$\mathcal{L}_\infty u = 0 \text{ in } \mathbb{R}^n. \tag{13}$$

*Assume that*

$$u(x) = O(|x|) \quad \text{as } |x| \to \infty,\ x \in \mathbb{R}^n, \tag{14}$$

*and $u$ is bounded above or below. Then, $u$ is a constant.*

The second application is a Radó-type result. As it is well known, the classical Radó's removability result [24] establishes that if $f$ is a complex-valued continuous function in an open set $\Omega$ in the complex plane and $f$ is holomorphic in $\Omega \setminus \{z \in \Omega : f(z) = 0\}$, then $f$ is holomorphic in the whole $\Omega$. For the case of harmonic functions, see [2]. The $p$-Laplacian case was treated in [14, 16]. For quasilinear elliptic and parabolic equations, this result was studied by Juutinen and Lindqvist [15] and for fully nonlinear equations by Takimoto [27]. For more details, see the references therein.

In this paper, we use the result in [15] to deepen the removability result on the level set $\{u = 0\}$ for the equation $\mathcal{L}_\infty u = 0$ yielding (smooth) classical solutions. We prove the following corollary.

**Corollary 1.5.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in C^1(\Omega)$ be a viscosity solution to*

$$\mathcal{L}_\infty u = 0 \quad \textit{in} \quad \Omega \setminus \{x \in \Omega : u(x) = 0\},$$

*then $u \in C^\infty(\Omega)$ and it is a classical solution to (10) in the whole $\Omega$.*

We observe that the $C^1$ regularity in the theorem above is sharp as can be seen by the example given by $u \in C^{0,1}(B_1)$ defined as $u(x) = |x_n|$.

Finally, we reach our last application, the Schwarz reflection principle for the equation $\mathcal{L}_\infty u = 0$. We highlight that our result also encompasses a regularity result by showing that $C^1$ solution up to flat boundary are indeed smooth. The result goes as follows.

**Corollary 1.6** (Schwarz reflection principle)**.** *Suppose $u \in C^0_{vfb}(\overline{B^+_R})$ be a viscosity solution to*

$$\mathcal{L}_\infty u = 0 \ \textit{in} \ B^+_R.$$

*Let $U$ be the odd reflection across the flat boundary, i.e.,*

$$U(x', x_n) = \begin{cases} u(x', x_n) & \textit{if } (x', x_n) \in B_R,\ x_n \geq 0 \\ -u(x', -x_n) & \textit{if } (x', x_n) \in B_R,\ x_n < 0. \end{cases}$$

*Then, $U \in C^0(\overline{B_R})$ and it is a viscosity solution to*

$$\mathcal{L}_\infty U = 0 \ \textit{in} \ B_R.$$

*Moreover, if $u \in C^1_{fb}(B^+_R)$ then $U \in C^\infty(B_R)$. In particular, $u \in C^\infty_{fb}(\overline{B^+_R})$.*

The plan of this paper is as follows: Since the regularity theory for (10) is used to prove the classification results, we begin by proving Theorem 1.2 in Section 2. In Section 3 a truncation argument is employed to introduce a suitable Lieberman's class of functions. Barriers are constructed in Section

4, which are used in Section 5 to prove the boundary gradient estimates. Section 5 presents a proof of Theorem 1.1. Finally, Section 7 deals with the proofs of the results regarding the applications.

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# 2 Proof of Theorem 1.2

In what follows we introduce various concepts of solutions to $\mathcal{L}_\infty u = 0$ that are used to prove the classification results. This set up the environment for our regularity result (Theorem 1.2). This will be used on several occasions in the proof to come.

**Definition 2.1.** *Let $\Omega \subset \mathbb{R}^n$ be an open set.*

1. *A function $u \in C^2(\Omega)$ is a classical solution to (10) if it satisfies the equation (10) pointwise everywhere in $\Omega$.*

2. *A function $u \in W^{2,p}_{loc}(\Omega)$, $1 \le p < \infty$, is a strong solution to (10) if it satisfies the equation (10) almost everywhere in $\Omega$.*

3. *Let*

$$\mathcal{W}(\Omega) = \left\{ u \in W^{1,1}_{loc}(\Omega) : \int_V 2e^{|\nabla u|^2} |\nabla u| \, dx < \infty \ \forall \, V \subset\subset \Omega \right\}.$$

*A function $u \in \mathcal{W}(\Omega)$ is a weak solution to (10) if*

$$\int_\Omega 2e^{|\nabla u|^2} \nabla u \nabla \varphi \, dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega),$$

*i.e., $u$ solves $\mathcal{L}_\phi(u) = 0$ in the sense of distributions, where $\phi(t) = 2te^{t^2}$ for $t \ge 0$.*

4. *A function $u \in C(\Omega)$ satisfies*

$$\mathcal{L}_\infty u \ge 0 \quad \text{in } \Omega \tag{15}$$

*in the viscosity sense if for every $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at $x_0 \in \Omega$ then $\mathcal{L}_\infty \varphi(x_0) \geq 0$. In this case, we say that $u$ is a viscosity subsolution to (10) in $\Omega$. A function $u \in C(\Omega)$ satisfies*

$$\mathcal{L}_\infty u \leq 0 \quad \text{in } \Omega \tag{16}$$

*in the viscosity sense if for every $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at $x_0 \in \Omega$ then $\mathcal{L}_\infty \varphi(x_0) \leq 0$. In this case, we say that $u$ is a viscosity supersolution to (10) in $\Omega$. If $u$ satisfies (15) and (16), we say that $u$ is a viscosity solution to (10) in $\Omega$.*

5. *A function $u \in W^{1,1}(\Omega)$ is a local minimizer of (9) if $J_\Omega(u) < \infty$ and*

$$J_V(u) \leq J_V(u + \varphi)$$

*for every $V \subset\subset \Omega$ and for every $\varphi \in W_0^{1,1}(V)$.*

In this paper, by a modulus of continuity we mean a nondecreasing function $\omega : [0, \infty) \to [0, \infty)$ such that $\lim_{t \to 0^+} \omega(t) = \omega(0) = 0$. We now recall the definition of the rank one map. Let $p, q \in \mathbb{R}^n$, we denote $p \otimes q$ the linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by $p \otimes q(v) = \langle q, v \rangle p$ for all $v \in \mathbb{R}^n$. It is easy to observe that $\|p \otimes q\| = \|p\| \|q\|$. In matrix terms, $p \otimes q = (a_{ij})$, $i, j = 1, \ldots, n$, $a_{ij} = p_i q_j$, where $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ are the coordinates representation in the canonical basis of $\mathbb{R}^n$. For more details we refer the reader to the Notation Section in [21] on page xviii.

For future use, we now present equivalent definitions of viscosity solutions in the next lemma, which is inspired by Proposition 2.4 in [10].

**Lemma 2.2.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then, the following are equivalent*

*a) $u$ is a viscosity subsolution to $\mathcal{L}_\infty(u) = 0$ in $\Omega$.*

*b) If $x_0 \in \Omega$, $A$ is an open neighborhood of $x_0$, $\varphi \in C^2(A)$ such that*

$$u \leq \varphi \text{ in } A \quad \text{and} \quad u(x_0) = \varphi(x_0). \tag{17}$$

*Then $\mathcal{L}_\infty(\varphi)(x_0) \geq 0$.*

*c) Same as b) with $\varphi \in C^2(A)$ replaced by $\varphi$ is a paraboloid.*

*Proof.* We start by proving that $a$) implies $b$). Indeed, since $A$ is a neighborhood of $x_0$, there exists $B_{2\delta}(x_0) \subset A$ for some $\delta > 0$. Now, we consider a cut-off function $\xi \in C^\infty(\mathbb{R}^n)$ such that so that $\xi \equiv 1$ in $B_\delta(x_0)$ and $supp(\xi) \subset B_{2\delta}(x_0)$. Now, $\psi := \varphi \cdot \xi \in C_c^2(B_{2\delta}(x_0))$. This way, denoting by $\psi^*$ the extension of $\psi$ by zero outside $B_{2\delta}(x_0)$, we see that $\psi^* \in C_c^2(\Omega)$.

