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Separating Path Systems for 2-Degenerate Graphs

(Extended abstract)

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Abstract

We prove that any connected 2-degenerate graph G with n vertices admits a family $\mathcal{P}$ of at most n paths that *strongly separates* $E(G)$, which means that for any pair of distinct edges e, f of G there is a path in $\mathcal{P}$ containing e and not f, and another path containing f and not e. Using this we show that n paths are also enough to strongly separate the edges of any n-vertex outerplanar graph and any connected subcubic graph except for K_4. For bipartite planar graphs with n vertices, we show that there is a separating family of at most $3n/2$ paths.

1 Introduction

Given a collection $\mathcal{P}$ of paths in a graph G, we say that two edges e, f of G are *separated* by $\mathcal{P}$ (and $\mathcal{P}$ *separates* e, f) if there are two paths P_e and P_f in $\mathcal{P}$ such that P_e contains e but not f, and P_f contains f but not e. We say $\mathcal{P}$ is a *strong separating path system* for G if $\mathcal{P}$ separates any pair of edges of G. If for any pair of edges e, f of G there is a path in $\mathcal{P}$ containing only one of e, f, then we say that $\mathcal{P}$ is a *weak separating path system* for G. The sizes of a minimum strong and a minimum weak separating path system for G are denoted, respectively, by $\mathrm{ssp}(G)$ and $\mathrm{wsp}(G)$.

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Falgas-Ravry, Kittipassorn, Korándi, Letzter, and Narayanan [4] conjectured that there are weak separating path systems with $O(n)$ paths for every n-vertex graph, and obtained interesting results for particular cases. Balogh, Csaba, Martin, and Pluhár [1] strengthened the conjecture to also apply to strong separating systems. Both groups observed that the $O(n \log n)$ bound holds for any n-vertex graph. A substantial improvement was obtained by Letzter [9], who proved that $\mathrm{ssp}(G) = O(n \log^* n)$ for any n-vertex graphs G. This conjecture was confirmed by Bonamy, Botler, Dross, Naia, and Skokan [2], who showed that $\mathrm{ssp}(G) \leq 19n$ for any n-vertex graph G.

The bound of $19n$ is not tight and, in fact, it could be true that $\mathrm{ssp}(G) \leq (1 + o(1))n$ for every connected n-vertex graph G, where the connectivity condition is necessary as the graph composed by $n/4$ disjoint copies of K_4 requires $5n/4$ paths to be strongly separated. Recently, the first and fourth authors together with Sanhueza-Matamala [5, 6] proved that $\mathrm{ssp}(K_n) = (1 + o(1))n$ and obtained upper bounds for $\mathrm{ssp}(G)$ for certain n-vertex αn-regular graphs. For example, they showed that $\mathrm{ssp}(K_{n/2,n/2}) \leq (\sqrt{5/2} - 1 + o(1))n$, which is best-possible up to the $o(1)$ term. More recently, the third author together with Stein [7] obtained weak and strong separating path systems for K_n with at most $n+1$ and $n+9$ paths respectively. From now on, we consider only strong separating path systems.

We say a subcubic graph is *strictly subcubic* if it is not cubic. A graph G is 2-*degenerate* if every subgraph of G contains a vertex of degree at most 2. We contribute to this line of research by proving that $\mathrm{ssp}(G) \leq n$ for every 2-degenerate n-vertex graph G. If all components of G contain at least three vertices, we remark that our separating family has the additional property that every edge is contained in exactly two paths, and that every vertex is the endpoint of exactly two paths. The class of 2-degenerate graphs includes, for instance, all connected series-parallel graphs, all connected strictly subcubic graphs, and all connected planar graphs of girth at least 6. Using this result, we also prove that $\mathrm{ssp}(G) \leq n$ for every connected subcubic n-vertex graph G except for K_4, that $\mathrm{ssp}(G) \leq 3n/2$ for every bipartite planar graph G, and that $\mathrm{ssp}(G) \leq 2n$ for every Hamiltonian planar graph (every 4-connected planar graph is Hamiltonian).

2 2-Degenerate graphs

A graph is *planar* if it can be drawn in the plane in such a way that its edges intersect only at their ends. Such a drawing is called a *planar drawing*. An interesting class of planar graphs are the *outerplanar*, which are those that admit a planar drawing for which all of its vertices lie in the outer face. A *k-degenerate graph* G is a graph such that every subgraph of G contains a vertex of degree at most k. Equivalently, there is an ordering of the vertices of G such that any vertex has at most k neighbours that appear earlier in the ordering.