Now, once $\psi^* = \varphi$ in $B_\delta(x_0)$, $u - \psi^*$ has a local maximum at $x_0$. Thus, once $D^2\psi^*(x_0) = D^2\varphi(x_0)$ and $\nabla\psi^*(x_0) = \nabla\varphi(x_0)$ and $u$ is a viscosity subsolution to $\mathcal{L}_\infty(u) = 0$ in $\Omega$, we conclude that, $\mathcal{L}_\infty(\varphi)(x_0) = \mathcal{L}_\infty(\psi^*)(x_0) \geq 0$. This proves $b$). It is immediate that $b$) implies $c$). Now let us prove that $c$) implies $a$). Indeed, let $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at $x_0$. By Taylor's expansion, we have

$$\varphi(x) = P_{x_0}(x) + o(|x - x_0|^2) \quad \text{as} \quad x \to x_0,$$

where $P_{x_0}(x) = \varphi(x_0) + \nabla\varphi(x_0)\cdot(x - x_0) + \frac{1}{2}(x - x_0)^t D^2\varphi(x_0)(x - x_0)$. From this, for small $\varepsilon \in (0, 1)$ given, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\varphi(x) \leq P_{x_0}(x) + \varepsilon|x - x_0|^2 =: P^\varepsilon_{x_0}(x) \quad \forall x \in B_\delta(x_0). \tag{18}$$

Now, reducing $\delta > 0$ if necessary, by assumption, we have

$$u(x) - \varphi(x) \leq u(x_0) - \varphi(x_0), \quad \forall x \in B_\delta(x_0).$$

This implies,

$$u(x) \leq \varphi(x) + (u(x_0) - \varphi(x_0)) \quad \forall x \in B_\delta(x_0).$$

Inequality above together with (18) implies that for all $x \in B_\delta(x_0)$,

$$u(x) \leq \varphi(x) + (u(x_0) - \varphi(x_0)) \leq P^\varepsilon_{x_0}(x) + (u(x_0) - \varphi(x_0)) =: Q^\varepsilon_{x_0}(x)$$

Clearly,

$$D^2Q^\varepsilon_{x_0}(x) = D^2P^\varepsilon_{x_0}(x) = D^2P_{x_0}(x) + 2\varepsilon I_n, \quad \nabla Q^\varepsilon_{x_0}(x) = \nabla\varphi(x_0) + 2\varepsilon(x - x_0)$$

Since $c$) holds, we conclude that

$$\begin{aligned} 0 \leq \mathcal{L}_\infty(Q^\varepsilon_{x_0})(x_0) &= \mathcal{L}_\infty(P^\varepsilon_{x_0})(x_0) \\ &= 2\Delta_\infty(P^\varepsilon_{x_0})(x_0) + \Delta P^\varepsilon_{x_0}(x_0) \\ &= 2\langle (D^2\varphi(x_0) + 2\varepsilon I_n)\cdot\nabla\varphi(x_0), \nabla\varphi(x_0)\rangle + \Delta\varphi(x_0) + 2n\varepsilon \\ &= \mathcal{L}_\infty(\varphi)(x_0) + 4\varepsilon|\nabla\varphi(x_0)|^2 + 2n\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain $\mathcal{L}_\infty(\varphi)(x_0) \geq 0$. This finishes the proof.

□

**Lemma 2.3.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $X \in C^0(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega)$ be a vector field satisfying*

$$|X(x) - X(y)| \leq \omega(|x - y|) \quad \forall\, x, y \in \Omega,$$

*where $\omega$ is a modulus of continuity. Set*

$$A(x) = X(x) \otimes X(x) + I_n = (X_i(x)X_j(x) + \delta_{ij}), \quad \forall\, i, j = 1, \dots, n, \forall\, x \in \Omega,$$

*where by $I_n$ we mean the $n \times n$-identity matrix and $X_1, \dots, X_n$ are the components of $X$, i.e., $X(x) = (X_1(x), \dots, X_n(x))$, $x \in \Omega$. Then,*

1. *$A$ is a uniformly elliptic (symmetric) matrix with ellipticity constants given by $1$ and $\left(1 + \|X\|^2_{L^\infty(\Omega)}\right)$, i.e.,*

$$|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \left(1 + \|X\|^2_{L^\infty(\Omega)}\right)|\xi|^2 \quad \forall\, x \in \Omega, \xi \in \mathbb{R}^n.$$

2. $\|A(x) - A(y)\| \le 2\|X\|_{L^\infty(\Omega)} \cdot \omega(|x - y|) \quad \forall\, x, y \in \Omega.$

*Proof.* For item (1), we observe that for every $\xi \in \mathbb{R}^n$, we have

$$\langle A(x)\xi, \xi\rangle = \langle X(x) \otimes X(x)\xi, \xi\rangle + |\xi|^2.$$

Thus,

$$(\|X\|^2_{L^\infty(\Omega)} + 1)|\xi|^2 \ge (|X(x)|^2 + 1)|\xi|^2 \ge \langle X(x) \otimes X(x)\xi, \xi\rangle + |\xi|^2 \ge |\xi|^2.$$

For item (2), for every $x, y \in \Omega$, we have

$$\begin{aligned}
&\|A(x) - A(y)\| = \|X(x) \otimes X(x) - X(y) \otimes X(y)\| \\
&\le \|X(x) \otimes X(x) - X(x) \otimes X(y)\| + \|X(x) \otimes X(y) - X(y) \otimes X(y)\| \\
&= \|X(x) \otimes (X(x) - X(y))\| + \|(X(x) - X(y)) \otimes X(y)\| \\
&= |X(x)||X(x) - X(y)| + |(X(x) - X(y)||X(y)| \\
&\le 2\|X\|_{L^\infty(\Omega)} \cdot \omega(|x - y|).
\end{aligned}$$

□

**Lemma 2.4.** *Let $V \subset \mathbb{R}^n$ be an open set, $u \in W^{1,p}(V) \cap L^\infty(V)$, where $1 \le p < \infty$, and $F \in C^1(\mathbb{R})$. Then the composite function $F(u) \in W^{1,p}(V)$ and $(F(u))_{x_i} = F'(u)u_{x_i}$ a.e. on $V$ for $i = 1, \dots, n$.*

*Proof.* Let $L = \|u\|_{L^\infty(V)}$ and $\varphi_L \in C^1(\mathbb{R})$, satisfying $0 \le \varphi_L \le 1$, $\varphi_L = 1$ on $[-2L, 2L]$, $\mathrm{supp}(\varphi_L) \subset (-3L, 3L)$ and $|\varphi_L'| \le 2/L$. Set $F_L = F \cdot \varphi_L$. We then have $F_L = F$ on $[-2L, 2L]$, $\mathrm{supp}(F_L) \subset (-3L, 3L)$, $F_L \in C^1(\mathbb{R})$, and $F_L' = F' \cdot \varphi_L + F \cdot \varphi_L'$. Since

$$\|F_L'\|_{L^\infty(\mathbb{R})} \le \sup_{[-3L,3L]} |F'| + \frac{2}{L} \sup_{[-3L,3L]} |F| < \infty,$$

by the chain rule [11, Theorem 4.4], $F_L(u) \in W^{1,p}(V)$ and

$$(F_L(u))_{x_i} = F_L'(u)u_{x_i} \quad \text{a.e. on } V \text{ for } i = 1, \ldots, n.$$

Since $|u(x)| \leq L$ a.e. on $V$, it follows that $F_L(u) = F(u)$ on $V$, and then we conclude the proof of the lemma. □

***Proof of Theorem 1.2.*** Let us prove that

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a) \iff (e),$$

which will establish the theorem.

We first prove that $(a) \Rightarrow (b)$. For this, we only observe that $C^2(\Omega) \subset W^{2,n}_{loc}(\Omega)$.

We now prove that $(b) \Rightarrow (c)$. Assume that $u \in W^{2,p}_{loc}(\Omega)$, $1 \leq p < \infty$, is a strong solution to (10). Let $\varphi \in C_c^\infty(\Omega)$ and $V \subset\subset \Omega$ such that supp$\varphi \subset V$. Using Lemma 2.4 to $F(t) = e^{t^2}$, we obtain $e^{|\nabla u|^2} \in W^{1,p}(V) \cap L^\infty(V)$. For each $i \in \{1, \ldots, n\}$, since $u_{x_i} \in W^{1,p}(V) \cap L^\infty(V)$, we have $e^{|\nabla u|^2}u_{x_i} \in W^{1,p}(V) \cap L^\infty(V)$, by Leibniz's rule in Sobolev space (see Theorem 4.4 in [11]). Using Lemma 2.4 once more, we have

$$\left(e^{|\nabla u|^2}u_{x_i}\right)_{x_i} = e^{|\nabla u|^2}\left(2\sum_{k=1}^n u_{x_k}u_{x_k x_i}u_{x_i} + u_{x_i x_i}\right), \quad \forall\, i \in \{1, \ldots, n\}.$$

Adding the identities above, we arrive at

$$-\text{div}\left(e^{|\nabla u|^2}\nabla u\right) = -e^{|\nabla u|^2}\left(2\Delta_\infty u + \Delta u\right) = 0 \quad \text{a.e. in } V.$$

Hence,

$$0 = -\int_V \varphi \text{div}\left(e^{|\nabla u|^2}\nabla u\right)\, dx = \int_\Omega e^{|\nabla u|^2}\nabla u \nabla\varphi\, dx,$$

which proves (c).

In order to show that $(c) \Rightarrow (d)$, assume that $u$ is a weak solution to (10). Let $V \subset\subset \Omega$ be an open set and

$$J_V(v) = \int_V e^{|\nabla v|^2}\, dx, \quad v \in W^{1,1}(V),$$

whenever $J_V(v)$ is finite. Let $v \in W^{1,1}(V)$ such that $v - u \in W_0^{1,1}(V)$. Using the convexity of the function $F(\xi) = e^{|\xi|^2}$, $\xi \in \mathbb{R}^n$, we obtain

$$J_V(v) = \int_V e^{|\nabla v|^2}\, dx \geq \int_V e^{|\nabla u|^2}\, dx + \int_V 2e^{|\nabla u|^2}\nabla u \nabla(v - u)\, dx. \tag{19}$$

Since $v - u \in W_0^{1,1}(V)$, there exists $\phi_k \in C_0^\infty(V)$ such that $\phi_k \to v - u$ in $W^{1,1}(V)$ as $k \to \infty$. Once $u$ is weak solution to (10), we have

$$\int_V 2e^{|\nabla u|^2} \nabla u \nabla \phi_k \, dx = 0 \quad \forall\, k \in \mathbb{N}.$$

Letting $k \to \infty$, we obtain

$$\int_V 2e^{|\nabla u|^2} \nabla u \nabla (v-u) \, dx = 0. \tag{20}$$

Combining (19) with (20) yields

$$J_V(v) = \int_V e^{|\nabla v|^2} \, dx \geq J_V(u)$$

for every $v \in W^{1,1}(V)$ such that $v - u \in W_0^{1,1}(V)$. This finishes the proof of $(d)$.

We now prove $(d) \Rightarrow (a)$. Suppose $u$ is a local minimizer of (9). From the main result in [19], it follows that $u \in C^2(\Omega)$ and is a classical solution to (10).

It remains to prove that $(a) \iff (e)$. We first prove that $(a) \Rightarrow (e)$. Assume that $u \in C^2(\Omega)$ is a classical solution to (10). Let $\varphi \in C^2(\Omega)$ be such that $u - \varphi$ has a local maximum at $x_0$, then $\nabla u(x_0) = \nabla \varphi(x_0)$ and the Hessian matrix $D^2(u - \varphi)(x_0)$ is negative-semidefinite, that is, $D^2 u(x_0) \leq D^2 \varphi(x_0)$. Thus,

$$\begin{aligned}
\mathcal{L}_\infty \varphi(x_0) &= 2\Delta_\infty \varphi(x_0) + \Delta \varphi(x_0) \\
&= 2\langle D^2\varphi(x_0)\nabla\varphi(x_0), \nabla\varphi(x_0)\rangle + \Delta\varphi(x_0) \\
&\geq 2\langle D^2 u(x_0)\nabla u(x_0), \nabla u(x_0)\rangle + \Delta u(x_0) \\
&= \mathcal{L}_\infty u(x_0) \\
&= 0,
\end{aligned}$$

therefore $u$ is a viscosity subsolution to (10). If we now assume that $u - \varphi$ has a local minimum at $x_0$, then $\nabla u(x_0) = \nabla \varphi(x_0)$ and $D^2 u(x_0) \geq D^2 \varphi(x_0)$. Repeating the argument above, we find

$$\mathcal{L}_\infty \varphi(x_0) \leq \mathcal{L}_\infty u(x_0) = 0,$$

which implies that $u$ is a viscosity supersolution to (10). Hence, $u$ is a viscosity solution to (10), and (e) is proved. We now prove that $(e) \Rightarrow (a)$. Let $u \in C^1(\Omega)$ be a viscosity solution to (10). We observe that the equation (10) can be written as

$$\operatorname{Tr}\big(A(x) D^2 u(x)\big) = 0, \quad x \in \Omega,$$

where $A(x) = X(x) \otimes X(x) \; + I_n$, with $I_n$ being the $n \times n$-identity matrix, $X(x) = \sqrt{2}\nabla u(x)$, and Tr is the trace operator. Let $V \subset\subset \Omega$, $\delta_V := dist(V, \partial\Omega)$, and $W = \{x \in \Omega : dist(x, \partial V) < \delta_V/4\} \subset\subset \Omega$. Since $u \in C^1(\Omega)$, $L_W := \|\nabla u\|_{L^\infty(W)} < \infty$. By item 1 of Lemma 2.3, $A$ is uniformly elliptic in $W$ with ellipticity constants given by 1 and $1 + 2L_W^2$. Set

$$\omega(r) := \sqrt{2} \sup_{x,y\in\overline{W},\, |x-y|<r} |\nabla u(x) - \nabla u(y)|.$$

Applying Lemma 2.3 to $X$ in $\overline{W}$, we have

$$|A(x) - A(y)| \leq 2L_W \cdot \omega(|x - y|), \quad \forall\, x, y \in \overline{W}.$$

As a consequence, for every $x_0 \in V$ and for every $r \leq r_0 := \delta_V/4$, we have

$$\left( ⨍_{B_r(x_0)} |A(x) - A(x_0)|^n \, dx \right)^{\frac{1}{n}} \leq 2L_W \cdot \omega(r).$$

Hence,

$$\lim_{r\to 0^+} \left( ⨍_{B_r(x_0)} |A(x) - A(x_0)|^n \, dx \right)^{\frac{1}{n}} = 0 \quad \text{uniformly for } x_0 \in V.$$

Now Caffarelli's $C^{1,\alpha}$-regularity theorem (Theorem 8.3 in [10]) implies that solution $u$ is $C^{1,\alpha}(V)$ for every $\alpha \in (0, 1)$, with $C^{1,\alpha}$-estimates in $W$ depending on $W$ itself and $\alpha$. Since $A \in C^\alpha(\overline{W})$, By Caffarelli-Schauder theory (Theorem 8.1 and Remark 3), $u \in C^{2,\alpha}(\overline{W})$ with estimates. Since $u \in C^2(V)$ and is a viscosity solution, Lemma 2.5 in [10] implies that $u$ is a classical solution. Combining this with the classical bootstrap argument, we conclude that $u \in C^\infty(V)$. As $V \subset\subset \Omega$ is arbitrary, we obtain that $u \in C^\infty(\Omega)$. This finishes the proof of $(a)$. □

# 3 Truncated equation

We begin with a truncation argument to introduce a suitable Lieberman's class. The purpose here is to recover the uniform ellipticity for Lipschitz solutions.

**Proposition 3.1.** *Let $g \in \mathcal{C}_\delta$ for some $\delta > 0$ and $L > 0$. Set*

$$\mathcal{D}_L(t) := \begin{cases} g'(t), & 0 < t < L \\ g'(L), & t \geq L. \end{cases}$$

*Then* $\mathcal{D}_L \in L^1[0,t]$ *for every* $t > 0$*. Assume now that*

$$\sup_{t\in(0,L]} Q_g(t) =: M_L < +\infty,$$

*and define*

$$g_{_L}(t) := \int_0^t \mathcal{D}_L(s)\, ds, \quad t \geq 0.$$

*Then* $g_{_L} \in \mathcal{C}_{\delta_L, g_0^L}$ *with* $\delta_L := \min\{1, \delta\}$ *and* $g_0^L := \max\{1, M_L\}$*.*

*Proof.* We start with the observation that $0 \leq \mathcal{D}_L \in L^\infty_{loc}(0,\infty)$. In order to prove the first claim, it is enough to show that $\mathcal{D}_L \in L^1[0,L]$. Indeed, let $0 < \varepsilon < L$. Then

$$\int_\varepsilon^L |\mathcal{D}_L(s)|\, ds = \int_\varepsilon^L g'(s)\, ds = g(L) - g(\varepsilon).$$