In this section, we prove that $\mathrm{ssp}(G) \leq n$ for every 2-degenerate n-vertex graph G. Then we will obtain upper bounds for outerplanar, subcubic, and bipartite planar graphs. Before going into our results, note that, since $\mathrm{ssp}(G) = \sum_{i=1}^{k} \mathrm{ssp}(G_i)$, where $G_1, \ldots, G_k$ are the connected components of G, if $\mathrm{ssp}(G_i) \leq |V(G_i)|$ for $i = 1, \ldots, k$, then $\mathrm{ssp}(G) \leq n$. Thus, we may restrict attention to connected graphs.

We refer to a path just by its sequence of vertices. That is, for a path with vertex set $\{v_1, \ldots, v_\ell\}$ where $v_i v_{i+1}$ is an edge for every $1 \leq i \leq \ell - 1$, we write $P = (v_1, \ldots, v_\ell)$.

Theorem 2.1. *For every connected* 2*-degenerate n-vertex graph G, we have* $\mathrm{ssp}(G) \leq n$.

In fact, we prove a stronger statement which will be useful for our other results and gives an extra structural property for our strong path system for 2-degenerate graphs.

Theorem 2.2. *For every connected* 2*-degenerate graph G on $n \geq 3$ vertices, there is a strong separating path system for G with n paths and the following two properties: (i) every edge lies in exactly two paths; (ii) there are exactly two paths ending at each vertex of G.*

Proof. The proof is by induction on n. If $n = 3$, then G is either a length-2 path or a triangle. If G is a length-2 path, say (v_1, v_2, v_3), the family of the three possible paths in G, by which we mean $\{(v_1, v_2, v_3), (v_1, v_2), (v_2, v_3)\}$, is a strong separating path system for G satisfying (i) and (ii). If G is a triangle with vertices $\{v_1, v_2, v_3\}$, then our strong separating path system satisfying (i) and (ii) is $\{(v_1, v_2, v_3), (v_2, v_3, v_1), (v_3, v_1, v_2)\}$. For the induction step, consider $n \geq 4$, and assume that the result holds for any connected 2-degenerate graph with fewer than n vertices.

First suppose that G has a vertex v of degree at most two for which $G - v$ is connected. If v has degree one, consider a separating path system $\mathcal{P}'$ of $G - v$ satisfying (i) and (ii), of size $|\mathcal{P}'| = n-1$, and let P_u be one of the paths in $\mathcal{P}'$ that ends at the neighbour u of v. Let P be the path that extends P_u to v. See Figure 1(a). The path family $\mathcal{P} = (\mathcal{P}' \setminus \{P_u\}) \cup \{P, (u, v)\}$ is a path system such that every edge lies in exactly two paths and there are exactly two paths ending at each vertex of G, that is, it satisfies (i) and (ii). It is separating because $e = (u, v)$ is a path in $\mathcal{P}$ and, by induction, any edge in $G - v$ lies in a path that is distinct from P and therefore avoids e. If v has degree two, let u and w be its neighbours, and let $\mathcal{P}'$ be a strong separating path system of $G - v$ satisfying (i) and (ii), of size $|\mathcal{P}'| = n-1$. Let P_u be one of the paths in $\mathcal{P}'$ that ends in u and let P_w be a path in $\mathcal{P}'$ that ends in w with the property that $P_w \neq P_u$. Let P_1 and P_2 be the paths that extend P_u and P_w to v using the edges uv and vw, respectively. See Figure 1(b). The path family $\mathcal{P} = (\mathcal{P}' \setminus \{P_u, P_w\}) \cup \{P_1, P_2, (u, v, w)\}$ is a path system with the property that every edge lies in exactly two paths and there are exactly two paths ending at each vertex of G. Note that P_1 and P_2 strongly separate uv from vw. The edges uv and vw are separated from the remaining edges by (u, v, w). Finally, by induction, any edge e' in G' lies in a path other than P_u (and so corresponds to a path in $\mathcal{P}$ that avoids uv) and lies in a path other than P_w (and so corresponds to a path in $\mathcal{P}$ that avoids vw).