Letting $\varepsilon \to 0^+$ and using the Monotone Convergence Theorem and the continuity of $g$ at 0, we conclude that $\mathcal{D}_L \in L^1[0,L]$. Furthermore, by the absolute continuity of the Lebesgue integral (Proposition 1.12 in [26]), we have

$$g_{_L}(0) = \lim_{t\to 0^+} g_{_L}(t) = 0.$$

Hence, $g_{_L} \in C^0[0,\infty)$. Now, by the Fundamental Theorem of Calculus,

$$g'_{_L}(t) = \begin{cases} g'(t), & 0 < t < L \\ g'(L), & t \geq L. \end{cases}$$

Hence $g_{_L} \in C^1(0,+\infty)$ and $g_{_L}$ satisfies $(g_1)$. Furthermore,

$$g_{_L}(t) = \int_0^t \mathcal{D}_L(s) ds > 0 \quad \text{if } t > 0$$

and for $t > L$ we have

$$\begin{aligned} g_{_L}(t) &= \int_0^L \mathcal{D}_L(s) ds + \int_L^t \mathcal{D}_L(s) ds \\ &= \int_0^L g'(s) ds + \int_L^t g'(L) ds \\ &= g(L) + g'(L)(t-L) \\ &\geq g'(L)(t-L). \end{aligned} \tag{21}$$

Hence,

$$\lim_{t\to+\infty} g_{_L}(t) = +\infty,$$

and we conclude that $g_{_L}$ satisfies $(g_2)$. We now observe that

(A) For $0 < t < L$, $g_{_L}(t) = \int_0^t g'(s)ds = g(t)$ and

$$Q_{g_{_L}}(t) = \frac{tg'_{_L}(t)}{g_{_L}(t)} = \frac{t\mathcal{D}_L(t)}{g_{_L}(t)} = \frac{tg'(t)}{g(t)} \geq \delta.$$

Moreover, by assumption $Q_{g_{_L}}(t) \leq M_L$ for every $t \in (0, L]$. Hence

$$\delta \leq Q_{g_{_L}}(t) \leq M_L \quad \forall\, t \in (0, L]. \tag{22}$$

(B) For $t \geq L$, by (21), we have

$$Q_{g_{_L}}(t) = \frac{tg'_{_L}(t)}{g_{_L}(t)} = \frac{tg'(L)}{g(L) + g'(L)(t-L)}.$$

Now,

$$Q'_{g_{_L}}(t) = \frac{Lg'(L)^2}{[g(L) + g'(L)(t-L)]^2}\left(\frac{g(L)}{Lg'(L)} - 1\right).$$

There are two cases to consider:

(a) $\dfrac{g(L)}{Lg'(L)} \geq 1$,

(b) $\dfrac{g(L)}{Lg'(L)} < 1$.

If (a) holds then $Q_{g_{_L}}$ is increasing in $[L, \infty)$. Thus,

$$\inf_{t\in[L,\infty)} Q_{g_{_L}}(t) = \lim_{t\to L^+} Q_{g_{_L}}(t) = Q_{g_{_L}}(L) = \frac{Lg'(L)}{g(L)} \geq \delta,$$
$$\sup_{t\in[L,\infty)} Q_{g_{_L}}(t) = \lim_{t\to\infty} Q_{g_{_L}}(t) = 1.$$

If (b) holds then $Q_{g_{_L}}$ is decreasing in $[L, \infty)$. Thus

$$\inf_{t\in[L,\infty)} Q_{g_{_L}}(t) = \lim_{t\to\infty} Q_{g_{_L}}(t) = 1,$$
$$\sup_{t\in[L,\infty)} Q_{g_{_L}}(t) = \lim_{t\to L^+} Q_{g_{_L}}(t) = Q_{g_{_L}}(L) \leq M_L.$$

Consequently,

$$\min\{1, \delta\} \leq Q_{g_{_L}}(t) \leq \max\{1, M_L\} \quad \forall\, t \geq L. \tag{23}$$

Now (22) and (23) together imply

$$\delta_L \leq Q_{g_{_L}}(t) \leq g_0^L \quad \forall\, t > 0,$$

and $g_{_L} \in \mathcal{C}_{\delta_L, g_0^L}$ as required.

□

**Remark 3.2.** *We observe that for* $\phi(t) = 2te^{t^2}$, *by (4),*

$$Q_\phi(t) = 1 + 2t^2 \geq 1 =: \delta.$$

*In particular,*

$$M_L := \sup_{t\in(0,L]} Q_\phi(t) = 1 + 2L^2.$$

*Therefore, the truncated function* $\phi_L \in \mathcal{C}_{\min\{\delta,1\},\max\{1,M_L\}} = \mathcal{C}_{1,M_L}$.

Now, we show the equivalence of solutions with the truncation equation.

**Proposition 3.3.** *Let* $g \in \mathcal{C}_\delta$ *for some* $\delta > 0$ *and assume that for some* $L > 0$, *we have*

$$M_L := \sup_{t\in(0,L]} Q_g(t) < +\infty.$$

*Assume that for some open set* $V \subset \mathbb{R}^n$ *the function* $u \in W^{1,\infty}(V)$ *with*

$$\|\nabla u\|_{L^\infty(V)} \leq L.$$

*Then,* $u$ *is a weak solution to*

$$\mathcal{L}_g(u) = \operatorname{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 \text{ in } V$$

*if only if* $u$ *is a weak solution to the truncated equation*

$$\mathcal{L}_{g_L}(u) = \operatorname{div}\left(\frac{g_L(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 \text{ in } V,$$

*where* $g_L \in \mathcal{C}_{\delta_L, g_0^L}$.

*Proof.* Since $u \in W^{1,\infty}(V)$, we have $g(|\nabla u|) \in L^\infty(V)$. Now, for every $\varphi \in C_0^\infty(V)$, we have

$$\int_V \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\nabla\varphi\, dx = \left[\int_{V\cap\{|\nabla u|>L\}} + \int_{V\cap\{|\nabla u|\leq L\}}\right]\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\nabla\varphi\, dx. \tag{24}$$

Since $u \in W^{1,\infty}(V)$ and $\|\nabla u\|_{L^\infty(V)} \leq L$, it follows that $|V\cap\{|\nabla u| > L\}| = 0$. Thus, once $g(|\nabla u|)|\nabla\varphi| \in L^1(V)$, we have

$$\int_{V\cap\{|\nabla u|>L\}} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\nabla\varphi\, dx = 0. \tag{25}$$

The proposition follows now from the identity

$$\int_{V\cap\{|\nabla u|\leq L\}} \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\nabla\varphi\, dx = \int_V \frac{g_L(|\nabla u|)}{|\nabla u|}\nabla u\nabla\varphi\, dx.$$

The proof is complete. □

# 4 Barrier

In this section, we construct barriers by borrowing ideas from the arguments employed in the proof of Theorem 1.7 in [23]. They are supersolutions for the operator $\mathcal{L}_\infty$. Moreover, we show they possess similar geometry as those barriers constructed in [8, 9], except they are flipped upside down.

**Proposition 4.1.** *Assume $\rho \in (0,1)$, $R > 0$ and $\mathcal{A}_{\rho,R} := B_R \setminus \overline{B_{\rho R}}$. Given $M \geq 0$, there exists $\Gamma_+^M \in C^\infty(\overline{\mathcal{A}_{\rho,R}})$ such that:*

*(i)* $\Gamma_+^M\Big|_{\partial B_R} = M$ *and* $\Gamma_+^M\Big|_{\partial B_{\rho R}} = 0$;

*(ii) There exists a constant $C > 0$ depending only $n$ and $\rho$ such that*

$$\mathcal{L}_\infty \Gamma_+^M \leq -C\frac{M}{R^2} \leq 0 \quad \text{in } \overline{\mathcal{A}_{\rho,R}};$$

*(iii) There exist $C_1, C_2 > 0$ depending only on $n$ and $\rho$ such that*

$$C_1\frac{M}{R}d(x,\partial B_{\rho R}) \leq \Gamma_+^M(x) \leq C_2\frac{M}{R}d(x,\partial B_{\rho R}) \quad \forall\, x \in \overline{\mathcal{A}_{\rho,R}};$$

$$C_1\frac{M}{R} \leq |\nabla\Gamma_+^M(x)| \leq C_2\frac{M}{R} \quad \forall\, x \in \overline{\mathcal{A}_{\rho,R}}.$$

*Proof.* The proof follows very closely the proof of Theorem 1.7 in [23]. Let

$$\Gamma_+^M(x) := M - \Gamma_R^M(x), \quad x \in \overline{\mathcal{A}_{\rho,R}}, \tag{26}$$

where $\Gamma_R^M$ is the function given by Theorem 1.7 in [23], i.e.,

$$\Gamma_R^M(x) = v(R - |x|), \quad x \in \overline{\mathcal{A}_{\rho,R}}, \tag{27}$$

where $v(t) := \alpha(e^{\beta t} - 1)$, $t \in \mathbb{R}$, and

$$\alpha := \frac{M}{e^{\beta(1-\rho)R} - 1} \quad \text{and} \quad \beta := \frac{2(n-1)}{\rho R}. \tag{28}$$

Clearly, $\Gamma_+^M \in C^\infty(\overline{\mathcal{A}_{\rho,R}})$. Moreover, since $\Gamma_R^M|_{\partial B_R} = 0$ and $\Gamma_R^M|_{\partial B_{\rho R}} = M$, we have $\Gamma_+^M\Big|_{\partial B_R} = M$ and $\Gamma_+^M\Big|_{\partial B_{\rho R}} = 0$, which proves $(i)$. By $(ii)$ of Theorem 1.7 in [23], there exists a constant $C > 0$ depending only on $n$ and $\rho$ such that

$$\mathcal{L}_\infty \Gamma_R^M \geq C\frac{M}{R^2} \geq 0 \quad \text{in } \overline{\mathcal{A}_{\rho,R}}.$$

Consequently,

$$\mathcal{L}_\infty \Gamma_+^M = \mathcal{L}_\infty \left( M - \Gamma_R^M(x) \right) \le -C \frac{M}{R^2} \le 0 \quad \text{in } \overline{\mathcal{A}_{\rho,R}},$$

which shows that $(ii)$ holds. Now, we prove $(iii)$. We set

$$\psi(t) := M - v(R-t), \quad t \in \mathbb{R}. \tag{29}$$

We observe that

$$\begin{aligned} &\psi(\rho R) = 0, \\ &\psi'(t) = v'(R-t) = \alpha\beta e^{\beta(R-t)} > 0, \forall t \in \mathbb{R}, \\ &\psi''(t) = -\alpha\beta^2 e^{\beta(R-t)} < 0, \forall t \in \mathbb{R}. \end{aligned}$$

Therefore, $\psi$ is a (strictly) concave function, which implies for all $t \in \mathbb{R}$

$$\psi(t) \le \psi(\rho R) + \psi'(\rho R)(t-\rho R) = \psi'(\rho R)(t-\rho R) = \alpha\beta e^{\beta(1-\rho)R}(t-\rho R). \tag{30}$$

By (26), (27), and (29),

$$\Gamma_+^M(x) = M - v(R-|x|) = \psi(|x|), \quad \forall\, x \in \overline{\mathcal{A}_{\rho,R}}.$$

This way, by (28) and (30), we obtain

$$\begin{aligned} \Gamma_+^M(x) &= \psi(|x|) \le \psi'(\rho R) \cdot (|x| - \rho R) \\ &= \alpha\beta e^{\beta(1-\rho)R} \cdot dist(x, \partial B_{\rho R}) \\ &\le \left( \frac{2(n-1) e^{2(n-1)(1-\rho)/\rho}}{\rho(e^{2(n-1)(1-\rho)/\rho} - 1)} \right) \cdot \frac{M}{R} \cdot d(x, \partial B_{\rho R}), \quad \forall\, x \in \overline{A_{\rho,R}}. \end{aligned}$$

On the other hand, given $t \in (\rho R, R)$, by the mean value theorem, there exists $\xi_t \in (\rho R, t)$ such that

$$\begin{aligned} \psi(t) &= \psi(t) - \psi(\rho R) = \psi'(\xi_t)(t - \rho R) \\ &= \alpha\beta e^{\beta(R-\xi_t)}(t-\rho R) \ge \alpha\beta(t-\rho R). \end{aligned} \tag{31}$$

Hence, by (31), we have

$$\begin{aligned} \Gamma_+^M(x) &= \psi(|x|) \ge \alpha\beta(|x| - \rho R) \\ &\ge \left( \frac{2(n-1)}{\rho(e^{2(n-1)(1-\rho)/\rho} - 1)} \right) \cdot \frac{M}{R} \cdot d(x, \partial B_{\rho R}), \quad \forall\, x \in \overline{A_{\rho,R}}. \end{aligned}$$

In order to show the gradient estimates in $(iii)$, by $(iii)$ of Theorem 1.7 in [23], we have

$$\frac{2(n-1)}{\rho[e^{2(n-1)(1-\rho)/\rho} - 1]} \frac{M}{R} \le |\nabla \Gamma_R^M(x)| \le \frac{2(n-1) e^{2(n-1)(1-\rho)/\rho}}{\rho[e^{2(n-1)(1-\rho)/\rho} - 1]} \frac{M}{R},$$

for every $x \in \overline{A_{\rho,R}}$. Since $|\nabla \Gamma_+^M(x)| = |\nabla \Gamma_R^M(x)|$, this finishes the proof. □

# 5 Boundary gradient estimates

The boundary estimates used in this paper are gathered in the following proposition. We single out item $(b)$ as it corresponds to the interior gradient estimate established in [23] and serves as the key element for establishing the classification result.

**Proposition 5.1** (Gradient estimates on the boundary)**.** *Let $u \in C^1(B_R^+) \cap C^0_{vfb}(\overline{B_R^+})$ be a viscosity solution to*

$$\mathcal{L}_\infty u = 0 \text{ in } B_R^+.$$

*Then, the following estimates hold*

$(a)$ $|u(x)| \leq C\left(\dfrac{\|u\|_{L^\infty(B_R^+)}}{R}\right) x_n,\ \forall\, x \in B^+_{R/2}$ *and for some* $C = C(n) > 0$.

$(b)$ $\|\nabla u\|_{L^\infty(B^+_{R/4})} \leq C\left(1 + \dfrac{\|u\|_{L^\infty(B_R^+)}}{R}\right)$ *for some* $C = C(n) > 0$.

$(c)$ *If*

$$|u(x)| \leq K|x| \quad \forall\, x \in B_R^+, \tag{32}$$

*then $u \in C^\infty(\overline{B^+_{R/4}})$ and it is a classical solution to $\mathcal{L}_\infty(u) = 0$ in $B^+_{R/4}$. Moreover, there exist $\alpha = \alpha(K,n) \in (0,1)$ and $C = C(K,n) > 0$ such that*

$$\|u\|^*_{C^{1,\alpha}\left(B^+_{R/8}\right)} \leq C\|u\|_{L^\infty\left(B^+_R\right)}, \tag{33}$$

*where*

$$\|u\|^*_{C^{1,\alpha}\left(B^+_r\right)} := \|u\|_{L^\infty\left(B^+_r\right)} + r\|\nabla u\|_{L^\infty\left(B^+_r\right)} + r^{1+\alpha}[\nabla u]_{C^{0,\alpha}\left(B^+_r\right)}.$$

*Proof.* We begin by proving the proposition for the case where $R = 1$. Invoking Theorem 1.2, the function $u$ is a classical solution to

$$\mathcal{L}_\infty u = 0 \text{ in } B_1^+.$$

Let $M := \|u\|_{L^\infty(B_1^+)}$. For simplicity of notation, we write $x = (x', x_n)$ instead of $x = (x_1, \dots, x_{n-1}, x_n)$. Define

$$Q = \left\{x = (x', x_n) \in \mathbb{R}^n_+ : |x'| \leq \frac{3}{4},\ 0 < x_n < \frac{1}{4}\right\}.$$

Given $x_0 = (x_0', -\frac{1}{16})$ with $|x_0'| \le \frac{3}{4}$, consider the function $\Gamma_+^M \in C^\infty(\overline{\mathcal{A}_{\frac{1}{2},\frac{2}{16}}})$ given by Proposition 4.1, where $\mathcal{A}_{\frac{1}{2},\frac{2}{16}} = B_{\frac{2}{16}}(x_0) \setminus \overline{B_{\frac{1}{16}}(x_0)}$. Thus,

$$u(z) \le \Gamma_+^M(z) \quad \forall z \in \partial \left( B_{\frac{2}{16}}(x_0) \cap \mathbb{R}_+^n \right),$$

and

$$\mathcal{L}_\infty \Gamma_+^M \le 0 = \mathcal{L}_\infty u \quad \text{in } B_{\frac{2}{16}}(x_0) \cap \mathbb{R}_+^n.$$

Due to the regularity theorem (Theorem 1.2), here we are entitled to use the comparison principle for classical solutions to quasilinear equations, namely, Theorem 10.1 in [13], to conclude that

$$u(z) \le \Gamma_+^M(z) \quad \forall\, z \in \overline{B_{\frac{2}{16}}(x_0) \cap \mathbb{R}_+^n}.$$