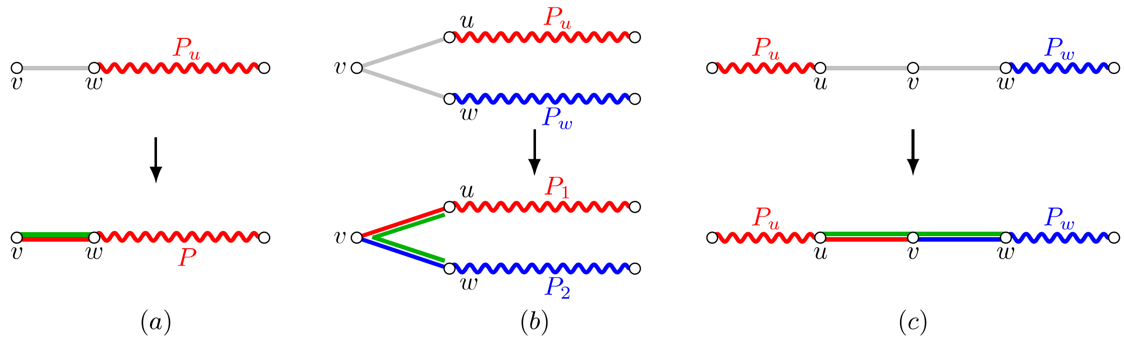

Figure 1: Illustration for the cases in the proof of Theorem 2.2.

Next suppose that the removal of any vertex v in G with degree at most two disconnects the graph. This means that the minimum degree of G is two and that the removal of a vertex v of degree two produces two components G_1 and G_2 of size n_1 and n_2, where $n_1 + n_2 = n - 1$

and $n_1, n_2 \geq 3$. By induction, let $\mathcal{P}_1$ and $\mathcal{P}_2$ be strong separating path systems for G_1 and G_2, respectively, with the required properties. Let u and w be the neighbours of v in G_1 and G_2, respectively, and consider paths P_u in $\mathcal{P}_1$ ending at u and P_w in $\mathcal{P}_2$ ending at w. Let P_1 and P_2 be the paths that extend P_u and P_w to v using the edges uv and vw, respectively. See Figure 1(c). Consider the path family $\mathcal{P} = (\mathcal{P}_1 \cup \mathcal{P}_2 \setminus \{P_u, P_w\}) \cup \{P_1, P_2, (u, v, w)\}$. It is easy to check that it is a strong separating path system for G satisfying properties (i) and (ii). $\square$

Consider a connected 2-degenerate graph G on $n \geq 3$ vertices, with non-neighbours u and v with $d_G(u) = d_G(v) = 2$. If u and v are the last two vertices considered in the induction above, from the last paragraph of the proof, we derive the following.

Fact 2.3. *Let G be a graph on $n \geq 3$ vertices whose components are strictly subcubic. If there are non-neighbours u and v in G with $d_G(u) = d_G(v) = 2$, then there is a strong separating path system for G that contains two length-2 paths, one with internal vertex u and the other with internal vertex v.*

Proof. Let $u, v \in V(G)$ be non-neighbours with $d_G(u) = d_G(v) = 2$. Denote the neighbours of u and v respectively by u_1, u_2 and v_1, v_2, and let G' be the graph obtained by removing u and v. Observe that G' is 2-degenerate because every subgraph of G' is also a subgraph of G. Apply Theorem 2.2 to each component of G' with at least three vertices. Using the obtained path systems, together with trivial path systems for possible components of G' with at most two vertices, one can obtain a strong separating path system $\mathcal{P}'$ for G' with $n-2$ paths such that there are two paths ending at u_1 and two paths ending at u_2. Then, we can take two distinct paths P_{u_1} and P_{u_2} ending respectively at u_1 and u_2. Similarly, let P_{v_1} and P_{v_2} be two distinct paths ending respectively at v_1 and v_2. Then, by extending P_{u_1} and P_{u_2} to u, and P_{v_1} and P_{v_2} to v, we just need to add the length-2 paths (u_1, u, u_2) and (v_1, v, v_2) to obtain the desired strong separating path system. $\square$

3 Corollaries of the result for 2-degenerate graphs

In this section we obtain some results that follow from the results in Section 2.

Theorem 3.1. *If G is a connected cubic n-vertex graph and it is not K_4, then $\operatorname{ssp}(G) \leq n$.*

Proof. As K_4 is the only connected cubic graph in which every edge lies in a triangle, we can take an edge $e = uv$ of G that belongs to no triangle. Then, each component of the graph $H = G - e$ is strictly subcubic, and u and v have disjoint neighbourhoods in H, which are denoted $\{u_1, u_2\}$ and $\{v_1, v_2\}$, respectively. From Fact 2.3, there is a strong separating path system $\mathcal{P}'$ for H with n paths that contains two paths, P_u and P_v, of length 2 with internal vertex u and v, respectively. Our aim is to show that these paths of length 2 may be re-routed as paths of length 3 using e in a way that preserves strong separation. See Figure 2. Recall that $\mathcal{P}'$ may be chosen so that every edge of H lies in exactly two paths and every vertex of H is the end of exactly two paths. Consider the two paths P_1 and P_2 starting at u, and assume without loss of generality that P_1 uses the edge uu_1 and P_2 uses the edge uu_2. Let Q_1 and Q_2 be the corresponding paths for v. If $P_1 = Q_1$, then $P_1 \neq Q_2$ and $P_2 \neq Q_1$. In this case, interchange Q_1 and Q_2 so that $P_1 \neq Q_1$ and $P_2 \neq Q_2$. Consider the path system $\mathcal{P} = \mathcal{P}' \setminus \{P_u, P_v\} \cup \{(u_1, u, v, v_1), (u_2, u, v, v_2)\}$. We claim that $\mathcal{P}$ is strong separating for G. It is easy to see that uv is strongly separated from any edge of H. Moreover, any two edges that were strongly separated in H are clearly strongly separated by the same paths (or by the paths that replaced them), except for the pairs $\{uu_1, vv_1\}$ and $\{uu_2, vv_2\}$, which were separated by P_u and P_v in $\mathcal{P}'$, but lie on the same new paths. However, our choice of re-routing guarantees that the pair $\{uu_1, vv_1\}$ is strongly separated by P_1 and Q_1, while the pair $\{uu_2, vv_2\}$ is strongly separated by P_2 and Q_2. □

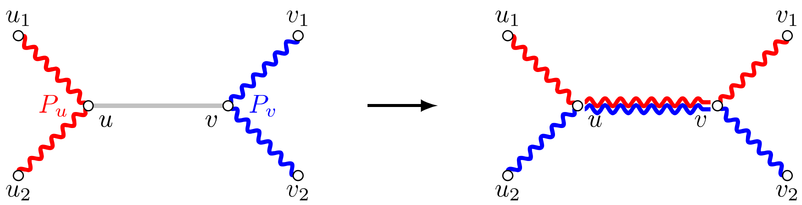

Figure 2: Illustration for the proof of Theorem 3.1.

We obtain the following direct corollary from Theorem 3.1.

Corollary 3.2. *If G is a subcubic n-vertex graph with k connected components isomorphic to K_4, then* $\mathrm{ssp}(G) \leq n + k$.

Proof. Since K_4 requires 5 paths to be strongly separated, we need $5k$ paths for the components isomorphic to K_4. Furthermore, in view of Theorem 3.1, we can strongly separate the remaining edges with $n - 4k$ paths, for a total of $5k + (n - 4k) = n + k$ paths. □

Outerplanar graphs always contain a vertex of degree at most 2, and their class is closed under (induced) subgraphs, so they are 2-degenerate. Therefore, the following is a direct corollary of Theorem 2.1.

Corollary 3.3. *If G is an outerplanar n-vertex graph, then* $\mathrm{ssp}(G) \leq n$.

The well-known $(4, 3)$-Conjecture by Chartrand, Geller, and Hedetniemi [3] states that every planar graph has an edge partition into two outerplanar graphs. This is known to hold for Hamiltonian planar graphs: one possible partition is the edges on the inside of a Hamilton cycle versus the edges on the outside (the edges on the cycle may be added to any of the parts). Tutte [10] proved that every 4-connected planar graph is Hamiltonian, so the $(4, 3)$-Conjecture holds for 4-connected planar graphs.

Corollary 3.4. *If G is a planar n-vertex graph that satisfies the $(4, 3)$-Conjecture, then* $\mathrm{ssp}(G) \leq 2n$.

Proof. Let $G = (V, E)$. Then its edge set may be split as $E = E_1 \cup E_2$, where $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ are outerplanar. By Corollary 3.3, there are strong separating path systems $\mathcal{P}_1$ and $\mathcal{P}_2$ of size n for H_1 and H_2, respectively. The path system $\mathcal{P}_1 \cup \mathcal{P}_2$ is clearly a strong separating path system for G. □

From [8], in every bipartite planar n-vertex graph, there are at most $n/2$ edges whose removal results in a 2-degenerate graph. This and Theorem 2.1 imply the following.

Corollary 3.5. *If G is a bipartite planar n-vertex graph, then* $\mathrm{ssp}(G) \leq 3n/2$.

Krisam [8] conjectured that, in every bipartite planar n-vertex graph G, the removal of $n/4$ edges would be enough to obtain a 2-degenerate graph, which would imply that $\mathrm{ssp}(G) \leq 5n/4$.

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