Thus, for every $x = (x_0', x_n)$, where $0 < x_n \le \frac{1}{16}$, we have $x \in \overline{B_{\frac{2}{16}}(x_0) \cap \mathbb{R}_+^n}$. By Proposition 4.1 $(iii)$, we have

$$u(x) \le \Gamma_+^M(x) \le C_2 M d(x, \partial B_{\frac{1}{16}}(x_0)) = C_2 M |x - (x_0', 0)| = C_2 M x_n,$$

for some constant $C_2 = C_2(n) > 0$. As $\mathcal{L}_\infty(-u) = -\mathcal{L}_\infty u$, we have $\mathcal{L}_\infty(\pm u) = 0$ in $B_1^+$. The argument above can also be applied to $-u$. Consequently, we conclude that

$$|u(x)| \le C_2 M x_n \quad \forall\, x \in Q^*, \tag{34}$$

where

$$Q^* = \left\{ x = (x', x_n) \in \mathbb{R}_+^n : |x'| \le \frac{3}{4},\ 0 < x_n \le \frac{1}{16} \right\}.$$

We claim that there exists $\widetilde{C_2} > 0$ such that

$$|u(x)| \le \widetilde{C_2} M x_n \quad \forall\, x \in Q_{\frac{1}{2}}^+, \tag{35}$$

where

$$Q_{\frac{1}{2}}^+ = \left\{ x = (x', x_n) \in \mathbb{R}_+^n : |x'| \le \frac{1}{2},\ 0 < x_n \le \frac{1}{2} \right\}.$$

Indeed, let $x \in Q_{\frac{1}{2}}^+$, we have two cases:

(i) $0 < x_n \le \frac{1}{16}$;

(ii) $x_n > \frac{1}{16}$.

For the case (i), $x \in Q^*$, and thus, $|u(x)| \le C_2 M x_n$ by (34). For the case (ii), if $x \in Q^+_{\frac{1}{2}}$,

$$|u(x)| \le M = 16\frac{M}{16} < 16Mx_n.$$

Taking $\widetilde{C_2} = \max\{C_2, 16\}$, we obtain (35), as claimed. In particular, since $B^+_{1/2} \subset Q^+_{\frac{1}{2}}$, we have

$$|u(x)| \le \widetilde{C_2} M x_n \quad \forall\, x \in B^+_{1/2}.$$

which proves $(a)$ in the case $R = 1$.
In order to prove $(b)$, let $y_0 = (y_0', y_{0n}) \in Q^+_{\frac{1}{4}}$, where

$$Q^+_{\frac{1}{4}} := \left\{ x = (x', x_n) \in \mathbb{R}^n_+ : |x'| \le \frac{1}{4},\ 0 < x_n \le \frac{1}{4} \right\}.$$

Applying the interior gradient estimates (Theorem 1.5 in [23]) in $B_{y_{0n}}(y_0)$, we obtain

$$|\nabla u(y_0)| \le \|\nabla u\|_{L^\infty(B_{\frac{y_{0n}}{2}}(y_0))} \le C_0 \left( 1 + \frac{\|u\|_{L^\infty(B_{y_{0n}}(y_0))}}{y_{0n}} \right), \tag{36}$$

where $C_0 = C_0(n) > 0$. It is easy to see that the following inclusion holds: $B_{y_{0n}}(y_0) \subset \overline{Q^+_{\frac{1}{2}}}$ for every $y_0 \in \overline{Q^+_{\frac{1}{4}}}$. Indeed, let $z = (z', z_n) \in B_{y_{0n}}(y_0)$. Thus

$$z_n \le 2y_{0n} \le 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

Moreover, and

$$|z'| \le |z' - y_0'| + |y_0'| \le |z - y_0| + |y_0'| < y_{0n} + |y_0'| \le \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Hence $B_{y_{0n}}(y_0) \subset \overline{Q^+_{\frac{1}{2}}}$. By (35), for every $x \in B_{y_{0n}}(y_0)$, we have

$$|u(x)| \le \widetilde{C_2} M x_n \le \widetilde{C_2} M 2 y_{0n}.$$

Consequently,

$$\frac{\|u\|_{L^\infty(B_{y_{0n}}(y_0))}}{y_{0n}} \le 2\widetilde{C_2} M. \tag{37}$$

From (36) and (37), we have

$$|\nabla u(y_0)| \le C(1 + M) \quad \forall\, y_0 \in Q^+_{\frac{1}{4}},$$

where $C := C_0(1 + 2\widetilde{C_2})$. Since $B^+_{\frac{1}{4}} \subset Q^+_{\frac{1}{4}}$, we have

$$|\nabla u(y)| \leq C(1 + M) \quad \forall y \in B^+_{\frac{1}{4}}.$$

Hence,

$$\|\nabla u\|_{L^\infty(B^+_{1/4})} \leq C\left(1 + \|u\|_{L^\infty(B^+_1)}\right), \tag{38}$$

which proves $(b)$ in the case $R = 1$.
Now, we prove $(c)$. By (32), we have

$$\sup_{B^+_1} |u| \leq K. \tag{39}$$

Combining $(b)$ with (39), we obtain

$$\|\nabla u\|_{L^\infty(B^+_{1/4})} \leq C\left(1 + \|u\|_{L^\infty(B^+_1)}\right) \leq C(1 + K) =: L. \tag{40}$$

By Theorem 1.2, $u$ is a weak solution to

$$\mathcal{L}_\phi(u) = \operatorname{div}(2e^{|\nabla u|^2}\nabla u) = 0 \quad \text{in } B^+_1,$$

where $\phi(t) = 2te^{t^2}$. Using Remark 3.2 with $L := C(1 + K)$, we have $\phi_L \in \mathcal{C}_{1,M_L}$. By Proposition 3.3 implies that $u$ is a weak solution to the truncated equation

$$\mathcal{L}_{\phi_L}(u) = \operatorname{div}\left(\frac{\phi_L(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0 \text{ in } B^+_{1/4},$$

and $u \in W^{1,\infty}(B^+_{1/4}) \cap C^0_{vfb}(\overline{B^+_{1/4}})$. By using the Schwarz reflection principle (Proposition 2.1 in [7]), the odd reflection $\tilde{u} \in W^{1,\infty}(B_{1/4})$ and it solves

$$\mathcal{L}_{\phi_L}(\tilde{u}) = \operatorname{div}\left(\frac{\phi_L(|\nabla \tilde{u}|)}{|\nabla \tilde{u}|}\nabla \tilde{u}\right) = 0 \text{ in } B_{1/4}, \tag{41}$$

in the distributional sense. By Lieberman's $C^{1,\alpha}$-regularity theory [18, Theorem 1.7], there exists $\alpha = \alpha(K, n) \in (0, 1)$ such that $\tilde{u} \in C^{1,\alpha}(B_{1/8})$ and

$$\|\tilde{u}\|_{C^{1,\alpha}(B_{1/8})} \leq C\|\tilde{u}\|_{L^\infty(B_{1/4})} = C\|u\|_{L^\infty(B^+_{1/4})},$$

where $C = C(K, n) > 0$. Once $\tilde{u} \equiv u$ in $\overline{B^+_{1/4}}$, we have

$$\|u\|_{C^{1,\alpha}(B^+_{1/8})} \leq \|\tilde{u}\|_{C^{1,\alpha}(B_{1/8})} \leq C\|\tilde{u}\|_{L^\infty(B^+_{1/4})} = C\|u\|_{L^\infty(B^+_{1/4})}. \tag{42}$$

This proves estimate (33) in the where $R = 1$. Finally, due to (41) and (40), the function $\tilde{u} \in C^{1,\alpha}(\overline{B_{1/4}})$ with $\|\nabla \tilde{u}\|_{L^\infty(B_{1/4})} \leq L$. By Proposition 3.3, the function $\tilde{u}$ is a weak solution to

$$\mathcal{L}_\phi(\tilde{u}) = \operatorname{div}\left(\frac{\phi(|\nabla \tilde{u}|)}{|\nabla \tilde{u}|}\nabla \tilde{u}\right) = 0 \text{ in } B_{1/4}.$$

By Theorem 1.2, $\tilde{u} \in C^\infty(B_{1/4})$ and it is classical solution to $\mathcal{L}_\infty(\tilde{u}) = 0$ in $B_{1/4}$. The same can be said about $u$ in $B^+_{1/4}$.
Let us prove the general case $R > 0$. Theorem 1.2 gives that $u \in C^\infty(B^+_R)$ and it is a classical to $\mathcal{L}_\infty(u) = 0$ in $B^+_R$. Now define the rescaled function

$$v(z) = \frac{u(Rz)}{R}, \quad z \in B^+_1.$$

Thus, $v \in C^\infty(B^+_1) \cap C^0_{vfb}(\overline{B^+_1})$. Moreover, since

$$\mathcal{L}_\infty(v(z)) = R\mathcal{L}_\infty(u(Rz)) \quad \forall\, z \in B^+_1.$$

Thus, $v \in C^1(B^+_1) \cap C^0_{vfb}(\overline{B^+_1})$ and it is classical solution to $\mathcal{L}_\infty(v) = 0$ in $B^+_1$. In particular, $v$ is a viscosity solution to $\mathcal{L}_\infty(v) =$ in $B^+_1$. We are now in a position to apply the previously discussed case when $R = 1$ to the function $v$. Items $(a)$ and $(b)$ applied to $v$ give the following estimates

$$|v(z)| \leq C\|v\|_{L^\infty(B^+_1)} z_n \quad \forall\, z \in B^+_{1/2}, \tag{43}$$

$$\|\nabla v\|_{L^\infty(B^+_{1/4})} \leq C\left(1 + \|v\|_{L^\infty(B^+_1)}\right). \tag{44}$$

Now we observe that

$$\|v\|_{L^\infty(B^+_1)} = \frac{1}{R}\|u\|_{L^\infty(B^+_R)}. \tag{45}$$

Hence, $(a)$ and $(b)$ follow readily translating back the estimates (43) and (44) taking into account (45). Regarding the item $(c)$, we observe that $v$ satisfies the growth condition (32). Now, $(c)$ applied to $v$, yields $v \in C^\infty(\overline{B^+_{1/4}})$. This clearly implies that $u \in C^\infty(\overline{B^+_{R/4}})$ since they differ essential by scaling. Additionally, item $(c)$ applied to $v$ gives

$$\|v\|_{C^{1,\alpha}\left(B^+_{1/8}\right)} \leq C\|v\|_{L^\infty\left(B^+_1\right)},$$

Finally, once more estimate (33) follows translating back the estimate above in terms of $u$ and recalling (45) and

$$\|v\|_{C^{1,\alpha}(B^+_{1/8})} = \frac{1}{R}\|u\|^*_{C^{1,\alpha}(B^+_{R/8})}$$

This finishes the proof of $(c)$ in the case $R > 0$. $\square$

# 6 Proof of Theorem 1.1

*Proof of Theorem 1.1.* Let $u \in C^1(\mathbb{R}^n_+) \cap C^0_{vfb}(\overline{\mathbb{R}^n_+})$ be a viscosity solution to (7) satisfying (8). We claim that there exists $L > 0$ such that

$$\sup_{\mathbb{R}^n_+} |\nabla u| \le L.$$

Indeed, the claim is trivial if $u \equiv 0$. Suppose $u \not\equiv 0$. By (8), there exist $K > 0$ and $R_0 > 0$ such that

$$|u(x)| \le K|x| \quad \forall\, x \in \mathbb{R}^n_+,\ |x| \ge R_0. \tag{46}$$

For every $R \ge R_0$, Proposition 5.1 (b) gives a positive constant $C_0 = C_0(n)$ for which

$$\|\nabla u\|_{L^\infty(B^+_{R/4})} \le C_0 \left(1 + \frac{\|u\|_{L^\infty(B^+_R)}}{R}\right). \tag{47}$$

By the regularity theorem (Theorem 1.2), $u$ is a classical solution to (7). Now, we can use the classical maximum principle for quasilinear equations (Theorem 10.3 in [13]). This way, since $u \not\equiv 0$, there exists $\xi_R \in \partial B^+_R \cap \mathbb{R}^n_+$ such that

$$\max_{\overline{B^+_R}} |u| \le \max_{\partial B^+_R} |u| = |u(\xi_R)|. \tag{48}$$

From (46) and (48), it follows that

$$\max_{\overline{B^+_R}} |u| = |u(\xi_R)| \le K|\xi_R| = KR \quad \forall\, R \ge R_0. \tag{49}$$

Combining (47) with (49), yields

$$\|\nabla u\|_{L^\infty(B^+_{R/4})} \le C_0 \left(1 + \frac{KR}{R}\right) = C_0\,(1 + K) \quad \forall\, R \ge R_0.$$

Letting $R \to \infty$, we obtain

$$\|\nabla u\|_{L^\infty(\mathbb{R}^n_+)} \le C_0\,(1 + K) =: L.$$

This finishes the proof of the claim. For every $x = (x', x_n) \in \mathbb{R}^n_+$, let $\bar{x} = (x', 0)$. By the mean value theorem, we have, for some $\xi_x \in [x, \bar{x}] \subset \overline{\mathbb{R}^n_+}$,

$$|u(x)| = |u(x) - u(\bar{x})| = |\nabla u(\xi_x) \cdot (x - \bar{x})| \le L x_n \le L|x|. \tag{50}$$

Let $R > 0$ and $x \in B^+_R$. Due to estimate (50), we can use Proposition 5.1 (c) to obtain

$$R^{1+\alpha} [\nabla u]_{C^\alpha(B^+_{R/8})} \le C \|u\|_{L^\infty(B^+_R)} \le CLR,$$

where $C = C(n, L) > 0$. Thus,

$$[\nabla u]_{C^\alpha(B^+_{R/8})} \leq \frac{CL}{R^\alpha}.$$

Letting $R \to \infty$, we obtain $[\nabla u]_{C^\alpha(\mathbb{R}^n_+)} = 0$, and hence $u$ is affine, that is, $u(x) = A \cdot x + b$, with $A = (A_1, \ldots, A_n)$. Since $b = u(0) = 0$, then $u(x) = A \cdot x$ for every $x \in \overline{\mathbb{R}^+_n}$. Once $0 = u(e_i) = A \cdot e_i = A_i$ for $i = 1, \ldots, n-1$. This way,

$$u(x) = A_n x_n = u(e_n) x_n \quad \forall\, x \in \overline{\mathbb{R}^n_+}.$$

This finishes the proof of Theorem 1.1. □

# 7 Proof of the results in the applications

This section is devoted to proving the results in the applications.

***Proof of Theorem 1.3***. Due to the regularity theorem (Theorem 1.2), $u$ is a classical solution to (13) in $\mathbb{R}^n$. Moreover, the interior gradient estimate for equation (13) (Theorem 1.5 in [23]) gives a universal constant $C_0 = C_0(n) > 0$ such that

$$\|\nabla u\|_{L^\infty(B_{R/2})} \leq C_0 \left(1 + \frac{\|u\|_{L^\infty(B_R)}}{R}\right). \tag{51}$$

By (12), there exist positive real numbers $K$ and $R_0$ such that

$$|u(x)| \leq K|x| \quad \text{whenever } x \in \mathbb{R}^n,\ |x| \geq R_0.$$

In particular,

$$\sup_{B_R} |u| \leq KR \quad \forall\, R \geq R_0. \tag{52}$$

Combining (51) with (52), yields

$$\|\nabla u\|_{L^\infty(B_{R/2})} \leq C_0 \left(1 + \frac{KR}{R}\right) = C_0\,(1 + K) \quad \forall\, R \geq R_0.$$

Letting $R \to \infty$, we obtain

$$\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \leq C_0\,(1 + K) =: L.$$

By Theorem 1.2 and Proposition 3.3, $u$ is a weak solution to the truncated equation

$$\mathcal{L}_{\phi_L}(u) = \operatorname{div}\left(\frac{\phi_L(|\nabla u|)}{|\nabla u|} \nabla u\right) = 0 \text{ in } \mathbb{R}^n,$$

where $\phi_L \in \mathcal{C}_{1,M_L}$, with $M_L = 1 + 2L^2$ (Remark 3.2). In particular, $u \in W^{1,\infty}(\mathbb{R}^n) \subset W^{1,\Phi_L}(B_R)$, where $\Phi_L(t) = \int_0^t \phi_L(s)ds$. From Lieberman's $C^{1,\alpha}$-regularity (Theorem 1.7 in [18]), there exist $\alpha = \alpha(L,n) \in (0,1)$ and $C = C(L,n) > 0$ such that

$$\|u\|^*_{C^{1,\alpha}\left(B_{R/2}\right)} \leq C\|u\|_{L^\infty(B_R)}. \tag{53}$$

From (12) and the maximum principle (Theorem 10.3 in [13]) applied to $\mathcal{L}_\infty u = 0$ in $B_R$,

$$\|u\|_{L^\infty(B_R)} = O(R) = o(R^{1+\alpha}) = R^{1+\alpha}o(1) \quad \text{as } R \to \infty. \tag{54}$$

Due to (53),

$$R^{1+\alpha}[\nabla u]_{C^{0,\alpha}\left(B_{R/2}\right)} \leq C\|u\|_{L^\infty(B_R)}. \tag{55}$$

From (54) and (55), we obtain

$$[\nabla u]_{C^{0,\alpha}\left(B_{R/2}\right)} = o(1) \quad \text{as } R \to \infty.$$

Let $x \in \mathbb{R}^n$. Take $R > 0$ so that $|x| \leq R/2$. Then,

$$|\nabla u(x) - \nabla u(0)| \leq [\nabla u]_{C^{0,\alpha}\left(B_{R/2}\right)}|x|^\alpha.$$

Letting $R \to \infty$ in the estimate above, we obtain $\nabla u(x) = \nabla u(0)$ for every $x \in \mathbb{R}^n$, and, as a consequence, $u$ is affine. This finishes the proof of Theorem 1.3. □

***Proof of Corollary 1.5***. Let $u \in C^1(\Omega)$ be a viscosity solution to

$$\mathcal{L}_\infty u = 0 \quad \text{in} \quad \Omega \setminus \{x \in \Omega : u(x) = 0\}.$$

By Theorem 2.3 in [15], $u$ is a viscosity solution in the whole $\Omega$. By Theorem 1.2, $u \in C^\infty(\Omega)$ and it is a classical solution to (10) in the whole $\Omega$. □

***Proof of Corollary 1.6***. The proof is inspired by Proposition 7.1 in [3]. We begin by proving that $\mathcal{L}_\infty U = 0$ in $B_R$ in viscosity sense. It is enough to test the definition of viscosity solutions in points belonging to $\overline{B_R^-}$. We only treat the subsolution case since the supersolution one is analogous. For that, let $x_0 \in \overline{B_R^-}$, $A$ be an open neighborhood of $x_0$, and $\phi \in C^2(B_R)$ such that

$$\phi(x) \geq U(x) \quad \forall\, x \in A, \quad \phi(x_0) = U(x_0). \tag{56}$$

We now divide the proof in two cases as follows.

**Case 1** ($x_0 \in B_R^-$): Let $\delta > 0$ such that $B_\delta(x_0) \subset\subset B_R^-$. Clearly

$$\phi(x) \geq U(x) \quad \forall\, x \in B_\delta(x_0), \quad \phi(x_0) = U(x_0).$$

Now, we put $\mathcal{R}$ the reflection $\mathcal{R}(x', x_n) = (x', -x_n)$ and set $\psi : B_\delta(x_0) \to \mathbb{R}$ by $\psi(x', x_n) = -\phi(\mathcal{R}(x', x_n)) = -\phi(x', -x_n)$. Clearly, $\mathcal{R}(\mathcal{R}(x)) = x$ for all $x \in \mathbb{R}^n$. Now, we claim that $\psi$ satisfies

$$\psi(x) \leq u(x) \quad \forall\, x \in B_\delta(\mathcal{R}(x_0)), \quad \psi(\mathcal{R}(x_0)) = u(\mathcal{R}(x_0)). \tag{57}$$

Indeed,

$$\psi(\mathcal{R}(x_0)) = -\phi(\mathcal{R}(\mathcal{R}(x_0))) = -U(x_0) = -(-u(\mathcal{R}(x_0))) = u(\mathcal{R}(x_0)).$$

Moreover, for all $x \in B_\delta(\mathcal{R}(x_0)) \subset\subset B_R^+$, we have $\mathcal{R}(x) \in \mathcal{R}(B_\delta(\mathcal{R}(x_0))) = B_\delta(x_0)$. This way, for every $x \in B_\delta(\mathcal{R}(x_0))$, we have

$$\psi(x) = -\phi(\mathcal{R}(x)) \leq -U(\mathcal{R}(x)) = -(-u(\mathcal{R}(\mathcal{R}(x)))) = u(x).$$

By (57) and the fact that $u$ is a viscosity solution, an immediate computations show

$$0 \geq \mathcal{L}_\infty \psi(\mathcal{R}(x_0)) = -\mathcal{L}_\infty \phi(x_0).$$

This finishes the proof of Case 1.

**Case 2** ($x_0 \in \partial B_R^- \cap \{x_n = 0\}$): Let $\delta > 0$ such that $B_\delta(x_0) \subset A$. We start observing that

$$\phi(x_0) = 0 \text{ and } \phi(z) \geq U(z) = 0 \quad \forall\, z \in B_\delta(x_0) \cap \{x_n = 0\} =: B'_\delta(x_0).$$

This way, $x_0$ is a local minimum of $\phi$ on $B'_\delta(x_0)$. In particular

$$\Delta_{n-1}\phi(x_0) = \sum_{i=1}^{n-1} \phi_{x_i x_i}(x_0) \geq 0. \tag{58}$$

Moreover, direct computation shows that

$$\Delta_\infty \phi(x_0) = \phi_{x_n x_n}(x_0)(\phi_{x_n}(x_0))^2.$$

From this, we obtain

$$\begin{aligned} \mathcal{L}_\infty \phi(x_0) &= 2\phi_{x_n x_n}(x_0)(\phi_{x_n}(x_0))^2 + \Delta_{n-1}\phi(x_0) + \phi_{x_n x_n}(x_0) \\ &\geq 2\phi_{x_n x_n}(x_0)(\phi_{x_n}(x_0))^2 + \phi_{x_n x_n}(x_0). \end{aligned}$$

In order to conclude the Case 2, it is enough to show that $\phi_{x_n x_n}(x_0) \geq 0$. We now start to do this. Let $0 < t < \delta$. By Taylor's expansion, we have

$$u(x_0 + te_n) = U(x_0 + te_n) \leq \phi(x_0 + te_n) = \phi_{x_n}(x_0)t + \frac{1}{2}\phi_{x_n x_n}(x_0)t^2 + o(t^2). \tag{59}$$

Now

$$U(x_0 - te_n) \leq \phi(x_0 - te_n) = -\phi_{x_n}(x_0)t + \frac{1}{2}\phi_{x_n x_n}(x_0)t^2 + o(t^2). \tag{60}$$

On the other hand, by definition of $U$ and (59), we have

$$U(x_0 - te_n) = -u(x_0 + te_n) \geq -\phi_{x_n}(x_0)t - \frac{1}{2}\phi_{x_n x_n}(x_0)t^2 - o(t^2). \tag{61}$$

From (60) and (61), we obtain

$$\phi_{x_n x_n}(x_0) \geq \frac{o(t^2)}{t^2} = o(1).$$

Letting $t \to 0^+$ in the last inequality, it follows that $\phi_{x_n x_n}(x_0) \geq 0$. This finishes the proof of Case 2.

For the second part, it is enough to prove that $U \in C^1(B_R)$. Clearly, $U \in C^1(B_R^+) \cap C^1(B_R^-)$. Hence, it only remains to prove that $U$ is differentiable and the gradient is continuous on every point belonging to $B'_R$. In order to do that, let us take $x_0 \in B'_R$. Since $u \in C^0_{vfb}(\overline{B_R^+}) \cap C^1(\overline{B_{R-\epsilon}^+})$ for every $0 < \epsilon < R$, we have

$$u(x) = u_{x_n}(x_0)x_n + o(|x - x_0|) \quad \forall\, x \in B_R^+. \tag{62}$$

Thus, if $x \in B_R^-$ then $\mathcal{R}(x) \in B_R^+$. By (62), we have

$$u(\mathcal{R}(x)) = u_{x_n}(x_0)(-x_n) + o(|\mathcal{R}(x) - x_0|) \quad \forall\, x \in B_R^-.$$

As $U(x) = -u(\mathcal{R}(x))$ for $x \in B_R^-$, we have

$$U(x) = u_{x_n}(x_0)x_n + o(|\mathcal{R}(x) - x_0|) \quad \forall\, x \in B_R^-.$$

Since $|\mathcal{R}(x) - x_0| = |x - x_0|$, we obtain

$$U(x) = u_{x_n}(x_0)x_n + o(|x - x_0|) \quad \forall\, x \in B_R^-. \tag{63}$$

From (62) and (63), we arrive at

$$U(x) = u_{x_n}(x_0)x_n + o(|x - x_0|) \quad \text{if } x = (x', x_n) \in B_R,\ x_n \neq 0.$$

From this, the conclusion of the corollary follows easily. □

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